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NONSTATIONARY LAMÉ SYSTEM WITHOUT DEFINITE SIGN ENERGY

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Abstract. This article concerns the existence and decay of solutions of a nonstationary Lamé system. This system has a nonlinear perturbation that produces an energy without definite sign. We consider displacement and traction conditions at the boundary and a general nonlinear boundary damping. We also obtain exponential decay of the energy.

1. Introduction

We consider an isotropic homogeneous elastic body that in its equilibrium position occupies a bounded set Ω of $\mathbb{R}^3$. Suppose that at the instant $t = 0$ an external force acts on the body and then stops. As a consequence of this, the particles of the body begin to oscillate. The motion of these small oscillations of the particles can be described by the nonstationary Lamé system

$$u''(x,t) - \mu\Delta u(x,t) - (\lambda + \mu)\nabla \operatorname{div} u(x,t) = 0, \quad x \in \Omega,\ t > 0, \tag{1.1}$$

where $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ denotes the displacement of the particle x of the body at the instant t, $u'(x,t) = \frac{\partial}{\partial t}u(x,t)$, $\Delta u(x,t) = (\Delta u_1(x,t), \Delta u_2(x,t), \Delta u_3(x,t))$, $\nabla = \big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\big)$, $\operatorname{div} u(x,t) = \sum_{i=1}^3 \frac{\partial u_i}{\partial x_i}(x,t)$ and λ, μ are the Lamé coefficients of the material of the body with $\mu > 0$ and $\lambda + \mu > 0$ (see, for example, Ciarlet [4], Duvaut and Lions [6] and Landau and Lifshitz [11]).

Existence of solutions of the mixed value problem for the system (1.1) can be found, for example, in Marsden and Hughes [17]. The decay of solutions for (1.1) with linear boundary conditions is analyzed in Caldas [3] and in Komornik [9]. The decay of the energy for some variations of (1.1) is investigated in Bociu, Derochers and Toundykov [2] and in Cordeiro, Santos and Raposo [5]. The inverse problem and observability for system (1.1) are studied in Belishev and Lasiecka [1] and in Imanuvilov and Yamamoto [7].

We introduce a nonlinear perturbation in (1.1) and consider a given nonlinear boundary damping acting on a part of the boundary of Ω. This in the $n-$dimensional frame work.

The objective of this paper is to investigate existence and decay of global solutions of the above mixed problem. Thus we consider an open bounded set Ω of $\mathbb{R}^n$

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whose boundary Γ, of class C^2, is constituted of two parts Γ_0 and Γ_1, both with positive Lebesgue measures, and $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$. The unit exterior normal vector at $x \in \Gamma$ is denoted by $\nu(x)$. In these conditions we have the problem

$$u''(x,t) - \mu\Delta u(x,t) - (\lambda+\mu)\nabla \operatorname{div} u(x,t) + |u(x,t)|^\rho = 0, \quad \text{in } \Omega \times (0,\infty); \tag{1.2}$$

$$u(x,t) = 0, \quad \text{on } \Gamma_0 \times (0,\infty); \tag{1.3}$$

$$\mu\frac{\partial u}{\partial \nu}(x,t) + (\lambda+\mu)\operatorname{div} u(x,t)\nu(x) + h(x,u'(x,t)) = 0, \quad \text{on } \Gamma_1 \times (0,\infty); \tag{1.4}$$

$$u(x,0) = u^0(x), \quad u^1(x,0) = u^1(x), \quad x \in \Omega, \tag{1.5}$$

where $u(x,t) = (u_1(x,t),\dots,u_n(x,t))$, $\Delta u = (\Delta u_1,\dots,\Delta u_n)$, $\nabla = \big(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\big)$, $\operatorname{div} u = \sum_{i=1}^n \frac{\partial u_i}{\partial x_i}$, $\frac{\partial u}{\partial \nu} = \big(\frac{\partial u_1}{\partial \nu},\dots,\frac{\partial u_n}{\partial \nu}\big)$, $|u|^\rho = (|u_1|^\rho,\dots,|u_n|^\rho)$ (ρ is a positive real number) and $h(x,u) = (h_1(x,u_1),\dots,h_n(x,u_n))$, with $h_i(x,s)$ is measurable in Γ_1 and continuous in $\mathbb{R}$, $i = 1,\dots,n$.

To obtain the existence of global solutions of the above problem we must overcome two serious difficulties. First, note that

$$\begin{aligned}
\int_\Omega |u(x,t)|^\rho . u'(x,t)\mathrm{d}x &= \sum_{i=1}^n \int_\Omega |u_i(x,t)|^\rho u_i'(x,t)\mathrm{d}x \\
&= \sum_{i=1}^n \frac{1}{\rho+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u_i(x,t)|^\rho u_i(x,t)\mathrm{d}x.
\end{aligned}$$

Then

$$\begin{aligned}
\int_0^t\int_\Omega |u(x,t)|^\rho . u'(x,t)\mathrm{d}x\mathrm{d}s &= \sum_{i=1}^n \frac{1}{\rho+1}\int_\Omega |u_i(x,t)|^\rho u_i(x,t)\mathrm{d}x \\
&\quad - \sum_{i=1}^n \frac{1}{\rho+1}\int_\Omega |u_i(x,0)|^\rho u_i(x,0)\mathrm{d}x.
\end{aligned}$$

Note that each term $\int_\Omega |u_i(x,t)|^\rho u_i(x,t)\mathrm{d}x$ does not have definite sign. Thus the energy method does not work in the present problem to obtain global solutions. To overcome this difficulty, we introduce a significative generalization of an idea of Tartar [24] (cf. [13, 19, 20, 21] for a direct application in [24]). This method simplify the potential well method due to Sattinger [22]. Of course, the norm of initial data u^0 and u^1 are related to ρ and this ρ depends on the embedding of Sobolev spaces.

The second difficulty is caused by the generality of the functions $h_i(x,s)$. Assuming that each $h_i(x,s)$ is strongly monotone in $\mathbb{R}$, $h_i(x,0) = 0$ a.e. in $x \in \Gamma_1$ and using an approximation of continuous function by Lipschitz continuous function (cf. [16, 23]), we overcome this difficulty. Also in this part we introduce a trace result given by a theorem for non-smooth functions.

In our approach on the existence of solutions we use the Galerkin method with an special basis in order to obtain a second a priori estimate of the approximate solutions. This choose is related to the boundary condition (1.4) at $t = 0$. In the passage to the limit on the nonlinear terms of the approximate problem, we use compactness arguments (see Lions [13]) and a result by Strauss [23].

The decay of solutions is derived by the multiplier method (see Komornik and Zuazua [10]). In our approach considered a boundary damping given by a strongly monotone Lipschitz continuous function. We note that Lasiecka and Tataru [12] and in Komornik and Zuazua [8] considered a wave equation with boundary damping given by a function $h(s)$ with $|h(s)| \leq L|s|$ for $|s|$ large enough.

2. Notation and main results

The notation introduced in the previous section for the n-dimensional case will now be complemented for establishing our results.

Let $L^2(\Omega)$ be the usual Hilbert space equipped with the following inner product and norm:

$$(f,g) = \int_\Omega f(x)g(x)\mathrm{d}x \quad \text{and} \quad |f|^2 = (f,f).$$

We use the Hilbert space

$$H^1_{\Gamma_0}(\Omega) := \{w \in H^1(\Omega) : w = 0 \text{ on } \Gamma_0\}$$

equipped with the following inner product and norm:

$$((u,z)) = \sum_{i=1}^{n} \int_\Omega \frac{\partial w}{\partial x_i}(x)\frac{\partial z}{\partial x_i}(x)\mathrm{d}x \quad \text{and} \quad \|w\|^2 = ((w,w)).$$

We use capital letters with double trace to represent the $n-$product of the same space. Thus $\mathbb{L}^2(\Omega) = (L^2(\Omega))^n$, $\mathbb{H}^1_0(\Omega) = (H^1_0(\Omega))^n$, $\mathbb{L}^2(\Gamma_1) = (L^2(\Gamma_1))^n$, $\mathbb{H}^{1/2}(\Gamma_1) = (H^{1/2}(\Gamma_1))^n$. Each of such space is endowed with its product topology. The dual of $\mathbb{H}^{1/2}(\Gamma_1)$ is denoted by $\mathbb{H}^{-1/2}(\Gamma_1)$.

Remark 2.1. We consider $\mathbb{H}^1_{\Gamma_0}(\Omega)$ with its usual product topology and $V = \mathbb{H}^1_{\Gamma_0}(\Omega)$ with the inner product

$$((u,v))_V = \mu((u,v)) + (\lambda+\mu)(\operatorname{div} u, \operatorname{div} v)_{L^2(\Omega)}.$$

We have

$$\mu^{1/2}\|u\|_{\mathbb{H}^1_{\Gamma_0}(\Omega)} \leq \|u\|_V \leq [\mu + n^2(\lambda+\mu)]^{1/2}\|u\|_{\mathbb{H}^1_{\Gamma_0}(\Omega)}, \ \forall u \in \mathbb{H}^1_{\Gamma_0}(\Omega).$$

We will use the notation $H = \mathbb{L}^2(\Omega)$ equipped with the inner product $(u,v)_H = \sum_{i=1}^n (u_i, v_i)$. Let B be the positive self-adjoint operator of H defined by triplet $\{V, H, ((u,v))_V\}$ (see Lions [14]). Then

$$(Bu, v) = ((u,v))_V, \quad \forall u \in D(B), \ \forall v \in V,$$

and

$$B = -\mu\Delta - (\lambda+\mu)\nabla \operatorname{div}, \quad D(B) = \{v \in V : Bu \in H, \gamma_1 u = 0 \text{ on } \Gamma_1\}, \tag{2.1}$$

where

$$\gamma_1 u = \mu\frac{\partial u}{\partial \nu} + (\lambda+\mu)(\operatorname{div}\, u)\nu. \tag{2.2}$$

Note that γ_1 is well defined (cf. Theorem 3.3).

We make the following restrictions on ρ:

$$\rho > 1 \quad \text{if } n = 1, 2, \tag{2.3}$$

$$\frac{n+1}{n} \leq \rho \leq \frac{n}{n-2} \quad \text{if } n \geq 3. \tag{2.4}$$

The notation $X \hookrightarrow Y$ indicates that the space X is continuously embedded in Y. From restrictions (2.4), we have

$$H^1_{\Gamma_0}(\Omega) \hookrightarrow L^{q^*}(\Omega) \hookrightarrow L^{2\rho}(\Omega) \hookrightarrow L^{\rho+1}(\Omega) \hookrightarrow L^{\rho}(\Omega),$$

$$L^{q^*}(\Omega) \hookrightarrow L^{n(\rho-1)}(\Omega),$$

where $q^* = \frac{2n}{n-1}$ if $n \geq 3$. Then there exist positive constants $k_0, \dots, k_5$ such that

$$\|w\|_{L^{\rho+1}(\Omega)} \leq k_0\|w\|, \quad \|w\|_{L^{\rho}(\Omega)} \leq k_1\|w\|, \tag{2.5}$$

$$\|w\|_{L^{2\rho}(\Omega)} \leq k_2\|w\|, \quad \|w\|_{L^{n(\rho)-1}(\Omega)} \leq k_3\|w\|, \tag{2.6}$$

$$\|w\|_{L^{q^*}(\Omega)} \leq k_4\|w\|, \quad |w| \leq k_5\|w\|, \tag{2.7}$$

for all $w \in H^1_{\Gamma_0}(\Omega)$. Note that $H^{1/2}_{\Gamma_0}(\Gamma_1) \hookrightarrow L^{q_1^*}(\Gamma_1)$ where $q_1^* = \frac{2(n-1)}{n-1}$, $n \geq 3$, and $q_1^* \geq \rho + 1$. Thus

$$H^1_{\Gamma_0}(\Omega) \hookrightarrow H^{1/2}(\Gamma_1) \hookrightarrow L^{q_1^*}(\Gamma_1) \hookrightarrow L^{\rho+1}(\Gamma_1) \hookrightarrow L^2(\Gamma_1).$$

Then there exist positive constants k_6 and k_7 such that

$$\|w\|_{L^{\rho+1}(\Gamma_1)} \leq k_6\|w\|, \quad \|w\|_{L^2(\Gamma_1)} \leq k_7\|w\|, \quad \text{for all } w \in H^1_{\Gamma_0}(\Gamma_1). \tag{2.8}$$

We assume that the function $h = (h_1, h_2, \dots, h_n)$ satisfies

$$h \in C^0(\mathbb{R}, \mathbb{L}^\infty(\Gamma_1)); \tag{2.9}$$

$$h_i(x, 0) = 0 \quad \text{a.e. for } x \text{ in } \Gamma_1,\ i = 1, 2, \dots, n; \tag{2.10}$$

$$[h_i(x, s) - h_i(x, r)](s - r) \geq d_0(s - r)^2, \quad \forall s, r \in \mathbb{R},\ \text{a.e. } x \text{ in } \Gamma_1, \tag{2.11}$$

where $i = 1, 2, \dots, n$ and d_0 is a positive constant.

