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ANTI-PERIODIC SOLUTIONS FOR SECOND ORDER DIFFERENTIAL INCLUSIONS

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Abstract. In this paper, we extend the existence results presented in [9] for L^p spaces to operator inclusions of Hammerstein type in $W^{1,p}$ spaces. We also show an application of our results to anti-periodic boundary-value problems of second-order differential equations with nonlinearities depending on u'.

1. Introduction

This paper concerns the second-order boundary-value problem

$$\begin{gathered} -u''(t) \in Au(t) + f(t, u(t), u'(t)) \quad \text{for a.e. } t \in [0,T] \\ u(0) = -u(T), \quad u'(0) = -u'(T), \end{gathered}$$

where $0 < T < \infty$, A is an m-dissipative multivalued mapping in a Hilbert space E and $f : [0,T] \times E^2 \to 2^E$. However, in this section, and in Section 2, we shall assume generally that E is a Banach space.

A function $u \in C^1([0,T];E)$ is said to be T-*anti-periodic* if $u(0) = -u(T)$ and $u'(0) = -u'(T)$. Note that there exists a close connection between the anti-periodic problem and the periodic one. Indeed, if $u \in W^{2,p}(0,T;E)$ $(1 \le p < \infty)$ is a T-anti-periodic solution of the inclusion

$$-u''(t) \in Au(t) + f(t, u(t), u'(t)) \quad \text{a.e. on } [0,T]$$

and A, f are odd in the following sense:

$$A(-x) = -Ax \quad \text{and} \quad f(t, -x, -y) = -f(t, x, y),$$

then the function

$$\widetilde{u}(t) = \begin{cases} u(t), & 0 \le t \le T \\ -u(t-T), & T < t \le 2T \end{cases}$$

belongs to $W^{2,p}(0,2T;E)$, is $2T$-periodic, i.e., $\widetilde{u}(0) = \widetilde{u}(2T)$, $\widetilde{u}'(0) = \widetilde{u}'(2T)$, and solves the inclusion

$$-\widetilde{u}''(t) \in A\widetilde{u}(t) + \widetilde{f}(t, \widetilde{u}(t), \widetilde{u}'(t)) \quad \text{a.e. on } [0,2T]$$

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where

$$\widetilde{f}(t,x,y)=\begin{cases} f(t,x,y), & 0\le t\le T\\ f(t-T,x,y), & T<t\le 2T.\end{cases}$$

The anti-periodic boundary value problem for various classes of evolution equations has been considered by Aftabizadeh-Aizicovici-Pavel [1], [2]; Aizicovici-Pavel [3], Aizicovici-Pavel-Vrabie [4], Cai-Pavel [6], Coron [7], Haraux [17] and Okochi [19, 20].

Let us denote by $|\cdot|$ the norm of E, by $|\cdot|_p$ the usual norm of $L^p(0,T;E)$ and by $|\cdot|_{1,p}$ the norm of $W^{1,p}(0,T;E)$, $|u|_{1,p}=\max\{|u|_p,|u'|_p\}$. One of the reasons of working with anti-periodic solutions is given by the following proposition.

Proposition 1.1. *If $u\in W^{1,p}(0,T;E)$ $(1\le p\le\infty)$ and $u(0)=-u(T)$, then*

$$|u(t)|\le \frac{1}{2}T^{\frac{p-1}{p}}|u'|_p,\quad t\in[0,T]. \tag{1.1}$$

Proof. Adding $u(t)=u(0)+\int_0^t u'(s)ds$ and $u(t)=u(T)-\int_t^T u'(s)ds$ we have

$$2u(t)=\int_0^t u'(s)ds-\int_t^T u'(s)ds.$$

Hence

$$2|u(t)|\le\int_0^t |u'(s)|ds+\int_t^T |u'(s)|ds=\int_0^T |u'(s)|ds.$$

Now Hölder's inequality gives (1.1). □

Let us denote

$$C_a^1=\{u\in C^1([0,T];E): u \text{ is } T\text{-anti-periodic}\}.$$

In what follows for a subset $K\subset E$, by $P_a(K)$ and $P_{kc}(K)$ we shall denote the family of all nonempty acyclic subsets of K and, respectively, the family of all nonempty compact convex subsets of K.

Recall that a metric space Ξ is said to be *acyclic* if it has the same homology as a single point space, and that Ξ is called an *absolute neighborhood retract* (ANR for short) if for every metric space Z and closed set $A\subset Z$, every continuous map $f:A\to\Xi$ has a continuous extension $\widehat{f}$ to some neighborhood of A. Note that every compact convex subset of a normed space is an ANR and is acyclic.

Our main abstract tools are: The Eilenberg-Montgomery fixed point theorem [13, 18]; a lemma of Petryshyn-Fitzpatrick [14]; and strong and weak compactness criteria in $L^p(0,T;E)$ (see [16] and [12]), where E is a general (non-reflexive) Banach space.

Theorem 1.2. *Let Ξ be acyclic and absolute neighborhood retract, Θ be a compact metric space, $\Phi:\Xi\to P_a(\Theta)$ be an upper semicontinuous map and $\Gamma:\Theta\to\Xi$ be a continuous single-valued map. Then the map $\Gamma\Phi:\Xi\to 2^{\Xi}$ has a fixed point.*

Lemma 1.3. *Let X be a Fréchet space, $D\subset X$ be closed convex and $N:D\to 2^X$. Then for each $\Omega\subset D$ there exists a closed convex set K, depending on N, D and Ω, with $\Omega\subset K$ and $\overline{\operatorname{conv}}(\Omega\cup N(D\cap K))=K$.*

Theorem 1.4. *Let $p\in[1,\infty]$. Let $M\subset L^p(0,T;E)$ be countable and suppose that there exists a $\nu\in L^p(0,T)$ with $|u(t)|\le\nu(t)$ a.e. on $[0,T]$ for all $u\in M$. Assume $M\subset C([0,T];E)$ if $p=\infty$. Then M is relatively compact in $L^p(0,T;E)$ if and only if*

(i) $\sup_{u\in M} |\tau_h u - u|_{L^p(0,T-h;E)} \to 0$ *as* $h \to 0$
(ii) $M(t)$ *is relatively compact in* E *for a.e.* $t \in [0,T]$.

Theorem 1.5. *Let* $p \in [1,\infty]$. *Let* $M \subset L^p(0,T;E)$ *be countable and suppose there exists* $\nu \in L^p(0,T)$ *with* $|u(t)| \le \nu(t)$ *a.e. on* $[0,T]$ *for all* $u \in M$. *If* $M(t)$ *is relatively compact in* E *for a.e.* $t \in [0,T]$, *then* M *is weakly relatively compact in* $L^p(0,T;E)$.

