Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 189, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXACTNESS OF THE NUMBER OF POSITIVE SOLUTIONS TO A SINGULAR QUASILINEAR PROBLEM

GIOVANNI ANELLO, LUCA VILASI

Abstract. We study the exact multiplicity of positive solutions to the one-dimensional Dirichlet problem

$$-(|u'|^{p-2}u')' = \lambda u^{s-1} - \mu u^{r-1} \quad \text{in }]0,1[$$
$$u(0) = u(1) = 0,$$

where $r \in]0,1[$, $p \in]1,+\infty[$, $r < s < p$ and $\lambda, \mu \in]0,+\infty[$. We shed light, in particular, on the case $r \in]0, \min\{s, p/(p+1)\}[$, completely determining the bifurcation diagram and solving some related open problems. Our approach relies upon quadrature methods.

1. Introduction

In this article we focus on the singular one-dimensional Dirichlet problem

$$\begin{gathered} -(|u'|^{p-2}u')' = \lambda u^{s-1} - \mu u^{r-1} \quad \text{in }]0,1[\\ u(0) = u(1) = 0, \end{gathered} \tag{1.1}$$

where $r \in]0,1[$, $p \in]1,+\infty[$, $r < s < p$ and λ, μ are positive real parameters. This model has captured a lot of attention over the latest years, both in the semilinear and in the quasilinear case (see for instance [4, 5, 7, 9, 10] and their references). In the framework of chemical reactions the nonlinearity in (1.1) assumes the meaning of a strong absorption (or endothermic) process with respect to the diffusion (see [7] and the references therein).

Our aim is to explore the exact number of positive classical solutions to (1.1) by means of quadrature methods. Such a number, established as one of the parameters of the problem varies, depends solely on the profile of a suitable integral function, as clarified later (see, besides the above references, also [1, 2, 3, 8] for applications of this procedure to several classes of one-dimensional problems).

Along this line of research, in [9] the authors studied the exact multiplicity of positive solutions to

$$\begin{gathered} -u'' = \beta - u^{r-1} \quad \text{in }]-L,L[\\ u(-L) = u(L) = 0, \end{gathered} \tag{1.2}$$

with $\beta, L > 0$ and $0 < r < 1$. This can be viewed as a particular case of (1.1) in which $p = 2$ and $s = 1$ (observe that, after a suitable change of variables, problem (1.1) can be always rescaled so that the parameters appear only in the endpoints

2010 *Mathematics Subject Classification.* 34B08, 34B16, 34B18.
Key words and phrases. Exactness; singular problem; positive solution; quadrature method.

Submitted April 19, 2018. Published November 20, 2018.

of the interval, see Theorem 2.9). In this paper the authors complemented the analysis carried out in [4], answering some open issues posed there and proving that, for an explicitly determined $L_0 > 0$, if $0 < r \leq 1/2$ there is no solution to (1.2) when $L < L_0$ and there is a unique solution for $L \geq L_0$. For $1/2 < r < 1$ there exists $L_{\min} < L_0$ such that, for $L < L_{\min}$ there is no solution to (1.2), for $L = L_{\min}$ there is a unique solution, for $L_{\min} < L \leq L_0$ there are exactly two solutions and for $L > L_0$ there exists exactly one solution (cf. [9, Theorem 1]). The picture depicted above basically reveals the existence of a critical exponent $r^* := 1/2$ below which one has uniqueness and above which one has multiplicity, and that two is the maximum number of possible positive solutions to (1.2).

The semilinear case with $s \neq 1$ was investigated in [10], where the author determined all possible bifurcation diagrams that can occur for different values of s and r, giving a short survey of the known results and pointing out the situations where the exact multiplicity of the solutions is not known. The main result is quite similar to the one obtained for problem (1.2); in this context the "limit" situation occurs when $s = 2 - 2r$: for $s \in]1, 2 - 2r]$ the bifurcation diagram is a monotone curve, while for $s \in]2 - 2r, 2[$ there exist an upper branch of positive solutions for $L > L_{\min}$ and a lower one stopping at $L = L_0 > L_{\min}$ (see [10, Theorem 3.2]). The case $0 < s < 1$ is left open; the monotonicity of the time-map associated with the problem is in fact proved only for $s \in]1, 2[$ (see [10, Lemma 3.1]).

The quasilinear counterpart (1.1) was the object of an in-depth study in [7], which generalizes the results of [6] to the singular framework. As clearly shown in this paper, when approaching (1.1) via quadrature methods, two different situations arise, depending on whether

$$\int_0^\delta \frac{dt}{(-F(t))^{\frac{p+1}{p}}} = +\infty \tag{1.3}$$

or

$$\int_0^\delta \frac{dt}{(-F(t))^{\frac{p+1}{p}}} < +\infty, \tag{1.4}$$

for some $\delta > 0$ small enough, where

$$F(t) := \frac{1}{s} t^s - \frac{1}{r} t^r \quad \text{for all } t > 0. \tag{1.5}$$

The validity of either (1.3) or (1.4) depends on the relationship between r and $\frac{p}{p+1}$; indeed, (1.3) (respectively, (1.4)) holds if and only if $r \geq \frac{p}{p+1}$ (respectively, $0 < r < \frac{p}{p+1}$) (see [7, Lemma 2 and Proposition 4]).