Remark 2.2. An example of functions $h_i(x, s)$, $i = 1, 2, \dots, n$, satisfying (2.9)–(2.11) is given by

$$h_i(x, s) = \delta(x)(s + |s|^\alpha s),\ x \in \Gamma_1,\ s \in \mathbb{R}$$

$\delta \in L^\infty(\Gamma_1)$, $\delta(x) \geq \delta_0 > 0$ and $\alpha > 1$, α constant.

Consider

$$\lambda^* = \Big(\frac{1}{4N}\Big)^{\frac{1}{\rho-1}} \quad \text{and} \quad N = \frac{nk_0^{\rho+1}}{(\rho+1)\mu^{\frac{\rho+1}{2}}}. \tag{2.12}$$

We make the following assumptions on the initial data u^0 and u^1:

$$u^0 \in D(B), \quad u^1 \in \mathbb{H}^1_0(\Omega), \tag{2.13}$$

$$\|u^0\|_V < \lambda^*, \tag{2.14}$$

$$\frac{1}{2}\|u^1\|_H^2 + \frac{1}{2}\|u^0\|_V^2 + \frac{nk_0^{\rho+1}}{(\rho+1)\mu^{\frac{\rho+1}{2}}}\|u^0\|_V^{\rho+1} < \frac{1}{4}(\lambda^*)^2. \tag{2.15}$$

Let A be the operator

$$A = -\mu\Delta - (\lambda + \mu)\nabla \operatorname{div}, \tag{2.16}$$

and the Hilbert space

$$W = \{u \in V : Au \in H\} \tag{2.17}$$

be provided with the inner product

$$(u, v)_W = ((u, v))_V + (Au, Av)_H. \tag{2.18}$$

Note that the operators A and B, introduced in (2.1) and (2.16), respectively, have the same form but $D(B)$ is contained in W.

Theorem 2.3. *Suppose that* (2.3), (2.4), (2.9)-(2.11) *and* (2.13)-(2.15) *hold. Then there exist a function u such that*

$$u \in L^\infty(0,\infty;V) \cap L^2_{\rm loc}(0,\infty;W), \tag{2.19}$$

$$u' \in L^\infty(0,\infty;H) \cap L^\infty_{\rm loc}(0,\infty;V), \tag{2.20}$$

$$u'' \in L^\infty_{\rm loc}(0,\infty;H), \quad \operatorname{div} u \in L^\infty(0,\infty;L^2(\Omega)), \tag{2.21}$$

satisfying

$$u'' - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u + |u|^\rho = 0 \quad \textit{in } L^2_{\rm loc}(0,\infty;H), \tag{2.22}$$

$$\gamma_1 u + h(\cdot,u') = 0 \ \textit{in}\, L^1_{\rm loc}(0,\infty;\mathbb{H}^{-1/2}(\Gamma_1)+\mathbb{L}^1(\Gamma_1)), \tag{2.23}$$

$$u(0)=u^0, \quad u'(0)=u^1. \tag{2.24}$$

It is worth noting that

1. (1) The energy E of system (2.22)–(2.24), which is defined in (2.42), does not has definite sign.
2. (2) The uniqueness of solution of Problem (2.22)–(2.24) is an open problem. The difficulty is due to the general assumption made about the function $h(\cdot,u')$.

To obtain the decay of the energy of the Problem (1.2)–(1.5), we make some restrictions on Γ and $h_i(x,s)$. This will lead us to a new theorem on the existence of solutions. The justification of this new theorem will be find in the proof of the decay of the energy, in Section 5. We assume that there is $x^0 \in \mathbb{R}^n$ such that

$$\Gamma_0 = \{x\in\Gamma : m(x)\cdot\nu(x)\le 0\} \quad \text{and} \quad \Gamma_1 = \{x\in\Gamma : m(x)\cdot\nu(x) > 0\}, \tag{2.25}$$

where $m(x)=x-x^0$, $x\in\Gamma$, and $x\cdot y$ is the inner product of $\mathbb{R}^n$. Let

$$R = \max\{||m(x)|| : x\in\Gamma_1\}, \tag{2.26}$$

$$0 < b_0 = \min\{m(x)\nu(x); x\in\Gamma_1\}. \tag{2.27}$$

Assume also that each $h_i(x,s)$ has the form

$$h_i(x,s) = [m(x)\cdot\nu(x)]p_i(s),$$

where $p_i(s)$ is strongly monotone and Lipschitiz continuous, namely,

$$[p_i(r)-p_i(s)](r-s) \ge d_0^*(r-s)^2, \quad \forall r,s\in\mathbb{R},\ p_i(0)=0,\ i=1,2,\dots,n, \tag{2.28}$$

$$|p_i(s)| \le L|s|, \quad \forall s\in\mathbb{R},\ i=1,2,\dots,n, \tag{2.29}$$

where d_0^* and L are positive constants.

Let

$$N_1 = \frac{1}{\mu^{\frac{\rho+1}{2}}}\Big\{\big[\frac{1}{\rho+1} + \big|\frac{2n}{\rho+1} - (n-1)\big|\big]k_0^{\rho+1}n + \frac{2R}{\rho+1}k_6^{\rho+1}n\Big\}, \tag{2.30}$$

where k_0 and k_6 were defined in (2.5) and (2.8), respectively. We consider the real number

$$\lambda_1^* = \Big(\frac{1}{4N_1}\Big)^{\frac{1}{\rho-1}}. \tag{2.31}$$

and introduce the hypotheses

$$u^0 \in D(B), \quad u^1 \in \mathbb{H}_0^1(\Omega), \tag{2.32}$$

$$\|u^0\|_V < \lambda_1^*, \tag{2.33}$$

$$\frac{1}{2}\|u^1\|_H^2 + \frac{1}{2}\|u^0\|_V^2 + \frac{nk_0^{\rho+1}}{(\rho+1)\mu^{\frac{\rho+1}{2}}}\|u^0\|_V^{\rho+1} < \frac{1}{4}(\lambda_1^*)^2. \tag{2.34}$$

Note that $\lambda_1^* < \lambda^*$, where λ^* was given in (2.12).

Theorem 2.4. *Assume* (2.3)*,* (2.4)*,* (2.28)*,* (2.29)*,* (2.32)*–*(2.34) *hold. Then there exist a unique function u in the class* (2.19)*–*(2.21) *such that u satisfies*

$$u'' - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u + |u|^\rho = 0 \quad \text{in } L^2_{\rm loc}(0,\infty;H), \tag{2.35}$$

$$\gamma_1 u + [m(\cdot)\nu(\cdot)]p(u') = 0 \quad \text{in } L^2_{\rm loc}(0,\infty;\mathbb{H}^{1/2}(\Gamma_1)), \tag{2.36}$$

$$u(0) = u^0,\ u'(0) = u^1. \tag{2.37}$$

We assume that the solution u given by Theorem 2.4 has the regularity

$$u \in L^2_{\rm loc}(0,\infty;\mathbb{H}^2(\Omega)) \tag{2.38}$$

Note that the u given by Theorem 2.4 is solution of an equation

$$\begin{gathered} Au = f \quad \text{in } \Omega\times(0,\infty) \quad (f \in L^2_{\rm loc}(0,\infty;H)),\\ u = 0 \quad \text{on } \Gamma_0\times(0,\infty),\\ \gamma_1 u = g \quad \text{on } \Gamma_1\times(0,\infty), \quad (g \in L^2_{\rm loc}(0,\infty;\mathbb{H}^{1/2}(\Gamma_1)). \end{gathered}$$

If Ω is a bounded domain of $\mathbb{R}^3$ with boundary $\Gamma \in C^2$, then u has regularity (2.38) (see Ciarlet [4, Theorem 63-6, p. 296]).

We consider the constants

$$M = \frac{1}{\mu^{1/2}}\left[4R + 2(n-1)k_5\right], \tag{2.39}$$

$$P = \frac{1}{\mu}(n-1)^2R^2L^2k_7^2 + \frac{1}{\mu}R^2L^2 + R, \tag{2.40}$$

$$\sigma = \min\Big\{\frac{1}{2M}, \frac{b_0d_0^*}{P}\Big\}, \tag{2.41}$$

where k_5 and k_7 were defined in (2.7) and (2.8), respectively. We introduce the energy

$$E(t) = \frac{1}{2}\|u'(t)\|_H + \frac{1}{2}\|u(t)\|_V + \frac{1}{\rho+1}(|u(t)|^\rho, u(t))_H, \quad t \ge 0. \tag{2.42}$$

Theorem 2.5. *Let u be the solution given by Theorem 2.4, and assume* (2.38) *holds. Then*

$$E(t) \le 3E(0)e^{-2\sigma t/3}, \quad \forall t \ge 0.$$

Before proving Theorem 2.3, we will show some previous results concerning to the trace $\gamma_1 u$ for a function $u \in W$ and on the approximation of the function h by a Lipchitz continuous function h_l.

3. Preliminary results

Let $\mathcal{O}$ be a star-shaped subset of $\mathbb{R}^n$. Consider the linear homotetic transformation $\sigma_\eta(x) = \eta x,\ \eta > 0$. Note that for $\eta > 1$,

$$\mathcal{O} \subset \overline{\mathcal{O}} \subset \sigma_\eta(\mathcal{O}). \tag{3.1}$$

We consider a vectorial function v defined in $\mathcal{O}$. For $\eta > 0$ introduce the function

$$\sigma_\eta \circ v : \sigma_\eta(\mathcal{O}) \to \mathbb{R}^n, \quad (\sigma_\eta \circ v)(y) = v(\sigma_{1/n}(y)).$$

Note that when $\eta > 1$, the domain of the function $\sigma_\eta \circ v$ contains the domain of w (see (3.1)).

Proposition 3.1. *Let $S \in \mathcal{D}'(\mathcal{O})$. Then*

(1) *$\sigma_\eta \circ S$ defined by*

$$\langle \sigma_\eta \circ S, \theta\rangle = \frac{1}{\eta^n}\langle S, \sigma_{\frac{1}{\eta}} \circ \theta\rangle, \quad \theta \in \mathcal{D}(\sigma_\eta(\mathcal{O}))$$

belongs to $\mathcal{D}'(\sigma_\eta(\mathcal{O}))$ $(\eta > 0)$.

