Strategies for Assessing Mathematical Knowledge for Teaching in Mathematics Content Courses

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Abstract: *In their practice, teachers must not only know how to solve mathematics problems; they must also be able to make sense of students' mathematical thinking, understand the organization and intent of curricular materials, and select contexts to motivate and highlight mathematical ideas. Similarly, mathematics content courses for prospective teachers (PTs) should not only seek to convey mathematical content; they should prepare PTs to use mathematical knowledge in ways that enhance school teaching and learning of the subject. Accordingly, mathematics teacher educators (MTEs) should assess not only the mathematics that PTs know but also whether this mathematical knowledge is organized in ways that are likely to support their teaching. In this article, we present some of the existing research on the assessment of mathematical knowledge for teaching and discuss ways in which MTEs can draw upon the work of elementary school teaching to help assess PTs' content knowledge and habits of mind. These include assessments that focus on using representations that occur in elementary textbooks, building mathematical arguments, selecting problems to bring out important ideas, and making sense of students' thinking.*

Keywords: non-traditional assessment, feedback, preservice elementary teachers, content courses, mathematical practices, mathematical knowledge for teaching

Introduction

In teacher education research, the term *mathematical knowledge for teaching* (MKT) describes understandings and skills that teachers use to facilitate students' access to mathematical ideas. Research has identified several empirically distinct subdomains of MKT (Figure 1) that are different from *common content knowledge* (CCK), the mathematical knowledge that adults are typically expected to have after completing school (Ball, Thames, &

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Phelps, 2008; Hill, Ball, & Schilling, 2008). For example, *specialized content knowledge* (SCK) consists of mathematical knowledge frequently deployed in teaching mathematics, but not typically used outside of this work. Although a well-defined decomposition of SCK remains elusive (Hoover, Mosvold, Ball, & Lai, 2016), examples of SCK include knowing distinctions among different models for operations, determining the conceptual origins of student errors, deciding whether a proposed solution approach for a class of problems will work in general, and identifying relationships among similar problems (Ball et al., 2008; Bair & Rich, 2011). *Knowledge at the mathematical horizon* refers to teachers' sense of the mathematical surroundings of the content they teach, as well as their understanding of major ideas and practices essential to the mathematics discipline (Ball & Bass, 2009; Zazkis & Mamolo, 2011). Mathematical knowledge for teaching also includes knowledge domains pertaining to how students learn mathematical ideas and how these ideas can be presented productively in classroom settings; these are characterized as *knowledge of content and students* (KCS) and *knowledge of content and teaching* (KCT), respectively. *Knowledge of curriculum* is also identified as a key component of MKT (Ball et al., 2008).

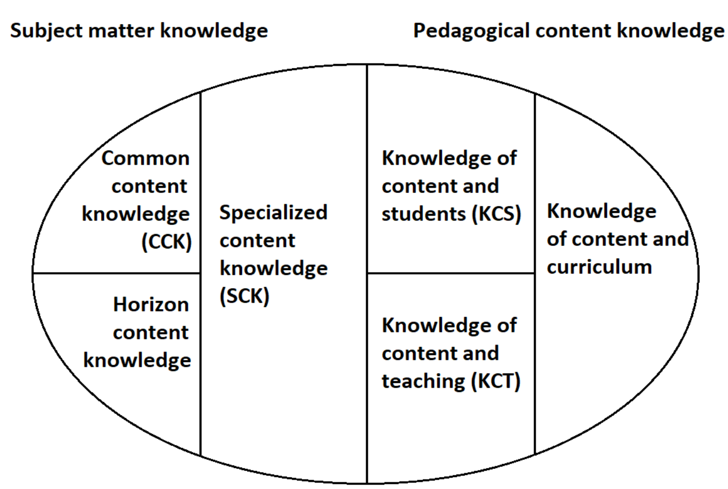


Figure 1. Subdomains of mathematical knowledge for teaching (MKT). Adapted from Hill, Ball, & Schilling (2008, p. 377).

As mathematics teacher educators (MTEs), we are interested in developing prospective teachers' understanding of the mathematical content they will teach as well as specialized mathematical and pedagogical knowledge useful in the everyday practice of teaching. In this article, we share some strategies for assessing facets of prospective (elementary) teachers' (PTs') emerging MKT that are distinct from common content knowledge. We focus on four specific examples of mathematical knowledge that we believe are useful in mathematics teaching: interpreting and using representations that appear in elementary mathematics curricula, building and critiquing arguments, analyzing the mathematical structure of problems that might be used in the classroom, and analyzing students' mathematical thinking. We do not claim that this is an exhaustive list of parts of the work of teaching that draw upon teachers' mathematical knowledge; rather, we choose to focus on these four because each uses mathematical content knowledge in a distinctive way, and because the four together represent a range of teaching practices from thinking about how to present material to students, to selecting specific problems that students will encounter, to making sense of their work on these problems. Moreover, these parts of the work of teaching draw from different subdomains of MKT: using representations and analyzing problems have both been cited as examples of tasks that use SCK (Bair & Rich, 2011; Hill, Schilling, & Ball, 2004), while making inferences about students' mathematical thinking from their work draws upon KCS. Mathematical practices such as argumentation have been characterized as part of knowledge at the mathematical horizon (Ball & Bass, 2009).

Assessments are used in a variety of ways and for a variety of purposes: to motivate learners and focus their attention on what is important, to provide feedback about their thinking, to identify what understandings and ideas might be within their zone of proximal development, and to gauge the effectiveness of teaching and identify parts of lessons that may need improvement (Stiggins, 2004). As MTEs, we often use assessment in mathematics content courses for PTs to encourage the development of their MKT, which includes not only knowledge of the mathematics content taught in elementary school, but also the mathematical habits of mind that help teachers plan and deliver instruction and understand how students grapple with novel ideas. Consequently, attention to the ways we expect teachers to engage in *doing* mathematics in their practice can benefit our assessment of PTs' MKT.