The analysis in the first case was performed and completed in [7, Theorem 2(a)–(e)]. It can be summarized as follows: there exist $\lambda_1 > \lambda_0 > 0$ such that problem (1.1) admits:

(i) no positive solutions if $\lambda < \lambda_0$;
(ii) a unique positive solution if $\lambda = \lambda_0$;
(iii) two positive solutions if $\lambda \in]\lambda_0, \lambda_1]$ and one of those corresponding to λ_1 is a compacton, i.e. its first derivative vanishes at $t = 0$ and $t = 1$;
(iv) a positive solution and a continuum of non-negative solutions compactly supported in $]0, 1[$ if $\lambda > \lambda_1$.

The item (iv) is particularly interesting as it states that the lower branch of the positive solutions can be continued to obtain non-negative solutions with compact support in $]0,1[$. Such a family of solutions, which vanishes on a positive-measured subset of $]0,1[$ called the "dead core", turns out to be generated by the compacton arising when $\lambda = \lambda_1$ (see [7, Theorem 2(e)] for an explicit expression of the members of this family).

We also point out that the positive solutions u of (ii), (iii) and (iv), excluding of course the compacton, satisfy Hopf's condition on the boundary, that is $u'(0) < 0$ and $u'(1) > 0$.

Condition (1.4) is more delicate to handle and the authors obtained only the following partial result: the fact (iv) continuing to hold, for $\lambda \in]\lambda_0, \lambda_1[$ there exists $\delta > 0$ such that there is a unique positive solution if $r \in]0, \delta[$ and there are exactly two positive solutions if $r \in \left]\frac{p}{p+1} - \delta, \frac{p}{p+1}\right[$. They were able to establish the exact number of positive solutions only when $s = 2r$ and this amounts to one if $p \leq \frac{2r}{1-r}$, two if $p > \frac{2r}{1-r}$ (see [7, Theorem 2(f)]). The reason of the success in the latter case relies on the fact that the derivative of the auxiliary function ψ defined by (2.2) at the unique zero t_0 of F can be expressed in closed form in terms of the Euler Beta function (see [7, Proposition 4]). And the knowledge of the sign of $\psi'(t_0)$, as explained later, is crucial to draw the bifurcation diagram of (1.1) when $0 < r < \frac{p}{p+1}$.

Driven by the analogy with the semilinear case, it is quite reasonable to conjecture the existence of a unique exponent $r^* \in \left]0, \frac{p}{p+1}\right[$ such that one has exactly one solution for $0 < r \leq r^*$ and exactly two for $r^* < r < \frac{p}{p+1}$ (see, in particular, the survey paper [5]). In this article we give a positive answer to the above conjecture, proving the uniqueness of the zero of the equation $\psi'(t_0(r)) = 0$ when $r \in]0, \min\{s, \frac{p}{p+1}\}[$ (Theorem 2.8) and determining therefore the exact multiplicity of solutions to (1.1) when (1.4) holds (Theorem 2.9). Our results are illustrated in the next section, preceded by some preliminary facts.

2. Results

Let F be the function defined by (1.5). Denoting by $F^{(i)}$, $i \in \mathbb{N} \cup \{0\}$, the i-th derivative of F (as usual, $F^{(0)} := F$), it is easy to prove the following property (cf. [2, Lemma 2.1]).

Lemma 2.1. *There exist $t_0, t_1 \in]0, +\infty[$, with $t_0 > t_1$, such that, for each $i = 0, 1$ one has*

$$\begin{gathered} F^{(i)}(t) < 0 \quad \text{if } t \in]0, t_i[, \\ F^{(i)}(t) > 0 \quad \text{if } t \in]t_i, +\infty[. \end{gathered} \tag{2.1}$$

For the sequel, it will be useful to know the behavior in $]0,1[$ of the real functions of the next three lemmas.

Lemma 2.2. *Let $x > 0$. The function $j(t) := \frac{\log t}{1-t^x}$ is increasing in $]0,1[$, with*

$$\lim_{t \to 1^-} j(t) = -\frac{1}{x}.$$

Proof. One has

$$j'(t) = \frac{t^{-1}(1-t^x) + x t^{x-1} \log t}{(1-t^x)^2} = t^x \frac{(t^{-x} - 1) + x \log t}{t(1-t^x)^2}, \quad \text{for all } t \in]0,1[.$$

Putting

$$\tilde{j}(t) := (t^{-x} - 1) + x \log t, \quad \text{for all } t \in]0,1],$$

one has $\tilde{j}(1) = 0$ and

$$\tilde{j}'(t) = -xt^{-x-1} + xt^{-1} = xt^{-1}(1 - t^{-x}) < 0, \quad \text{for all } t \in]0,1[.$$

Hence, $\tilde{j}(t) > 0$ for all $t \in]0,1[$, and so is $j'(t)$. The limit follows by an immediate application of L'Hôpital's rule. □