(2) $\frac{\partial}{\partial y_i}(\sigma_\eta \circ S) = \eta\sigma_\eta \circ \big(\frac{\partial}{\partial y_i} S\big)$ *$(\eta > 0)$.*

(3) *If $\eta > 1$, $\eta \to 1$, the restriction to $\mathcal{O}$ of $\sigma_\eta \circ S$ converges in the distribution sense to S.*

(4) *If $v \in L^p(\mathcal{O})$ $(1 \le p < \infty)$, $\sigma_\eta \circ v \in L^p(\sigma_\eta(\mathcal{O}))$ $(\eta > 0)$. For $\eta > 1$, $\eta \to 1$, the restriction to $\mathcal{O}$ of $\sigma_\eta \circ v$ converges to v in $L^p(\mathcal{O})$.*

The proof of the above proposition can be found in Temam [25].

Theorem 3.2. *The space $(\mathcal{D}(\overline{\Omega}))^n$ is dense in W.*

Proof. Let U be an open set of $\mathbb{R}^n$ with boundary ∂U of class C^2. We introduce the Hilbert space

$$X(U) = \{u \in \mathbb{H}^1(\Omega) : Au \in \mathbb{L}^2(\Omega)\}$$

equipped with the scalar product

$$(u, v)_{X(U)} = (u, v)_{\mathbb{H}^1(U)} + (Au, Au)_{\mathbb{L}^2(U)}.$$

The proof will be divided into four steps:

Step 1. By truncation and regularization (see Lions [14]), we prove that $(\mathcal{D}(\mathbb{R}^n))^n$ is dense in $X(\mathbb{R}^n)$.

Step 2. Let $(U_l)_{1\le l\le m}$ be an open cover of Γ_0 and Γ_1 with $U_l^+ = \Omega \cap U_l$ star-shaped with respect to one of its points, $l = 1, 2, \dots, m$. Let $(\varphi_l)_{0\le l\le m}$ be a C^∞ partition of unity subordinate to the open convering Ω, $(U_l)_{1\le l\le n}$ of $\overline{\Omega}$. Thus

$$\varphi_0(x) + \sum_{l=1}^{n} \varphi_l(x) = 1, \quad \forall x \in \overline{\Omega},\ \varphi_0 \in \mathcal{D}(\Omega), \varphi_l \in \mathcal{D}(U_l),\ l = 1, 2, \dots, m.$$

Considering $u \in W$,

$$u = \varphi_0 u + \sum_{l=1}^{m} \varphi_l u. \tag{3.2}$$

We use the notation $v_l = \varphi_l u$ for $l = 0, 1, \dots, m$.

Analysis of $v_0 = (v_{01}, \dots, v_{0n})$. Represent by U_0 an open of $\mathbb{R}^n$ such that $\operatorname{supp}\varphi_0 \subset U_0 \subset \Omega$ is contained in U_0 and U_0 is star-shaped with respect to one of its points. After translation, we consider U_0 as being star-shaped with respect to $0 \in \mathbb{R}^n$. We define

$$\sigma_\eta \circ v_0 = (\sigma_\eta \circ v_{01}, \dots, \sigma_\eta \circ v_{0n}), \quad \eta > 1.$$

Then by (3.1) and Proposition 3.1 part 1, we have that $\sigma_\eta \circ v_0$ is defined in $\sigma_\eta(U_0)$. Consider $\psi \in \mathcal{D}(\sigma_\eta(U_0))$ such that $\psi = 1$ on U_0, and

$$w_{0\eta} = \psi[\sigma_\eta \circ v_0],\ \eta > 1. \tag{3.3}$$

Then $\operatorname{supp} w_{0\eta}$ is contained in $\sigma_\eta(U_0)$. By Proposition 3.1 part 2, we obtain

$$\frac{\partial}{\partial x_i} w_{0\eta} = \frac{\partial \psi}{\partial x_i}[\sigma_\eta \circ v_0] + \eta\psi\Big(\sigma_\eta \circ \frac{\partial v_0}{\partial x_i}\Big), \tag{3.4}$$

$$A w_{0\eta} = f(\eta)\psi[\sigma_\eta \circ A v_0] + \sum_{\substack{1\le|\alpha|\le 2\\ |\beta|\le 1}} g_\beta(\eta) P_\alpha(\psi)[\sigma_\eta \circ Q_\beta(v_0)], \tag{3.5}$$

where $f(\eta)$ and $g_\beta(\eta)$ are real functions of η with $f(\eta) \to 1$ as $\eta \to 1$, α and β, are multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ and P_α, Q_β are partial differential operators.

By the two preceding equalities, we obtain that $w_{0\eta} \in X(\sigma_\eta(U_0))$. Consider $\widetilde{w_{0\eta}}$ the extension of $w_{0\eta}$ by zero outside of $\sigma_\eta(U_0)$. Then $\widetilde{w_{0\eta}} \in X(\mathbb{R}^n)$. By the first part, we have that $\widetilde{w_{0\eta}}$ can be approximated in $X(\mathbb{R}^n)$ by functions of $(\mathcal{D}(\mathbb{R}^n))^n$. Consequently

$$w_{0\eta} \text{ can be approximate in } X(\sigma_\eta(U_0)) \text{ by functions of } (\mathcal{D}(\overline{\sigma_\eta(U_0)}))^n. \tag{3.6}$$

By (3.3)–(3.5) we have

$$\begin{gathered} w_{0\eta}\big|_{U_0} = \sigma_\eta \circ v_0\big|_{U_0}, \\ \frac{\partial w_{0\eta}}{\partial x_i}\big|_{U_0} = \eta\sigma_\eta \circ \frac{v_0}{\partial x_i}\big|_{U_0}, \\ A w_{0\eta}\big|_{U_0} = f(\eta)[\sigma_\eta \circ A v_0]\big|_{U_0}. \end{gathered}$$

Then by Proposition 3.1 part 3, we obtain

$$\begin{gathered} w_{0\eta}\big|_{U_0} \to v_0 \quad \text{in } \mathbb{L}^2(\Omega) \text{ as } \eta \to 1, \\ \frac{\partial w_{0\eta}}{\partial x_i}\big|_{U_0} \to \frac{\partial v_0}{\partial x_i} \quad \text{in } \mathbb{L}^2(\Omega) \text{ as } \eta \to 1, \\ A w_{0\eta}\big|_{U_0} \to A v_0 \quad \text{in } \mathbb{L}^2(\Omega) \text{ as } \eta \to 1. \end{gathered}$$

By (3.6) and the last three convergences we conclude that v_0 can be approximated in $X(U_0)$ by functions of $\mathcal{D}(\overline{U_0})$.

Step 3. Analysis of v_l, $l = 1, 2, \dots, m$. In this case we apply similar arguments to those used previously for v_0. Thus we take U_l^+ instead of U_0. We can assume the U_l^+ is star-shaped with respect to $0 \in \mathbb{R}^n$. Consider $\sigma_\eta(U_l^+)$ instead of $\sigma_\eta(U_0)$. We introduce

$$\psi \in \mathcal{D}(\sigma_\eta(U_l^+) \quad \text{with } \psi \equiv 1 \text{ on } U_l^+.$$

Consider $w_{l\eta} = \psi[\sigma_\eta \circ v_l]$, $\eta > 1$. Then

$$\begin{gathered} w_{l\eta} \in X(\sigma_\eta(U_l^+)), \\ \operatorname{supp} w_{l\eta} \text{ is contained in } \sigma_\eta(U_l^+), \\ \widetilde{w_{l\eta}} \text{ belongs to } X(\mathbb{R}^n), \\ w_{l\eta}\big|_{U_l^+} \to v_l \quad \text{in } X(U_l^+) \text{ as } \eta \to 1. \end{gathered}$$

Thus v_l can be approximated in $X(U_l^+)$ by a function of $(\mathcal{D}(\overline{U_l^+}))^n$. By (3.2) and the above results we conclude that $v \in W$ can be approximated in $X(\Omega)$ by functions of $(\mathcal{D}(\overline{\Omega}))^n$.

Step 4. Theorem follows because $X(\Omega)$ and W have equivalent norms in W. $\square$

Theorem 3.3. *There exists a linear continuous map $\gamma_1 : W \to \mathbb{H}^{-1/2}(\Gamma_1)$, $u \mapsto \gamma_1 u$ such that*

$$\gamma_1 u = \mu \frac{\partial u}{\partial \nu} + (\lambda + \mu)(\operatorname{div} u)\nu, \quad \forall u \in (\mathcal{D}(\overline{\Omega}))^n. \tag{3.7}$$

Furthermore

$$(Au, v)_H = ((u, v))_V - \langle \gamma_1 u, v\rangle_{\mathbb{H}^{-1/2}(\Gamma_1) \times \mathbb{H}^{1/2}(\Gamma_1)}, \quad \forall u \in W,\ v \in V. \tag{3.8}$$

Proof. Let $u \in (\mathcal{D}(\overline{\Omega}))^n$ and $v \in V$. By Gauss's Divergence Theorem, we obtain

$$(Au, v)_H = ((u, v))_V - (\gamma_1 u, v)_{\mathbb{L}^2(\Gamma_1)}. \tag{3.9}$$

Then

$$|(\gamma_1 u, v)_{\mathbb{L}^2(\Gamma_1)}| \le \|u\|_V \|v\|_V + C\|Au\|_H \|v\|_V .$$

Therefore,

$$|\langle \gamma_1 u, v\rangle_{\mathbb{H}^{-1/2}(\Gamma_1) \times \mathbb{H}^{1/2}(\Gamma_1)}| \le C\|u\|_W \|v\|_V . \tag{3.10}$$

Let $\xi \in \mathbb{H}^{1/2}(\Gamma_1)$, then by the Trace Theorem there is $v \in V$ such that $\gamma_0 v = \xi$ and the map

$$\mathbb{H}^{1/2}(\Gamma_1) \to V,\ \xi \mapsto v$$

is continuous. By the above map and (3.10), we obtain

$$\big|\langle \gamma_1 u, \xi\rangle_{\mathbb{H}^{-1/2}(\Gamma_1) \times \mathbb{H}^{1/2}(\Gamma_1)}\big| \le C\|u\|_W \|\xi\|_{\mathbb{H}^{1/2}(\Gamma_1)}, \quad \forall u \in (\mathcal{D}(\overline{\Omega}))^n, \xi \in \mathbb{H}^{1/2}(\Gamma_1).$$

By this inequality and the density of $(\mathcal{D}(\overline{\Omega}))^n$ in W given by Theorem 3.2, we obtain (3.7). The equality (3.9) and Theorem 3.2 provide (3.8). □

Proposition 3.4. *Let h be a function satisfying* (2.9)–(2.11). *Then for each $i = 1, \dots, n$ there exists a sequence (h_{il}) of functions of $C^0(\mathbb{R}, L^\infty(\Gamma_1))$ such that*

$$h_{il}(x, 0) = 0 \quad \text{for a.e. } x \text{ in } \Gamma_1; \tag{3.11}$$

$$[h_{il}(x, s) - h_{il}(x, r)](s - r) \ge d_0(s - r)^2, \quad \forall s, r \in \mathbb{R} \text{ for a.e. } x \in \Gamma_1; \tag{3.12}$$

$$\text{There exists a function } c_l \in L^\infty(\Gamma_1) \text{ satisfying} \tag{3.13}$$

$$|h_{il}(x, s) - h_{il}(x, r)| \le c_l|s - r|, \quad \forall s, r \in \mathbb{R}, \text{for a.e. } x \text{ in } \Gamma_1;$$

$$(h_{il}) \text{ converges to } h_i \text{ uniformly on boundary sets for } \mathbb{R} \text{ for a.e. } x \in \Gamma_1. \tag{3.14}$$

4. Proof of Theorem 2.3

In this section, we will prove the existence of solution of problem (2.22)–(2.24).

