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INITIAL-VALUE PROBLEMS FOR LINEAR DISTRIBUTED-ORDER DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. We solve the Cauchy problem for inhomogeneous distributed-order equations in a Banach space with a linear bounded operator in the right-hand side, with respect to the distributed Caputo derivative. First we find the solution by using the unique solvability theorem for the Cauchy problem. Then the results obtained are applied to the analysis of a distributed-order system of ordinary differential equations. Then we study an analogous equation, but with degenerate linear operator at the distributed derivative, which is called a degenerate equation. The pair of linear operators in the equation is assumed to be relatively bounded. For the two types of initial-value problems, we obtain the existence and uniqueness of a solution, and derive its form. Abstract results for the degenerate equations are used in the study of initial-boundary value problems with distributed order in time equations with polynomials of self-adjoint elliptic differential operator with respect to the spatial derivative.

1. Introduction

At the end of the previous and the beginning of this century, the interest in differential equations with distributed fractional derivatives has increased; see for example the works by Nakhushev [20, 21], Caputo [5, 6], and Pskhu [25, 26]. Such equations began to appear in various applied problems describing certain physical or technical processes: in the theory of viscoelasticity [19], in the kinetic theory [27], and so on (see, e.g., [2, 3, 5, 6]). At the same time, equations with distributed fractional derivatives began to be investigated from the mathematical point of view: unique solvability, qualitative behavior of solutions [1, 15], and numerical solutions of the corresponding boundary-value problems [7, 8]. We note the following works: Pskhu [25, 26] on the solvability and qualitative properties of both ordinary differential equations of distributed order, and the diffusion equation of distributed order in time; Umarov and Gorenflo [31] on the unique solvability of multipoint problems, including the Cauchy problem, to the equation with a distributed Caputo derivative in time and with pseudodifferential operators with respect to the space

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variables; and Kochubei [16] on the solvability of initial-boundary value problems to the multidimensional diffusion equation of distributed order in time.

This article consists of two parts. In the first part we consider the Cauchy problem for distributed-order equation with Caputo derivative

$$\int_a^b \omega(\alpha)\mathrm{D}_t^\alpha z(t)d\alpha = Az(t) + g(t), \quad t \in [0,T). \tag{1.1}$$

Here $m-1 < b < m \in \mathbb{N}$, $0 \leq a < b$, the operator A is linear and bounded on the Banach space $\mathfrak{Z}$, $T > 0$, $g : [0,T) \to \mathfrak{Z}$. In Section 2 we study the homogeneous equation, and in Section 3 the inhomogeneous equation. Unique solvability theorems for the Cauchy problem are proved, the form of the solution is obtained. The deduced general results are applied then to systems of distributed-order ordinary differential equations.

In the second part of this article, we consider the equation

$$\int_a^b \omega(\alpha)\mathrm{D}_t^\alpha Lx(t)d\alpha = Mx(t) + f(t), \quad t \in [0,T) \tag{1.2}$$

with the degenerate operator $L : \mathfrak{X} \to \mathfrak{Y}$, i.e. $\ker L \neq \{0\}$, and operator $M : D_M \to \mathfrak{Y}$ being (L,p)-bounded linear closed and densely defined in $\mathfrak{X}$ [29]. Here $m-1 < b < m \in \mathbb{N}$, $0 \leq a < b$, $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces, $T > 0$, $f{:}[0,T) \to \mathfrak{Y}$. For two types of initial value problems to equation (1.2), we obtain theorems for existence and uniqueness of a solution, and derive the form of the solution. Here we apply the theorem on the Cauchy problem for equation (1.1). Abstract results for (1.2) are used for the research of initial-boundary value problems unique solvability for distributed order in time equations with polynomials of self-adjoint elliptic differential operator with respect to the spatial variables.

This work is a continuation of the paper [28], in which the solvability of (1.1) with $b \leq 1$ with the unique Cauchy condition was studied. The results here develop the theory of resolving operators families for the distributed-order equations using the Laplace transform. This is done in the spirit of the operator semigroup theory [14] and its generalizations for integral evolution equations [17, 24], fractional order evolution equations [4, 18], including degenerate fractional order evolution equations, i.e. equations with a degenerate operator at the highest order derivative [9, 10, 11, 12, 13, 22, 23].

2. Cauchy problem for a homogeneous equation

For $\beta > 0$, $t > 0$ denote $g_\beta(t) := t^{\beta-1}/\Gamma(\beta)$,

$$J_t^\beta h(t) := \int_0^t g_\beta(t-s)h(s)ds = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}h(s)ds.$$

Let $m-1 < \alpha \leq m \in \mathbb{N}$, D_t^m is the usual m-th order derivative, D_t^α is the Caputo fractional derivative (see in details, for example, in [4]), i.e.

$$D_t^\alpha h(t) := D_t^m J_t^{m-\alpha}\Big(h(t) - \sum_{k=0}^{m-1} h^{(k)}(0)g_{k+1}(t)\Big).$$

Let $\overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{0\}$, and $\mathfrak{Z}$ be a Banach space. The Laplace transform of the function $h : \overline{\mathbb{R}}_+ \to \mathfrak{Z}$ is denoted by $\mathfrak{L}[h]$. The formula for the Laplace transform of the Caputo fractional derivative has the form

$$\mathfrak{L}[D_t^\alpha h](\lambda) = \lambda^\alpha \mathfrak{L}[h](\lambda) - \sum_{k=0}^{m-1} \lambda^{\alpha-k-1} h^{(k)}(0). \tag{2.1}$$

Denote by $\mathcal{L}(\mathfrak{Z})$ a Banach space of all linear continuous operators from $\mathfrak{Z}$ to $\mathfrak{Z}$. For $A \in \mathcal{L}(\mathfrak{Z})$ consider the Cauchy problem

$$z^{(k)}(0) = z_k, \quad k = 0, 1, \dots, m-1, \tag{2.2}$$

to the distributed-order equation

$$\int_a^b \omega(\alpha) \mathrm{D}_t^\alpha z(t) d\alpha = Az(t), \quad t \geq 0, \tag{2.3}$$

where D_t^α is the Caputo fractional derivative, $m-1 < b \leq m \in \mathbb{N}$, $0 \leq a < b$, ω:$(a,b) \to \mathbb{C}$. By a solution of problem (2.2), (2.3) we mean a function $z \in C^{m-1}(\overline{\mathbb{R}}_+; \mathfrak{Z})$, such that there exist $\int_a^b \omega(\alpha) \mathrm{D}_t^\alpha z(t) d\alpha \in C(\overline{\mathbb{R}}_+; \mathfrak{Z})$ and equalities (2.2) and (2.3) are satisfied.

We denote

$$\gamma := \cup_{k=1}^3 \gamma_k, \quad \gamma_1 := \{\lambda \in \mathbb{C} : |\lambda| = r_0, \arg\lambda \in (-\pi, \pi)\},$$
$$\gamma_2 := \{\lambda \in \mathbb{C} : \arg\lambda = \pi, \lambda \in [-r_0, -\infty)\},$$
$$\gamma_3 := \{\lambda \in \mathbb{C} : \arg\lambda = -\pi, \lambda \in (-\infty, -r_0]\},$$
$$W_c^d(\lambda) := \int_c^d \omega(\alpha)\lambda^\alpha d\alpha, \quad a_k := \max\{a, k\},$$
$$Z_k(t) := \frac{1}{2\pi i} \int_\gamma \frac{e^{\lambda t}}{\lambda^{k+1}} W_{a_k}^b(\lambda) \left(W_a^b(\lambda) I - A\right)^{-1} d\lambda, \quad k = 0, 1, \dots, m-1.$$

Denote by $E(K, a; \mathfrak{Z})$ the set of functions $x : \overline{\mathbb{R}}_+ \to \mathfrak{Z}$, for which there exist $K > 0$, $a \geq 0$ such that

$$\|z(t)\|_{\mathfrak{Z}} \leq K e^{at} \quad \forall t \in \overline{\mathbb{R}}_+ .$$

Also we will use the denotation

$$E(\mathfrak{Z}) := \cup_{K>0} \cup_{a \geq 0} E(K, a; \mathfrak{Z}).$$

Theorem 2.1. *Let $A \in \mathcal{L}(\mathfrak{Z})$, $z_k \in \mathfrak{Z}$, $k = 0, 1, \dots, m-1$, and for some $\beta > 1$ $W_a^b(\lambda)$, $W_k^b(\lambda)$, $k = 0, 1, \dots, m-1$, are holomorphic functions on the set $S_\beta := \{\lambda \in \mathbb{C} : |\lambda| \geq \beta, \arg\lambda \in (-\pi, \pi)\}$, satisfying the conditions*

$$\exists C_1 > 0 \quad \exists \delta > 0 \text{ such that } |W_a^b(\lambda)| \geq C_1 |\lambda|^{m-1+\delta}, \forall \lambda \in S_\beta, \tag{2.4}$$

$$\begin{gathered} \exists C_2 > 0 \text{ such that } |W_a^k(\lambda)||W_a^b(\lambda)|^{-1} \leq C_2 |\lambda|^{k-m+1-\delta} \\ \forall k \in \{0, 1, \dots, m-1\} \; \forall \lambda \in S_\beta, \end{gathered} \tag{2.5}$$

with $r_0 = \max\{\beta, (2\|A\|_{\mathcal{L}(\mathfrak{Z})}/C_1)^{1/\delta}\}$, $z_k \in \mathfrak{Z}$, $k = 0, 1, \dots, m-1$. Then the function $z(t) = \sum_{k=0}^{m-1} Z_k(t) z_k$ is a unique solution to (2.2), (2.3) in the space $E(\mathfrak{Z})$.