In this article we outline some general principles from the literature on assessment and discuss how these might apply to the assessment of PTs' MKT in mathematics content courses. We then offer specific strategies – both from research on elementary teacher education and ones that we have tested in our own classrooms – for assessing PTs' MKT. Some of these strategies for assessment parallel the recommendations for developing PTs' MKT offered in Kuennen and Beam (2020). While our discussion focuses on content courses, we emphasize that assessment of MKT can (and should) also occur in other contexts, such as methods courses and field experiences, as these contexts offer different windows into the knowledge that PTs use for developing their professional practice. Our focus on content courses is particularly strategic because content courses, in contrast to other courses that comprise teacher preparation programs, are most likely to resemble conventional college-level mathematics classes. Furthermore, because the "apprenticeship of observation" – PTs' long history of observing the work of teaching from the perspective of students – has a strong pull on the practices of novice teachers (Lortie, 1975; Borg, 2004), we believe that content courses offer a powerful opportunity to model evidence-based assessment practices.

Assessment: General Principles and Best Practices

Assessment should not merely serve the evaluation purpose (summative); it should serve as a foundation for ongoing learning for both PTs and MTEs (formative/informative). For PTs, assessment can provide indicators of areas that deserve further study. For MTEs, assessment can help determine whether lessons are helping PTs develop desired understandings and habits of mind and suggest directions for instructional improvement.

Summative assessment is used to measure and report on students' performance, often at the end of a unit or course. On the other hand, formative assessment is a process in which instructors use students' work on tasks to gain information about students' progress that can be used for instructional planning, while students receive feedback that can help them make strategic progress toward learning goals (Heritage, 2007; Sadler, 1989). Formative assessments may be different not only in timing but also in kind from summative assessments (Harlen & James, 1997); what works well as a formative assessment task may not be appropriate for summative assessment.

Because students tend to focus on what is assessed, assessments need to be of sufficient frequency and duration to engage them in challenging intellectual activity for a substantial amount of time (Gibbs & Simpson, 2005). The effectiveness of formative assessment in supporting learning depends upon the quantity and quality of assessment tasks used and the timeliness and actionability of feedback provided. For formative assessment to realize its potential to significantly boost academic achievement, instructors must use assessment as a foundation for decision-making, rather than relying solely on preconceived notions of students' learning trajectories (Black & Wiliam, 1998). For assessments to support robust understanding, they must target conceptual understanding rather than superficial and rote learning (Davis, 1992;

Hiebert & Lefevre, 1986); and instructors must shift their focus from grading the assessment to learning from assessment and learning overall. For example, number talks (Harris, 2011; Humphreys & Parker, 2015) offer a great formative assessment tool that can focus attention on conceptual understanding and provide information for instructors on *how* students are thinking.

Feedback on assessment tasks can assume a variety of forms and serve different purposes. Often, in mathematics courses, students receive feedback in the form of grades; however, these can shift their focus away from learning and undermine both interest and academic performance. Instead, feedback should be focused on attributes of their performance, timely enough to allow them to process the information while they still remember their work, and specific enough to support them in making adjustments. Feedback about the processes, strategies, and self-regulatory mechanisms that students can employ are more powerful than feedback about the outcomes of specific tasks (Hattie & Timperley, 2007). Therefore, feedback in content courses should regularly focus on learning processes and self-monitoring strategies, such as checking for errors, checking the reasonableness of answers, and adjusting the problem-solving approach when initial efforts do not succeed.

Assessment and Mathematical Knowledge for Teaching

Elliott and colleagues (2009) suggest reinforcing the relevance of mathematical activities in teacher education settings by invoking and developing specialized content knowledge (SCK), because teachers are aware of the necessity and relevance of specialized content knowledge in the work of teaching. Similarly, we argue that invoking aspects of the work of teaching in content courses can heighten the relevance of tasks for PTs, particularly because many PTs take mathematics content courses prior to completing their student teaching and fieldwork, and are enthusiastic about learning how they will use their knowledge in their future classrooms.

Assessments can place PTs in situations that simulate specific tasks of teaching, such as presenting an example problem, responding to a student error, or illustrating an alternative solution approach.

When constructing assessment tasks that draw upon teaching situations, MTEs should consider possible roles that the teaching context of a task can play in assessment. In well-designed tasks, a description of a teaching situation can focus PTs on facets of mathematical knowledge relevant to the context (Phelps & Howell, 2016). For example, a task might present a list of problems and ask which would target a specific mathematical conception. In this case, the teaching context serves to direct the PTs to determine which problem has the most suitable mathematical structure for addressing the specified concept, rather than focusing on other considerations such as problem difficulty, readability, or relevance to students' lives and interests. Such framing helps to maintain the focus of assessment on PTs' application of content knowledge to teaching situations.

Some textbooks used in mathematics content courses (such as Bassarear & Moss, 2016; Beam et al., 2018; Beckmann, 2018) incorporate "teaching situations" into course activities; these texts use mathematical representations and artifacts of student thinking to situate mathematics tasks in the work of teaching, providing opportunities for PTs to develop MKT (Lai & Patterson, 2017). However, even without these resources, MTEs can develop and assess PTs' MKT. A useful starting point is to consider the following questions, which have served as catalysts for our own thinking about developing MKT in content courses, and lead to the four foci for assessment discussed in the subsequent sections of this article.

1. How is a topic likely to be presented in an elementary classroom? What verbal, symbolic, and visual representations appear in elementary school textbooks used to teach this topic?

2. Beyond knowledge of topics in the elementary mathematics curriculum, what mathematical dispositions and habits of mind do PTs need to be able to formulate classroom-appropriate mathematical arguments and support students in developing arguments of their own?
3. What problems are likely to be propitious for surfacing students' conceptions about this topic? What behind-the-scenes mathematical work do PTs need to do to select these problems?
4. What ideas and strategies occur in students' work, and what mathematical work do PTs need to do to make sense of these?