Now, we recall the following definition: A map $\psi : [a,b] \times D \to 2^Y \setminus \{\emptyset\}$, where $D \subset X$ and $(X, |\cdot|_X)$, $(Y, |\cdot|_Y)$ are two Banach spaces, is said to be (q,p)-*Carathéodory* ($1 \le q \le \infty$, $1 \le p \le \infty$) if

(C1) $\psi(.,x)$ is strongly measurable for each $x \in D$
(C2) $\psi(t,.)$ is upper semicontinuous for a.e. $t \in [a,b]$
(C3) (a) if $1 \le p < \infty$, there exists $\nu \in L^q(a,b;\mathbb{R}_+)$ and $d \in \mathbb{R}_+$ such that $|\psi(t,x)|_Y \le \nu(t) + d|x|_X^p$ a.e. on $[a,b]$, for all $x \in D$
(b) if $p = \infty$, for each $\rho > 0$ there exists $\nu_\rho \in L^q(a,b;\mathbb{R}_+)$ such that $|\psi(t,x)|_Y \le \nu_\rho(t)$ a.e. on $[a,b]$, for all $x \in D$ with $|x|_X \le \rho$.

2. A General Existence Principle

The aim of this section is to extend the general existence principles given in [10] for inclusions in $L^p(0,T;E)$, to inclusions in $W^{1,p}(0,T;E)$. Here again E a Banach space with norm $|\cdot|$. This extension allows us to consider boundary-value problems for second order differential inclusions with u' dependence perturbations and, by this, it complements the theory from [8], [9] and [10].

Let $p \in [1,\infty]$ and $q \in [1,\infty[$. Let $r \in]1,\infty]$ be the conjugate exponent of q, that is $1/q + 1/r = 1$. Let $g : [0,T] \times E^2 \to 2^E$ and let $G : W^{1,p}(0,T;E) \to 2^{L^q(0,T;E)}$ be the Nemytskii set-valued operator associated to g, p and q, given by

$$G(u) = \{w \in L^q(0,T;E) : w(s) \in g(s,u(s),u'(s)) \text{ a.e. on } [0,T]\}. \tag{2.1}$$

Also consider a single-valued nonlinear operator

$$S : L^q(0,T;E) \to W^{1,p}(0,T;E).$$

We have the following existence principle for the operator inclusion

$$u \in SG(u), \quad u \in W^{1,p}(0,T;E). \tag{2.2}$$

Theorem 2.1. *Let* K *be a closed convex subset of* $W^{1,p}(0,T;E)$, U *a convex relatively open subset of* K *and* $u_0 \in U$. *Assume*

(H1) $SG : \overline{U} \to P_a(K)$ *has closed graph and maps compact sets into relatively compact sets*
(H2) $M \subset \overline{U}$, M *closed,* $M \subset \overline{\text{conv}}(\{u_0\} \cup SG(M))$ *implies that* M *is compact*
(H3) $u \notin (1-\lambda)u_0 + \lambda SG(u)$ *for all* $\lambda \in]0,1[$ *and* $u \in \overline{U} \setminus U$.

Then (2.2) *has a solution in* $\overline{U}$.

Proof. Let $D = \overline{\text{conv}}(\{u_0\} \cup SG(\overline{U}))$. Clearly $u_0 \in D \subset K$. Let $P : K \to \overline{U}$ be given by $P(u) = u$ if $u \in \overline{U}$ and $P(u) = \overline{u}$ if $u \in K \setminus \overline{U}$, where $\overline{u} = (1-\lambda)u_0 + \lambda u \in \overline{U} \setminus U$, $\lambda \in]0,1[$. Note P is single-valued, continuous and maps closed sets into closed sets. Let $\widetilde{N} : D \to P_a(K)$, $\widetilde{N}(u) = SGP(u)$. It is easy to see that $\widetilde{N}(D) \subset D$, the graph of $\widetilde{N}$ is closed and $\widetilde{N}$ maps compact sets into relatively compact sets. Let D_0 be a closed convex set with $D_0 = \overline{\text{conv}}(\{u_0\} \cup \widetilde{N}(D_0 \cap D))$ whose existence is guaranteed by Lemma 1.3. Since $\widetilde{N}(D) \subset D$ we have $D_0 \subset D$ and so $D_0 = \overline{\text{conv}}(\{u_0\} \cup \widetilde{N}(D_0))$. Using the definition of P, we obtain

$$P(D_0) \subset \text{conv}(\{u_0\} \cup D_0) = \overline{\text{conv}}(\{u_0\} \cup \widetilde{N}(D_0)) = \overline{\text{conv}}(\{u_0\} \cup SG(P(D_0))).$$

In addition, since D_0 is closed, $P(D_0)$ is also closed. Now (H2) guarantees that $P(D_0)$ is compact. Since SG maps compact sets into relatively compact sets, we have that $\widetilde{N}(D_0)$ is relatively compact. Then Mazur's Lemma guarantees that D_0 is compact. Now apply the Eilenberg-Montgomery Theorem with $\Xi = \Theta = D_0$, $\Phi = \widetilde{N}$ and $\Gamma =$identity of D_0, to deduce the existence of a fixed point $u \in D_0$ of $\widetilde{N}$. If $u \notin \overline{U}$, then $P(u) = (1-\lambda)u_0 + \lambda u = (1-\lambda)u_0 + \lambda SG(P(u))$ for some $\lambda \in]0,1[$. Since $P(u) \in \overline{U} \setminus U$, this contradicts (H3). Thus $u \in \overline{U}$, so $u = SG(u)$ and the proof is complete. □

Remark 2.2. Additional regularity for the solutions of (2.2) depends on the values of S. In particular if the values of S are in C_a^1 then so are all solutions of (2.2).

In what follows K will be a closed linear subspace of $W^{1,p}(0,T;E)$, $u_0 = 0$ and U will be the open ball of K,

$$U = \{u \in K : \|u\| < R\}$$

with respect to an equivalent norm $\|.\|$ on K. For $p \in [1,\infty]$ denote

$$\mu_p := \sup\{\frac{|u|_{1,p}}{\|u\|} : u \in K,\ u \neq 0\}, \quad \mu_0 := \sup\{\frac{|u|_\infty}{\|u\|} : u \in K,\ u \neq 0\}.$$

Note that μ_p and μ_0 are finite because of the equivalence of norms $\|\cdot\|$ and $|\cdot|_{1,p}$ on K and the continuously embedding of $W^{1,p}(0,T;E)$ into $C([0,T];E)$.

Now we give sufficient conditions on S and g in order that the assumptions (H1)-(H2) be satisfied.