Proof. For $\lambda \in \gamma$ with the given r_0 the inequality $|W_a^b(\lambda)| \geq 2\|A\|_{\mathcal{L}(\mathfrak{Z})}$ holds. Then there exists $(W_a^b(\lambda) I - A)^{-1} \in \mathcal{L}(\mathfrak{Z})$, and for $k = 0, 1, \dots, m-1$,

$$\|W_{a_k}^b(\lambda) \left(W_a^b(\lambda) I - A\right)^{-1}\|_{\mathcal{L}(\mathfrak{Z})} \leq \frac{|W_{a_k}^b(\lambda)|}{|W_a^b(\lambda)|} \frac{1}{1 - \frac{\|A\|_{\mathcal{L}(\mathfrak{Z})}}{|W_a^b(\lambda)|}} \leq 2(1 + C_2). \tag{2.6}$$

Indeed, by condition (2.5),

$$\frac{|W^b_{a_k}(\lambda)|}{|W^b_a(\lambda)|}=\Big|1-\frac{W^{a_k}_a(\lambda)}{W^b_a(\lambda)}\Big|\le 1+C_2r_0^{a_k-m+1-\delta}\le 1+C_2.$$

Here $W^{a_k}_a\equiv 0$, if $k\le a$. Thus, at $t>0$ the integrals $Z_k(t)$ converge for $k=0,1,\dots,m-1$.

Let $R>r_0$,

$$\begin{gathered}\Gamma_R=\cup_{k=1}^4\Gamma_{k,R},\\ \Gamma_{1,R}=\gamma_1,\quad \Gamma_{2,R}=\{\lambda\in\mathbb{C}:|\lambda|=R,\,\arg\lambda\in(-\pi,\pi)\},\\ \Gamma_{3,R}=\{\lambda\in\mathbb{C}:\arg\lambda=\pi,\,\lambda\in[-r_0,-R]\},\\ \Gamma_{4,R}=\{\lambda\in\mathbb{C}:\arg\lambda=-\pi,\,\lambda\in[-R,-r_0]\},\end{gathered}$$

and let Γ_R be the closed loop, oriented counter-clockwise. Consider also the contours

$$\begin{gathered}\Gamma_{5,R}=\{\lambda\in\mathbb{C}:\arg\lambda=\pi,\,\lambda\in(-R,-\infty)\},\\ \Gamma_{6,R}=\{\lambda\in\mathbb{C}:\arg\lambda=-\pi,\,\lambda\in(-\infty,-R)\}.\end{gathered}$$

Then $\gamma=\Gamma_{5,R}\cup\Gamma_{6,R}\cup\Gamma_R\setminus\Gamma_{2,R}$.

For $t\ge 0$, $k=0,1,\dots,m-1$, $l=0,1,\dots,k-1$,

$$Z_k^{(l)}(t)=\frac{1}{2\pi i}\int_\gamma\frac{e^{\lambda t}}{\lambda^{k+1-l}}W^b_{a_k}(\lambda)\left(W^b_a(\lambda)I-A\right)^{-1}\,d\lambda,$$

$Z_k^{(l)}(0)=0$, by the Cauchy Theorem

$$\frac{1}{2\pi i}\int_{\Gamma_R}\frac{1}{\lambda^{k+1-l}}W^b_{a_k}(\lambda)\left(W^b_a(\lambda)I-A\right)^{-1}\,d\lambda=0,$$

and by inequality (2.6), we have

$$\begin{gathered}\Big\|\int_{\Gamma_{2,R}}\frac{1}{\lambda^{k+1-l}}W^b_{a_k}(\lambda)\left(W^b_a(\lambda)I-A\right)^{-1}\,d\lambda\Big\|_{\mathcal{L}(\mathfrak{Z})}\le\frac{4\pi(1+C_2)}{R},\\ \Big\|\int_{\Gamma_{s,R}}\frac{1}{\lambda^{k+1-l}}W^b_{a_k}(\lambda)\left(W^b_a(\lambda)I-A\right)^{-1}d\lambda\Big\|_{\mathcal{L}(\mathfrak{Z})}\le\frac{2(1+C_2)}{R},\quad s=5,6.\end{gathered}$$

Therefore, the integrals in the two last inequalities tend to zero as $R\to\infty$, and

$$\begin{aligned}Z_k^{(l)}(0)&=\lim_{R\to\infty}\frac{1}{2\pi i}\Big(\int_{\Gamma_R}-\int_{\Gamma_{2,R}}+\int_{\Gamma_{5,R}}+\int_{\Gamma_{6,R}}\Big)\frac{W^b_{a_k}(\lambda)}{\lambda^{k+1-l}}\left(W^b_a(\lambda)I-A\right)^{-1}\,d\lambda\\ &=0.\end{aligned}$$

For $t>0$ and $k=0,1,\dots,m-1$, we have

$$\begin{aligned}Z_k^{(k)}(t)&=\frac{1}{2\pi i}\int_\gamma\frac{e^{\lambda t}}{\lambda}\frac{W^b_{a_k}(\lambda)}{W^b_a(\lambda)}\sum_{k=0}^\infty W^b_a(\lambda)^{-k}A^k\,d\lambda\\ &=\frac{1}{2\pi i}\int_\gamma\frac{e^{\lambda t}}{\lambda}\Big(1-\frac{W^{a_k}_a(\lambda)}{W^b_a(\lambda)}\Big)d\lambda I+\frac{1}{2\pi i}\int_\gamma\frac{e^{\lambda t}}{\lambda}\frac{W^b_{a_k}(\lambda)}{W^b_a(\lambda)}\sum_{k=1}^\infty W^b_a(\lambda)^{-k}A^k\,d\lambda\\ &=I-\frac{1}{2\pi i}\int_\gamma\frac{e^{\lambda t}}{\lambda}\frac{W^{a_k}_a(\lambda)}{W^b_a(\lambda)}d\lambda I+\frac{1}{2\pi i}\int_\gamma\frac{e^{\lambda t}}{\lambda}\frac{W^b_{a_k}(\lambda)}{W^b_a(\lambda)}\sum_{k=1}^\infty W^b_a(\lambda)^{-k}A^k\,d\lambda.\end{aligned}$$

For $t \in [0,1]$ and $\lambda \in \gamma$, by conditions (2.4) and (2.5) we have

$$\Big|\frac{e^{\lambda t}}{\lambda}\frac{W_a^{a_k}(\lambda)}{W_a^b(\lambda)}\Big| \le \frac{C_2 e^{r_0}}{|\lambda|^{1+\delta}},$$

$$\Big\|\frac{e^{\lambda t}}{\lambda}\frac{W_{a_k}^b(\lambda)}{W_a^b(\lambda)}\sum_{k=1}^{\infty} W_a^b(\lambda)^{-k}A^k\Big\|_{\mathcal{L}(3)} \le \frac{4C_1^{-1}(1+C_2)e^{r_0}\|A\|_{\mathcal{L}(3)}}{|\lambda|^{m+\delta}};$$

therefore,

$$\Big|\frac{1}{2\pi i}\int_\gamma \frac{e^{\lambda t}}{\lambda}\frac{W_a^{a_k}(\lambda)}{W_a^b(\lambda)}d\lambda\Big| \le \frac{C_2 e^{r_0}}{r_0^\delta} + \frac{C_2 e^{r_0}}{\pi\delta r_0^\delta},$$

$$\begin{aligned}
&\Big\|\frac{1}{2\pi i}\int_\gamma \frac{e^{\lambda t}}{\lambda}\frac{W_{a_k}^b(\lambda)}{W_a^b(\lambda)}\sum_{k=1}^{\infty} W_a^b(\lambda)^{-k}A^k\,d\lambda\Big\|_{\mathcal{L}(3)}\\
&\le \frac{4C_1^{-1}(1+C_2)e^{r_0}\|A\|_{\mathcal{L}(3)}}{r_0^{m-1+\delta}} + \frac{4C_1^{-1}(1+C_2)e^{r_0}\|A\|_{\mathcal{L}(3)}}{\pi\delta r_0^{m-1+\delta}}
\end{aligned}$$