Proof of Theorem 2.3. We employ the Faedo-Galerkin's method with a special basis of V (see [16] or [18] for other special basis). Let (u_l^1) be a sequence of $(\mathcal{D}(\Omega))^n$ such that

$$u_l^1 \to u^1 \quad \text{in } \mathbb{H}_0^1(\Omega). \tag{4.1}$$

Fix $l \in \mathbb{N}$. With u^0 and u_l^1 we constructed a basis

$$\{w_1^l, w_2^l, \dots\} \tag{4.2}$$

of V such that u^0, u_l^1 belong to the subspace $[w_1^l, w_2^l]$ generated by w_1^l and w_2^l.

Consider the approximation (h_{il}) of h_i $(i = 1, \dots, n)$ given by Proposition 3.4. Here we will denote $(h_l) = (h_{1l}, \dots, h_{nl})$.

Remark 4.1. Since $u^0 \in D(B)$ and $u_l^1 \in \mathbb{H}_0^1(\Omega)$, we have $\gamma_1 u = 0$ and $h_l(\cdot, u_l^1) = 0$ on Γ_1. Thus

$$\gamma_1(u^0) + h_l(\cdot, u_l^1) = 0 \quad \text{in } \Gamma_1, \forall l \in \mathbb{N}.$$

Consider $V_m^l = [w_1^l, \dots, w_m^l]$ the subspace of V generated by the first m vectors of the basis of (4.2). Let us find the approximate solution $u_{lm}(t) \in V_m^l$ of Problem (1.2)–(1.5), that is,

$$u_{lm}(t) = \sum_{j=1}^{m} g_{lmj}(t) w_j^l,$$

where $u_{lm}(t) \in V_m^l$ is the solution of the system

$$\begin{aligned} &(u_{lm}''(t),\, v)_H + ((u_{lm}(t),\, v))_V + (|u_{lm}(t)|^\rho,\, v)_H \\ &+ (h_l(\cdot, u_{lm}'(t)),\, v)_{\mathbb{L}^2(\Gamma_1)} = 0, \quad \forall v \in V_m^l \end{aligned} \tag{4.3}$$

$$u_{lm}(0) = u^0, \quad u_{lm}'(0) = u_l^1. \tag{4.4}$$

System (4.3)–(4.4) has a solution on an interval $[0,\, t_{lm})$ with $t_{lm} < T$. This solution can be extended to the interval $[0,\, T]$ as a consequence of the a priori estimates that shall be proved.

First estimate. Taking $v = u_{lm}'(t) \in V_m^l$ in (4.3), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u_{lm}'(t)\|_H^2 + \frac{1}{2}\frac{d}{dt}\|u_{lm}(t)\|_V^2 + (|u_{lm}(t)|^\rho, u_{lm}'(t))_V \\ &+ (h_l(\cdot, u_{lm}'(t)), u_{lm}'(t))_{\mathbb{L}^2(\Gamma_1)} = 0. \end{aligned} \tag{4.5}$$

Then

$$(|u_{lm}(t)|^\rho, u_{lm}'(t))_H = \frac{1}{\rho+1}\frac{d}{dt}(|u_{lm}(t)|^\rho, u_{lm}(t))_H$$

and by (3.12),

$$(h_l(\cdot, u_{lm}'(t)), u_{lm}'(t))_{\mathbb{L}^2(\Gamma_1)} \ge d_0 \|u_{lm}'(t)\|_{\mathbb{L}^2(\Gamma_1)}^2.$$

Remark 4.2. As $h(x,u) = (h_1(x,u_1), \dots, h_n(x,u_n))$ where $h_i(x,s)$ is measurable in Γ_1 and continuous in $\mathbb{R}$, $i = 1, \dots, n$, by (3.12) we have

$$\begin{aligned} ((h_l(\cdot, u_{lm}'(t)), u_{lm}'(t))_{\mathbb{L}^2(\Gamma_1)} &= \sum_{i=1}^{n} (h_{il}(\cdot, u_{kmi}'(t)),\, u_{kmi}'(t)) \\ &\ge \sum_{i=1}^{n} d_0 |u_{kmi}'(t)|^2 = d_0 \|u_{km}'(t)\|_{\mathbb{L}^2(\Gamma_1)}^2. \end{aligned} \tag{4.6}$$

Putting the above two expressions in (4.5) and then integrating on $[0,t]$, $0 < t < t_{lm}$, we obtain

$$\begin{aligned} &\frac{1}{2}\|u_{lm}'(t)\|_H^2 + \frac{1}{2}\|u_{lm}(t)\|_V^2 + \frac{1}{\rho+1}(|u_{lm}(t)|^\rho, u_{lm}(t))_H \\ &+ d_0 \int_0^t \|u_{lm}'(s)\|_{\mathbb{L}^2(\Gamma_1)}^2 \mathrm{d}s \\ &\le \frac{1}{2}\|u_l^1\|_H^2 + \frac{1}{2}\|u^0\|_V^2 + \frac{1}{\rho+1}(|u^0|^\rho, u^0)_H. \end{aligned} \tag{4.7}$$

Next, our goal is to determine for which $t \in (0, t_{lm})$ the first member of (4.7) becomes non-negative. By (2.5) and Remark 2.1, we obtain

$$\Big|\frac{1}{\rho+1}(|u_{lm}(t)|^\rho, u_{lm}(t))_H\Big| \le \frac{1}{\rho+1}\|u_{lm}(t)\|^{\rho+1}_{\mathbb{L}^{\rho+1}(\Omega)} \le \frac{nk_0^{\rho+1}}{(\rho+1)\mu^{\frac{\rho+1}{2}}}\|u_{lm}(t)\|_V^{\rho+1}.$$

Also

$$\Big|\frac{1}{\rho+1}(|u^0|^\rho, u^0)_H\Big| \le \frac{nk_0^{\rho+1}}{(\rho+1)\mu^{\frac{\rho+1}{2}}}\|u^0\|_V^{\rho+1}.$$

Then by (2.15) we find a positive real number τ such that

$$\frac{1}{2}\|u_l^1\|_H^2 + \frac{1}{2}\|u^0\|_V^2 + \frac{nk_0^{\rho+1}}{(\rho+1)\mu^{\frac{\rho+1}{2}}}\|u^0\|_V^{\rho+1} < \tau < \frac{1}{4}(\lambda^*)^2, \quad \forall l \ge l_0^*. \tag{4.8}$$

Taking into account the three inequalities in (4.7), we obtain

$$\begin{aligned} &\frac{1}{2}\|u'_{lm}(t)\|_H^2 + \frac{1}{2}\|u_{lm}(t)\|_V^2 - \frac{nk_0^{\rho+1}}{(\rho+1)\mu^{\frac{\rho+1}{2}}}\|u_{lm}(t)\|_V^{\rho+1} \\ &+ \ d_0\int_0^t \|u'_{lm}(s)\|^2_{\mathbb{L}^2(\Gamma_1)}\mathrm{d}s \\ &\le \tau < \frac{1}{4}(\lambda^*)^2,\ \forall l \ge l_0^*, \quad \forall t \in [0, t_{lm}). \end{aligned} \tag{4.9}$$

We analyze for which $t \in (0, t_{lm})$, we would have

$$\frac{1}{4}\|u_{lm}(t)\|_V^2 - \frac{nk_0^{\rho+1}}{(\rho+1)\mu^{\frac{\rho+1}{2}}}\|u_{lm}(t)\|_V^{\rho+1} \ge 0.$$

Motivated by the above inequality, we consider the function

$$J(\lambda) = \frac{1}{4}\lambda^2 - \frac{nk_0^{\rho+1}}{(\rho+1)\mu^{\frac{\rho+1}{2}}}\lambda^{\rho+1}, \quad \lambda \ge 0.$$

We find that

$$J(\lambda) = \lambda^2\big[\frac{1}{4} - \frac{nk_0^{\rho+1}}{(\rho+1)\mu^{\frac{\rho+1}{2}}}\lambda^{\rho-1}\big]$$

which implies

$$J(\lambda) \ge 0 \text{ if } 0 \le \lambda \le \big[\frac{(\rho+1)\mu^{\frac{\rho+1}{2}}}{4nk_0^{\rho+1}}\big]^{\frac{1}{\rho-1}} = \lambda^*. \tag{4.10}$$

To continue the proof we need the following result.

Lemma 4.3. *We have*

$$\|u_{lm}(t)\|_V < \lambda^*, \quad \forall t \in [0, \infty),\ \forall l \ge l_0,\ \forall m \in \mathbb{N}.$$

Proof. Fix $m \in \mathbb{N}$. We argue by contradiction. Suppose that there exists $t_1 \in (0, t_{lm})$ such that $\|u_{lm}(t_1)\|_V \le \lambda^*$ and let $\theta(t) = \|u_{lm}(t)\|_V$. As θ is continuous on $[0, t_1]$, by the Intermediate Value Theorem we have that there exists $\tau_1 \in (0, t_1]$ such that $\theta(\tau_1) = \lambda^*$. Let

$$t^* = \inf\{\tau \in (0, t_{lm}) : \theta(\tau) = \lambda^*\}.$$

We have

$$\theta(t^*) = \lambda^* \quad \text{because } \theta \text{ is continuous on } [0, t_{lm}); \tag{4.11}$$

$$0 < t^* < t_{lm} \quad \text{because } \theta(0) = \|u^0\|_V < \lambda^*; \tag{4.12}$$

$$\theta(t) < \lambda^*, \quad \forall t \in [0, t^*). \tag{4.13}$$

By (4.10) and (4.13), we obtain

$$J(\|u_{lm}(t)\|_V\|) \geq 0, \quad \forall t \in [0, t^*).$$

Putting this inequality in (4.8), we find

$$\frac{1}{4}\|u_{lm}(t)\|_V^2 \leq \tau < \frac{1}{4}(\lambda^*)^2, \quad \forall t \in [0, t^*).$$

Taking the limit in this inequality as $t \to t^*$, $t < t^*$ and using (4.11), we arrive to a contradiction. Thus the lemma is proved. □

By (4.8), Lemma 4.3 and (4.10), we obtain

$$\begin{aligned} &\frac{1}{2}\|u'_{lm}(t)\|_H^2 + \frac{1}{4}\|u_{lm}(t)\|_V^2 + d_0 \int_0^t \|u'_{lm}(s)\|_{\mathbb{L}^2(\Gamma_1)} \mathrm{d}s \\ &< \frac{1}{4}(\lambda^*)^2, \quad \forall t \in [0, \infty),\ \forall l \geq l_0,\ \forall m \in \mathbb{N}. \end{aligned} \tag{4.14}$$

Thus

$$(u_{lm}) \text{ is bounded in } L^\infty(0, \infty; V)\ \forall l \geq l_0,\ \forall m \in \mathbb{N}; \tag{4.15}$$

$$(u'_{lm}) \text{ is bounded in } L^\infty(0, \infty; H),\ \forall l \geq l_0,\ \forall m \in \mathbb{N}; \tag{4.16}$$

$$(u'_{lm}) \text{ is bounded in } L^2(0, \infty; \mathbb{L}^2(\Gamma_1)),\ \forall l \geq l_0,\ \forall m \in \mathbb{N}. \tag{4.17}$$