(S1) There exists a function $k : [0,T]^2 \to \mathbb{R}_+$ with $k(t,.) \in L^r(0,T)$ and a constant $L > 0$ such that

$$|S(w_1)(t) - S(w_2)(t)| \le \int_0^T k(t,s)|w_1(s) - w_2(s)|ds$$

for a.e. $t \in [0,T]$, and $|S(w_1)' - S(w_2)'|_p \le L|w_1 - w_2|_q$ for all $w_1, w_2 \in L^q(0,T;E)$

(S2) $S : L^q(0,T;E) \to K$ and for every compact convex subset C of E, S is sequentially continuous from $L_w^1(0,T;C)$ to $W^{1,p}(0,T;E)$. (Here $L_w^1(0,T;C)$ stands for $L^1(0,T;C)$ endowed with the weak topology of $L^1(0,T;E)$)

(G1) $g : [0,T] \times E^2 \to P_{kc}(E)$

(G2) $g(.,z)$ has a strongly measurable selection on $[0,T]$, for every $z \in E^2$

(G3) $g(t,.)$ is upper semicontinuous for a.e. $t \in [0,T]$

(G4) If $1 \le p < \infty$, then $|g(t,z_1,z_2)| \le \nu(t)$ for a.e. $t \in [0,T]$ and all $z_1, z_2 \in E$ with $|z_1| \le \mu_0 R$; if $p = \infty$, then $|g(t,z_1,z_2)| \le \nu(t)$ for a.e. $t \in [0,T]$ and all $z_1, z_2 \in E$ with $|z_1| \le \mu_\infty R$ and $|z_2| \le \mu_\infty R$. Here $\nu \in L^q(0,T;\mathbb{R}_+)$.

(G5) For every separable closed subspace E_0 of the space E, there exists a (q,∞)-Carathéodory function $\omega : [0,T] \times [0,\mu_0 R] \to \mathbb{R}_+$, $\omega(t,0) = 0$, such that for almost every $t \in [0,T]$,

$$\beta_{E_0}(g(t,M,E_0) \cap E_0) \le \omega(t, \beta_{E_0}(M))$$

for every set $M \subset E_0$ satisfying $|M| \le \mu_0 R$, and $\varphi = 0$ is the unique solution in $L^\infty(0,T;[0,\mu_0 R])$ to the inequality

$$\varphi(t) \le \int_0^T k(t,s)\omega(s,\varphi(s))ds \quad \text{a.e. on } [0,T]. \tag{2.3}$$

Here β_{E_0} is the ball measure of non-compactness on E_0. (Recall that for a bounded set $A \subset E_0$, $\beta_{E_0}(A)$ is the infimum of $\varepsilon > 0$ for which A can be covered by finitely many balls of E_0 with radius not greater than ε)

(SG) For every $u \in \overline{U}$ the set $SG(u)$ is acyclic in K.

Remark 2.3. If S has values in C_a^1 then a sufficient condition for (S1) is to exist a function $\theta \in L^r(0,T;\mathbb{R}_+)$ such that

$$|S(w_1)' - S(w_2)'|_p \le \int_0^T \theta(s)|w_1(s) - w_2(s)|ds$$

for all $w_1, w_2 \in L^q(0,T;E)$.

Indeed, using Proposition 1.1 and Hölder's inequality, we immediately see that (S1) is satisfied with $k(t,s) = \frac{1}{2}T^{\frac{p-1}{p}}\theta(s)$ and $L = |\theta|_r$.

Remark 2.4. In case that $k(t,.) \in L^\infty(0,T)$ for a.e. $t \in [0,T]$, we may assume that ω in (G5) is a $(1,\infty)$-Carathéodory function (in order that the integral in (2.3) be defined).

As in [10] we can prove the following existence result.

Theorem 2.5. *Assume (S1)-(S2), (G1)-(G5) and (SG) hold. In addition assume (H3). Then* (2.2) *has at least one solution u in $K \subset W^{1,p}(0,T;E)$ with $\|u\| \le R$.*

The proof is based on Theorem 2.1 and consists in showing that conditions (H1)-(H2) are satisfied. We shall use the following analog of [10, lemma 4.4].

Lemma 2.6. *Assume (S1), (S2). Let M be a countable subset of $L^q(0,T;E)$ such that $M(t)$ is relatively compact for a.e. $t \in [0,T]$ and there is a function $\nu \in L^q(0,T;\mathbb{R}_+)$ with $|u(t)| \le \nu(t)$ a.e. on $[0,T]$, for every $u \in M$. Then the set $S(M)$ is relatively compact in $W^{1,p}(0,T;E)$. In addition S is continuous from M equipped with the relative weak topology of $L^q(0,T;E)$ to $W^{1,p}(0,T;E)$ equipped with its strong topology.*

Proof. Let $M = \{u_n : n \ge 1\}$ and let $\varepsilon > 0$ be arbitrary. As in the proof of [10, lemma 4.3], we can find functions $\widehat{u}_{n,k}$ with values in a compact $\overline{B}_k \subset E$ ($\overline{B}_k$ being a closed ball of a $k$dimensional subspace of E) such that

$$|u_n - \widehat{u}_{n,k}|_q \le \varepsilon$$

for every $n \ge 1$. Then assumption (S1) implies

$$|S(u_n) - S(\widehat{u}_{n,k})|_p \le ||k(t,.)|_r|_p |u_n - \widehat{u}_{n,k}|_q \le \varepsilon ||k(t,.)|_r|_p, \tag{2.4}$$

$$|S(u_n)' - S(\widehat{u}_{n,k})'|_p \le L|u_n - \widehat{u}_{n,k}|_q \le \varepsilon L. \tag{2.5}$$

On the other hand, according to Theorem 1.5, the set $\{\widehat{u}_{n,k} : n \ge 1\} \subset L^q(0,T;E)$ is weakly relatively compact in $L^q(0,T;E)$. Then assumption (S2) guarantees that $\{S(\widehat{u}_{n,k}) : n \ge 1\}$ is relatively compact in $W^{1,p}(0,T;E)$. Hence from (2.4) and (2.5) we see that $\{S(\widehat{u}_{n,k}) : n \ge 1\}$ is a relatively compact $\varepsilon\varrho$-net of $S(M)$ with respect to the norm $|\cdot|_{1,p}$, where $\varrho = \max\{L, ||k(t,.)|_r|_p\}$. Since ε was arbitrary we conclude that $S(M)$ is relatively compact in $W^{1,p}(0,T;E)$.