Consequently, the integrals converge uniformly with respect to $t \in [0,1]$. By continuity

$$\begin{aligned}
Z_k^{(k)}(0) &= I - \frac{1}{2\pi i}\int_\gamma \frac{1}{\lambda}\frac{W_a^{a_k}(\lambda)}{W_a^b(\lambda)}d\lambda I + \frac{1}{2\pi i}\int_\gamma \frac{1}{\lambda}\frac{W_{a_k}^b(\lambda)}{W_a^b(\lambda)}\sum_{k=1}^{\infty} W_a^b(\lambda)^{-k}A^k\,d\lambda\\
&= I - \lim_{R\to\infty}\frac{1}{2\pi i}\Big(\int_{\Gamma_R} - \int_{\Gamma_{2,R}} + \int_{\Gamma_{5,R}} + \int_{\Gamma_{6,R}}\Big)\frac{1}{\lambda}\frac{W_a^{a_k}(\lambda)}{W_a^b(\lambda)}d\lambda I\\
&\quad + \lim_{R\to\infty}\frac{1}{2\pi i}\Big(\int_{\Gamma_R} - \int_{\Gamma_{2,R}} + \int_{\Gamma_{5,R}} + \int_{\Gamma_{6,R}}\Big)\frac{1}{\lambda}\frac{W_{a_k}^b(\lambda)}{W_a^b(\lambda)}\sum_{k=1}^{\infty} W_a^b(\lambda)^{-k}A^k\,d\lambda\\
&= I.
\end{aligned}$$

By the Cauchy Theorem,

$$-\frac{1}{2\pi i}\int_{\Gamma_R}\frac{1}{\lambda}\frac{W_a^{a_k}(\lambda)}{W_a^b(\lambda)}d\lambda I + \frac{1}{2\pi i}\int_{\Gamma_R}\frac{1}{\lambda}\frac{W_{a_k}^b(\lambda)}{W_a^b(\lambda)}\sum_{k=1}^{\infty} W_a^b(\lambda)^{-k}A^k\,d\lambda = 0,$$

and

$$\Big|\frac{1}{2\pi i}\int_{\Gamma_{2,R}}\frac{1}{\lambda}\frac{W_a^{a_k}(\lambda)}{W_a^b(\lambda)}d\lambda\Big| \le \frac{C_2}{R^\delta},$$

$$\Big|\frac{1}{2\pi i}\int_{\Gamma_{s,R}}\frac{1}{\lambda}\frac{W_a^{a_k}(\lambda)}{W_a^b(\lambda)}d\lambda\Big| \le \frac{C_2}{2\pi\delta R^\delta},$$

$$\Big\|\frac{1}{2\pi i}\int_{\Gamma_{2,R}}\frac{1}{\lambda}\frac{W_{a_k}^b(\lambda)}{W_a^b(\lambda)}\sum_{k=1}^{\infty} W_a^b(\lambda)^{-k}A^k\,d\lambda\Big\|_{\mathcal{L}(3)} \le \frac{4C_1^{-1}(1+C_2)\|A\|_{\mathcal{L}(3)}}{R^\delta},$$

$$\Big\|\frac{1}{2\pi i}\int_{\Gamma_{s,R}}\frac{1}{\lambda}\frac{W_{a_k}^b(\lambda)}{W_a^b(\lambda)}\sum_{k=1}^{\infty} W_a^b(\lambda)^{-k}A^k\,d\lambda\Big\|_{\mathcal{L}(3)} \le \frac{2C_1^{-1}(1+C_2)\|A\|_{\mathcal{L}(3)}}{\pi\delta R^\delta},$$

for $s = 5, 6$.

For $t \ge 0$, $k = 0, 1, \dots, m-1$, $l = k+1, k+2, \dots, m-1$ we have

$$Z_k^{(l)}(t) = \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{l-k-1}W_{a_k}^b(\lambda)\left(W_a^b(\lambda)I - A\right)^{-1}\,d\lambda$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{l-k-1}\left(W_a^b(\lambda)-W_a^{a_k}(\lambda)\right)\left(W_a^b(\lambda)I-A\right)^{-1}d\lambda \\
&= \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{l-k-1}d\lambda I+\frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{l-k-1}\left(W_a^b(\lambda)I-A\right)^{-1}A\,d\lambda \\
&\quad -\frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{l-k-1}W_a^{a_k}(\lambda)\left(W_a^b(\lambda)I-A\right)^{-1}d\lambda.
\end{aligned}
$$

Hence, $Z_k^{(l)}(0)=0$, since by the Cauchy Theorem,

$$
\begin{gathered}
\frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{l-k-1}d\lambda=0, \\
\frac{1}{2\pi i}\int_{\Gamma_R} e^{\lambda t}\lambda^{l-k-1}\left(W_a^b(\lambda)I-A\right)^{-1}A\,d\lambda=0, \\
\frac{1}{2\pi i}\int_{\Gamma_R} e^{\lambda t}\lambda^{l-k-1}W_a^{a_k}(\lambda)\left(W_a^b(\lambda)I-A\right)^{-1}d\lambda=0,
\end{gathered}
$$

and under conditions (2.4) and (2.5),

$$
\begin{gathered}
\Big\|\int_{\Gamma_{2,R}}\lambda^{l-k-1}\left(W_a^b(\lambda)I-A\right)^{-1}A\,d\lambda\Big\|_{\mathcal{L}(\mathfrak{Z})}\le\frac{4\pi C_1^{-1}\|A\|_{\mathcal{L}(\mathfrak{Z})}}{R^\delta}, \\
\Big\|\int_{\Gamma_{s,R}}\lambda^{l-k-1}\left(W_a^b(\lambda)I-A\right)^{-1}A\,d\lambda\Big\|_{\mathcal{L}(\mathfrak{Z})}\le\frac{2C_1^{-1}\|A\|_{\mathcal{L}(\mathfrak{Z})}}{\delta R^\delta}, \\
\Big\|\int_{\Gamma_{2,R}}\lambda^{l-k-1}W_a^{a_k}(\lambda)\left(W_a^b(\lambda)I-A\right)^{-1}d\lambda\Big\|_{\mathcal{L}(\mathfrak{Z})}\le\frac{4\pi C_2}{R^\delta}, \\
\Big\|\int_{\Gamma_{s,R}}\lambda^{l-k-1}W_a^{a_k}(\lambda)\left(W_a^b(\lambda)I-A\right)^{-1}d\lambda\Big\|_{\mathcal{L}(\mathfrak{Z})}\le\frac{2C_2}{\delta R^\delta},\quad s=5,6.
\end{gathered}
$$

Thus, $Z_k\in C^{m-1}(\overline{\mathbb{R}}_+;\mathcal{L}(\mathfrak{Z}))$, $k=0,1,\dots,m-1$, the function $z(t)=\sum_{k=0}^{m-1}Z_k(t)z_k$ satisfy the Cauchy conditions (2.2).

By construction, and estimate (2.5), we have

$$
\|Z_k(t)\|_{\mathcal{L}(\mathfrak{Z})}\le\frac{C_2+1}{\pi}\int_\gamma\frac{e^{t\operatorname{Re}\lambda}}{|\lambda|}\,ds\le K_ke^{r_0t},
$$

because

$$
\begin{gathered}
\frac{1}{\pi}\int_{\gamma_1}\frac{e^{t\operatorname{Re}\lambda}}{|\lambda|}\,ds\le\frac{e^{r_0t}}{\pi}\int_0^{2\pi}e^{r_0t(\cos\varphi-1)}\,d\varphi\le 2e^{r_0t}, \\
\frac{1}{\pi}\int_{\gamma_k}\frac{e^{t\operatorname{Re}\lambda}}{|\lambda|}\,ds\le\frac{e^{r_0t}}{\pi}\int_{-\infty}^{-r_0}\frac{e^x}{|x|}dx\le C_3e^{r_0t},\quad k=2,3,\quad t\ge 1.
\end{gathered}
$$

Therefore, we can take

$$
K_k=2(C_2+1)+2C_2\max\Big\{\frac{1}{\pi}\int_{-\infty}^{-r_0}\frac{e^x}{|x|}dx,\max_{t\in[0,1]}e^{-r_0t}\|Z_k(t)\|_{\mathcal{L}(\mathfrak{Z})}\Big\}.
$$

Thus, $\|z(t)\|_{\mathfrak{Z}}\le e^{r_0t}\sum_{k=0}^{m-1}K_k\|z_k\|_{\mathfrak{Z}}$, i.e. $x\in E(\mathfrak{Z})$.