Second estimate. We differentiate the approximate equation (4.3) and then we take $v = u''_{lm}(t)$. We obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u''_{lm}(t)\|_H^2 + \frac{1}{2}\frac{d}{dt}\|u'_{lm}(t)\|_V \\ &+ \big(\rho|u_{lm}(t)|^{\rho-2}u_{lm}(t)u'_{lm}(t), u''_{lm}(t)\big)_H \\ &+ (h'_l(\cdot, (u'_{lm}(t))u''_{lm}(t), u''_{lm}(t))_{\mathbb{L}^2(\Gamma_1)} = 0. \end{aligned} \tag{4.18}$$

From Hölder inequality applied to $\frac{1}{n} + \frac{1}{q^*} + \frac{1}{2} = 1$, (2.6), (2.7) and estimate (4.15), we find that

$$|(\rho|u_{lm}(t)|^{\rho-2}u_{lm}(t)u'_{lm}(t), u''_{lm}(t))_H| \leq C\|u'_{lm}(t)\|_V^2 + \frac{1}{2}\|u''_{lm}(t)\|_H^2,$$

where $C > 0$ is a constant independent of $l \in \mathbb{N}$ and $m \in \mathbb{N}$. By (3.12) of Proposition 3.4, we obtain

$$(h'_l(\cdot, u'_{lm}(t))u''_{lm}(t), u''_{lm}(t))_{\mathbb{L}^2(\Gamma_1)} \geq d_0\|u''_{lm}(t)\|_{\mathbb{L}^2(\Gamma_1)}^2.$$

Taking into account the last two inequalities in (4.18), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u''_{lm}(t)\|_H^2 + \frac{1}{2}\frac{d}{dt}\|u'_{lm}(t)\|_V + d_0\|u''_{lm}(t)\|_{\mathbb{L}^2(\Gamma_1)}^2 \\ &\leq C\|u'_{lm}(t)\|_V^2 + \frac{1}{2}\|u''_{lm}(t)\|_H^2. \end{aligned} \tag{4.19}$$

Third estimate. We make $t = 0$ in (4.3) and then take $v = u''_{lm}(0)$. We obtain

$$\|u''_{lm}(0)\|_H^2 + ((u^0, u''_{lm}(0)))_V + (|u^0|^\rho, u''_{lm}(0))_H + (h_l(\cdot, u_l^1), u''_{lm}(0))_{\mathbb{L}^2(\Gamma_1)} = 0.$$

From Remark 4.1 it follows that

$$\|u''_{lm}(0)\|_H^2 + (Bu^0, u''_{lm}(0)_H + (|u^0|^\rho, u''_{lm}(0))_H = 0. \tag{4.20}$$

In this part the choose of the special basis (4.2) is crucial.

By applying hypothesis (2.13) and inequality (2.6) in (4.20), we have

$$\|u''_{lm}(0)\|_H^2 \le C, \forall l \ge l_0, \quad \forall m \in \mathbb{N}. \tag{4.21}$$

Consider a real number $T > 0$. Integrating both sides of inequality (4.19) on $[0, t]$, $0 \le t \le T$, and taking into account the estimate (4.21) and the convergence (4.1), we obtain

$$\begin{aligned} &\frac{1}{2}\|u''_{lm}(t)\|_H^2 + \frac{1}{2}\|u'_{lm}(t)\|_V^2 + d_0 \int_0^t \|u''_{lm}(s)\|^2_{\mathbb{L}^2(\Gamma_1)} \mathrm{d}s \\ &\le C + \int_0^t [C\|u'_{lm}(s)\|_V^2 + \frac{1}{2}\|u''_{lm}(s)\|_H^2] \mathrm{d}s, \quad 0 \le t \le T, \end{aligned} \tag{4.22}$$

where the constant $C > 0$ is independent of $l \ge l_0$, $m \in \mathbb{N}$ and $T > 0$.

By Gronwall Lemma and noting that $T > 0$ was arbitrary, by (4.22) we obtain

$$(u'_{lm}) \text{ is bounded in } L^\infty_{\rm loc}(0, \infty; V), \ \forall l \ge l_0, \ \forall m \in \mathbb{N}; \tag{4.23}$$

$$(u''_{lm}) \text{ is bounded in } L^\infty_{\rm loc}(0, \infty; H), \ \forall l \ge l_0, \ \forall m \in \mathbb{N}; \tag{4.24}$$

$$(u''_{lm}) \text{ is bounded in } L^2_{\rm loc}(0, \infty; \mathbb{L}^2(\Gamma_1)), ; \forall l \ge l_0, \ \forall m \in \mathbb{N}. \tag{4.25}$$

Pass to limit in m. In what follows of the paper it will be understood that various subsequences of the principal sequence will be considered. Also the diagonal process will be applied to obtain convergence in all $(0, \infty)$.

By (4.15)–(4.17) and (4.23)–(4.25), we find that there exists a subsequence of (u_{lm}), still denoted by (u_{lm}), and a function u_l such that

$$u_{lm} \to u_l \quad \text{weak star in } L^\infty(0, \infty; V); \tag{4.26}$$

$$u'_{lm} \to u'_l \quad \text{weak star in } L^\infty(0, \infty; H) \cap L^\infty_{\rm loc}(0, \infty; V); \tag{4.27}$$

$$u''_{lm} \to u''_l \quad \text{weak star in } L^\infty_{\rm loc}(0, \infty; H); \tag{4.28}$$

$$u''_{lm} \to u''_l \quad \text{weak in } L^2_{\rm loc}(0, \mathbb{L}^2(\Gamma_1)). \tag{4.29}$$

It follows from (4.27) that

$$u'_{lm} \to u'_l \quad \text{weak star in } L^\infty_{\rm loc}(0, \infty; \mathbb{H}^{1/2}(\Gamma_1)). \tag{4.30}$$

As the embedding of V in H is compact, we obtain by (4.26) and (4.27) that

$$u_{lm} \to u_l \quad \text{in } L^\infty_{\rm loc}(0, \infty; H).$$

Thus

$$u_{lm}(x, t) \to u_l(x, t) \quad \text{a.e. in } \Omega \times (0, T).$$

On the other hand, by (4.15) and (2.6), we find that

$$(|u_{lm}|^\rho)_{m\in\mathbb{N}} \text{ is bounded in } L^2(0, T; H).$$

These two last results, Lions Lemma [13] and the diagonal process imply that

$$|u_{lm}|^\rho \ \to |u_l|^\rho \quad \text{weak in } L^2_{\rm loc}(0, \infty; H). \tag{4.31}$$

In a similar way, noting that the embedding of $\mathbb{H}^{1/2}(\Gamma_1)$ in $\mathbb{L}^2(\Gamma_1)$ is compact, by (4.30) and (4.29), we obtain

$$h_l(x, u'_{lm}) \to h_l(x, u'_l) \quad \text{for a.e. in } \Gamma_1 \times (0, T) \tag{4.32}$$

and by (3.13) and (4.23) we obtain that

$$(h_l(\cdot, u'_{lm})) \text{ is bounded in } L^2(0, T; \mathbb{L}^2(\Gamma_1)).$$

Therefore

$$h_l(x, u'_{lm}) \to h_l(x, u'_l) \quad \text{weak in } L^2_{\rm loc}(0, \infty; \mathbb{L}^2(\Gamma_1)). \tag{4.33}$$

Convergences (4.26)–(4.29), (4.31) and (4.32) permit us to pass to the limit as $m \to \infty$ in approximate equation (4.3). Thus for $\theta \in \mathcal{D}(\Omega)$ and noting that (4.2) is a base of V, we obtain

$$\begin{aligned}
&\int_0^\infty (u''_l(t),\, \theta(t)v)_H \mathrm{d}t + \int_0^\infty ((u_l(t),\, \theta(t)v))_V \mathrm{d}t \\
&+ \int_0^\infty (|u_l(t))|^\rho,\, \theta(t)v)_V \mathrm{d}t + \int_0^\infty (h_l(\cdot, u'_l(t)),\, \theta(t)v)_{\mathbb{L}^2(\Gamma_1)} \mathrm{d}t = 0.
\end{aligned}$$

As the set $\{\theta v; \theta \in \mathcal{D}(0,\infty), v \in V\}$ is total in $L^2(0,\infty;V)$, the above inequality implies

$$\begin{aligned}
&\int_0^\infty (u''_l(t), \varphi)_H \mathrm{d}t + \int_0^\infty ((u_l(t),\, \varphi))_V \mathrm{d}t + \int_0^\infty (|u_l(t))|^\rho,\, \varphi)_V \theta(t) \mathrm{d}t \\
&+ \int_0^\infty (h_l(\cdot, u'_l(t)), \varphi)_{\mathbb{L}^2(\Gamma_1)} \mathrm{d}t = 0, \quad \forall \varphi \in L^2(0,\infty;V),
\end{aligned} \tag{4.34}$$

and $\operatorname{supp}\varphi$ is bounded in$[0,\infty)$.

Taking $\varphi \in \mathcal{D}((0,\infty) \times (\Omega)^n)$ in (4.33), we obtain

$$u''_l + Au_l + |u_l|^\rho = 0 \quad \text{in } \mathcal{D}'((0,\infty) \times (\Omega)^n).$$

As u''_l and $|u_l|^\rho$ belong to $L^2_{\rm loc}(0,\infty;H)$, we obtain

$$u''_l + Au_l + |u_l|^\rho = 0 \quad \text{in } L^2_{\rm loc}(0,\infty;H). \tag{4.35}$$

From now on , φ denotes a function satisfying conditions (4.33). We take the inner product of H with φ in both of sides of (4.35). We deduce

$$\int_0^\infty (u''_l(s), \varphi)_H \mathrm{d}s + \int_0^\infty (Au_l(s), \varphi)_H \mathrm{d}s + \int_0^\infty (|u_l(s)|^\rho, \varphi)_H \mathrm{d}s = 0. \tag{4.36}$$

Note that

$$u_l \in L^\infty(0,\infty;V) \quad \text{and} \quad Au_l \in L^2_{\rm loc}(0,\infty;H).$$

Then by Theorem 3.3, Part (3.8), we obtain $\gamma_1 u_l \in L^2_{\rm loc}(0,\infty;\mathbb{H}^{-1/2}(\Gamma_1))$ and thus

$$\int_0^\infty (Au_l(s), \varphi)_H \mathrm{d}s = \int_0^\infty ((u_l(s), \varphi))_V \mathrm{d}s - \int_0^\infty \langle \gamma_1 u_l(s), \varphi\rangle_{Y' \times Y} \mathrm{d}s,$$

where $Y = \mathbb{H}^{1/2}(\Gamma_1)$. Replacing this equality in (4.36), we deduce

$$\begin{aligned}
&\int_0^\infty (u''_l(s), \varphi)_H \mathrm{d}s + \int_0^\infty ((u_l(s), \varphi))_V \mathrm{d}s - \int_0^\infty \langle \gamma_1 u_l(s), \varphi\rangle_{Y' \times Y} \mathrm{d}s \\
&+ \int_0^\infty (|u_l(s)|^\rho, \varphi)_H \mathrm{d}s = 0.
\end{aligned} \tag{4.37}$$

Comparing (4.33) and (4.37) and taking into account the regularity of $h_l(\cdot, u'_l)$, we find

$$\gamma_1 u_l + h_l(\cdot, u'_l) = 0 \text{ in } L^2_{\rm loc}(0,\infty,\mathbb{L}^2(\Gamma_1)). \tag{4.38}$$