Now suppose that the sequence $(w_m)_m$ converges weakly in $L^q(0,T;E)$ to w and $w_m \in M$ for all $m \geq 1$. In view of the relative compactness of $S(M)$, we may assume that $(S(w_m))_m$ converges in K towards some function $v \in K$. We have to prove

$$v = S(w).$$

For an arbitrary number $\varepsilon > 0$, we have already seen that the proof of [10, lemma 4.3] provides a compact set P_ε and a sequence $(w_m^\varepsilon)_m$ of P_ε-valued functions satisfying,

$$|w_m - w_m^\varepsilon|_q \leq \varepsilon \tag{2.6}$$

for every $m \geq 1$. Now the sequence $(w_m^\varepsilon)_m$ being weakly relatively compact in $L^q(0,T,E)$, a suitable subsequence $(w_{m_j}^\varepsilon)_j$ must be weakly convergent in $L^q(0,T,E)$ towards some w^ε. Then Mazur's Lemma and (2.6) provide

$$|w - w^\varepsilon|_q \leq \varepsilon. \tag{2.7}$$

The triangle inequality yields

$$\begin{aligned} |v - S(w)|_p \leq\ & |v - S(w_{m_j})|_p + |S(w_{m_j}) - S(w_{m_j}^\varepsilon)|_p \\ & + |S(w_{m_j}^\varepsilon) - S(w^\varepsilon)|_p + |S(w^\varepsilon) - S(w)|_p \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} |v' - S(w)'|_p \leq\ & |v' - S(w_{m_j})'|_p + |S(w_{m_j})' - S(w_{m_j}^\varepsilon)'|_p \\ & + |S(w_{m_j}^\varepsilon)' - S(w^\varepsilon)'|_p + |S(w^\varepsilon)' - S(w)'|_p. \end{aligned} \tag{2.9}$$

Passing to the limit when j goes to infinity in (2.8), (2.9) and using assumption (S2) we obtain

$$|v - S(w)|_p \leq \limsup_j |S(w_{m_j}) - S(w_{m_j}^\varepsilon)|_p + |S(w^\varepsilon) - S(w)|_p, \tag{2.10}$$

$$|v' - S(w)'|_p \leq \limsup_j |S(w_{m_j})' - S(w_{m_j}^\varepsilon)'|_p + |S(w^\varepsilon)' - S(w)'|_p\,. \tag{2.11}$$

According to (2.6) and (2.7) we deduce from (2.10), (2.11) and assumption (S1) that

$$|v - S(w)|_p \leq 2\varepsilon ||k(t,.)|_r|_p, \quad |v' - S(w)'|_p \leq 2\varepsilon L.$$

Hence $|v - S(w)|_{1,p} \leq 2\,\varepsilon\varrho$. Since ε was arbitrary we must have $v = S(w)$ and the proof is complete. □

Proof of Theorem 2.5. (a) First we show that $G(u) \neq \emptyset$ and so $SG(u) \neq \emptyset$ for every $u \in \overline{U}$. Indeed, since g takes nonempty compact values and satisfies (G2)-(G3), for each strongly measurable function $u : [0,T] \to E^2$ there exists a strongly measurable selection w of $g(.,u(.))$ (see [11], Proof of Proposition 3.5 (a)). Next, if $u \in L^p(0,T;E^2)$, (G4) guarantees $w \in L^q(0,T;E)$. Hence $w \in G(u)$.
(b) The values of SG are acyclic according to assumption (SG).
(c) The graph of SG is closed. To show this, let $(u_n, v_n) \in \text{graph}\,(SG)$, $n \geq 1$, with $|u_n - u|_{1,p}$, $|v_n - v|_{1,p} \to 0$ as $n \to \infty$. Let $v_n = S(w_n)$, $w_n \in L^q(0,T;E); w_n \in G(u_n)$. Since $|u_n - u|_{1,p} \to 0$, we may suppose that for every $t \in [0,T]$, there exists a compact set $C \subset E^2$ with $\{(u_n(t), u_n'(t)); n \geq 1\} \subset C$. Furthermore, since g is upper semicontinuous by (G3) and has compact values, we have that $g(t,C)$ is compact. Consequently, $\{w_n(t) : n \geq 1\}$ is relatively compact in E. If we also take into account (G4) we may apply Theorem 1.5 to conclude that

(at least for a subsequence) (w_n) converges weakly in $L^q(0,T;E)$ to some w. As in [15, p. 57], since g has convex values and satisfies (G3), we can show that $w \in G(u)$. Furthermore, by using Lemma 2.6 and a suitable subsequence we deduce $S(w_n) \to S(w)$. Thus $v = S(w)$ and so $(u,v) \in$ graph (SG).

(d) We show that $SG(M)$ is relatively compact for each compact $M \subset \overline{U}$. Let $M \subset \overline{U}$ be a compact set and (v_n) be any sequence of elements of $SG(M)$. We prove that (v_n) has a convergent subsequence. Let $u_n \in M$ and $w_n \in L^q(0,T;E)$ with

$$v_n = S(w_n) \quad \text{and} \quad w_n \in G(u_n).$$

The set M being compact, we may assume that $|u_n - u|_{1,p} \to 0$ for some $u \in \overline{U}$. As above, there exists a $w \in G(u)$ with $w_n \to w$ weakly in $L^q(0,T;E)$ (at least for a subsequence) and $S(w_n) \to S(w)$. Hence $v_n \to S(w)$ as we wished. Now (c) and (d) guarantee (H1).

(e) Finally, we check (H2). Suppose $M \subset \overline{U}$ is closed and $M \subset \overline{\text{conv}}(\{0\} \cup SG(M))$. To prove that M is compact it suffices that every sequence (u_n^0) of M has a convergent subsequence. Let $M_0 = \{u_n^0 : n \geq 1\}$. Clearly, there exists a countable subset $M_1 = \{u_n^1 : n \geq 1\}$ of M, $w_n^1 \in G(u_n^1)$ and $v_n^1 = S(w_n^1)$ with $M_0 \subset \overline{\text{conv}}(\{0\} \cup V^1)$, where $V^1 = \{v_n^1 : n \geq 1\}$. Furthermore, there exists a countable subset $M_2 = \{u_n^2 : n \geq 1\}$ of M, $w_n^2 \in G(u_n^2)$ and $v_n^2 = S(w_n^2)$ with $M_1 \subset \overline{\text{conv}}(\{0\} \cup V^2)$, where $V^2 = \{v_n^2 : n \geq 1\}$, and so on. Hence for every $k \geq 1$ we find a countable subset $M_k = \{u_n^k : n \geq 1\}$ of M and correspondingly $w_n^k \in G(u_n^k)$ and $v_n^k = S(w_n^k)$ such that $M_{k-1} \subset \overline{\text{conv}}(\{0\} \cup V^k)$, with $V^k = \{v_n^k : n \geq 1\}$. Let $M^* = \bigcup_{k \geq 0} M_k$. It is clear that M^* is countable, $M_0 \subset M^* \subset M$ and $M^* \subset \overline{\text{conv}}(\{0\} \cup V^*)$, where $V^* = \bigcup_{k \geq 1} V^k$. Since M^*, V^* and $W^* := \{w_n^k : n \geq 1,\, k \geq 1\}$ are countable sets of strongly measurable functions, we may suppose that their values belong to a separable closed subspace E_0 of E. Since $|w_n^k(t)| \leq \nu(t)$ where $\nu \in L^q(0,T)$, then [10, Lemma 4.3] guarantees