Lemma 2.3. *Let $0 < x < y$. The function $k(t) := \frac{1-t^y}{t^x - t^y}$ is decreasing in $]0,1[$, with*

$$\lim_{t\to 1^-} k(t) = \frac{y}{y-x}.$$

Proof. One has

$$\begin{aligned} k'(t) &= \frac{-yt^{y-1}(t^x - t^y) - (1-t^y)(xt^{x-1} - yt^{y-1})}{(t^x - t^y)^2} \\ &= -\frac{t^{x-1}}{(t^x - t^y)^2}((y-x)t^y + x - yt^{y-x}), \end{aligned}$$

for any $t \in]0,1[$. If we define

$$\tilde{k}(t) := (y-x)t^y + x - yt^{y-x}, \quad \text{for any } t \in]0,1],$$

we get $\tilde{k}(1) = 0$ and

$$\tilde{k}'(t) = y(y-x)t^{y-1} - y(y-x)^{y-x-1} = y(y-x)t^{y-x-1}(t^x - 1) < 0,$$

for all $t \in]0,1[$. As a result, $\tilde{k}(t) > 0$ for all $t \in]0,1[$ and so $k'(t) < 0$ for all $t \in]0,1[$. □

Lemma 2.4. *Let $y \in]0,+\infty[$ and $t \in]0,1[$. The function $l(x) := \frac{y-x}{t^x - t^y}$ is increasing in $[0,y[$ and*

$$\lim_{x\to y} l(x) = -\frac{1}{t^y \log t}.$$

Proof. One has

$$\begin{aligned} l'(x) &= (t^x - t^y)^{-2}(-(t^x - t^y) - (y-x)t^x \log t) \\ &= (t^x - t^y)^{-2} t^x (t^{y-x} - 1 - (y-x)\log t), \end{aligned}$$

for all $x \in]0,y[$. Fix $x \in [0,y[$ and put $\tilde{l}(\tau) := \tau^{y-x} - 1 - (y-x)\log\tau$, for all $\tau \in]0,1[$. One has $\tilde{l}(1) = 0$ and $\tilde{l}'(\tau) = \tau^{-1}(y-x)(\tau^{y-x} - 1) \le 0$ for all $\tau \in]0,1[$. Therefore $\tilde{l}(\tau) \ge 0$ for all $\tau \in]0,1[$ and so $l'(x) = (t^x - t^y)^{-2} t^x \tilde{l}(t) \ge 0$ for all $x \in]0,y[$. The limit is a consequence of L'Hôpital's rule. □

The essential part of our approach is the analysis of the following function associated with (1.1):

$$\psi(\sigma) := \int_0^\sigma \frac{dt}{(F(\sigma) - F(t))^{1/p}}, \quad \text{for each } \sigma \ge t_0. \tag{2.2}$$

In [7] the authors proved the following basic facts concerning the profile of ψ (cf. [7, Propositions 1 and 4]).

Lemma 2.5. *Let $r \in \left]0, \frac{p}{p+1}\right[$, $p \in]1,+\infty[$, $r < s < p$ and let $\psi: [t_0,+\infty[\to \mathbb{R}$ be the function defined by (2.2). Then:*

(i) *$\psi \in C^0([t_0,+\infty)) \cap C^1((t_0,+\infty))$;*
(ii) *$\lim_{\sigma\to+\infty} \psi(\sigma) = +\infty$;*

(iii) *if* $s \geq \frac{p}{p+1}$, *there exists* $\delta > 0$ *small enough such that*

$$0 < \psi'(t_0) < +\infty \quad \text{if } r \in]0, \delta[;$$
$$-\infty < \psi'(t_0) < 0 \quad \text{if } r \in \Big] \frac{p}{p+1} - \delta, \frac{p}{p+1} \Big[;$$

(iv) *if* $s = 2r$ *then*

$$0 < \psi'(t_0) < +\infty \quad \text{if } p < \frac{2r}{1-r};$$
$$\psi'(t_0) = 0 \quad \text{if } p = \frac{2r}{1-r};$$
$$-\infty < \psi'(t_0) < 0 \quad \text{if } p > \frac{2r}{1-r}.$$

For $s < p/(p+1)$, Lemma 2.5 gives no information about the sign of $\psi'(t_0)$ when r is near s. The next result provides this missing detail.

Lemma 2.6. *Assume* $s < \frac{p}{p+1}$. *Then, there exists* $\delta > 0$ *such that* $\psi'(t_0) > 0$, *for each* $r \in]s - \delta, s[$.

Proof. For any $r \in]0, s[$, we can write

$$\psi'(t_0) = \frac{1}{p} \big(\frac{s}{r}\big)^{-\frac{r}{p(s-r)}} r^{1/p} (s-r)^{-\frac{1}{p}} \hat{\phi}(r),$$

where

$$\hat{\phi}(r) := \int_0^1 \Big(\frac{s-r}{t^r - t^s}\Big)^{1/p} \frac{(p-s)(1-t^s) - (p-r)(1-t^r)}{t^r - t^s} dt.$$

For each $t \in]0, 1[$, using L'Hôpital's rule, one has

$$\begin{aligned}
&\lim_{r \to s^-} \Big(\frac{s-r}{t^r - t^s}\Big)^{1/p} \frac{(p-s)(1-t^s) - (p-r)(1-t^r)}{t^r - t^s} \\
&= \Big(-\frac{1}{t^s \log t}\Big)^{1/p} \frac{1 - t^s + (p-s) t^s \log t}{t^s \log t} \\
&= (-\log t)^{-\frac{p+1}{p}} t^{-s/p} [1 - t^{-s} - (p-s) \log t].
\end{aligned}$$