Under the condition $\operatorname{Re}\mu>r_0$ we have the equality

$$
\mathfrak{L}[x](\mu)=\sum_{k=0}^{m-1}\frac{1}{2\pi i}\int_\gamma\frac{W_{a_k}^b(\lambda)}{\lambda^{k+1}(\mu-\lambda)}\left(W_a^b(\lambda)I-A\right)^{-1}z_kd\lambda.
$$

By (2.5) these integrals converge and

$$\lim_{R\to\infty}\sum_{k=0}^{m-1}\frac{1}{2\pi i}\int_{\Gamma_{s,R}}\frac{W^b_{a_k}(\lambda)}{\lambda^{k+1}(\mu-\lambda)}\left(W^b_a(\lambda)I-A\right)^{-1}z_k d\lambda=0,\quad s=2,5,6.$$

Therefore, by the Cauchy integral formula,

$$\begin{aligned}\mathfrak{L}[x](\mu)&=\lim_{R\to\infty}\sum_{k=0}^{m-1}\frac{1}{2\pi i}\int_{\Gamma_R}\frac{W^b_{a_k}(\lambda)}{\lambda^{k+1}(\mu-\lambda)}\left(W^b_a(\lambda)I-A\right)^{-1}z_k d\lambda\\&=\sum_{k=0}^{m-1}\frac{W^b_{a_k}(\mu)}{\mu^{k+1}}\left(W^b_a(\mu)I-A\right)^{-1}z_k.\end{aligned}$$

Hence, $\mathfrak{L}[x](\mu)$ has a holomorphic extension on $\{\mu\in\mathbb{C}:|\mu|>r_0,\ \arg\mu\in(-\pi,\pi)\}$, because the resolvent of the operator A is holomorphic there.

Further, using formula (2.1) for the Laplace transform, we can write

$$\begin{aligned}&\mathfrak{L}\Big[\int_a^b\omega(\alpha)\mathrm{D}^\alpha_t z(t)d\alpha\Big](\mu)\\&=\sum_{k=0}^{m-1}\frac{W^b_{a_k}(\mu)}{\mu^{k+1}}W^b_a(\mu)\left(W^b_a(\mu)I-A\right)^{-1}z_k-\sum_{k=0}^{m-1}\frac{W^b_{a_k}(\mu)}{\mu^{k+1}}z_k\\&=A\sum_{k=0}^{m-1}\frac{W^b_{a_k}(\mu)}{\mu^{k+1}}(\mu)\left(W^b_a(\mu)I-A\right)^{-1}z_k=A\mathfrak{L}[x](\mu).\end{aligned}$$

Here the commutation of an operator and its resolvent was taken into account. We can apply the inverse Laplace transform on the both parts of the equality and obtain equality (2.3) in all continuity points of function x, i.e. for all $t\geq 0$. It was proved, that $x\in C(\overline{\mathbb{R}}_+;\mathfrak{Z})$, hence, by the continuity of the operator A, the right-hand side of equation (2.3) is continuous on $\overline{\mathbb{R}}_+$, and therefore, the left-hand side of the equation is continuous also and the function x is a solution of problem (2.2), (2.3).

If there are two solutions z_1, z_2 of problem (2.2), (2.3) from the class $E(\mathfrak{Z})$, then their difference $y=z_1-z_2\in E(\mathfrak{Z})$ is a solution of equation (2.3) and satisfy the initial conditions $y^{(k)}(0)=0$, $k=0,1,\dots,m-1$. Applying the Laplace transform to the both sides of equation (2.3) gives the equality $W^b_a(\lambda)\mathfrak{L}[y](\lambda)=A\mathfrak{L}[y](\lambda)$. Therefore, for $|\lambda|>\beta$ we have $\mathfrak{L}[y](\lambda)\equiv 0$. It means that $y\equiv 0$. $\square$

Remark 2.2. Under the conditions of Theorem 2.1, the families of operators

$$\{Z_k(t)\in\mathcal{L}(\mathfrak{Z}):t\in\overline{\mathbb{R}}_+\},\quad k=0,1,\dots,m-1,$$

and, therefore, the solution $z(t)=\sum_{k=0}^{m-1}Z_k(t)z_k$ of problem (2.2), (2.3) have holomorphic extensions to the right half-plane $\{t\in\mathbb{C}:\operatorname{Re}t>0\}$. Indeed, as was seen in the proof of Theorem 2.1 that the integrals

$$\frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{l-k-1}W^b_{a_k}(\lambda)\left(W^b_a(\lambda)I-A\right)^{-1}d\lambda,\quad k=0,1,\dots,m-1,$$

with $l\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$, converge uniformly on arbitrary compact set from the right half-plane $\{t\in\mathbb{C}:\operatorname{Re}t>0\}$.

Remark 2.3. We can consider equation (2.3) with $a < 0$, where $\mathrm{D}_t^\alpha := J_t^{-\alpha}$ for $\alpha < 0$ by definition. Then Theorem 2.1 on the unique solution of (2.2), (2.3) is valid also, and $a_k = k$, $k = 0, 1, \dots, m-1$.

Remark 2.4. It is easy to show, that, for example, the functions $\omega(\alpha) = \alpha^n$, $n \in \mathbb{N}$, or $\omega(\alpha) = c^\alpha$, $c > 0$, satisfy conditions of Theorem 2.1.

The following general assertion holds.

Proposition 2.5. *Let a function $\omega : (a, b) \to \mathbb{R}$ be bounded, and for some $\varepsilon \in (0, b-a)$ in the left ε-neighborhood of the point b, it does not change the sign and there exists $c_1 > 0$ such that for all $\alpha \in (b-\varepsilon, b)$ we have $|\omega(\alpha)| \geq c_1$. Then conditions (2.4) and (2.5) with arbitrary $\delta \in (0, b-m+1)$ hold.*

Proof. If ω is bounded in some interval $(c, d) \subset (a, b)$, then

$$\Big|\int_c^d \omega(\alpha)\lambda^\alpha d\alpha\Big| \leq |\lambda|^d (d-c) \sup_{c<\alpha<d} |\omega(\alpha)| . \tag{2.7}$$

Also, for c sufficiently close to b,

$$\begin{aligned}
&\Big|\int_c^b \omega(\alpha)\lambda^\alpha d\alpha\Big| \\
&= \Big|\int_c^b \omega(\alpha)|\lambda|^\alpha e^{i\alpha \arg\lambda} d\alpha\Big| \\
&= \Big(\Big|\int_c^b \omega(\alpha)|\lambda|^\alpha \cos(\alpha\arg\lambda)d\alpha\Big|^2 + \Big|\int_c^b \omega(\alpha)|\lambda|^\alpha \sin(\alpha\arg\lambda)d\alpha\Big|^2\Big)^{1/2} \\
&\geq \frac{1}{\sqrt{2}}\Big(\Big|\int_c^b \omega(\alpha)|\lambda|^\alpha \cos(\alpha\arg\lambda)d\alpha\Big| + \Big|\int_c^b \omega(\alpha)|\lambda|^\alpha \sin(\alpha\arg\lambda)d\alpha\Big|\Big) \\
&= \frac{1}{\sqrt{2}}\int_c^b |\omega(\alpha)||\lambda|^\alpha (|\cos(\alpha\arg\lambda)| + |\sin(\alpha\arg\lambda)|)d\alpha \\
&\geq \frac{1}{\sqrt{2}}\int_c^b |\omega(\alpha)||\lambda|^\alpha d\alpha,
\end{aligned}$$

because $\omega(\alpha)$, $\cos(\alpha\arg\lambda)$ and $\sin(\alpha\arg\lambda)$ do not change the sign for α from a sufficiently small left neighborhood of the point b, and

$$|\cos(\alpha\arg\lambda)| + |\sin(\alpha\arg\lambda)| = \sqrt{1 + 2|\cos(\alpha\arg\lambda)||\sin(\alpha\arg\lambda)|} \geq 1.$$

Therefore, for sufficiently small $\varepsilon_1 \in (0, \min\{\varepsilon, 2(b-m+1)\})$ and for sufficiently large $|\lambda|$, we have

$$\begin{aligned}
&\Big|\int_a^b \omega(\alpha)\lambda^\alpha d\alpha\Big| \\
&\geq \frac{1}{\sqrt{2}}\int_{b-\varepsilon_1}^b |\omega(\alpha)|\,|\lambda|^\alpha d\alpha - |\lambda|^{b-\varepsilon_1}(b-\varepsilon_1-a) \sup_{a<\alpha<b-\varepsilon_1} |\omega(\alpha)| \\
&\geq \frac{c_1}{\sqrt{2}}\frac{|\lambda|^b - |\lambda|^{b-\varepsilon_1}}{\ln|\lambda|} - C_1|\lambda|^{b-\varepsilon_1} \\
&\geq 2C_1|\lambda|^{b-\varepsilon_1/2} - C_1|\lambda|^{b-\varepsilon_1} = C_1|\lambda|^{b-\varepsilon_1/2}.
\end{aligned} \tag{2.8}$$

Denotting $\delta = b - m + 1 - \varepsilon_1/2 > 0$, the above inequality implies condition (2.4).