Assessing Mathematical Processes and Practices

Regardless of how many mathematics content courses PTs take during their teacher preparation program, they will not be able to study all the mathematics that they will encounter in their teaching. Therefore, in addition to gaining content knowledge such as mathematical concepts, procedures, and representations, PTs also need to gain independence and confidence as mathematics learners. Teacher education programs can do this by helping PTs acquire *mathematical practices*, which we define as the mathematical community's ways of accessing, understanding, and developing mathematical ideas. Such practices include the ways we unpack mathematical arguments, solve problems, validate results, generate and represent new ideas, and communicate our thinking. These practices may enable PTs to relearn mathematical concepts and procedures that they do not already understand or may have forgotten (Bernander, Szydlik, & Seaman, 2020; Seaman & Szydlik, 2007). Furthermore, the Association of Mathematics Teacher Educators' *Standards for Preparing Teachers of Mathematics* states:

> Well-prepared beginning teachers of mathematics have solid and flexible knowledge of mathematical processes and practices, recognizing that these are tools used to solve

> problems and communicate ideas. … The mathematical knowledge of well-prepared beginning teachers of mathematics includes ability to use mathematical and statistical processes and practices to solve problems. (AMTE, 2017, p. 9)

The importance of developing mathematical practices is further highlighted by the fact that leading US educational organizations and school standards mandate that students develop such practices. In *Principles and Standards for School Mathematics* (NCTM, 2000), the National Council of Teachers of Mathematics describes 5 practices: problem solving, reasoning & proof, communication, connections, and representations. Additionally, the Common Core State Standards for Mathematics (NGA & CCSSO, 2010) identify eight standards for mathematical practice: make sense of problems and persevere in solving them, reason abstractly and quantitatively, construct viable arguments and critique the reasoning of others, model with mathematics, use appropriate tools strategically, attend to precision, look for and make use of structure, and look for and express regularity in repeated reasoning. As *The Mathematical Education of Teachers II* (CBMS, 2012) states, "...although those standards were written for K-12 students, they apply to all who do mathematics, including elementary teachers." (p. 24). PTs need to develop these practices, because they must understand and develop them in their own classrooms.

Assessing PTs' mathematical practices is essential because PTs tend to place value, effort, and attention primarily on things that are graded (Gibbs & Simpson, 2005). It is therefore critical that MTEs not only work on incorporating mathematical practices into classroom activities and discussions, but also make them an ongoing focus of meaningful formative and summative assessment. All learning is contextual, so for mathematical practices to become independent of specific content, they must be developed consistently throughout PTs' experiences in various contexts (Schoenfeld, 1992).

We would recommend that MTEs who want to begin assessing practices first focus on the practices of problem solving and justification (proof) because of their importance to the mathematical community (Ross, 1998, Tall, 1998 Wu, 1996) and their relation to how people view what constitutes mathematics. They can then add additional practices based on trends in their PT's needs. When assessing a mathematical practice, it is valuable to first consider what proficiency looks like for that practice. As we assess our own PTs' mathematical practices, we ask ourselves, "How will I recognize this practice? What will I observe?" We then create situations and problems in which PTs will need to engage in that practice.

In the following sections, we share strategies for assessing how PTs use representations, build and critique arguments, select problems based on their mathematical structure, and analyze student thinking. Many of these strategies have evolved over the course of our own work as MTEs. Two of the authors of this article teach at a regional state university that offers eight credit hours of mathematics content for PTs, taught in three courses which focus on number systems, geometry, probability, and data analysis. Courses are problem-based and student-driven, and practices are explicitly discussed and integrated throughout content coverage. PTs are typically in the first or second year of their program, and separate courses on methodology are provided later in their programs in the university's college of education.

Throughout these sections we use the word "task" to refer to a summative or formative assessment that might be used in a content course for PTs. We use the word "problem" for a problem or question that might be posed to elementary school students.

Interpreting and Using Representations that Appear in Elementary Mathematics Curricula

Part of the work of teaching elementary mathematics is to select and generate appropriate representations in response to student questions and confusion (Ball, 1990). Research on cognitively guided instruction has suggested that teachers support students in developing their own representations of mathematical ideas (Carpenter, Fennema, & Franke, 1996). Accordingly, content courses should provide frequent opportunities to make sense of representations commonly used in the elementary classroom, as well as of representations that students might produce. When possible, PTs should be asked to interpret as well as produce instances of these representations and use them to solve problems. For example, an assessment of PTs' use of representations for fraction addition might ask them to show how to use a number line diagram to compute the sum 3/4 + 5/8. To correctly assess PTs' understanding of how to represent addition on the number line, task instructions should emphasize the importance of using the number line representation to perform the addition, rather than merely drawing a picture of the two addends on the number line and then using a symbolic method to add them.

In addition to assessing fluency in using representations as tools for problem solving, tasks can assess understanding of important features and complexities of the representations themselves. For example, an assessment task on the meaning of the fraction *a*/*b* might give a diagram such as the one in Figure 2 and ask PTs to explain how the diagram could be interpreted as representing each of the fractions 5/8, 5/4, and 5/2. This task has an important mathematical purpose: assessing whether PTs can demonstrate flexibility in their choice of unit (whole) when discussing a fraction problem. This flexibility with respect to units becomes particularly important when teachers address division of fractions; for example, when one divides 3 by 1/4, it is useful to think of the division as determining how many "units" are in 3 when we consider 1/4 to be one unit. Additionally, the task helps to challenge and broaden the "part-whole" interpretation that predominates many US teachers' understanding of fractions (Moseley, Okamoto, & Ishida, 2007); since the diagram shows five pieces shaded and allows PTs to vary the value of each piece, the task encourages the development of the interpretation of *a*/*b* as "*a* pieces of size 1/*b*" called for by the Common Core standards (NGA & CCSSO, 2010). While the task addresses important mathematical content knowledge, it also addresses PTs' understanding that representations can convey different information depending on different starting assumptions. In this case, PTs must grapple with the fact that visual fraction diagrams are ambiguous prior to the explicit selection of a unit.