Moreover, thanks to Lemma 2.4, we have the estimate

$$\begin{aligned}
&\Big| \Big(\frac{s-r}{t^r - t^s}\Big)^{1/p} \frac{(p-s)(1-t^s) - (p-r)(1-t^r)}{t^r - t^s} \Big| \\
&= \Big(\frac{s-r}{t^r - t^s}\Big)^{1/p} \Big| p - r - \frac{s-r}{t^r - t^s} (1 - t^s) \Big| \\
&\leq t^{-s/p} (-\log t)^{-\frac{1}{p}} \Big(p - t^{-s} \frac{1 - t^s}{\log t}\Big),
\end{aligned} \tag{2.3}$$

and the function

$$]0, 1[\ni t \mapsto t^{-s/p} (-\log t)^{-\frac{1}{p}} \Big(p - t^{-s} \frac{1 - t^s}{\log t}\Big)$$

is summable in $[0, 1]$.

Then, by Lebesgue's dominated convergence theorem, the function $\hat{\phi}$ admits limit as $r \to s^-$ and

$$\lim_{r \to s^-} \hat{\phi}(r) = \int_0^1 (-\log t)^{-\frac{p+1}{p}} t^{-s/p} [1 - t^{-s} - 1 - (p-s) \log t] dt := \bar{\phi}(s).$$

Let us to evaluate $\bar\phi(s)$. An integration by parts yields

$$\begin{aligned}
\bar\phi(s) &= \int_0^1 (-\log t)^{-\frac{p+1}{p}} t^{-s/p}(1-t^{-s}-(p-s)\log t)dt\\
&= \int_0^1 (-\log t)^{-\frac{p+1}{p}} t^{-1}\big(t^{1-\frac{s}{p}}-t^{1-s-\frac{s}{p}}\big)dt + (p-s)\int_0^1 (-\log t)^{-\frac{1}{p}} t^{-s/p}dt\\
&= \big[p(-\log t)^{-\frac{1}{p}}\big(t^{1-\frac{s}{p}}-t^{1-s-\frac{s}{p}}\big)\big]_0^1\\
&\quad - p\int_0^1 (-\log t)^{-\frac{1}{p}}\big[\big(1-\frac{s}{p}\big)t^{-s/p}-\big(1-s-\frac{s}{p}\big)t^{-s-\frac{s}{p}}\big]dt\\
&\quad + (p-s)\int_0^1 (-\log t)^{-\frac{1}{p}} t^{-s/p}dt\\
&= \int_0^1 (-\log t)^{-\frac{1}{p}}(p-s(p+1))t^{-s\frac{p+1}{p}}dt > 0.
\end{aligned}$$

This completes the proof. ∎

Now, we add a new piece of information on the shape of ψ showing that it possesses at most one critical point in $]t_0,+\infty[$.

Lemma 2.7. *The equation $\psi'(\sigma)=0$ admits at most one solution in $]t_0,+\infty[$.*

Proof. After a change of variables, we can write ψ as

$$\psi(\sigma) = \int_0^1 \frac{\sigma}{(F(\sigma)-F(\sigma t))^{1/p}}dt = \int_0^1 \frac{dt}{(\eta_t(\sigma))^{1/p}}, \tag{2.4}$$

for any $\sigma\in[t_0,+\infty[$, where

$$\eta_t(\sigma) := \frac{1}{s}(1-t^s)\sigma^{s-p} - \frac{1}{r}(1-t^r)\sigma^{r-p}$$

for all $t\in[0,1]$ and for all $\sigma\in[t_0,+\infty[$. Then, the first two derivatives of ψ take the form

$$\begin{aligned}
\psi'(\sigma) &= -\frac{1}{p}\int_0^1 (\eta_t(\sigma))^{-\frac{1}{p}-1}\Big(\frac{s-p}{s}(1-t^s)\sigma^{s-p-1} - \frac{r-p}{r}(1-t^r)\sigma^{r-p-1}\Big)dt,\\
\psi''(\sigma) &= \frac{1}{p}\big(1+\frac{1}{p}\big)\int_0^1 (\eta_t(\sigma))^{-\frac{1}{p}-2}\Big(\frac{s-p}{s}(1-t^s)\sigma^{s-p-1}\\
&\quad - \frac{r-p}{r}(1-t^r)\sigma^{r-p-1}\Big)^2 dt\\
&\quad - \frac{1}{p}\int_0^1 (\eta_t(\sigma))^{-\frac{1}{p}-1}\Big(\frac{s-p}{s}(s-p-1)(1-t^s)\sigma^{s-p-2}\\
&\quad - \frac{r-p}{r}(r-p-1)(1-t^r)\sigma^{r-p-2}\Big)dt.
\end{aligned}$$