From inequalities (2.4) and (2.7) it follows, that for $k > a$

$$|W_a^k(\lambda)||W_a^b(\lambda)|^{-1} \le C_1^{-1}|\lambda|^{k-m+1-\delta}(m-1-a)\sup_{a<\alpha<b}|\omega(\alpha)|.$$

Hence, condition (2.5) is obtained. □

Corollary 2.6. *Let $\omega \in C([a,b];\mathbb{R})$ and $\omega(b) \ne 0$. Then conditions (2.4), (2.5) with arbitrary $\delta \in (0, b-m+1)$ hold.*

Indeed, all assumptions of Proposition 2.5 hold.

2.1. **Example.** Consider the problem

$$\frac{\partial^k v}{\partial t^k}(s,0) = v_k(s), \quad s \in \Omega,\ k = 0,1,\dots,m-1, \tag{2.9}$$

$$\int_a^b \omega(\alpha)\mathrm{D}_t^\alpha v(s,t)d\alpha = \int_\Omega K(s,\xi)Bv(\xi,t)d\xi, \quad (s,t) \in \Omega \times \overline{\mathbb{R}}_+. \tag{2.10}$$

Here $\Omega \subset \mathbb{R}^d$ is a bounded region, $a < b$, $0 < m-1 < b \le m$, $\omega : (a,b) \to \mathbb{R}$, B is a $(n \times n)$-matrix, $K : \Omega \times \Omega \to \mathbb{R}^n$ are given, $v(s,t) = (v_1(s,t), v_2(s,t), \dots, v_n(s,t))$ is an unknown vector-function.

We take $\mathfrak{Z} = L_2(\Omega)^n$, $(Aw)(s) = \int_\Omega K(s,\xi)Bw(\xi)d\xi$ for vector-function $w = (w_1, w_2, \dots, w_n) \in L_2(\Omega)^n$. Then $A \in \mathcal{L}(L_2(\Omega)^n)$, and if the function ω satisfies the conditions of Theorem 2.1, problem (2.9), (2.10) has a unique solution from the class $E(L_2(\Omega)^n)$.

3. Inhomogeneous equation

A solution of problem (2.2) for the equation

$$\int_a^b \omega(\alpha)\mathrm{D}_t^\alpha z(t)d\alpha = Az(t) + g(t), \quad t \in [0,T), \tag{3.1}$$

where D_t^α is the Caputo fractional derivative, $m-1 < b \le m \in \mathbb{N}$, $a \in [0,b)$, $\omega : (a,b) \to \mathbb{C}$, $T > 0$, $g \in C([0,T];\mathfrak{Z})$, is called a function $z \in C^{m-1}([0,T);\mathfrak{Z})$, such that there exists $\int_a^b \omega(\alpha)\mathrm{D}_t^\alpha z(t)d\alpha \in C([0,T);\mathfrak{Z})$ and equalities (2.2) and (3.1) are valid. Denote

$$Z(t) := \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\left(W_a^b(\lambda)I - A\right)^{-1} d\lambda. \tag{3.2}$$

This integral converges at $t > 0$.

Lemma 3.1. *Let $A \in \mathcal{L}(\mathfrak{Z})$, $g \in C([0,T];\mathfrak{Z})$, and for some $\beta > 1$ $W_a^b(\lambda)$ is holomorphic function on the set $S_\beta := \{\lambda \in \mathbb{C} : |\lambda| \ge \beta, \arg\lambda \in (-\pi,\pi)\}$, satisfying the condition (2.4), $r_0 = \max\{\beta, (2\|A\|_{\mathcal{L}(\mathfrak{Z})}/C_1)^{1/\delta}\}$. Then the function $z_g(t) = \int_0^t Z(t-s)g(s)ds$ is a unique solution to problem (2.2), (3.1) with $z_k = 0$, $k = 0,1,\dots,m-1$, from the class $E(\mathfrak{Z})$.*

Proof. It is easy to show that the integrals

$$Z^{(k)}(t) := \frac{1}{2\pi i}\int_\gamma \lambda^k e^{\lambda t}\left(W_a^b(\lambda)I - A\right)^{-1} d\lambda, \quad k = 0,1,\dots,m-1,$$

converge uniformly with respect to t on every compact set from the half-plane $\{t \in \mathbb{C} : \operatorname{Re} t > 0\}$, therefore, $Z(t)$ can be holomorphically extended onto this half-plane. A more difficult question is the behavior of this functions at zero. Let us consider it.

For $t \in [0,1]$ we have

$$\int_{\gamma_1} \frac{e^{t \operatorname{Re}\lambda}}{|\lambda|^{m-1+\delta}}\, ds \le 2\pi r_0^{2-m-\delta} e^{r_0},$$
$$\int_{\gamma_k} \frac{e^{t \operatorname{Re}\lambda}}{|\lambda|^{m-1+\delta}}\, ds \le \int_{-\infty}^{-r_0} \frac{dx}{|x|^{m-1+\delta}} = \frac{r_0^{2-m-\delta}}{2-m-\delta},$$

$k = 2,3$, for $b > 1$ and, consequently, $m \ge 2$. Hence, integral (3.2) converges uniformly with respect to $t \in [0,1]$, and there exists the limit

$$\lim_{t \to 0+} Z(t) = \frac{1}{2\pi i} \int_\gamma \left(W_a^b(\lambda) I - A\right)^{-1} d\lambda := Z(0).$$

Analogously for $k = 1, 2, \dots, m-2$ we have the limit

$$\lim_{t \to 0+} Z^{(k)}(t) = \frac{1}{2\pi i} \int_\gamma \lambda^k \left(W_a^b(\lambda) I - A\right)^{-1} d\lambda := Z^{(k)}(0).$$

Moreover,

$$\begin{aligned} Z^{(k)}(0) &= \frac{1}{2\pi i} \int_\gamma \lambda^k \left(W_a^b(\lambda) I - A\right)^{-1} d\lambda \\ &= \frac{1}{2\pi i} \Big(\int_{\Gamma_R} + \int_{\Gamma_{5,R}} + \int_{\Gamma_{6,R}} - \int_{\Gamma_{2,R}} \Big) \lambda^k \left(W_a^b(\lambda) I - A\right)^{-1} d\lambda \to 0 \end{aligned}$$

as $R \to \infty$ by the Cauchy Theorem and estimates

$$\|\lambda^k \left(W_a^b(\lambda) I - A\right)^{-1}\|_{\mathcal{L}(\mathfrak{Z})} \le \frac{2}{|\lambda|^{1+\delta}}, \quad k = 0, 1, \dots, m-2$$

(see the proof of Theorem 2.1). Thus, $Z^{(k)}(0) = 0$ for $k = 0, 1, \dots, m-2$.

It remains to consider $\lim_{t \to 0+} Z^{(m-1)}(t)$. For $t \in [0,1]$ we have

$$\int_{\gamma_1} \frac{e^{t \operatorname{Re}\lambda}}{|\lambda|^{\delta}}\, ds \le 2\pi r_0^{1-\delta} e^{r_0},$$
$$\int_{\gamma_k} \frac{e^{t \operatorname{Re}\lambda}}{|\lambda|^{\delta}}\, ds \le \int_{-\infty}^{-r_0} \frac{e^{tx} dx}{|x|^{\delta}} = t^{\delta-1} \int_{tr_0}^{+\infty} \frac{e^{-y} dy}{y^\delta} \le \Gamma(1-\delta) t^{\delta-1}, \quad k = 2,3,$$

Thus, $\|Z^{(m-1)}(t)\|_{\mathcal{L}(\mathfrak{Z})} = O(t^{\delta-1})$ as $t \to 0+$.