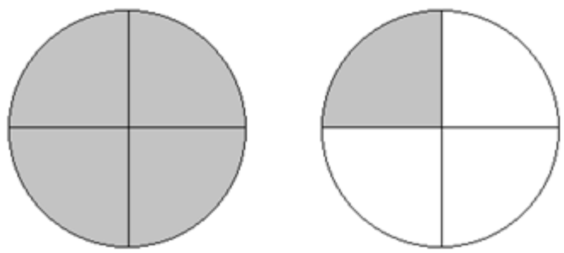


Figure 2. A visual representation of a fraction with an unspecified unit.

In addition to assessing PTs' skill in using representations to solve problems, content courses should assess their proficiency at using classroom-appropriate representations to convey information, such as definitions of mathematical terms and their logical implications. We have assisted with the development of a formative assessment activity in which PTs read mathematical definitions of the terms *parallelogram*, *rectangle*, *rhombus*, and *square*, and use these definitions to create Venn diagrams that correctly represent inclusion relationships among these classes of quadrilaterals. They then use the definitions and the Venn diagrams to complete statements such as "A square is _____ a rectangle" with the words "always," "sometimes," or "never." We include this activity in our course because Venn diagrams are used as representations of set inclusion relationships in some geometry curricula (Gavin, 2001; Kimmins & Winters, 2015); we wish to ensure that PTs are equipped to interpret these visual representations and present them to students with clarity.

Prospective teachers work on this activity in small groups, so that MTEs can observe how PTs resolve mathematical disagreements as well as how they translate definitions into other representations. We observed that while PTs frequently answer "always-sometimes-never" questions correctly, they often have difficulty representing inclusion relationships correctly in a Venn diagram that follows standard mathematical conventions (each quadrilateral should fit in a unique region of the diagram; each region of the diagram should represent a nonempty set of quadrilaterals), as illustrated by the diagram on the left in Figure 3. In some cases, they reassess their Venn diagrams and find that they are inconsistent with their answers to the "always-sometimes-never" questions, leading to revision of the diagrams, as shown in the diagram on the right in Figure 3. This suggests that while the task addresses common content knowledge by assessing PTs' understanding of inclusion relationships among classes of quadrilaterals, which marks the "order" or "informal deduction" level in van Hiele's hierarchy of geometric thinking (Crowley, 1987; Usiskin, 1982; van Hiele & van Hiele-Geldof, 1958), the process of representing definitions and inclusion relationships visually is a distinct skill. We claim that this skill is integral to the work of elementary mathematics teaching because it enables teachers to present information in a format that is concise and avoids some of the limitations of verbal explanations. The quadrilateral Venn diagram activity provides an opportunity to assess this skill; and as PTs' revisions of Venn diagrams illustrate, it may even include opportunities for self-assessment.

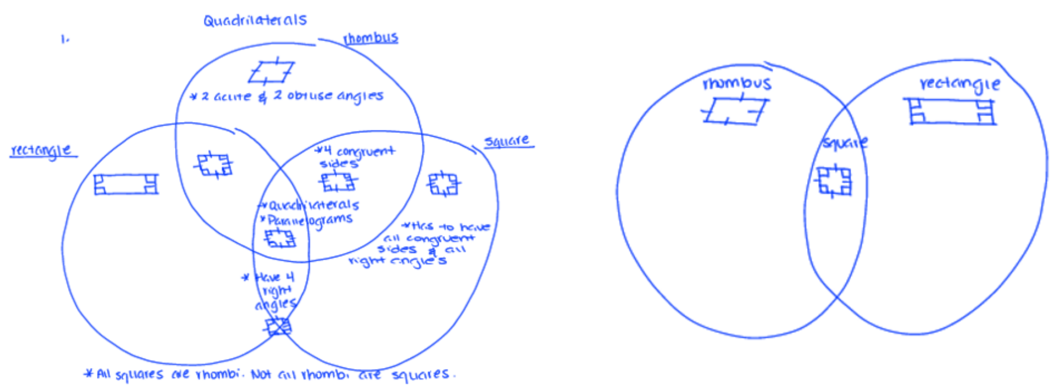


Figure 3. PTs' group work on the quadrilateral Venn diagram task, before and after revision.

Building and Critiquing Arguments

Proof is arguably the most highly valued activity in the mathematical community, serving as the primary method of determining and communicating whether claims are true or false. Because mathematical truths are established by logical reasoning, not authority (Harel & Sowder, 1998), making and critiquing mathematical arguments are critical practices for teachers to develop. Proving is, however, very complicated, involving many different practices, knowledge, and skills; acknowledging this, researchers have advocated that teachers provide a continuum of practices that balance honest representation of mathematical practices with students' current level of preparation and understanding (Stylianides, 2007).

Primarily, a proof consists of a logical, deductive argument that uses what is known to be true to convince the community that the result is correct. While there are many ways to decompose the practice of mathematical argumentation into practices that would build toward proof, we assess proficiency using these benchmarks: a) understanding the difference between explaining *why* a result is correct versus *how* it was found, b) recognizing when general arguments are needed versus when an example or counterexample will suffice, c) incorporating correct assumptions and definitions to argue a point, d) identifying what criteria would be sufficient to argue the claim, and e) providing sufficient details to convince a reasonable skeptic.

One way to assess the practice of making mathematical arguments is to have PTs solve a novel problem and submit a formal solution called a "write-up." We use the phrase "novel problem" to describe a problem for which significant reasoning and justification are needed, and that we consider unlikely to be familiar to most of the PTs we teach. In our write-ups we require that PTs include four distinct sections: a description of the problem, the strategies used to solve the problem, the solution to the problem, and an argument for why the solution makes sense. In many cases, PTs do not make the distinction between describing *how* they solved a problem and justifying *why* the solution is correct; when asked to justify *why*, they often appeal to authority (e.g., "the textbook tells us to use this formula") rather than to deductive reasoning (Simon & Blume, 1996). Asking them to provide separate sections on explanation of strategy and justification both encourages attention to this important distinction and reveals their understanding of justification.