$$\beta_{E_0}(M^*(t)) \leq \beta_{E_0}(V^*(t)) = \beta_{E_0}(S(W^*)(t)) \leq \int_0^T k(t,s)\beta_{E_0}(W^*(s))ds,$$

while (G5) gives

$$\beta_{E_0}(W^*(s)) \leq \beta_{E_0}(g(s, M^*(s), E_0) \cap E_0) \leq \omega(s, \beta_{E_0}(M^*(s))). \tag{2.12}$$

It follows that

$$\beta_{E_0}(M^*(t)) \leq \int_0^T k(t,s)\omega(s, \beta_{E_0}(M^*(s)))ds.$$

Moreover the function $\varphi(t) = \beta_{E_0}(M^*(t))$ belongs to $L^\infty(0,T;[0,\mu_0 R])$. Consequently, $\varphi \equiv 0$, and so

$$\varphi(t) = \beta_{E_0}(M^*(t)) = 0$$

a.e. on $[0,T]$. Let (v_i^*) be any sequence of V^* and let (w_i^*) be the corresponding sequence of W^*, with $v_i^* = S(w_i^*)$ for all $i \geq 1$. Then, as at step (c), (w_i^*) has a weakly convergent subsequence in $L^q(0,T;E)$, say to w. Also (2.12) together with $\omega(t,0) = 0$ implies that the set $\{w_i^*(t) : i \geq 1\}$ is relatively compact for a.e. $t \in [0,T]$. From Lemma 2.6 we then have that the corresponding subsequence of $(S(w_i^*)) = (v_i^*)$ converges to $S(w)$ in $W^{1,p}(0,T;E)$. Hence V^* is relatively compact. Now Mazur's Lemma guarantees that the set $\overline{\text{conv}}(\{0\} \cup V^*)$ is compact and so its subset M^* is relatively compact too. Thus M_0 possesses a convergent subsequence as we wished. Now the result follows from Theorem 2.1. □

3. The Anti-Periodic Solution Operator

For the rest of this paper E will be a real Hilbert space of inner product $(.,.)$ and norm $|.|$. Consider the anti-periodic boundary value problem

$$\begin{gathered} -u'' - \varepsilon u' \in Au + g(t,u,u') \quad \text{a.e. on } [0,T] \\ u(0) = -u(T), \quad u'(0) = -u'(T), \end{gathered} \tag{3.1}$$

in E, where $\varepsilon \in \mathbb{R}$ and $A : D(A) \subset E \to 2^E \setminus \{\emptyset\}$ is an odd m-dissipative nonlinear operator.

Let us consider the *anti-periodic solution operator* associated to A and ε,

$$S : L^2(0,T;E) \to H^2(0,T;E) \cap C_a^1$$

defined by $S(w) := u$, where u is the unique solution of

$$\begin{gathered} -u'' - \varepsilon u' \in Au + w \quad \text{a.e. on } [0,T] \\ u(0) = -u(T), \quad u'(0) = -u'(T)\,. \end{gathered} \tag{3.2}$$

The operator S is well defined as it follows from Theorem 3.1 in Aftabizadeh-Aizicovici-Pavel [1]. It is clear that any fixed point u of $N := SG$, where G is the Nemytskii set-valued operator given by (2.1) with $p = q = 2$, is a solution for (3.1).

Theorem 3.1. *The above operator S satisfies* (S1) *and* (S2) *for $p = q = 2$ and $K = \overline{C_a^1}$ in $H^1(0,T;E)$ with norm $\|u\| = |u'|_2$.*

Proof. (I) We first show that S satisfies (S1). Let $w_1, w_2 \in L^2(0,T;E)$ and denote $u_i = S(w_i)$, $i = 1,2$. Then $-u_i'' - \varepsilon u_i' = v_i + w_i$, where $v_i(t) \in Au_i(t)$ a.e. on $[0,T]$. One has

$$-(u_1 - u_2)''(t) - \varepsilon(u_1 - u_2)'(t) = (v_1 - v_2)(t) + (w_1 - w_2)(t).$$

Multiplying by $(u_1 - u_2)(t)$ and using that A dissipative, we obtain

$$\begin{aligned} &-(|u_1(t) - u_2(t)|^2)'' + 2|u_1'(t) - u_2'(t)|^2 - \varepsilon(|u_1(t) - u_2(t)|^2)' \\ &\le 2(w_1(t) - w_2(t), u_1(t) - u_2(t)). \end{aligned} \tag{3.3}$$

Consequently,

$$|u_1(t) - u_2(t)|^2 \le 2\int_0^T G(t,s)(w_1(s) - w_2(s), u_1(s) - u_2(s))ds. \tag{3.4}$$

Here G is the Green function of the differential operator $-u'' - \varepsilon u'$ corresponding to the anti-periodic boundary conditions. This yields

$$|S(w_1)(t) - S(w_2)(t)| \le m\int_0^T |w_1(s) - w_2(s)|ds \tag{3.5}$$

where $m = 2\max_{(t,s)\in[0,T]^2} G(t,s)$. From (3.3) by integration we obtain

$$\int_0^T |u_1' - u_2'|^2 ds \le \int_0^T (w_1 - w_2, u_1 - u_2)ds.$$

This together with (3.5) yields

$$|S(w_1)' - S(w_2)'|_2 \le \sqrt{mT}|w_1 - w_2|_2.$$

(II) The fact that S satisfies (S2) is achieved in several steps: (1) We first show that the graph of S is sequentially closed in $L_w^2(0,T;E) \times H^1(0,T;E)$. In this order, let $w_j \to w$ weakly in $L^2(0,T;E)$ and $S(w_j) \to u$ strongly in $H^1(0,T;E)$. Then $(w_j - w, S(w_j) - S(w)) \to 0$ strongly in $L^1(0,T;\mathbb{R})$. Now (3.4) implies

$$|S(w_j)(t) - S(w)(t)| \to 0 \quad \text{as } j \to \infty.$$

Hence $S(w) = u$.
(2) For each positive integer n we let

$$J_n = \big(J - \frac{1}{n}A\big)^{-1}, \quad A_n = n(J_n - J),$$

where J is the identity map of E. We also consider the operator $S_n : L^2(0,T;E) \to H^2(0,T;E) \cap C_a^1$, given by $S_n(w) = u_n$, where u_n is the unique solution of

$$\begin{gathered} -u_n'' - \varepsilon u_n' = A_n u_n + w \quad \text{a.e. on } [0,T] \\ u_n(0) = -u_n(T), \quad u_n'(0) = -u_n'(T)\,. \end{gathered} \tag{3.6}$$