Further, for $k = 0, 1, \dots, m-2$ we have

$$z_g^{(k)}(t) = 0 + \int_0^t Z^{(k)}(t-s) g(s) ds,$$

$\|z_g^{(m-1)}(t)\| \le C t^\delta \to 0$ as $t \to 0+$. Thus, zero initial conditions (2.2) are fulfilled.

Define $g(t) = 0$ for $t \ge T$, then we have the convolution $z_g = X * g$, and $\mathfrak{L}[z_g] = \mathfrak{L}[X]\mathfrak{L}[g]$. Arguing as in the proof of Theorem 2.1, we obtain $\mathfrak{L}[X](\mu) = \left(W_a^b(\mu) I - A\right)^{-1}$. From condition (2.4) it follows that

$$\|\left(W_a^b(\lambda) I - A\right)^{-1}\|_{\mathcal{L}(\mathfrak{Z})} \le \frac{2C_1^{-1}}{|\lambda|^{m-1+\delta}}, \quad \|\frac{1}{\mu - \lambda}\left(W_a^b(\lambda) I - A\right)^{-1}\|_{\mathcal{L}(\mathfrak{Z})} \le \frac{C}{|\lambda|^{m+\delta}},$$

for $m+\delta>1$. Hence,

$$\begin{aligned}\mathfrak{L}\Big[\int_a^b \omega(\alpha)\mathrm{D}_t^\alpha z_g d\alpha\Big](\mu) &= W_a^b(\mu)\left(W_a^b(\mu)I-A\right)^{-1}\mathfrak{L}[g](\mu)\\ &= \mathfrak{L}[g](\mu)+A\left(W_a^b(\mu)I-A\right)^{-1}\mathfrak{L}[g](\mu).\end{aligned}$$

Acting by the inverse Laplace transform on the both sides of this equality, obtain

$$\int_a^b \omega(\alpha)\mathrm{D}_t^\alpha z_g(t)d\alpha = g(t)+A(X*g)(t)=g(t)+Az_g(t)$$

due to the continuity of the linear operator A.

The proof of the solution uniqueness reduces in an obvious way to the proof of uniqueness for the homogeneous equation. □

The next theorem follows from Theorem 2.1 and Lemma 3.1.

Theorem 3.2. *Let $A\in\mathcal{L}(\mathfrak{Z})$, $g\in C([0,T);\mathfrak{Z})$, $z_k\in\mathfrak{Z}$, $k=0,1,\dots,m-1$, and for some $\beta>1$ $W_a^b(\lambda)$, $W_k^b(\lambda)$, $k=0,1,\dots,m-1$, are holomorphic functions on the set $S_\beta:=\{\lambda\in\mathbb{C}:|\lambda|\geq\beta,\arg\lambda\in(-\pi,\pi)\}$, satisfying conditions* (2.4), (2.5), *$r_0=\max\{\beta,(2\|A\|_{\mathcal{L}(\mathfrak{Z})}/C_1)^{1/\delta}\}$. Then the function*

$$z(t)=\sum_{k=0}^{m-1}Z_k(t)z_k+\int_0^t Z(t-s)g(s)ds$$

is a unique solution to problem (2.2), (3.1) *from the class $E(\mathfrak{Z})$.*

4. Degenerate distributed-order equation

We present some results from [29] for (L,σ)-bounded operators, which are necessary for further considerations.

Let $\mathfrak{X}$, $\mathfrak{Y}$ be Banach spaces, $\mathcal{L}(\mathfrak{X};\mathfrak{Y})$ be the Banach space of linear continuous operators, acting from $\mathfrak{X}$ into $\mathfrak{Y}$, $\mathcal{C}l(\mathfrak{X};\mathfrak{Y})$ be the set of all linear closed densely defined in the space $\mathfrak{X}$ operators, acting into $\mathfrak{Y}$, $\mathcal{L}(\mathfrak{X};\mathfrak{X}):=\mathcal{L}(\mathfrak{X})$, $\mathcal{C}l(\mathfrak{X};\mathfrak{X}):=\mathcal{C}l(\mathfrak{X})$.

Let $L\in\mathcal{L}(\mathfrak{X};\mathfrak{Y})$, $\ker L\neq\{0\}$, $M\in\mathcal{C}l(\mathfrak{X};\mathfrak{Y})$ has a domain D_M. Since M is a closed operator, we can consider D_M as the Banach space with the graph norm of the operator M. We also use the notation $\rho^L(M):=\{\lambda\in\mathbb{C}:(\lambda L-M)^{-1}\in\mathcal{L}(\mathfrak{Y};\mathfrak{X})\}$, $\sigma^L(M):=\mathbb{C}\backslash\rho^L(M)$, $R_\lambda^L(M):=(\lambda L-M)^{-1}L$, $L_\lambda^L(M):=L(\lambda L-M)^{-1}$.

An operator M is called *(L,σ)-bounded*, if $\sigma^L(M)\subset\{\lambda\in\mathbb{C}:|\lambda|\leq a\}$ for some $a>0$. In this case there exist projections

$$P:=\frac{1}{2\pi i}\int_\gamma R_\lambda^L(M)d\lambda\in\mathcal{L}(\mathfrak{X}),\quad Q:=\frac{1}{2\pi i}\int_\gamma L_\lambda^L(M)d\lambda\in\mathcal{L}(\mathfrak{Y}),$$

where $\gamma=\{\lambda\in\mathbb{C}:|\lambda|=a+1\}$. Denote by $\mathfrak{X}^0$ ($\mathfrak{Y}^0$) the kernel $\ker P$ ($\ker Q$), and by $\mathfrak{X}^1$ ($\mathfrak{Y}^1$) the image $\operatorname{im}P$ ($\operatorname{im}Q$) of the projection P (Q). Let M_k (L_k) be the restriction of the operator M (L) on $D_{M_k}:=\mathfrak{X}^k\cap D_M$ ($\mathfrak{X}^k$), $k=0,1$.

Theorem 4.1 ([29])**.** *Let an operator M be (L,σ)-bounded. Then*

- (i) $\mathfrak{X}=\mathfrak{X}^0\oplus\mathfrak{X}^1$, $\mathfrak{Y}=\mathfrak{Y}^0\oplus\mathfrak{Y}^1$;
- (ii) $L_k\in\mathcal{L}(\mathfrak{X}^k;\mathfrak{Y}^k)$, $k=0,1$, $M_0\in\mathcal{C}l(\mathfrak{X}^0;\mathfrak{Y}^0)$, $M_1\in\mathcal{L}(\mathfrak{X}^1;\mathfrak{Y}^1)$;
- (iii) *there exist operators* $M_0^{-1}\in\mathcal{L}(\mathfrak{Y}^0;\mathfrak{X}^0)$ *and* $L_1^{-1}\in\mathcal{L}(\mathfrak{Y}^1;\mathfrak{X}^1)$.

Denote $G := M_0^{-1}L_0 \in \mathcal{L}(\mathfrak{X}^0)$. For $p \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ an operator M is called *(L,p)-bounded*, if it is (L,σ)-bounded and $G^p \neq 0$, $G^{p+1} = 0$.

Let us consider the distributed-order equation

$$\int_a^b \omega(\alpha) \mathrm{D}_t^\alpha Lx(t)d\alpha = Mx(t) + f(t), \quad t \in [0,T), \tag{4.1}$$

where D_t^α is the Caputo fractional derivative, $m-1 < b \leq m \in \mathbb{N}$, $a < b$, $\omega : (a,b) \to \mathbb{R}$, $f \in C([0,T);\mathfrak{Y})$. Equation (4.1) is called degenerate, because it is supposed that $\ker L \neq \{0\}$.

A function x:$[0,T) \to D_M$ is called a solution of equation (4.1), if $Mx \in C([0,T);\mathfrak{Y})$, there exists $\int_a^b \omega(\alpha)\mathrm{D}_t^\alpha Lx(t)d\alpha \in C([0,T);\mathfrak{Y})$ and equality (4.1) is valid. A solution x of (4.1) is called a solution to the Cauchy problem

$$x^{(k)}(0) = x_k, \quad k = 0,1,\dots,m-1, \tag{4.2}$$

for equation (4.1), if $x \in C^{m-1}([0,T);\mathfrak{X})$ satisfies conditions (4.2).