Now, fix $K\in]p+1-s,p+1-r[$ and put

$$\chi(\sigma) := \sigma\psi''(\sigma) + K\psi'(\sigma), \tag{2.5}$$

for all $\sigma\in[t_0,+\infty[$. For any $\sigma\in[t_0,+\infty[$ we can deduce the lower estimate

$$\begin{aligned}
\chi(\sigma) &> -\frac{1}{p}\int_0^1 (\eta_t(\sigma))^{-\frac{1}{p}-1}\Big(\frac{s-p}{s}(s-p-1)(1-t^s)\sigma^{s-p-1}\\
&\quad - \frac{r-p}{r}(r-p-1)(1-t^r)\sigma^{r-p-1}\Big)dt
\end{aligned}$$

$$
\begin{aligned}
&- \frac{K}{p}\int_0^1 (\eta_t(\sigma))^{-\frac{1}{p}-1}\Big(\frac{s-p}{s}(1-t^s)\sigma^{s-p-1} - \frac{r-p}{r}(1-t^r)\sigma^{r-p-1}\Big)dt \\
&= \frac{1}{p}\int_0^1 (\eta_t(\sigma))^{-\frac{1}{p}-1}\Big(\frac{p-s}{s}(K-p-1+s)(1-t^s)\sigma^{s-p-1} \\
&- \frac{p-r}{r}(K-p-1+r)(1-t^r)\sigma^{r-p-1}\Big)dt
\end{aligned}
$$

and the choice of K forces $\chi(\sigma) > 0$. At this point, solving (2.5) for ψ', one has

$$
\psi'(\sigma) = \frac{1}{\sigma^K}\Big(\int_{t_0}^{\sigma} \tau^{K-1}\chi(\tau)d\tau + c\Big)
$$

for all $\sigma \in [t_0, +\infty[$ and for a suitable constant $c \in \mathbb{R}$. In view of the positivity of χ, ψ' admits at most one zero in $]t_0, +\infty[$ and the conclusion is achieved. □

Lemma 2.5 leaves the question whether $r \mapsto \psi'(t_0(r))$ changes sign at most once in $]0, \min\{s, \frac{p}{p+1}\}[$ open. In the next theorem, we answer this question affirmatively. This allows us to completely compute the number of solutions to $(P_{\lambda,\mu})$ as the exponent r varies and, in particular, to extend the result in [10] to the case $s > 0$.

Theorem 2.8. *Let $p \in]1, +\infty[$ and $s \in]0, p[$. Denote by ϕ:$]0, \min\{s, \frac{p}{p+1}\}[\to \mathbb{R}$ the function*

$$
\phi(r) := \psi'(t_0(r)), \quad \text{for each } r \in]0, \min\{s, \frac{p}{p+1}\}[. \tag{2.6}
$$

Then

- (i) *if $s \geq \frac{p}{p+1}$, the equation $\phi(r) = 0$ admits exactly one zero, say r^*, in $]0, \frac{p}{p+1}[$, with ϕ positive in $]0, r^*[$ and negative in $]r^*, \frac{p}{p+1}[$;*
- (ii) *if $s < \frac{p}{p+1}$, the function ϕ is positive in $]0, s[$.*

Proof. To obtain conclusions (i) and (ii) it is sufficient to show that ϕ changes sign at most once in $]0, \min\{s, \frac{p}{p+1}\}[$. Indeed, if this is true, since $\phi \in C^0(]0, \min\{s, \frac{p}{p+1}\}[)$, conclusion (i) follows from (iii) of Lemma 2.5, and conclusion (ii) is a consequence of Lemma 2.6.

Taking (2.4) into account we obtain

$$
\begin{aligned}
\psi'(\sigma) &= \frac{\psi(\sigma)}{\sigma} + \int_0^1 \frac{\sigma(F'(\sigma) - tF'(\sigma t))}{p(F(\sigma) - F(\sigma t))^{\frac{p+1}{p}}}dt \\
&= \frac{1}{p}\int_0^1 \frac{p(F(\sigma) - F(\sigma t)) - \sigma(F'(\sigma) - tF'(\sigma t))}{(F(\sigma) - F(\sigma t))^{\frac{p+1}{p}}}dt \\
&= \frac{1}{p}\int_0^1 \frac{\frac{p-s}{s}\sigma^s(1-t^s) - \frac{p-r}{r}\sigma^r(1-t^r)}{(F(\sigma) - F(\sigma t))^{\frac{p+1}{p}}}dt
\end{aligned}
$$

for any $\sigma \in]t_0, +\infty[$. Taking the limit as $\sigma \to t_0(r)$ in the above expression and bearing in mind that $t_0(r) = (\frac{s}{r})^{1/(s-r)}$, we therefore obtain

$$
\begin{aligned}
\phi(r) &= \frac{1}{p}\int_0^1 \frac{\frac{p-s}{s}t_0(r)^s(1-t^s) - \frac{p-r}{r}t_0(r)^r(1-t^r)}{(-F(t_0(r)t))^{\frac{p+1}{p}}}dt \\
&= \frac{1}{p}t_0(r)^{-\frac{r}{p}}\int_0^1 \frac{\frac{p-s}{s}t_0(r)^{s-r}(1-t^s) - \frac{p-r}{r}(1-t^r)}{\big(\frac{t^r}{r} - t_0(r)^{s-r}\frac{t^s}{s}\big)^{\frac{p+1}{p}}}dt
\end{aligned}
$$

$$= \frac{1}{p}\left(\frac{s}{r}\right)^{-\frac{r}{p(s-r)}} r^{1/p} \int_0^1 \frac{(p-s)(1-t^s)-(p-r)(1-t^r)}{(t^r-t^s)^{\frac{p+1}{p}}} dt$$
$$= \frac{1}{p}\left(\frac{s}{r}\right)^{-\frac{r}{p(s-r)}} r^{1/p}(s-r) \int_0^1 \frac{\frac{p-s}{s-r}(1-t^s)-\frac{p-r}{s-r}(1-t^r)}{(t^r-t^s)^{\frac{p+1}{p}}} dt.$$