A sample write-up and examples of PT justifications are given below. The write-up problem is typical of those presented early in the first content course that PTs take. The responses are common responses we have observed.

Problem:

Suppose that you have a bunch of red blocks and a bunch of white blocks. How many different towers can you make that are three blocks high? (For example, one possible tower could have red on top, white in the middle, and red on the bottom.)

Directions:

Work on and solve the problem individually, then submit a write-up with the following (clearly labelled) parts:

1. A paragraph explaining and clarifying the problem. (Convince me you understand the question and define ambiguous terms or notation.)
2. A paragraph or two reflecting on your problem-solving strategies and how you made decisions about how to approach the problem. (How did you initially attack the problem? What types of things did you do to gain insights, even if they did not pan out? Any conjectures you made along the way should be discussed as well as any data gathered, tables created, or sketches made.)
3. The solution to the problem.
4. An explanation of why the solution is correct. (Why does your solution make sense mathematically? Argue that it is a complete solution, i.e. there are no other solutions, and prove it is correct.)

The following are PTs' justifications for why there are exactly 8 possible towers:

A: "When you think about it, you have 2 colors. When you start you can make 4 towers with the 2 colors. Then if you do the opposite of those towers, you have 4 more so this could be represented as 2x4. 2 colors times 4 towers, which would then equal 8 towers."

B: "R can be in the top middle or bottom with W filling in the other slots to make 3. Or W can be in the top middle or bottom with R filling in the other slots. It can also be RR on the top or bottom with W filling in the last slot. Or WW on the top or bottom with R filling in the rest. Or simply just all one color. I went through all the possible outcomes and it equaled out to 8 possibilities."

C: "The solution to the tower problem is correct because if you put it into the formula 3x2= 6+2= 8. The answer is 8 because with all the possibilities listed out and then drawn out I received the answer of 8 towers."

As discussed above, one of the critical benchmarks that we want PTs to attain is to distinguish between a description of *how* one solves a problem and an explanation of *why* the solution is correct. Our approach in cultivating PTs' skill at explaining and justifying mathematical solutions is informed by our experience as content course instructors; Hallman-Thrasher, Rhodes, and Schultz (2020) offer some additional strategies for helping PTs learn about attributes of sound mathematical explanations and gain confidence in constructing them.

In the case of the towers task above, response A justifies its finding by merely reiterating the approach used in solving the problem. A solid argument could be fashioned from her description using mathematical relationships; however, the PT does not provide this justification. We would point out to this PT that their description is telling us about the process they used to come up with the answer 8. Our feedback for the PT would look something like this, "I am not convinced that you have found all the possibilities. How do you know that all 8 of your towers are different? Why are there only 8? How do you know you found them all?"

We also expect PTs to make claims and back them using information drawn from the context. The explanation in response B partially draws from the problem's context. It attempts to argue that all possibilities have been exhausted but does not recognize the overlap in the cases described, nor does it provide a justification that these are the only options. Our feedback to this student would be similar to that provided for student A: "This is a nice, systematic approach. What I don't see is a connection between your approach and the number of towers. How does that approach result in 8 towers and no more? As you do this, be careful. Some of the towers are listed twice in your description."

Finally, we want PTs to recognize holes and unjustified claims in their arguments. The lack of this recognition is illustrated in response C. While the response attempts to justify the finding that there are 8 possible towers, the argument is based solely on a formula that is not explained. Our feedback to this PT would state, "You have an interesting formula here, but it's unclear as to why it would apply. Your argument expects the reader to take your word for it. When making mathematical arguments, aim for convincing a skeptic, someone who won't just take your word for it. You might consider addressing questions like: Where does the 3 come from? Why is it 3? What is the difference between the two in 2x3 and the 2 that you are adding onto that? Each of these 2s represents something different, you need to explain this." We note that we would also comment on response C's nonstandard use of the equals sign; this would address the mathematical practice of attending to precision.

In addition to constructing their own mathematical arguments, PTs benefit from experience with analyzing and critiquing arguments developed by others, potentially including some constructed by elementary students (Max & Welder, 2020). To offer PTs opportunities to make sense of arguments constructed by others, we frequently ask PTs to edit peers' solutions. Here, PTs evaluate and provide constructive feedback on problem write-ups prior to instructor evaluation. When assessing their ability to critique others' reasoning, we observe whether PTs understand the other person's perspective, and then directly address the other person's reasoning, explaining why it is or is not correct and offering specific suggestions for improvement, not just providing an alternative approach (often their own). We have found that this works best if they are given specific questions to answer, such as:

- Describe two aspects of the write-up that were done well.
- Give two specific suggestions to improve the write-up.
- Written Communication: Is the writing communicated clearly in a well-organized manner?
- Precision: Are the definitions used appropriately? Is the deductive argument complete and easy to follow? Are all diagrams accurate and readable?
- Make Sense of Problems: Does the description of the problem clearly indicate that the author has made sense of the problem? Does the description include all relevant definitions and techniques used?
- Problem Solving: Does the description of the problem-solving strategies clearly describe the problem-solving process?
- Justification: Is the solution complete? Is the solution correct? Does the solution clearly justify why the answer makes sense?

In general, we find that peer assessment has several advantages. First, it increases the frequency with which PTs receive feedback on their work. Second, it provides opportunities for them to see and experience different perspectives as well as effective and ineffective arguments. Third, it provides opportunities to practice making sense of unpolished mathematical reasoning, a common task of teaching in student-centered classrooms. Finally, it gives PTs some agency in the assessment process, shifting mathematical authority and responsibility toward them as future teachers.