Pass to limit in l. Estimates (4.15)–(4.17) and (4.23)–(4.25) are independent of $l \geq l_0$. Then as in (4.26)–(4.29), (4.30) and (4.31), we obtain that there exist a subsequence of (u_l), still denoted by (u_l), and a function u such that

$$u_l \to u \quad \text{weak star in } L^\infty(0,\ \infty; V); \tag{4.39}$$

$$u_l' \to u' \quad \text{weak star in } L^\infty(0,\infty; H) \cap L^\infty_{\rm loc}(0,\infty; V); \tag{4.40}$$

$$u_l'' \to u'' \quad \text{weak star in } L^\infty_{\rm loc}(0,\infty; H); \tag{4.41}$$

$$|u_l|^\rho \to |u|^\rho \quad \text{weakly in } L^2_{\rm loc}(0,\infty; H); \tag{4.42}$$

$$u_l'(x,t) \to u'(x,t) \quad \text{a.e. } (x,t) \in \Gamma_1 \times (0,T). \tag{4.43}$$

Convergence (4.41) follows from the compact embedding of $\mathbb{H}^{1/2}(\Gamma_1)$ in $\mathbb{L}^2(\Gamma_1)$. Take the limit in (4.33). Then by convergences (4.39)–(4.41) and applying similar arguments used to obtain (4.35), we deduce

$$u'' + Au + |u|^\rho = 0 \quad \text{in } L^2_{\rm loc}(0,\infty; H). \tag{4.44}$$

By estimate (4.15) and equation (4.35), we obtain that

$$\begin{gathered}(u_l) \text{ is bounded in } L^\infty(0,\infty; V),\\ (Au_l) \text{ is bounded in } L^2_{\rm loc}(0,\infty; H).\end{gathered}$$

Then by Theorem 3.3, we obtain

$$\gamma_1 u_l \to \gamma_1 u \quad \text{in } L^2_{\rm loc}(0,\infty; \mathbb{H}^{-1/2}(\Gamma_1)). \tag{4.45}$$

Fix $(x,t) \in \Gamma_1 \times (0,T)$. Then by convergences (4.43) and condition (3.14) of Proposition 3.4, we deduce

$$h_l(x, u_l'(x,t)) \to h(x, u'(x,t)) \quad \text{a.e. } x \text{ in } \Gamma_1 \times (0,T). \tag{4.46}$$

On the other hand, by estimates (4.15), (4.23), (4.16), and (4.24), we find that

$$(u_l) \text{ is bounded in } C^0([0,T]; V);\ \forall T > 0; \tag{4.47}$$

$$(u_l') \text{ is bounded in } C^0([0,T]; H);\ \forall T > 0. \tag{4.48}$$

By (4.33) and noting that each $h_i(x,s)$ is increasing in s, we obtain

$$\begin{aligned}0 &\leq \int_0^T (h_l(\cdot, u_l'), u_l')_{\mathbb{L}^2(\Gamma_1)} {\rm d}t\\ &= -\frac{1}{2}\|u_l'(T)\|_H^2 + \frac{1}{2}\|u_l^1\|_H^2 - \frac{1}{2}\|u_l(T)\|_V^2 + \frac{1}{2}\|u^0\|_H^2 - \int_0^T (|u_l|^\rho, u_l')_H {\rm d}t.\end{aligned}$$

Then by (4.47), (4.48) and (4.42), and (4.40), we have

$$0 \leq \int_0^T (h_l(\cdot, u_l'), u_l')_{\mathbb{L}^2(\Gamma_1)} {\rm d}t \leq C(T). \tag{4.49}$$

It follows from (4.46), (4.49) and a results due to Strauss [23] that

$$h_l(\cdot, u_l') \to h(\cdot, u') \quad \text{in } L^1(0,T; \mathbb{L}^1(\Gamma_1)).$$

As $T > 0$ was arbitrary it follows that

$$h_l(\cdot, u_l') \to h(\cdot, u') \quad \text{in } L^1_{\rm loc}(0,\infty; \mathbb{L}^1(\Gamma_1)). \tag{4.50}$$

Taking the limit in (4.38) and using (4.45) and (4.50), we find that

$$\gamma_1 u + h(\cdot, u') = 0 \quad \text{in } L^1_{\rm loc}(0,\infty; \mathbb{H}^{-1/2}(\Gamma_1) + \mathbb{L}^1(\Gamma_1)).$$

By equation (4.44), we deduce that $u \in L^2_{\rm loc}(0,\infty;W)$. Convergences (4.39)-(4.41) provide the initial conditions (2.24). Thus the proof is complete. $\square$

Remark 4.4. The proof of the existence of solutions of Theorem 2.4 follows by applying similar arguments used to obtain Theorem 2.3. In this case, we consider $J_1(\lambda) = \frac{1}{4}\lambda^2 - N_1\lambda^{\rho+1}$, $\lambda \geq 0$ and

$$h_{il}(x,s) = [m(x)\cdot\nu(x)]p_i(s), \quad \forall l \in \mathbb{N},\ i = 1,\dots,n.$$

Note that $\lambda_1^* < \lambda^*$. The uniqueness of solution is derived by the energy method.

5. Proof of Theorem 2.3

By (2.16) we have

$$(Au)_i = -\mu\Delta u_i - (\lambda+\mu)\frac{\partial}{\partial x_i}\operatorname{div} u, \quad i = 1,\dots,n. \tag{5.1}$$

Proposition 5.1. *Let $u \in \mathbb{H}^2(\Omega)$. Then*

$$\begin{aligned}
&\sum_{i=1}^n 2(-(Au)_i, m\nabla u_i)\\
&= \mu(n-2)\sum_{i=1}^n \int_\Omega |\nabla u_i|^2 \mathrm{d}x - \mu\sum_{i=1}^n\int_\Gamma |\nabla u_i|^2(m\cdot\nu)\mathrm{d}\Gamma\\
&\quad + 2\mu\sum_{i=1}^n\int_\Gamma \frac{\partial u_i}{\partial\gamma}(m\cdot\nabla u_i)\mathrm{d}\Gamma + (\lambda+\mu)(n-2)\int_\Omega(\operatorname{div} u)^2\mathrm{d}x\\
&\quad - (\lambda+\mu)\int_\Gamma(\operatorname{div} u)^2(m\cdot\nu)\mathrm{d}\Gamma + 2(\lambda+\mu)\sum_{i=1}^n\int_\Gamma(\operatorname{div} u)(m\cdot\nabla u_i)\nu_i d\Gamma\\
&= \sum_{i=1}^6 M_i,
\end{aligned}$$

where,

- $M_1 = \mu(n-2)\sum_{i=1}^n\int_\Omega|\nabla u_i|^2\mathrm{d}x$;
- $M_2 = \mu\sum_{i=1}^n\int_\Gamma|\nabla u_i|^2(m\cdot\nu)\mathrm{d}\Gamma$;
- $M_3 = 2\mu\sum_{i=1}^n\int_\Gamma\frac{\partial u_i}{\partial\gamma}(m\cdot\nabla u_i)\mathrm{d}\Gamma$;
- $M_4 = (\lambda+\mu)(n-2)\int_\Omega(\operatorname{div} u)^2\mathrm{d}x$;
- $M_5 = -(\lambda+\mu)\int_\Gamma(\operatorname{div} u)^2(m\cdot\nu)\mathrm{d}\Gamma$;
- $M_6 = 2(\lambda+\mu)\sum_{i=1}^n\int_\Gamma(\operatorname{div} u)(m\cdot\nabla u_i)\nu_i d\Gamma$.

Proof. Expression (5.1) provides

$$\begin{aligned}
&\sum_{i=1}^n 2(-(Au)_i, m\cdot\nabla u_i)\\
&= \sum_{i=1}^n 2\mu(\Delta u_i, m\cdot\nabla u_i) + \sum_{i=1}^n 2(\lambda+\mu)\Big(\frac{\partial}{\partial x_i}\operatorname{div} u, m\cdot\nabla u_i\Big).
\end{aligned} \tag{5.2}$$

By the Rellich identity (see Komornik-Zuazua [8]), we have

$$\sum_{i=1}^n 2\mu(\Delta u_i, m\cdot\nabla u_i) = M_1 + M_2 + M_3. \tag{5.3}$$

On the other hand

$$\Big(\frac{\partial}{\partial x_i}\operatorname{div} u, m\cdot\nabla u_i\Big) = -\Big(\operatorname{div} u, \frac{\partial}{\partial x_i}(m\cdot\nabla u_i)\Big) + \int_\Gamma (\operatorname{div} u)(m\cdot\nabla u_i)\nu_i \mathrm{d}\Gamma. \quad (5.4)$$

Also

$$\frac{\partial}{\partial \nu_i}(m\cdot\nabla u_i) = \frac{\partial u_i}{\partial x_i} + \sum_{l=1}^n m_l \frac{\partial}{\partial x_l}\Big(\frac{\partial u_i}{\partial x_i}\Big),$$

$$\begin{aligned}
&\sum_{i=1}^n \Big(\operatorname{div} u, \frac{\partial}{\partial x_i}(m\cdot\nabla u_i)\Big)\\
&= \int_\Omega (\operatorname{div} u)^2 \mathrm{d}x + \sum_{l=1}^n \int_\Omega (\operatorname{div} u) m_l \frac{\partial}{\partial x_l}[\operatorname{div} u]\mathrm{d}x\\
&= \int_\Omega (\operatorname{div} u)^2 \mathrm{d}x - \frac{n}{2}\int_\Omega (\operatorname{div} u)^2 \mathrm{d}x + \frac{1}{2}\int_\Gamma (\operatorname{div} u)^2 (m\cdot\nu)\mathrm{d}\Gamma.
\end{aligned}$$

Plugging the last expression in (5.4), we obtain

$$\sum_{i=1}^n 2(\lambda+\mu)\Big(\frac{\partial}{\partial x_i}\operatorname{div} u, m\cdot\nabla u_i\Big) = M_4 + M_5 + M_6. \quad (5.5)$$

The proposition follows from (5.3) and (5.5). □

Proof of Theorem 2.5. We take the inner product of H in both sides of (2.35) with u'. Then by (2.27) and (2.36) we find that

$$E'(l) \le -\tau_0 \|u'(t)\|^2_{\mathbb{L}^2(\Gamma_1)}, \quad (5.6)$$

where $\tau_0 = b_0 d_0^*$, with d_0^* defined in (2.28). We introduce the perturbed energy

$$E_\varepsilon(t) = E(t) + \varepsilon\alpha(t),\ t\ge 0,\ \varepsilon>0, \quad (5.7)$$

where

$$\alpha(t) = \sum_{i=1}^n 2(u_i'(t), m\cdot\nabla u_i(t)) + (n-1)\sum_{i=1}^n (u_i'(t), u_i(t)). \quad (5.8)$$

I. Equivalence between $E_\varepsilon(t)$ and $E(t)$. First of all, we note that

$$\frac{1}{4}\|u(t)\|_V + \frac{1}{\rho+1}(|u(t)|^\rho, u(t))_H \ge 0, \quad \forall t\ge 0. \quad (5.9)$$

In fact, since

$$|(|u(t)|^\rho, u(t))_H| \le n k_0^{\rho+1}\|u(t)\|^{\rho+1}_{\mathbb{H}^1_{\Gamma_0}(\Omega)} \le \frac{n k_0^{\rho+1}}{\mu^{\frac{\rho+1}{2}}}\|u(t)\|_V^{\rho+1},$$

it follows that

$$\Big|\frac{1}{\rho+1}(|u(t)|^{\rho+1}, u(t))_H\Big| \le \frac{1}{\mu^{\frac{\rho+1}{2}}}\frac{n k_0^{\rho+1}}{\rho+1}\|u(t)\|_V^{\rho+1}. \quad (5.10)$$