Then

$$-|u_k''|^2 - \varepsilon(u_k', u_k'') = (A_k u_k, u_k')' - ((A_k u_k)', u_k') + (w, u_k'').$$

Since A_k is dissipative, we have

$$((A_k u_k)', u_k') = \lim_{h\to 0} \frac{1}{h^2}(A_k u_k(t+h) - A_k u_k(t), u_k(t+h) - u_k(t)) \le 0\,.$$

Hence

$$|u_k''|^2 \le -(A_k u_k, u_k')' - (w, u_k'') - \frac{\varepsilon}{2}(|u_k'|^2)'.$$

By integration, since A_k is odd and u_k is anti-periodic, it follows

$$|u_k''|_2^2 = \int_0^T |u_k''|^2 dt \le -\int_0^T (w, u_k'')dt \le \frac{1}{2}(|w|_2^2 + |u_k''|_2^2)\,.$$

Consequently,

$$|u_k''|_2 \le |w|_2. \tag{3.7}$$

Using $2|u'|^2 = (|u|^2)'' - 2(u'', u)$ and $(|u|^2)' = 2(u', u)$ we obtain

$$\begin{aligned} &2\int_0^T |u_k' - u_m'|^2 dt \\ &= (|u_k - u_m|^2)'(T) - (|u_k - u_m|^2)'(0) - 2\int_0^T (u_k'' - u_m'', u_k - u_m)dt \\ &= -2\int_0^T (u_k'' - u_m'', u_k - u_m)dt\,. \end{aligned} \tag{3.8}$$

On the other hand

$$\begin{aligned} &(u_k'' - u_m'', u_k - u_m) \\ &= -\big(A_k u_k - A_m u_m, u_k - u_m\big) - \varepsilon(u_k' - u_m', u_k - u_m) \\ &= -\big(A_k u_k - A_m u_m,\ J_k u_k - J_m u_m + \frac{1}{k}A_k u_k - \frac{1}{m}A_m u_m\big) - \varepsilon(u_k' - u_m', u_k - u_m) \end{aligned}$$

and since $A_k u_k \in AJ_k u_k$, $A_m u_m \in AJ_m u_m$ and A is dissipative, we obtain

$$-(u_k'' - u_m'', u_k - u_m) \le (A_k u_k - A_m u_m,\ \frac{1}{k}A_k u_k - \frac{1}{m}A_m u_m) + \frac{\varepsilon}{2}(|u_k - u_m|^2)'.$$

From (3.6) and (3.7), also applying Proposition 1.1 to u_k', we see that

$$|A_k u_k|_2 \le |u_k''|_2 + |w|_2 + |\varepsilon||u_k'|_2 \le |u_k''|_2 + |w|_2 + |\varepsilon|\frac{T}{2}|u_k''|_2 \le (2 + |\varepsilon|\frac{T}{2})|w|_2.$$

Then

$$-\int_0^T (u_k''-u_m'',u_k-u_m)dt \le 2(2+|\varepsilon|\frac{T}{2})^2|w|_2^2(\frac{1}{k}+\frac{1}{m}).$$

This together with (3.8) shows that

$$\int_0^T |u_k'-u_m'|^2dt \le 2(2+|\varepsilon|\frac{T}{2})^2|w|_2^2(\frac{1}{k}+\frac{1}{m}). \tag{3.9}$$

Thus there exists $u\in K$ with $u_k\to u$ in K. From (3.9), letting $m\to\infty$ we have

$$|u_k'-u'|_2^2 \le \frac{2}{k}(2+|\varepsilon|\frac{T}{2})^2|w|_2^2. \tag{3.10}$$

Now we show that u is the solution of (3.2). Since (u_k'') is bounded in $L^2(0,T;E)$ and (u_k'') converges to $w'=u''$ in $\mathcal{D}'(0,T;E)$, we may conclude that

$$u_k''\to u'' \quad \text{weakly in } L^2(0,T;E). \tag{3.11}$$

Let $\mathcal{A}$ be the realization of A in $L^2(0,T;E)$, i.e., $\mathcal{A}: L^2(0,T;E)\to 2^{L^2(0,T;E)}$,

$$\mathcal{A}u=\{v\in L^2(0,T;E): v(t)\in Au(t) \text{ a.e. on } [0,T]\}.$$

Then $(\mathcal{A}_k u)(t)=A_k u(t)$ a.e. on $[0,T]$, so that (3.11) implies that

$$\mathcal{A}_k u_k\to -u''-\varepsilon u'-w \ \text{ weakly in } L^2(0,T;E).$$

Since $u_k\to u$ strongly in $L^2(0,T;E)$ and $\mathcal{A}$ is m-dissipative in $L^2(0,T;E)$, this implies (see Barbu [5], Proposition II. 3.5) $u\in D(\mathcal{A})$ and $[u,-u''-\varepsilon u'-w]\in\mathcal{A}$. Thus, u is the solution of (3.2), i.e., $u=S(w)$. Now from (3.10) we see that for each bounded set $M\subset L^2(0,T;E)$ and every $\epsilon>0$, there exists a k_0 such that

$$\|S_k(w)-S(w)\|\le\epsilon \quad \text{for all } k\ge k_0 \text{ and } w\in M. \tag{3.12}$$

Hence $S_{k_0}(M)$ is an ϵ-net for $S(M)$.