We provide students with feedback on these issues through written comments or through a rubric such as the one shown in Table 1:

Table 1

Rubric for Problem Write-Ups

Practice	Proficient Understanding	Almost Proficient	Basic Understanding	Not Yet	Unevidenced
Making Mathematical Arguments	Argument provides clear and complete justification by using sound logic, correct assumptions, and an appropriate level of detail. It would convince a reasonable skeptic.	Argument provides sound justification, with only minor errors or missing details, still convincing a reasonable skeptic.	Argument provides some critical justification, but may have significant errors, gaps, or oversights. May convince a friend, but not a skeptic.	Attempts to convince without using appropriate justification (e.g. only showing computations, telling how the problem was solved, giving examples).	Problem not solved or did not attempt to justify the solution.
Attending to Precision	Work and explanations are clear and show an appropriate level of accuracy, including the correct and accurate use of terminology, definitions, diagrams, and calculations (as appropriate).	Language, definitions, calculations, and diagrams are accurate, with only minor flaws that have little impact on the clarity or accuracy of the work.	Work or explanations have a few errors or inaccuracies impacting, while other critical parts of the work are correct and accurate.	Major flaws or inaccuracies in the work severe enough to stop important progress or even suggest a lack of essential understanding of terminology, methods, or representations.	Problem not attempted.
Problem Solving	Effective strategies used and described which led to a full and complete solution of the problem.	Effective strategies used or described, leading to a correct solution, with only a few minor oversights or omissions.	Specific strategies used or described but only partially leading to important insights and progress on the problem.	Work and explanations show ineffective approaches with little to no progress or an incomplete understanding of the problem.	Problem not attempted.
Analyzing Others' Arguments	Critique shows a full understanding and careful analysis of others' reasoning by posing useful comments or questions, justifying the solution or approach, and identifying any flaws.	Critique shows a full understanding of others' reasoning, including any critical flaws; though it might not fully justify or correct it.	Critique shows understanding of parts of others' reasoning, but fails to understand and address critical parts, such as the approach's ability or flaws.	Critique shows an unsuccessful attempt to understand others' work or arguments.	No critique provided.

Analyzing the Mathematical Structure of Problems

Elementary mathematics instruction is most successful in developing students' persistence and problem-solving skills when teachers select problems that build on students' prior knowledge (Anderson, 1989; Stein, Grover, & Henningsen, 1996). The skill of selecting problems that develop key mathematical ideas while building on students' prior understandings is an important, yet often invisible, part of mathematical knowledge for teaching. Through assessments of PTs' problem selection skills, MTEs can make this knowledge more visible to PTs and stimulate discussion of mathematical concepts in ways that feel relevant to them.

For example, a formative assessment task for PTs, modeled after a task developed by Hill et al. (2008, p. 400), might ask them to select one of several decimal comparison questions to highlight the relative size difference between hundredths and thousandths (Figure 4). While discussions of problem-selection exercises like these often invoke knowledge of content and students (KCS) due to considerations of what students are likely to do or understand, they also present opportunities to assess how PTs analyze the mathematical structure of problems and select problems that are likely to build on students' prior understandings, which we claim are components of specialized content knowledge (SCK). For example, PTs might point out that while the second comparison question can be solved simply by comparing the digits in each place and the third can be solved by comparing digits in the thousandths place, the first and fourth questions call for thinking about the size comparison between hundredths and thousandths. For the fourth, students might reason that while 0.823 contains two extra thousandths, 0.831 contains one extra hundredth, which is equivalent to ten thousandths; thus 0.831 is greater. A similar argument could be used on the first comparison question. This problem-selection exercise also presents an opportunity to assess whether PTs anticipate that a student might compare 0.823 and 0.831 simply by comparing the numerals to the right of the decimal points as though they represent whole numbers, and thus argue that the first is more likely to assess the targeted proficiency.

Replace each question mark (?) with one of the symbols <, =, or >.

(1) 0.438 ? 0.44

(2) 0.315 ? 0.436

(3) 0.679 ? 0.675

(4) 0.823 ? 0.831

Figure 4. Four decimal comparison questions.

Tasks like the following from Ball et al. (2008) that ask PTs to select problems can assess whether PTs understand models for operations.

Which of the following story problems can be used to represent 1¼ divided by ½?

a) You want to split 1¼ pies evenly between two families. How much should each family get?

b) You have $1.25 and may soon double your money. How much money would you end up with?

c) You are making some homemade taffy and the recipe calls for 1¼ cups of butter. How many sticks of butter (each stick = ½ cup) will you need? (p. 400)

Of the story problems, (a) has an answer of 1¼ divided by 2 rather than 1¼ divided by ½, and therefore can be eliminated. Problems (b) and (c) both have answers numerically equivalent to 1¼ divided by ½, but (b) is more directly a representation of the operation 1¼ times 2 while (c) is an illustration of a quotative (or measurement) interpretation of 1 ¼ divided by ½. While the question asks students to analyze story problems, a common task in elementary teaching, it assesses students' understanding of the conceptual distinction between multiplication by 2 and division by ½, and their knowledge of the quotative model for fraction division ("how many halves make up 1¼"). In this instance, classroom mathematical tasks draw from the work of teaching but also serve as a window into PTs' understanding of mathematical structure.

The work on cognitively guided instruction (Carpenter, Fennema, & Franke, 1996) offers some examples of how problems in the elementary mathematics curriculum can clarify distinctions among different models and meanings for addition and subtraction. Following this framework, MTEs can assess PTs' understandings of these distinctions by asking them to generate problems that illustrate certain models. For example, an assessment task might ask PTs to write a contextual problem that can be solved using the comparison model for subtraction and whose answer is 11 – 5. Responses to such a task can help MTEs assess whether PTs have understood the distinction between the comparison model for subtraction and other models (such as the take-away model), and whether they can state a subtraction problem following this model in a clear and precise way. Although a deep knowledge of the distinction among different models for the same operation is not needed by most people outside the teaching profession, it is essential for teaching elementary mathematics (Ball, Thames, & Phelps, 2008).