In light of the above manipulations, to conclude the proof it is equivalent to show that $\tilde{\phi}(r) = 0$ has at most one solution in $]0, \min\{s, \frac{p}{p+1}\}[$, where

$$\tilde{\phi}(r) := \int_0^1 \frac{\frac{p-s}{s-r}(1-t^s)-\frac{p-r}{s-r}(1-t^r)}{(t^r-t^s)^{\frac{p+1}{p}}} dt \tag{2.7}$$

for any $r \in]0, \min\{s, \frac{p}{p+1}\}[$. Now, the integrand function in (2.7) can be expressed as

$$\frac{\frac{p-s}{s-r}(1-t^s)-\frac{p-r}{s-r}(1-t^r)}{(t^r-t^s)^{\frac{p+1}{p}}} = (1-t^s)^{-\frac{1}{p}} g(t,r)$$

for all $t \in]0,1[$, where

$$g(t,r) := h(t,r)\big(\frac{p-r}{s-r} - h(t,r)^p\big)$$

and

$$h(t,r) := \Big(\frac{1-t^s}{t^r-t^s}\Big)^{1/p}. \tag{2.8}$$

By an easy calculation we get, for all $t \in]0,1[$,

$$\frac{\partial h}{\partial r}(t,r) = -\frac{h(t,r)}{p} \cdot \frac{\log t}{1-t^{s-r}}$$

and therefore

$$\begin{aligned}
\frac{\partial g}{\partial r}(t,r) &= \frac{\partial h}{\partial r}(t,r)\Big(\frac{p-r}{s-r} - h(t,r)^p\Big) \\
&\quad + h(t,r)\Big(\frac{p-s}{(s-r)^2} - p h(t,r)^{p-1}\frac{\partial h}{\partial r}(t,r)\Big) \\
&= -\frac{h(t,r)}{p} \cdot \frac{\log t}{1-t^{s-r}}\Big(\frac{p-r}{s-r} - h(t,r)^p\Big) \\
&\quad + h(t,r)\Big(\frac{p-s}{(s-r)^2} + h(t,r)^p \frac{\log t}{1-t^{s-r}}\Big) \\
&= \Big(\frac{p+1}{p}\Big)\frac{\log t}{1-t^{s-r}} h(t,r)^{p+1} \\
&\quad - \Big(\frac{p-r}{p(s-r)} \cdot \frac{\log t}{1-t^{s-r}} - \frac{p-s}{(s-r)^2}\Big) h(t,r).
\end{aligned}$$

Now, consider the function

$$\alpha(t) := \big(1 - \frac{r}{p}\big)\frac{1}{s-r} - \big(1+\frac{1}{p}\big)\frac{1-t^s}{t^r-t^s}, \quad t \in]0,1[.$$

In view of Lemma 2.3, α is strictly increasing in $]0,1[$ and one has

$$\lim_{t\to 0^+} \alpha(t) = -\infty,$$
$$\lim_{t\to 1^-} \alpha(t) = \big(1-\frac{r}{p}\big)\frac{1}{s-r} - \big(1+\frac{1}{p}\big)\frac{s}{s-r}.$$

Two possible situations, related to the range of s, occur.

(a) If $s \geq \frac{p}{p+1}$, then $\alpha(t) < 0$ for all $t \in]0,1[$;

(b) if $0 < s < \frac{p}{p+1}$, then $\alpha(t) < 0$ for all $t \in]0,1[$ if $p-(p+1)s < s$ and $p-(p+1)s < r < s$, while the equation $\alpha(t) = 0$ admits a unique root in $]0,1[$, say $t(r)$, provided that $0 < r < \min\{s, p-(p+1)s\}$. Moreover, α is negative in $]0,t(r)[$ and positive in $]t(r),1[$.

Notice that in case (b), by the implicit function theorem the function $]0,s[\ni r \mapsto t(r)$ is continuous in $]0,\min\{s,p-(p+1)s\}[$. Moreover, when $p-(p+1)s \in]0,s[$, it is easy to see that

$$\lim_{r \to p-(p+1)s^-} t(r) = 1. \tag{2.9}$$

Next, define $M(r) := 1$, for all $r \in]0,s[$, if $s \geq \frac{p}{p+1}$; for $s < \frac{p}{p+1}$ set instead

$$M(r) := \begin{cases} -\frac{(s-r)\log t(r)}{1-t(r)^{s-r}} & \text{if } 0 < r < \min\{s, p-(p+1)s\} \\ 1 & \text{if } p-(p+1)s < s \text{ and } p-(p+1)s \leq r < s. \end{cases}$$