Let B be the operator, defined as

$$(Bx)(t) := \int_a^b \omega(\alpha)\mathrm{D}_t^\alpha x(t)d\alpha \tag{4.3}$$

on functions x:$[0,T) \to \mathfrak{X}$, such that the right-hand side of (4.3) has meaning.

Lemma 4.2. *Let $H \in \mathcal{L}(\mathfrak{X})$ is a nilpotent operator of a power not greater than $p \in \mathbb{N}_0$, $(BH)^k h \in C([0,T);\mathfrak{X})$, $k = 0,1,\dots,p$. Then there exists a unique solution of the equation*

$$\int_a^b \omega(\alpha)\mathrm{D}_t^\alpha Hw(t)d\alpha = w(t) + h(t), \quad t \in [0,T), \tag{4.4}$$

and it has the form

$$w(t) = -\sum_{k=0}^{p} [(BH)^k h](t). \tag{4.5}$$

Proof. If w is a solution of (4.4), then $w + h + BHh = BHw + BHh = (BH)^2 w$ for $t \in [0,T)$. The last expression is defined, because BHw and BHh is defined also. Analogously obtain $w+h+BHh+(BH)^2h = (BH)^2w+(BH)^2h = (BH)^3w$. Continuing these arguments, we obtain

$$w + \sum_{k=0}^{p}(BH)^k h = (BH)^{p+1}w = B^{p+1}H^{p+1}h \equiv 0,$$

since $H^{p+1} = 0$. Hence, the solution has form (4.5). Therefore, there exists a solution of equation (4.4), and it is unique. $\square$

Define operators

$$X_k(t) := \frac{1}{2\pi i}\int_\gamma \frac{e^{\lambda t}}{\lambda^{k+1}} W_{a_k}^b(\lambda) R_{W_a^b(\lambda)}^L(M)d\lambda, \quad a_k = \max\{a,k\},\ k = 0,\dots,m-1,$$

$$X(t) := \frac{1}{2\pi i}\int_\gamma e^{\lambda t} R_{W_a^b(\lambda)}^L(M)d\lambda.$$

From Theorem 4.1 it follows that

$$R_{W_a^b(\lambda)}^L(M) = (W_a^b(\lambda)I - L_1^{-1}M_1)^{-1}P + (W_a^b(\lambda)G - I)^{-1}G(I-P). \tag{4.6}$$

Theorem 4.3. *Let $p \in \mathbb{N}_0$, an operator M be (L,p)-bounded, $f \in C([0,T];\mathfrak{Y})$, $(BG)^l M_0^{-1}(I-Q)f \in C^{m-1}([0,T];\mathfrak{X})$, $l = 0,1,\dots,p$, and for some $\beta > 1$ $W_a^b(\lambda)$, $W_k^b(\lambda)$, $k = 0,1,\dots,m-1$, are holomorphic functions on the set*

$$S_\beta := \{\lambda \in \mathbb{C} : |\lambda| \geq \beta, \arg\lambda \in (-\pi,\pi)\},$$

satisfying conditions (2.4), (2.5), $r_0 = \max\{\beta, (2\|L_1^{-1}M_1\|_{\mathcal{L}(\mathfrak{X}^1)}/C_1)^{1/\delta}\}$, $x_k \in \mathfrak{X}$, $k = 0,1,\dots,m-1$, *such that*

$$(I-P)x_k = -D_t^k|_{t=0} \sum_{l=0}^{p} [(BG)^l M_0^{-1}(I-Q)f](t). \tag{4.7}$$

Then the function

$$x(t) = \sum_{k=0}^{m-1} X_k(t)x_k + \int_0^t X(t-s)L_1^{-1}Qf(s)\,ds - \sum_{l=0}^{p}[(BG)^l M_0^{-1}(I-Q)f](t) \tag{4.8}$$

is a unique solution to the Cauchy problem (4.1), (4.2) *from the class* $E(\mathfrak{X})$.

Proof. By Theorem 4.1, problem (4.1), (4.2) can be reduced to the two Cauchy problems

$$\int_a^b \omega(\alpha) \mathrm{D}_t^\alpha v(t) d\alpha = L_1^{-1}M_1 v(t) + L_1^{-1}Qf(t), \quad t \in [0,T), \tag{4.9}$$

$$v^{(k)}(0) = Px_k, \quad k = 0,1,\dots,m-1, \tag{4.10}$$

and

$$\int_a^b \omega(\alpha) \mathrm{D}_t^\alpha G w(t) d\alpha = w(t) + M_0^{-1}(I-Q)f(t), \quad t \in [0,T), \tag{4.11}$$

$$w^{(k)}(0) = (I-P)x_k, \quad k = 0,1,\dots,m-1, \tag{4.12}$$

on the subspaces $\mathfrak{X}^1$ and $\mathfrak{X}^0$ respectively. Here $v(t) := Px(t)$, $w(t) := (I-P)x(t)$. Problem (4.9), (4.10) is uniquely solvable by Theorem 3.2, and its solution has the form

$$\begin{aligned} v(t) &= \sum_{k=0}^{m-1} \frac{1}{2\pi i} \int_\gamma \frac{e^{\lambda t}}{\lambda^{k+1}} W_{a_k}^b(\lambda)(W_a^b(\lambda)I - L_1^{-1}M_1)^{-1} d\lambda P x_k \\ &\quad + \int_0^t \frac{1}{2\pi i} \int_\gamma e^{\lambda(t-s)} (W_a^b(\lambda)I - L_1^{-1}M_1)^{-1} d\lambda L_1^{-1}Qf(s) ds \\ &= \sum_{k=0}^{m-1} X_k(t)Px_k + \int_0^t X(t-s)L_1^{-1}Qf(s)ds \end{aligned}$$

because of equality (4.6). Then (4.11) has the unique solution

$$w(t) = -\sum_{l=0}^{p} [(BG)^l M_0^{-1}(I-Q)f](t).$$

It satisfies conditions (4.12), if and only if conditions (4.7) are valid. □

It is obvious, that for the problem

$$(Px)^{(k)}(0) = x_k, \quad k = 0,1,\dots,m-1, \tag{4.13}$$

the next unique solvability theorem without additional conditions (4.7) is true.

Note that the definition of equation (4.1) solution implies the inclusion $Lx \in C^{m-1}([0,T];\mathfrak{Y})$, therefore, $Px \equiv L_1^{-1}LPx \equiv L_1^{-1}QLx \in C^{m-1}([0,T];\mathfrak{X})$ and conditions (4.13) have the meaning for every solution of (4.1). Thus, a solution of (4.1) is called a solution of problem (4.1), (4.13), if it satisfies condition (4.13).

Theorem 4.4. *Let $p \in \mathbb{N}_0$, an operator M be (L,p)-bounded, $f \in C([0,T];\mathfrak{Y})$, $(BG)^l M_0^{-1}(I-Q)f \in C([0,T];\mathfrak{X})$, $l = 0,1,\dots,p$, and for some $\beta > 1$ $W_a^b(\lambda)$, $W_k^b(\lambda)$, $k = 0,1,\dots,m-1$, are holomorphic functions on the set*

$$S_\beta := \{\lambda \in \mathbb{C} : |\lambda| \geq \beta, \arg\lambda \in (-\pi,\pi)\},$$

satisfying conditions (2.4), (2.5), $r_0 = \max\{\beta, (2\|L_1^{-1}M_1\|_{\mathcal{L}(\mathfrak{X}^1)}/C_1)^{1/\delta}\}$, $x_k \in \mathfrak{X}^1$, $k = 0,1,\dots,m-1$. *Then function* (4.8) *is a unique solution to problem* (4.1), (4.13) *from the class $E(\mathfrak{X})$.*

5. Applications to boundary-value problems

Let $P_n(\lambda) = \sum_{i=0}^n c_i\lambda^i$, $Q_n(\lambda) = \sum_{i=0}^n d_i\lambda^i$, $c_i, d_i \in \mathbb{C}$, $i = 0,1,\dots,n$, $c_n \neq 0$. Let $\Omega \subset \mathbb{R}^d$ be a bounded region with a smooth boundary $\partial\Omega$, operators pencil $A, B_1, B_2, \dots, B_r$ be regularly elliptic [30], where

$$(Au)(s) = \sum_{|q|\leq 2r} a_q(s)D_s^q u(s), \quad a_q \in C^\infty(\overline{\Omega}),$$

$$(B_l u)(s) = \sum_{|q|\leq r_l} b_{l_q}(s)D_s^q u(s), \quad b_{l_q} \in C^\infty(\partial\Omega), \quad l = 1,2,\dots,r,$$

$D_s^q = D_{s_1}^{q_1}D_{s_2}^{q_2}\dots D_{s_d}^{q_d}$, $D_{s_i}^{q_i} = \partial^{q_i}/\partial s_i^{q_i}$, $q = (q_1,q_2,\dots,q_d) \in \mathbb{N}_0^d$. Define the operator $A_1 \in \mathcal{C}l(L_2(\Omega))$ with domain $D_{A_1} = H^{2r}_{\{B_l\}}(\Omega)$ [30] by the equality $A_1u = Au$. Let A_1 be self-adjoint operator and it has a bounded from the right spectrum. Then the spectrum $\sigma(A_1)$ of the operator A_1 is real, discrete and condensed at $-\infty$. Let $0 \notin \sigma(A_1)$, $\{\varphi_k : k \in \mathbb{N}\}$ is an orthonormal in $L_2(\Omega)$ system of the operator A_1 eigenfunctions, numbered in according to nonincreasing of the corresponding eigenvalues $\{\lambda_k : k \in \mathbb{N}\}$, taking into account their multiplicity.