Analyzing Students' Mathematical Thinking

Elementary teachers must use their knowledge to make sense of students' mathematical productions, form hypotheses about what students understand, and identify possible directions for further development (Carpenter & Lehrer, 1999). In some cases, teachers must analyze problem-solving approaches different from the ones that may be considered "standard" and assess whether these approaches are robust.

Hill, Ball, and Schilling (2008, p. 400) present an example of a task designed to assess a teacher's facility for analyzing student errors and identifying conceptions that may lead to those errors. The reader is presented with three hypothetical students' work on different problems, each involving the addition of three two-digit numbers. The reader is asked to determine which solutions "have the same kind of error." We would recommend that MTEs also ask their PTs to explain their reasoning behind their choice in this task. Two of the errors result from "carrying a one" in the tens place, even though the digits in the ones column sum to more than 20, requiring the grouping of two tens rather than just one. The other error is the result of an addition mistake in the ones column. Tasks like this can be used to assess PTs' skill at identifying the conceptual sources of student errors, especially if they are asked to justify their selection and explain how they might help the students who have the same type of error. A teacher who says that he would advise students to carry a two rather than a one demonstrates a different understanding of the underlying mathematics from one who says that she would help students think about grouping twenty ones into two tens.

Tasks can also go a step farther and ask PTs to describe how they might address specific instances of student work. For example, the following task (Figure 5) asks PTs to respond to hypothetical third grade students' work (Beam et. al., 2018, p. 76). When assessing work on this task, we look for PTs to: (a) attempt to understand each child's approach to solving the problem; (b) recognize if the child's reasoning is mathematically sound; and (c) acknowledge the value of the child's thinking by not discrediting the solution or proposing a change in strategy but rather working with the child's understanding to correct mistakes.

Your third graders show you the following work. Figure out what each child is doing and decide exactly what you would do and say as his or her teacher.

Lana's work: $\begin{array}{r} 36 \\ \underline{+\,17} \\ 40 \\ \underline{13} \\ 53 \end{array}$

Jose's work: $\begin{array}{r} 36 \\ \underline{+\,17} \\ 413 \end{array}$

Emily's work: $\begin{array}{r} 36 \\ \underline{+\,17} \\ 43 \end{array}$

Max's work: $\begin{array}{r} 36 \\ \underline{+\,17} \end{array}$

Max says, "37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53."

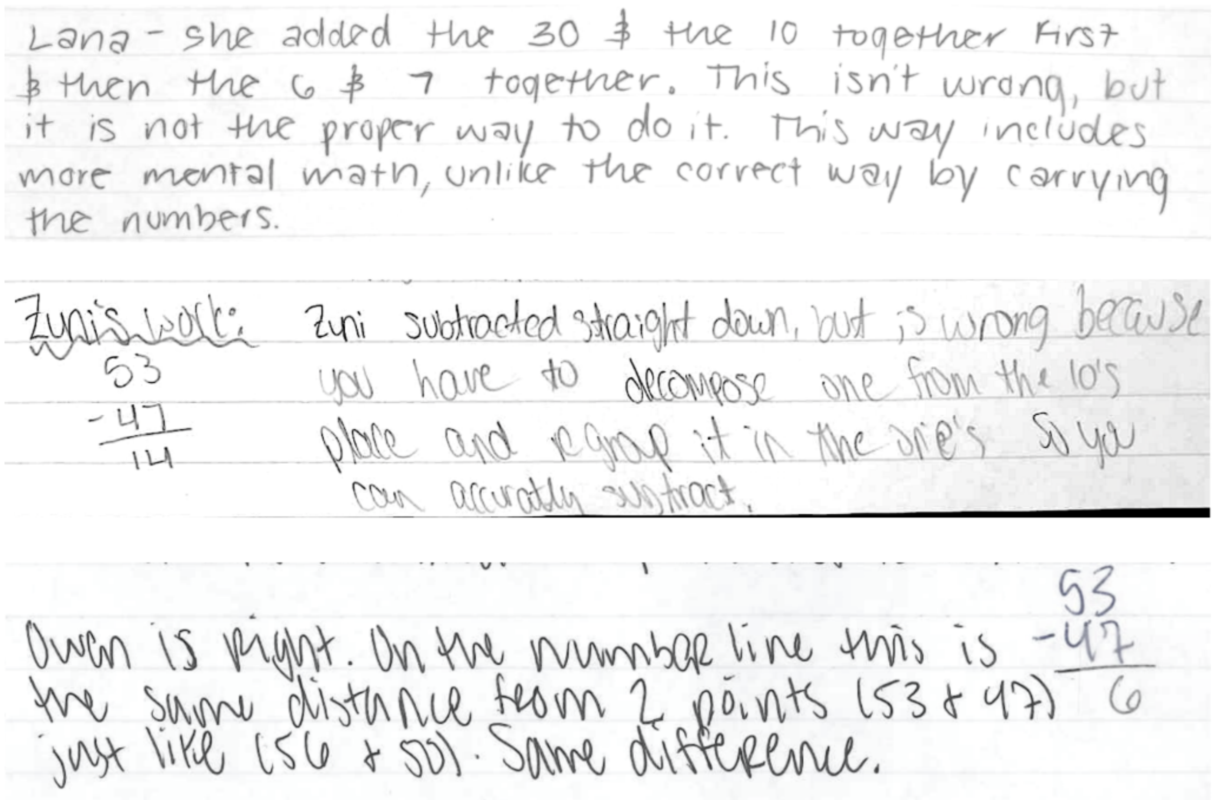
Zuni's work: $\begin{array}{r} 53 \\ \underline{-\,47} \\ 14 \end{array}$

Owen's work $\begin{array}{r} 53 \\ \underline{-\,47} \end{array}$

Owen explains, "This is the same as 56 minus 50 and so the answer is 6."