As a function of the variable r, M turns out to be defined in the whole interval $]0,s[$ and it is continuous there as, by (2.9) and Lemma 2.2, one has

$$\lim_{r \to p-(p+1)s^-} M(r) = 1, \quad \text{whenever } p-(p+1)s \in]0,s[.$$

Also, observe that again by Lemma 2.2 we can infer that

$$M(r) \geq 1, \quad \text{for all } r \in]0,s[, \tag{2.10}$$

and that the function

$$\beta(t) := \frac{M(r)}{s-r} + \frac{\log t}{1-t^{s-r}}, \quad t \in]0,1[,$$

satisfies the same properties (a) and (b) as the function α. Hence,

$$\alpha(t)\beta(t) > 0, \quad \text{for all but at most one } t \in]0,1[. \tag{2.11}$$

Therefore, taking (2.8) into account and denoting for brevity $h(t,r)$ by h, we obtain

$$\begin{aligned}
&M(r)\frac{p+1}{p(s-r)}g(t,r) - \frac{\partial g}{\partial r}(t,r) \\
&= -\frac{p+1}{p}\Big(\frac{M(r)}{s-r} + \frac{\log t}{1-t^{s-r}}\Big)h^{p+1} \\
&\quad + \Big(\frac{p-r}{p(s-r)} \cdot \frac{\log t}{1-t^{s-r}} - \frac{p-s}{(s-r)^2} + M(r)\frac{(p+1)(p-r)}{p(s-r)^2}\Big)h \\
&= \Big[-\big(1+\frac{1}{p}\big)\Big(\frac{M(r)}{s-r} + \frac{\log t}{1-t^{s-r}}\Big)\frac{1-t^s}{t^r-t^s} + \big(1-\frac{r}{p}\big)\frac{1}{s-r} \cdot \frac{\log t}{1-t^{s-r}} \\
&\quad + \frac{1}{(s-r)^2}\big(M(r)(p-r) - (p-s)\big) + \big(1-\frac{r}{p}\big)\frac{M(r)}{(s-r)^2}\Big]h \\
&= \Big[-\big(1+\frac{1}{p}\big)\Big(\frac{M(r)}{s-r} + \frac{\log t}{1-t^{s-r}}\Big)\frac{1-t^s}{t^r-t^s} + \big(1-\frac{r}{p}\big)\frac{1}{s-r}\Big(\frac{\log t}{1-t^{s-r}} \\
&\quad + \frac{M(r)}{s-r}\Big) + \frac{M(r)(p-r)-(p-s)}{(s-r)^2}\Big]h \\
&= \Big\{\Big(\frac{M(r)}{s-r} + \frac{\log t}{1-t^{s-r}}\Big)\Big[-\big(1+\frac{1}{p}\big)\frac{1-t^s}{t^r-t^s} + \big(1-\frac{r}{p}\big)\frac{1}{s-r}\Big]
\end{aligned}$$

$$+\frac{M(r)(p-r)-(p-s)}{(s-r)^2}\Big\}h.$$

Noticing that by (2.10) one has

$$\frac{M(r)(p-r)-(p-s)}{(s-r)^2} \geq \frac{1}{s-r} > 0,$$

and recalling (2.11), we can continue the previous chain of computations to finally arrive at

$$\begin{aligned} &M(r)\frac{p+1}{p(s-r)}g(t,r) - \frac{\partial g}{\partial r}(t,r) \\ &\geq \Big(\frac{M(r)}{s-r} + \frac{\log t}{1-t^{s-r}}\Big)\Big[-\big(1+\frac{1}{p}\big)\frac{1-t^s}{t^r-t^s} + \big(1-\frac{r}{p}\big)\frac{1}{s-r}\Big]h \\ &= \beta(t)\alpha(t)h > 0 \end{aligned} \tag{2.12}$$

for all but at most one $t \in]0,1[$. Multiplying (2.12) by $(1-t^s)^{-\frac{1}{p}}$ and integrating between 0 and 1, we obtain

$$M(r)\frac{p+1}{p(s-r)}\tilde{\phi}(r) - \tilde{\phi}'(r) > 0$$

for any $r \in]0, \min\{s, \frac{p}{p+1}\}[$. If we set

$$\omega(r) := M(r)\frac{p+1}{p(s-r)}\tilde{\phi}(r) - \tilde{\phi}'(r) \tag{2.13}$$

and solve (2.13) for $\tilde{\phi}$ we get, for a fixed $r_0 \in]0, \min\{s, \frac{p}{p+1}\}[$,

$$\tilde{\phi}(r) = e^{\frac{p+1}{p}\int_{r_0}^r \frac{M(\sigma)}{s-\sigma}d\sigma}\Big(-\int_{r_0}^r \omega(\sigma)e^{-\frac{p+1}{p}\int_{r_0}^\sigma \frac{M(\tau)}{s-\tau}d\tau}d\sigma + c\Big)$$

for some $c \in \mathbb{R}$. Since $\omega > 0$, $\tilde{\phi}$ may vanish at most once in $]0, \min\{s, \frac{p}{p+1}\}[$ and the proof is complete. □

Using all the information obtained up to now, it is not difficult to derive the following result concerning the exactness of the number of positive solutions to (1.1).