Consider the initial-boundary value problem

$$\frac{\partial^k u}{\partial t^k}(s,0) = u_k(s), \quad k = 0,1,\dots,m-1,\ s \in \Omega, \tag{5.1}$$

$$B_l A^k u(s,t) = 0, \quad k = 0,1,\dots,n-1,\ l = 1,2,\dots,r,\ (s,t) \in \partial\Omega \times [0,T), \tag{5.2}$$

$$\int_a^b \omega(\alpha)\mathrm{D}_t^\alpha P_n(A)u(s,t)d\alpha = Q_n(A)u(s,t) + f(s,t), \quad (s,t) \in \Omega \times [0,T), \tag{5.3}$$

where D_t^α is the Caputo fractional derivative, $m-1 < b \leq m \in \mathbb{N}$, $a \in [0,b)$, $\omega : (a,b) \to \mathbb{R}$, $f \in \Omega \times [0,T) \to \mathbb{R}$. Set

$$\begin{aligned}\mathfrak{X} = \big\{u \in H^{2rn}(\Omega) : B_l A^k u(s) = 0,\ k = 0,1,\dots,n-1,\\ l = 1,2,\dots,r,\ x \in \partial\Omega\big\},\end{aligned} \tag{5.4}$$

$$\mathfrak{Y} = L_2(\Omega), \quad L = P_n(A), \quad M = Q_n(A). \tag{5.5}$$

Then $L, M \in \mathcal{L}(\mathfrak{X};\mathfrak{Y})$ and problem (5.1), (5.3) is presented in the form (4.1), (4.2).

Theorem 5.1. [9]. *Let the spectrum $\sigma(A_1)$ does not contain zero point and common roots of the polynomials $P_n(\lambda)$ and $Q_n(\lambda)$, and denotations* (5.4), (5.5) *are valid. Then the operator M is $(L,0)$-bounded,*

$$\sigma^L(M)=\{\mu\in\mathbb{C}:\mu=Q_n(\lambda_k)/P_n(\lambda_k),P_n(\lambda_k)\neq 0\},$$

$\mathfrak{X}^0=\mathfrak{Y}^0=\operatorname{span}\{\varphi_k:P_n(\lambda_k)=0\}$, *$\mathfrak{X}^1$ is the closure of* $\operatorname{span}\{\varphi_k:P_n(\lambda_k)\neq 0\}$ *in the norm of the space $\mathfrak{X}$, $\mathfrak{Y}^1$ is the closure of the same set in $L_2(\Omega)$.*

Theorem 5.2. *Let the spectrum $\sigma(A_1)$ does not contain zero point and common roots of the polynomials $P_n(\lambda)$ and $Q_n(\lambda)$, $f\in C^{m-1}([0,T);L_2(\Omega))$, and for some $\beta>1$ $W_a^b(\lambda)$, $W_k^b(\lambda)$, $k=0,1,\dots,m-1$, are holomorphic functions on the set $S_\beta:=\{\lambda\in\mathbb{C}:|\lambda|\geq\beta,\arg\lambda\in(-\pi,\pi)\}$, satisfying conditions* (2.4), (2.5),

$$r_0=\max\Big\{\beta,\Big(2C_1^{-1}\cdot\sup_{P_n(\lambda_k)\neq 0}\frac{Q_n(\lambda_k)}{P_n(\lambda_k)}\Big)^{1/\delta}\Big\},$$

$u_k\in\mathfrak{X}$, $k=0,1,\dots,m-1$. If $P_n(\lambda_l)=0$, then

$$Q_n(\lambda_l)\langle u_k,\varphi_l\rangle=-D_t^k|_{t=0}\langle f(\cdot,t),\varphi_l\rangle,\quad k=0,1,\dots,m-1. \tag{5.6}$$

Then there exists a unique solution of problem (5.1)*)–*(5.3) *from the class $E(\mathfrak{X})$.*

Proof. By Theorem 5.1, $p=0$; hence, $G=0$. Conditions (5.6) mean (4.7) for this case. It remains to apply Theorem 4.3. Here we use the evident equality

$$\|L_1^{-1}M_1\|_{\mathcal{L}(\mathfrak{X}^1)}=\sup_{P_n(\lambda_k)\neq 0}\frac{Q_n(\lambda_k)}{P_n(\lambda_k)}.$$

This supremum is finite because the power of Q_n not greater than n. □

Theorem 5.1 implies the equalities $\mathfrak{X}^0=\ker P=\ker L$, $\operatorname{im}L=\operatorname{im}L_1=\mathfrak{Y}^1$, therefore, initial condition (4.13) can be represented in the equivalent form

$$(Lx)^{(k)}(t)=y_k=Lx_k\in\mathfrak{X}^1,\quad k=0,1,\dots,m-1.$$

For equation (5.3) they have the form

$$\frac{\partial^k P_n(A)u}{\partial t^k}(s,0)=u_k(s),\quad k=0,1,\dots,m-1,\ s\in\Omega. \tag{5.7}$$

Theorem 5.3. *Let the spectrum $\sigma(A_1)$ do not contain zero point and common roots of the polynomials $P_n(\lambda)$ and $Q_n(\lambda)$, $f\in C([0,T);L_2(\Omega))$, and for some $\beta>1$ $W_a^b(\lambda)$, $W_k^b(\lambda)$, $k=0,1,\dots,m-1$, are holomorphic functions on the set $S_\beta:=\{\lambda\in\mathbb{C}:|\lambda|\geq\beta,\arg\lambda\in(-\pi,\pi)\}$, satisfying conditions* (2.4), (2.5),

$$r_0=\max\Big\{\beta,\Big(2C_1^{-1}\cdot\sup_{P_n(\lambda_k)\neq 0}\frac{Q_n(\lambda_k)}{P_n(\lambda_k)}\Big)^{1/\delta}\Big\},$$

$u_k\in\mathfrak{X}$, $k=0,1,\dots,m-1$. If $P_m(\lambda_l)=0$, then

$$\langle u_k,\varphi_l\rangle=0,\quad k=0,1,\dots,m-1. \tag{5.8}$$

Then there exists a unique solution of problem (5.2), (5.3), (5.7) *from the class $E(\mathfrak{X})$.*

Proof. Conditions (5.8) mean that $u_k\in\mathfrak{X}^1$, $k=0,1,\dots,m-1$. Theorem 4.4 implies the required statement. □

Remark 5.4. If $P_n(\lambda_k) \neq 0$ for all $k \in \mathbb{N}$, then conditions (5.1) equivalent to (5.7), and the unique solvability of the corresponding initial-boundary value problems follows from Theorem 3.2.

Let $n = 1$, $P_1(\lambda) = a - \lambda$, $Q_1(\lambda) = b\lambda + c$, $Au = \Delta u$, $r = 1$, $B_1 = I$, $f \equiv 0$. Then problem (5.1)–(5.3) has the form

$$\int_a^b \omega(\alpha) \mathrm{D}_t^\alpha (a - \Delta) u(s,t) d\alpha = b\Delta u(s,t) + cu(s,t), \quad (s,t) \in \Omega \times \overline{\mathbb{R}}_+,$$

$$u(s,t) = 0, \quad (s,t) \in \partial\Omega \times \overline{\mathbb{R}}_+,$$

$$\frac{\partial^k u}{\partial t^k}(s,0) = u_k(s), \quad k = 0,1,\dots,m-1,\ s \in \Omega.$$

Conditions (5.7) become

$$\frac{\partial^k (a - \Delta) u}{\partial t^k}(s,0) = u_k(s), \quad k = 0,1,\dots,m-1,\ s \in \Omega.$$

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