Figure 5. Student work analysis task.

Figure 6 shows portions of PT responses to this task. In the response to Lana, we see that the PT may privilege the US standard algorithm for addition and view alternative approaches as undesirable. Her assertion that "it is not the proper way to do it" discredits a child's correct strategy and suggests that there is only one correct way to solve an addition problem. In the response to Zuni, the PT does not address why the student's method is wrong, but only how to perform the subtraction correctly. She does not evaluate the reasonableness or mathematical correctness of the student's process, nor does she demonstrate that she fully understands Zuni's approach. On the other hand, although brief, the response to Owen shows that the PT understands that the child's reasoning is mathematically viable by recognizing the "same difference" strategy, that the difference (or distance) between two numbers is maintained when the same amount is added to both subtrahend and minuend. The PT's response also acknowledges the value of the student's approach by not proposing an alternative strategy.



Lana - she added the 30 & the 10 together first & then the 6 & 7 together. This isn't wrong, but it is not the proper way to do it. This way includes more mental math, unlike the correct way by carrying the numbers.

Zuni's work: 53 − 47 = 14. Zuni subtracted straight down, but is wrong because you have to decompose one from the 10's place and regroup it in the one's so you can accurately subtract.

53 − 47 = 6. Owen is right. On the number line this is the same distance from 2 points (53 & 47) just like (56 & 50). Same difference.

Figure 6. Sample PT responses to student work analysis task.

Another approach to assessing the knowledge entailed in eliciting and making sense of student thinking has been developed by the Assessing Teaching Practice (@Practice) Project (Shaughnessy, Boerst, & Ball, 2015; Shaughnessy & Boerst, 2018). The authors use simulated student encounters in which a PT first examines a copy of work produced by a "standardized student" with a predefined written response to a problem and a scripted set of responses to related questions, then has five minutes to interact with the "student" and probe their thinking. An assessor then interviews the PT to elicit his/her understanding of the student's thinking. In this final stage, the PT might answer questions about what the student understands, or how the student would likely respond to a related problem.

The @Practice Project's work with simulated student encounters provides evidence that asking participants to consider and comment on a hypothetical student strategy can reveal how

PTs think about the affordances or limitations of specific problem-solving approaches. In one study (DeFino, Prawat, & Shaughnessy, 2017), a simulation asked PTs to interact with a student who had developed a strategy for calculating the area of a rectangle by skip-counting rows of square units, and subsequently asked them to identify a shape for which the strategy would not work (because the number of unit squares in each row was not constant). This task again simulates a specific aspect of the work of teaching: pressing students to test the generalizability of a solution strategy. While all PTs in the study could identify a shape for which the skip-counting strategy does not work, only about half clearly explained why the strategy would fail in that case. Using a task like this in an assessment context might afford an MTE the opportunity to observe how PTs speak or write about alternative algorithms and clarify points that are revealed to be challenging for students. If teaching simulations are not available, MTEs can still emulate some features of this assessment by asking PTs questions that push beyond their initial reactions to a student's work. For example, a written assessment task might present a sample of student work that contains an error and asks a PT to state a question which, if asked of the student, might help the PT understand more precisely how the student is thinking; another task might ask the PT to state another problem that the student might be able to solve correctly, or that might replicate the error. Although many PTs take content courses prior to experiences that allow direct interaction with student thinking, tasks like these in which they map out possible responses to student work can open fruitful conversations about common mathematical conceptions and problem selection that help develop prospective teachers' KCS and KCT.

Conclusion: Assessment as a Catalyst for Prospective Teachers' Learning

In using the work of teaching elementary mathematics as a source of inspiration for assessment in content courses, we aim to stimulate professional growth both for PTs and for

MTEs. For PTs, formative assessment tasks that draw upon the work of teaching can provide additional motivation for the mathematics they are learning; they can also offer opportunities for PTs to learn mathematical principles and habits of mind that operate more broadly than the discrete concepts and topics they explore in a content course. For example, tasks that ask PTs to use visual representations to illustrate mathematical ideas and processes often highlight conceptual difficulties that sometimes accompany the use of such representations, such as the ambiguity of the unit in a fraction diagram or the hidden assumptions implicitly conveyed by a Venn diagram. We hypothesize that teachers who are attuned to these difficulties are better prepared to assist students in navigating them. Tasks that ask PTs to respond to student thinking, especially when accompanied with small-group and whole-class discussion, can provide compelling opportunities for MTEs to reinforce productive habits such as acknowledging the value of students' diverse ways of thinking and making sense of students' own mathematical ideas rather than quickly diverting them toward a traditional solution method. We have found that tasks involving unusual student approaches, even when the student work embedded in the tasks is artificial, are excellent reminders to PTs that "The answer to the question, 'Why didn't this work?', is not, 'You should have done it this other way.'" Thus these tasks can address PTs' emerging pedagogical knowledge in addition to their content knowledge.

For MTEs, observing PTs' work on assessment tasks can help us identify directions to support their development. It also helps us to improve our own instruction: on the quadrilateral Venn diagram task, our observations helped us to realize that PTs did not have a shared interpretation of Venn diagrams; some PTs' initial response was to place properties of different types of quadrilaterals in each region, rather than having each region represent a set of quadrilaterals. Based on our observations, we improved the task by naming the representation a "classification chart" rather than a Venn diagram, and providing an introduction on classification charts in a non-mathematical context at the start of the lesson. Assessment can provide us with evidence of how PTs' mathematical knowledge might help them respond to tasks of teaching. For instance, PTs' difficulties in making sense of hypothetical student arguments may reveal a need for MTEs to incorporate student thinking into content courses, so that PTs can practice recognizing the mathematical ideas in students' work. Similar uses of assessment can help identify areas in which lessons can be fine-tuned to develop PTs' mathematical knowledge and opportunities to strengthen links between content knowledge and the work of teaching.

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