Theorem 2.9. *Let $r^* \in \big]0, \frac{p}{p+1}\big[$ be as in Theorem 2.8 and let*

$$\lambda_1 := \mu^{\frac{p-s}{p-r}}\Big(2\Big(\frac{p-1}{p}\Big)^{1/p}\int_0^{t_0}\frac{dt}{(-F(t))^{1/p}}\Big)^{\frac{p(s-r)}{p-r}}. \tag{2.14}$$

Then:

- (i) *if $s \geq \frac{p}{p+1}$ and $r \in]0, r^*]$, or if $s < \frac{p}{p+1}$, problem (1.1) admits no positive solution for $\lambda < \lambda_1$ and a unique positive solution for $\lambda \geq \lambda_1$;*
- (ii) *if $s \geq \frac{p}{p+1}$ and $r \in \big]r^*, \frac{p}{p+1}\big[$, there exists $\lambda_0 \in]0, \lambda_1[$ such that problem (1.1) admits no positive solution for $\lambda < \lambda_0$; a unique solution for $\lambda = \lambda_0$; two solutions for $\lambda_0 < \lambda \leq \lambda_1$ and one solution for $\lambda > \lambda_1$.*

Remark 2.10. As pointed out in the introduction, the solution corresponding to λ_1 in case (i), and one of the solutions corresponding to λ_1 in case (ii), are of compacton-type. In both cases the compacton generates, for $\lambda > \lambda_1$, a continuum of non-negative solutions to (1.1) compactly supported in $]0,1[$. In addition, the positive solutions, excluding the compactons, satisfy Hopf's condition at $t=0$ and $t=1$.

Proof of Theorem 2.9. Let $L>0$ and consider the problem

$$\begin{gathered} -(|u'|^{p-2}u')' = u^{s-1} - u^{r-1} \quad \text{in }]0,L[\\ u(0)=u(L)=0. \end{gathered} \tag{2.15}$$

It is not difficult to verify that the rescaling

$$u(x) = \Big(\frac{\mu}{\lambda}\Big)^{\frac{1}{s-r}} v\Big(\mu^{\frac{s-p}{p(s-r)}}\lambda^{\frac{p-r}{p(s-r)}}x\Big) \tag{2.16}$$

turns any solution u to (1.1) into a solution v to (2.15), with

$$L := \mu^{\frac{s-p}{p(s-r)}}\lambda^{\frac{p-r}{p(s-r)}}, \tag{2.17}$$

and vice versa. Therefore let us focus on problem (2.15). It is well-known that any positive solution u to (2.15) is symmetric in $[0,L]$ and therefore

$$\sigma := \|u\|_\infty = u\big(\frac{L}{2}\big).$$

Multiplying the first equation in (2.15) by u' and integrating from $x\in]0,L/2[$ to $L/2$ we arrive at

$$\frac{p-1}{p}|u'(x)|^p = F(\sigma) - F(u(x)). \tag{2.18}$$

If we plug the value $x=0$ in the above relation, we deduce that $F(\sigma)\geq 0$, and, in light of Lemma 2.1, that $\sigma \geq t_0$. Being u increasing in $]0,L/2[$, from (2.18) one has

$$u'(x) = \Big(\frac{p}{p-1}\Big)^{1/p}(F(\sigma)-F(u(x)))^{1/p} \tag{2.19}$$

and therefore

$$\int_0^x \frac{u'(y)}{(F(\sigma)-F(u(y)))^{1/p}}dy = \Big(\frac{p}{p-1}\Big)^{1/p}x.$$

Putting $u(y)=t$ we get

$$\int_0^{u(x)} \frac{dt}{(F(\sigma)-F(t))^{1/p}} = \Big(\frac{p}{p-1}\Big)^{1/p}x, \tag{2.20}$$

which evaluated at $x=L/2$ yields

$$\psi(\sigma) = \frac{L}{2}\Big(\frac{p}{p-1}\Big)^{1/p}. \tag{2.21}$$

Hence the number of positive solutions to (2.15) amounts to the number of solutions to (2.21), as L varies in $]0,+\infty[$.

Starting from these premises, let us prove (i). If $s\geq \frac{p}{p+1}$ and $r\in]0,r^*]$, or if $s<\frac{p}{p+1}$, thanks to Theorem 2.8, ψ is increasing in a right neighbourhood of t_0 and, by Lemmas 2.5 and 2.7, it cannot admit any maximum. So the number of solutions to (2.21) is zero if $L<L_1 := 2(\frac{p-1}{p})^{1/p}\psi(t_0)$, one if $L\geq L_1$. Now, let us pass to (ii). If $s\geq \frac{p}{p+1}$ and $r\in]r^*,\frac{p}{p+1}[$, again due to Theorem 2.8 and Lemmas 2.5 and 2.7, ψ has a global minimum in $]t_0,+\infty[$. So (2.21) will admit no solution for $L<L_0 := 2\left(\frac{p-1}{p}\right)^{1/p}\min_{]t_0,+\infty[}\psi$, one solution for $L=L_0$ and $L>L_1$ and two solutions for $0<L_0\leq L_1$. The conclusion then follows using the rescaling (2.16)-(2.17). $\square$

Acknowledgements. The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

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Giovanni Anello
Department of Mathematics and Computer Sciences, Physical Sciences and Earth Sciences, University of Messina, Viale F. Stagno d'Alcontres 31, 98166 Messina, Italy
E-mail address: ganello@unime.it

Luca Vilasi
Department of Mathematics and Computer Sciences, Physical Sciences and Earth Sciences, University of Messina, Viale F. Stagno d'Alcontres 31, 98166 Messina, Italy
E-mail address: lvilasi@unime.it