Since

$$-\frac{1}{\mu^{\frac{\rho+1}{2}}}\frac{n k_0^{\rho+1}}{\rho+1} > -N_1,$$

with N_1 defined in (2.30), we obtain

$$\frac{1}{4}\|u(t)\|_V^2 - \frac{1}{\mu^{\frac{\rho+1}{2}}}\frac{nk_0^{\rho+1}}{\rho+1}\|u(t)\|_V^{\rho+1} \geq \frac{1}{4}\|u(t)\|_V^2 - N_1\|u(t)\|_V^{\rho+1} \geq 0, \quad \forall t \geq 0, \tag{5.11}$$

because

$$J_1(\lambda) = \frac{1}{4}\lambda^2 - N_1\lambda^{\rho+1} \geq 0, \quad \forall 0 \leq \lambda \leq \lambda_1^*, \text{ and } 0 \leq \|u(t)\|_V < \lambda_1^*$$

(see Remark 4.4). Inequalities (5.10) and (5.11) provide (5.9). Then by (5.9) we find that

$$E(t) \geq \frac{1}{4}\|u'(t)\|_H^2 + \frac{1}{4}\|u(t)\|_V^2. \tag{5.12}$$

On the other hand, we have

$$|\alpha(t)| \leq 2R\|u'(t)\|_H\|u(t)\|_{\mathbb{H}^1_{\Gamma_0}(\Omega)} + (n-1)\|u'(t)\|_H\|u(t)\|_H.$$

Thus

$$|\alpha(t)| \leq \frac{R}{\mu^{1/2}}\left(\|u(t)\|_H^2 + \|u(t)\|_V^2\right) + \frac{(n-1)k_5}{\mu^{1/2}}\Big((\frac{1}{2}\|u'(t)\|_H^2 + \frac{1}{2}\|u(t)\|_V^2\Big),$$

which implies

$$|\alpha(t)| \leq M\Big(\frac{1}{4}\|u'(t)\|_H^2 + \frac{1}{4}\|u(t)\|_V^2\Big). \tag{5.13}$$

By (5.12) and (5.13), we obtain $|\alpha(t)| \leq ME(t)$, $\forall t \geq 0$. Thus

$$|E_\varepsilon(t) - E(t)| = \varepsilon|\alpha(t)| \leq \varepsilon ME(t), \quad \forall t \geq 0.$$

Choosing $\varepsilon_1 = \frac{1}{2M}$, we have

$$\frac{1}{2}E(t) \leq E_\varepsilon(t) \leq \frac{3}{2}E(t), \quad \forall t \geq 0, \ \forall 0 < \varepsilon \leq \varepsilon_1. \tag{5.14}$$

II. Relation between $E'_\varepsilon(t)$ and $E(t)$. By (5.8) we obtain

$$\begin{aligned} \alpha'(t) &= \sum_{i=1}^n 2(u_i''(t), m\cdot\nabla u_i(t)) + \sum_{i=1}^n 2(u_i'(t), m\cdot\nabla u_i'(t)) \\ &\quad + (n-1)\sum_{i=1}^n (u_i''(t), u_i(t)) + (n-1)\sum_{i=1}^n |u_i'(t)|^2 \\ &= D(t) + F(t) + G(t) + I(t). \end{aligned} \tag{5.15}$$

We have

$$\begin{aligned} D(t) &= \sum_{i=1}^n 2(-(Au(t))_i, m\cdot\nabla u_i(t)) - \sum_{i=1}^n 2(|u_i(t)|^\rho, m\cdot\nabla u_i'(t)) \\ &= D_1(t) + D_2(t). \end{aligned} \tag{5.16}$$

Analysis of $D_2(t)$. We find

$$\begin{aligned} (|u_i(t)|^\rho, m\cdot\nabla u_i(t)) &= \sum_{j=1}^n \int_\Omega |u_i(t)|^\rho m_j \frac{\partial u_i(t)}{\partial x_j} \mathrm{d}x \\ &= \sum_{j=1}^n \int_\Omega m_j \frac{1}{\rho+1}[\frac{\partial}{\partial x_j}|u_i(t)|^\rho u_i(t)]\mathrm{d}x \end{aligned}$$

$$= -\frac{n}{\rho+1}\int_\Omega |u_i(t)|^\rho u_i(t)\mathrm{d}x + \frac{1}{\rho+1}\int_\Gamma |u_i(t)|^\rho u_i(t)(m\cdot\nu)\mathrm{d}\Gamma.$$

Then

$$D_2(t) = \frac{2n}{\rho+1}\sum_{i=1}^n \int_\Omega |u_i(t)|^\rho u_i(t)\mathrm{d}x - \frac{2}{\rho+1}\sum_{i=1}^n \int_\Gamma |u_i(t)|^\rho u_i(t)(m\cdot\nu)\mathrm{d}\Gamma.$$

This result and (5.16) provide

$$\begin{aligned} D(t) = D_1(t) &+ \frac{2n}{\rho+1}\sum_{i=1}^n \int_\Omega |u_i(t)|^\rho u_i(t)\mathrm{d}x \\ &- \frac{2}{\rho+1}\sum_{i=1}^n \int_\Gamma |u_i(t)|^\rho u_i(t)(m\cdot\nu)\mathrm{d}\Gamma. \end{aligned} \tag{5.17}$$

Note that $D_1(t)$ is given by Proposition 5.1.

Analysis of $F(t)$. We have

$$\begin{aligned} (u_i'(t), m\cdot\nabla u_i'(t)) &= \sum_{j=1}^n \int_\Omega m_j \frac{1}{2}\frac{\partial}{\partial x_j}(u'(t))^2\mathrm{d}x \\ &= -\frac{n}{2}\int_\Gamma (u_i'(t))^2\mathrm{d}x + \frac{1}{2}\int_\Gamma (u_i'(t))^2(m\cdot\nu)\mathrm{d}\Gamma. \end{aligned}$$

Thus

$$F(t) = -n\sum_{i=1}^n |u_i'(t)|^2 + \int_\Gamma \Big(\sum_{i=1}^n (u_i'(t))^2\Big)(m\cdot\nu)\mathrm{d}\Gamma. \tag{5.18}$$

Analysis of $G(t)$. We obtain

$$\begin{aligned} (u_i''(t), u_i(t)) &= \mu(\Delta u_i(t), u_i(t)) + (\lambda+\mu)\Big(\frac{\partial}{\partial \nu_i}\operatorname{div} u(t), u_i(t)\Big) - (|u_i(t)|^\rho, u_i(t)) \\ &= l_1(t) + l_2(t) + l_3(t), \end{aligned}$$

where

$$\begin{gathered} l_1(t) = -\mu\int_\Omega |\nabla u_i(t)|^2\mathrm{d}x + \mu\int_\Gamma \frac{\partial u_i(t)}{\partial\nu}u_i(t)\mathrm{d}\Gamma, \\ l_2(t) = -(\lambda+\mu)\int_\Omega (\operatorname{div} u(t))\frac{\partial u(t)}{\partial x_i}\mathrm{d}x + (\lambda+\mu)\int_\Gamma (\operatorname{div} u(t))u_i(t)\nu_i\mathrm{d}\Gamma. \end{gathered}$$

Then

$$\begin{aligned} G(t) = &-(n-1)\mu\sum_{i=1}^n \int_\Omega |\nabla u_i(t)|^2\mathrm{d}x + (n-1)\mu\sum_{i=1}^n \int_\Gamma \frac{\partial u_i(t)}{\partial\nu}u_i(t)\mathrm{d}\Gamma \\ &- (n-1)(\lambda+\mu)\int_\Omega (\operatorname{div} u(t))^2\mathrm{d}x \\ &+ (n-1)(\lambda+\mu)\int_\Gamma (\operatorname{div} u(t))\Big(\sum_{i=1}^n u_i(t)\nu_i\Big)\mathrm{d}\Gamma \\ &- (n-1)\sum_{i=1}^n (|u_i(t)|^\rho, u_i(t)). \end{aligned} \tag{5.19}$$

By (5.4), Proposition (5.1), (5.17)–(5.19), we find that

$$\alpha'(t) = D(t) + F(t) + G(t) + I(t),$$

where

$$\begin{aligned} &D(t)\\ &= \mu(n-2)\sum_{i=1}^{n}\int_{\Omega}|\nabla u_i(t)|^2\mathrm{d}x - \mu\sum_{i=1}^{n}\int_{\Gamma}|\nabla u_i(t)|^2(m\cdot\nu)\mathrm{d}\Gamma\\ &\quad + 2\mu\sum_{i=1}^{n}\int_{\Gamma}\frac{\partial u_i(t)}{\partial\nu}(m\cdot\nabla u_i(t))\mathrm{d}\Gamma\\ &\quad + (\lambda+\mu)(n-2)\int_{\Omega}(\operatorname{div}u(t))^2\mathrm{d}x - (\lambda+\mu)\int_{\Gamma}(\operatorname{div}u(t))^2(m\cdot\nu)\mathrm{d}\Gamma\\ &\quad + 2(\lambda+\mu)\sum_{i=1}^{n}\int_{\Gamma}(\operatorname{div}u(t))(m\cdot\nabla u_i(t))\nu_i\mathrm{d}\Gamma + \frac{2n}{\rho+1}\sum_{i=1}^{n}(|u_i(t)|^{\rho},u_i(t))\\ &\quad - \frac{2}{\rho+1}\sum_{i=1}^{n}\int_{\Gamma_1}|u_i(t)|^{\rho}u_i(t)(m\cdot\nu)\mathrm{d}\Gamma_1, \end{aligned} \tag{5.20}$$

$$F(t) = -n\sum_{i=1}^{n}|u_i(t)|^2 + \int_{\Gamma}\Big[\sum_{i=1}^{n}(u_i'(t))^2\Big](m\cdot\nu)\mathrm{d}\Gamma, \tag{5.21}$$

$$\begin{aligned} G(t) &= -(n-1)\mu\sum_{i=1}^{n}\int_{\Omega}|\nabla u_i(t)|^2\mathrm{d}x + (n-1)\mu\sum_{i=1}^{n}\int_{\Gamma_1}\frac{\partial u_i(t)}{\partial\nu}u_i(t)\mathrm{d}\Gamma_1\\ &\quad - (n-1)(\lambda+\mu)\int_{\Omega}(\operatorname{div}u(t))^2\mathrm{d}x\\ &\quad + (n-1)(\lambda+\mu)\int_{\Gamma}(\operatorname{div}u(t))\Big(\sum_{i=1}^{n}u_i(t)\nu_i\Big)\mathrm{d}\Gamma\\ &\quad - (n-1)\sum_{i=1}^{n}(|u_i(t)|^{\rho},u_i(t)) \end{aligned} \tag{5.22}$$

and

$$I(t) = (n-1)\sum_{i=1}^{n}|u_i'(t)|^2. \tag{5.23}$$

The goal is to transform (5.15)–(5.18) into an inequality of the form

$$\alpha'(t) \le -E(t) - \Big(\frac{1}{4}\|u(t)\|_V^2 - N_1\|u(t)\|_V^{\rho+1}\Big) + P\|u'(t)\|^2_{\mathbb{L}^2(\Gamma_1)}$$

and then to find conditions to have

$$\frac{1}{4}\|u(t)\|_V^2 - N_1\|u(t)\|_V^{\rho+1} \ge 0, \quad \forall t\ge 0.$$

This last inequality motivates the introduction of Theorem 2.4.