(3) Now we consider a compact convex subset C of E and a countable set $M\subset L^2(0,T;C)$. We shall prove that for each n, the set $S_n(M)$ is relatively compact in K, equivalently, the set $S_n(M)'$ is relatively compact in $L^2(0,T;E)$. Then, also taking into account (3.12), by Hausdorff's Theorem we shall deduce that $S(M)$ is relatively compact in K as desired. We shall apply Theorem 1.4 to $S_n(M)'$. From (3.12) and assumption (S1) we see that for each n and any bounded $M\subset L^2(0,T;E)$, the set $S_n(M)$ is bounded in K. In addition, using

$$u_n(t)=\int_0^T G(t,s)[A_nu_n(s)+w(s)]ds$$

and the Lipschitz property of A_n, we obtain

$$\begin{aligned}|\tau_h u_n'-u_n'|_2^2 &\le \int_0^T\Big(\int_0^T |G_t(t+h,s)-G_t(t,s)|[2n|u_n(s)|+|w(s)|]ds\Big)^2dt\\ &\le (2n|u_n|_2+|w|_2)^2\int_0^T\int_0^T |G_t(t+h,s)-G_t(t,s)|^2dsdt.\end{aligned}$$

This implies

$$\sup_{w\in M}|\tau_hS_n(w)'-S_n(w)'|_{L^2(0,T-h;E)}\to 0 \quad \text{as } h\to 0. \tag{3.13}$$

We claim that $S_n(M)'(t)$ is relatively compact in E for every $t\in[0,T]$. Indeed, for any $w\in M$, the unique solution $u_n=S_n(w)$ of (3.6) satisfies

$$-u_n''-\varepsilon u_n'+nu_n=nJ_nu_n+w \quad \text{a.e. on } [0,T].$$

If we denote by $\widetilde{G}$ the Green function of the operator $-u''-\varepsilon u'+nu$ corresponding to the boundary conditions $u(0)=-u(T)$, $u'(0)=-u'(T)$, then

$$u_n(t)=\int_0^T \widetilde{G}(t,s)[nJ_nu_n(s)+w(s)]ds. \tag{3.14}$$

Using a result by Heinz, the nonexpansivity of J_n and the inclusion $M(s)\subset C$ a.e. on $[0,T]$, from (3.14), we obtain

$$\beta_0(S_n(M)(t))\le n\int_0^T \widetilde{G}(t,s)\beta_0(S_n(M)(s))ds. \tag{3.15}$$

Here β_0 is the ball measure of non-compactness corresponding to a suitable separable closed subspace of E. Let

$$\varphi(t)=\beta_0(S_n(M)(t))\,,\quad v(t)=\int_0^T \widetilde{G}(t,s)\varphi(s)ds.$$

We have

$$-v''-\varepsilon v'+nv=\varphi,\quad v(0)=-v(T),\quad v'(0)=-v'(T).$$

According to (3.15), $\varphi\le nv$. Hence $-v''-\varepsilon v'\le 0$. Also since $v\ge 0$ we have $v(0)=v(T)=0$. The maximum principle for the operator $-u''-\varepsilon u'$ implies $v\le 0$ on $[0,T]$. Hence $v\equiv 0$. Thus $\beta_0(S_n(M)(t))=0$ for all $t\in[0,T]$, that is $S_n(M)(t)$ is relatively compact in E. As a result, $S_n(M)$ is relatively compact in $C([0,T];E)$. Next from (3.14) we have

$$u_n'(t)=\int_0^T \widetilde{G}_t(t,s)[nJ_nu_n(s)+w(s)]ds,$$

whence $S_n(M)'(t)$ is relatively compact in E. This together with (3.13) via Theorem 1.4 implies that $S_n(M)'$ is relatively compact in $L^2(0,T;E)$. □

4. Superlinear Inclusions

In this section we establish an existence result for the anti-periodic problem

$$\begin{gathered}-u''-\varepsilon u'-s(u)\in Au+h(t,u,u')\quad\text{a.e. on }[0,T]\\ u(0)=-u(T),\quad u'(0)=-u'(T)\end{gathered} \tag{4.1}$$

in the Hilbert space E, where $\varepsilon>0$, $A:D(A)\subset E\to 2^E\setminus\{\emptyset\}$ is odd m-dissipative, $s:E\to E$ is continuous with a possible superlinear growth, and $h:[0,T]\times E^2\to 2^E$. Let $G:H^1(0,T;E)\to 2^{L^2(0,T;E)}$ be the Nemytskii set-valued operator associated with $g(t,x,y)=s(x)+h(t,x,y)$, that is

$$G(u)=\{v\in L^2(0,T;E):v=s(u)+w,\ w\in \text{sel}\,_{L^2}h(.,u,u')\},$$

and let S be the anti-periodic solution operator associated to A and ε, already defined in Section 3.

The next result concerns condition (H3) and gives sufficient conditions to obtain a priori bounds of solutions.

Theorem 4.1. *Assume that the following conditions hold:*

(i) *There exist two even real functions ϕ, ψ such that $\psi\in C^1(E;\mathbb{R})$ and $A=-\partial\phi$ and $s=\psi'$, where $\partial\phi$ stands for the subdifferential of ϕ*

(ii) *There are* $a, b \in \mathbb{R}_+$ *and* $\alpha, \gamma \in [1, 2[$, $\beta \in [0, 2[$ *with* $\beta + \gamma < 2$ *such that*

$$-(z, y) \leq a|y|^{\alpha} + b|x|^{\beta}|y|^{\gamma} \tag{4.2}$$

for all $x, y \in E$, $z \in h(t, x, y)$, *and for a.e.* $t \in [0, T]$.

Then there exists a constant $R > 0$ *such that* $\|u\| = |u'|_2 < R$ *for any solution* u *of*

$$u \in \lambda SG(u) \tag{4.3}$$

and every $\lambda \in]0, 1[$.

Proof. Let u be any non-zero solution of (4.3) for some $\lambda \in]0, 1[$. Let $u_\lambda := \frac{1}{\lambda}u$. Then $u = \lambda u_\lambda$ and

$$u_\lambda = S(w), \quad w \in G(u)$$

that is

$$\begin{gathered}
-u_\lambda'' - \varepsilon u_\lambda' \in Au_\lambda + w, \\
w = s(u) + v, \\
v \in \text{sel }_{L^2} h(., u, u').
\end{gathered}$$

Hence

$$-u_\lambda'' - s(u) - \varepsilon u_\lambda' - v \in Au_\lambda.$$

Multiplying by $u' = \lambda u_\lambda'$ and using the formula $(Au_\lambda, u_\lambda') = -(\phi(u_\lambda))'$ (see [5, p. 189]), we obtain

$$\frac{\lambda}{2}(|u_\lambda'|^2)' + (\psi(u))' + \frac{\varepsilon}{\lambda}|u'|^2 + (v, u') = \lambda(\phi(u_\lambda))'.$$

Thus,

$$\big(\frac{\lambda}{2}|u_\lambda'|^2 + \psi(u) - \lambda\phi(u_\lambda)\big)' + \frac{\varepsilon}{\lambda}|u'|^2 = -(v, u').$$

By integration from 0 to T and taking into account the anti-periodic boundary conditions and the fact that ϕ and ψ are even, we deduce

$$\varepsilon|u'|_2^2 < \frac{\varepsilon}{\lambda}|u'|_2^2 = -\int_0^T (v(t), u'(t))dt.$$