By reducing similar terms in (5.20)–(5.23), we obtain

$$\begin{aligned} \alpha'(t) &= -\|u'(t)\|_H^2 - \mu\|u(t)\|^2_{\mathbb{H}^1_{\Gamma_0}(\Omega)} - (\lambda+\mu)|\operatorname{div}u(t)|^2\\ &\quad + \frac{2n}{\rho+1}\sum_{i=1}^{n}(|u_i(t)|^{\rho},u_i(t)) - (n-1)\sum_{i=1}^{n}(|u_i(t)|^{\rho},u_i(t)) + Q(t), \end{aligned} \tag{5.24}$$

where

$$
\begin{aligned}
Q(t) &= -\mu \sum_{i=1}^{n} \int_{\Gamma} |\nabla u_i(t)|^2 (m\cdot\nu) \mathrm{d}\Gamma + 2\mu \sum_{i=1}^{n} \int_{\Gamma} \frac{\partial u_i(t)}{\partial \nu} (m\cdot \nabla u_i(t)) \mathrm{d}\Gamma \\
&\quad - (\lambda+\mu) \int_{\Gamma} (\operatorname{div} u(t))^2 (m\cdot\nu) \mathrm{d}\Gamma \\
&\quad + 2(\lambda+\mu) \sum_{i=1}^{n} \int_{\Gamma} (\operatorname{div} u(t))^2 \nu_i (m\cdot\nabla u_i(t)) \mathrm{d}\Gamma \\
&\quad - \frac{2}{\rho+1} \sum_{i=1}^{n} \int_{\Gamma} |u_i(t)|^{\rho} u_i(t) (m\cdot\nu) \mathrm{d}\Gamma + \sum_{i=1}^{n} \int_{\Gamma} (u_i'(t))^2 (m\cdot\nu) \mathrm{d}\Gamma \\
&\quad + (n-1)\mu \sum_{i=1}^{n} \int_{\Gamma} \big(\frac{\partial u_i}{\partial \nu}\big) u_i(t) \mathrm{d}\Gamma \\
&\quad + (n-1)(\lambda+\mu) \sum_{i=1}^{n} \int_{\Gamma} [(\operatorname{div} u(t))\nu_i] u_i(t) \mathrm{d}\Gamma \\
&= \sum_{j=1}^{8} q_j(t).
\end{aligned}
\tag{5.25}
$$

By (5.24), we have

$$
\begin{aligned}
\alpha'(t) &= -2E(t) + \frac{2}{\rho+1}(|u(t)|^{\rho}, u(t))_H + \frac{2n}{\rho+1}(|u(t)|^{\rho}, u(t))_H \\
&\quad - (n-1)(|u(t)|^{\rho}, u(t))_H + Q(t),
\end{aligned}
$$

which implies

$$
\begin{aligned}
\alpha'(t) &\le -E(t) - \frac{1}{2}\|u(t)\|_V^2 + \frac{1}{\rho+1}(|u(t)|^{\rho}, u(t))_H \\
&\quad + \frac{2n}{\rho+1}(|u(t)|^{\rho}, u(t))_H - (n-1)(|u(t)|^{\rho}, u(t))_H + Q(t).
\end{aligned}
\tag{5.26}
$$

By (2.5), we obtain

$$
|(|u_i(t)|^{\rho}, u_i(t))| \le k_0^{\rho+1} \|u(t)\|_{\mathbb{H}^1_{\Gamma_0}(\Omega)}^{\rho+1}.
$$

Then

$$
\begin{aligned}
&\Big|\frac{1}{\rho+1}(|u(t)|^{\rho}, u(t))_H + \frac{2n}{\rho+1}(|u(t)|^{\rho}, u(t))_H - (n-1)(|u(t)|^{\rho}, u(t))_H\Big| \\
&\le \omega \|u(t)\|_{\mathbb{H}^1_{\Gamma_0}(\Omega)}^{\rho+1},
\end{aligned}
\tag{5.27}
$$

where

$$
\omega = \Big|\frac{1+2n}{\rho+1} - (n-1)\Big| n k_0^{\rho+1}. \tag{5.28}
$$

From (5.26) and (5.27) we obtain

$$
\alpha'(t) \le -E(t) - \frac{1}{2}\|u(t)\|_V^2 + \omega \|u(t)\|_{\mathbb{H}^1_{\Gamma_0}(\Omega)}^{\rho+1} + Q(t). \tag{5.29}
$$

Let us analyze (5.25). Note that $\frac{\partial u_i(t)}{\partial x_j} = \frac{\partial u_i(t)}{\partial \nu}\nu_j$ in Γ_0 (see Lions [15]). Since

$$q_1(t) = -\mu \sum_{i=1}^n \int_{\Gamma_0} \Big(\frac{\partial u_i(t)}{\partial \nu}\Big)^2 (m\cdot\nu)\mathrm{d}\Gamma - \mu \sum_{i=1}^n \int_{\Gamma_1} |\nabla u_i(t)|^2 (m\cdot\nu)\mathrm{d}\Gamma,$$

$$q_2(t) = 2\mu \sum_{i=1}^n \int_{\Gamma_0} \Big(\frac{\partial u_i(t)}{\partial \nu}\Big)^2 (m\cdot\nu)\mathrm{d}\Gamma + 2\mu \int_{\Gamma_1} \frac{\partial u_i(t)}{\partial \nu}(m\cdot\nabla u_i(t))\mathrm{d}\Gamma,$$

by noting that $m\cdot\nu\le 0$ in Γ_0, we find that

$$q_1(t)+q_2(t) \quad \le -\mu \sum_{i=1}^n \int_{\Gamma_1} |\nabla u_i(t)|^2 (m\cdot\nu)\mathrm{d}\Gamma + 2\mu \sum_{i=1}^n \int_{\Gamma_1} \frac{\partial u_i(t)}{\partial \nu}(m\cdot\nabla u_i(t))\mathrm{d}\Gamma. \tag{5.30}$$

Therefore,

$$q_3(t) \le -(\lambda+\mu)\int_{\Gamma_0} \Big(\sum_{i=1}^n \frac{\partial u_i(t)}{\partial \nu}\nu_i\Big)^2 (m\cdot\nu)\mathrm{d}\Gamma,$$

and

$$\begin{aligned} q_4(t) &= 2(\lambda+\mu)\int_{\Gamma_0} \Big(\sum_{j=1}^n \frac{\partial u_j(t)}{\partial \nu}\nu_j\Big)^2 (m\cdot\nu)\mathrm{d}\Gamma \\ &\quad + 2(\lambda+\mu)\sum_{i=1}^n \int_{\Gamma_1} [(\operatorname{div} u(t))\nu_i](m\cdot\nabla u_i(t))\mathrm{d}\Gamma. \end{aligned}$$

Then

$$q_3(t)+q_4(t) \le 2(\lambda+\mu)\sum_{i=1}^n \int_{\Gamma_1} [(\operatorname{div} u)\nu_i](m\cdot\nabla u_i(t))\mathrm{d}\Gamma. \tag{5.31}$$

By observing that $\mu\frac{\partial u_i(t)}{\partial \nu} + (\lambda+\mu)(\operatorname{div} u(t))\nu_i + (m\cdot\nu)h_i(u_i'(t)) = 0$ on Γ_1, it follows from (5.26) and (5.27) that

$$\begin{aligned} &q_1(t)+\dots+q_4(t) \\ &\le -\mu \sum_{i=1}^n \int_{\Gamma_1} |\nabla u_i(t)|^2 (m\cdot\nu)\mathrm{d}\Gamma + 2\sum_{i=1}^n \int_{\Gamma_1} [-(m\cdot\nu)h_i(u_i'(t))](m\cdot\nabla u_i(t))\mathrm{d}\Gamma. \end{aligned}$$

However

$$\begin{aligned} &\Big|2\int_{\Gamma_1} [-(m\cdot\nu)h_i(u_i'(t))](m\cdot\nabla u_i(t))\mathrm{d}\Gamma\Big| \\ &\le \frac{1}{\mu}R^3L^2 \int_{\Gamma_1} |u_i(t)|^2\mathrm{d}\Gamma + \mu\int_{\Gamma_1} |\nabla u_i(t)|^2 (m\cdot\nu)\mathrm{d}\Gamma. \end{aligned}$$

Then the last two inequalities provide

$$q_1(t)+\dots+q_4(t) \le \frac{1}{\mu}R^3L^2\|u'(t)\|^2_{\mathbb{L}^2(\Gamma_1)}. \tag{5.32}$$

By noting that $u_i = 0$ in Γ_0, we find that

$$\begin{aligned} q_7(t)+q_8(t) &= (n-1)\sum_{i=1}^n \int_{\Gamma_1} [-(m\cdot\nu)h_i(u_i'(t))]u_i(t)\mathrm{d}\Gamma \\ &\le \frac{1}{\mu}(n-1)^2R^2L^2k_7^2\|u'(t)\|^2_{\mathbb{L}^2(\Gamma_1)} + \frac{\mu}{4}\|u(t)\|^2_{\mathbb{H}^1_{\Gamma_0}(\Omega)}, \end{aligned}$$

where k_7 was defined in (2.8). Thus

$$q_7(t)+q_8(t)\leq \frac{1}{\mu}(n-1)^2R^2L^2k_7^2\|u'(t)\|^2_{\mathbb{L}^2(\Gamma_1)}+\frac{1}{4}\|u(t)\|^2_V. \tag{5.33}$$

We have

$$\begin{aligned}|q_5(t)|&\leq \frac{2}{\rho+1}R\sum_{i=1}^n\int_{\Gamma_1}|u_i(t)|^{\rho+1}\mathrm{d}\Gamma\\ &\leq \frac{2R}{\rho+1}k_6^{\rho+1}\sum_{i=1}^n\|u_i(t)\|^{\rho+1}_{\mathbb{H}^1_{\Gamma_0}(\Omega)}\\ &\leq \frac{2R}{\rho+1}k_6^{\rho+1}n\|u(t)\|^{\rho+1}_{\mathbb{H}^1_{\Gamma_0}(\Omega)}.\end{aligned} \tag{5.34}$$

Since $u_i'=0$ on Γ_0, it follows that

$$|q_6(t)|\leq R\sum_{i=1}^n\int_{\Gamma_1}(u_i'(t))^2d\Gamma=R\|u'(t)\|^2_{\mathbb{L}^2(\Gamma_1)}. \tag{5.35}$$

Taking into account (5.32)–(5.35) in (5.29), we obtain

$$\alpha'(t)\leq -E(t)-\big[\frac{1}{4}\|u(t)\|^2_V-N_1\|u(t)\|^{\rho+1}_V\big]+P\|u'(t)\|^2_{\mathbb{L}^2(\Gamma_1)}, \tag{5.36}$$

where N_1 and P were defined in (2.30) and (2.40)), respectively. By applying Theorem 2.4 to (5.36), we find that

$$\alpha'(t)\leq -E(t)+P\|u'(t)\|^2_{\mathbb{L}^2(\Gamma_1)}. \tag{5.37}$$

Now let us to return to the perturbed energy $E_\varepsilon(t)$ given in (5.7). By (5.6) and (5.37), we obtain

$$E_\varepsilon'(t)=E'(t)+\varepsilon\alpha'(t)\leq -\varepsilon E(t)-(\tau_0-\varepsilon P)\|u'\|^2_{\mathbb{L}^2(\Gamma_1)}.$$

Choosing $0<\varepsilon_2\leq \frac{\tau_0}{P}$, we have

$$E_\varepsilon'(t)\leq -\varepsilon E(t),\quad \forall 0<\varepsilon\leq\varepsilon_2. \tag{5.38}$$

Thus for $\sigma=\min\{\frac{1}{2M},\frac{\tau_0}{\rho}\}$ we have that (5.9) and (5.38) hold for all $0<\varepsilon\leq\sigma$. By (5.38) and (5.9), we obtain

$$E_\varepsilon'(t)\leq -\frac{2}{3}\sigma E_\varepsilon(t),$$

which implies

$$E_\varepsilon(t)\leq E_\varepsilon(0)e^{-\frac{2}{3}\sigma t}.$$

This inequality and (5.9) provide Theorem 2.5. □

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