Now using (4.2) and (1.1) we obtain

$$\begin{aligned}
\varepsilon|u'|_2^2 &< a|u'|_\alpha^\alpha + b\int_0^T |u|^\beta |u'|^\gamma dt \\
&\leq a|u'|_\alpha^\alpha + b(\frac{1}{2}|u'|_1)^\beta \int_0^T |u'|^\gamma dt \\
&= a|u'|_\alpha^\alpha + b\frac{1}{2^\beta}|u'|_1^\beta |u'|_\gamma^\gamma.
\end{aligned}$$

Since $\alpha, \gamma \in [1, 2[$ there are constants c_1, c_2 such that $|u'|_\alpha \leq T^{\frac{2-\alpha}{2\alpha}}|u'|_2$ and $|u'|_\gamma \leq T^{\frac{2-\gamma}{2\gamma}}|u'|_2$. In addition $|u'|_1 \leq T^{\frac{1}{2}}|u'|_2$. Consequently, one has

$$\varepsilon|u'|_2^2 < C_1|u'|_2^\alpha + C_2|u'|_2^{\beta+\gamma},$$

where the constants C_1, C_2 (independent of u and λ) are:

$$C_1 = aT^{\frac{2-\alpha}{2}}, \quad C_2 = b\frac{1}{2^\beta}T^{\frac{2+\beta-\gamma}{2}}.$$

Now the conclusion follows since $\alpha < 2$ and $\beta + \gamma < 2$. $\square$

Remark 4.2. The above result is also true if $\alpha = 2$ or $\beta + \gamma = 2$ provided that a, respectively b, is sufficiently small.

Now we are ready to state the main result of this section.

Theorem 4.3. *Let E be a Hilbert space, $\varepsilon > 0$, $s : E \to E$, $A : E \to 2^E$ and $h : [0,T] \times E^2 \to 2^E$. Assume:*

- (i) *$s = \psi'$ for some even function $\psi \in C^1(E;\mathbb{R})$, and s sends bounded sets into bounded sets*
- (ii) *A is an m-dissipative mapping with $A = -\partial\phi$ for some even real function ϕ*
- (iii) *$h : [0,T] \times E^2 \to P_{kc}(E)$, $h(.,z)$ has a strongly measurable selection on $[0,T]$ for every $z \in E^2$, $h(t,.)$ is upper semicontinuous for a.e. $t \in [0,T]$, and for each $\tau > 0$ there exists $\nu \in L^2(0,T)$ with $|h(t,z)| \le \nu(t)$ for a.e. $t \in [0,T]$ and all $z = (z_1,z_2) \in E^2$ with $|z_1| \le \tau$; in addition there are $a,b \in \mathbb{R}_+$ and $\alpha,\gamma \in [1,2[$ and $\beta \in [0,\infty[$ such that*
$$-(z,y) \le a|y|^\alpha + b|x|^\beta|y|^\gamma$$
for all $x,y \in E$, $z \in h(t,x,y)$, and for a.e. $t \in [0,T]$
- (iv) *There exists $R > 0$ with*
$$\varepsilon R^2 \ge aT^{\frac{2-\alpha}{2}}R^\alpha + b\frac{1}{2^\beta}T^{\frac{2+\beta-\gamma}{2}}R^{\beta+\gamma} \tag{4.4}$$
such that for every separable closed subspace E_0 of E, there exists a $(1,\infty)$-Carathéodory function $\omega : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for almost every $t \in [0,T]$,
$$\beta_{E_0}(g(t,M,E_0) \cap E_0) \le \omega(t,\beta_{E_0}(M))$$
(where $g(t,x,y) = s(x) + h(t,x,y)$) for every bounded set $M \subset E_0$, and $\varphi = 0$ is the unique solution in $L^\infty(0,T;\mathbb{R}_+)$ to the inequality
$$\varphi(t) \le m\int_0^T \omega(s,\varphi(s))ds \quad \text{a.e. on } [0,T] \tag{4.5}$$
- (v) *SG has acyclic values.*

Then (4.1) has at least one solution $u \in H^2(0,T;E) \cap C_a^1$ with $\|u\| \le R$.

Remark 4.4. (a) Note that we do not assume $\beta + \gamma < 2$, so the perturbation term $h(t,u,u')$ can have a superlinear growth in u; inequality (4.4) guarantees that $\|u\| \ne R$ for each solution of (4.3) and $\lambda \in]0,1[$. This does not exclude the existence of solutions with $\|u\| > R$.
(b) However, according to Theorem 4.1, if $\beta + \gamma < 2$, then there exists a sufficiently large constant $R_0 > 0$ such that (4.4) holds with equality. In this case R_0 is a bound for all solutions to (4.3).
(c) Sufficient conditions for (v) can be found in [10]. For example (v) always holds if A is single-valued.

5. Applications

In this section we are concerned with two applications of Theorem 4.3 to partial differential inclusions.

(I) First we look for a function $u = u(t,x) = u(t)(x)$ solving the problem

$$\begin{gathered}-u_{tt} - \varepsilon u_t + \sigma\Delta_x^{-1}(|u|^{p-2}u) + u \in h(t,u,u_t) \quad \text{a.e. on } [0,T] \\ u(t,.) \in H_0^1(\Omega) \quad \text{for a.e. } t \in [0,T] \\ u(0,x) = -u(T,x), \quad u_t(0,x) = -u_t(T,x) \quad \text{a.e. on } \Omega.\end{gathered} \tag{5.1}$$

Here Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 3$, $2 < p < 2^* = \frac{2n}{n-2}$, $\varepsilon > 0$, $\sigma \in \mathbb{R}$ and $\Delta_x : H_0^1(\Omega) \to H^{-1}(\Omega)$ is the Laplacian. Also by $|\cdot|$ we mean here the absolute value of a real number.

In this setting we let $E = H_0^1(\Omega)$ with the inner product $(u,v)_{H_0^1(\Omega)} = \int_\Omega \nabla u \cdot \nabla v dx$ and norm $|u|_{H_0^1(\Omega)} = (\int_\Omega |\nabla u|^2 dx)^{\frac{1}{2}}$, $A(u) = -u$ with $D(A) = H_0^1(\Omega)$ and $s(u) = -\sigma\Delta_x^{-1}(|u|^{p-2}u)$. Note that the conditions (i) and (ii) in Theorem 4.3 hold with

$$\phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx \quad \text{and} \quad \psi(u) = \frac{\sigma}{p}\int_\Omega |u|^p dx.$$

Also note that for any bounded $M \subset H_0^1(\Omega)$ the set $s(M)$ is relatively compact in $H_0^1(\Omega)$, that is $\beta_{H_0^1(\Omega)}(s(M)) = 0$. Here $\beta_{H_0^1(\Omega)}$ is the ball measure of non-compactness in $H_0^1(\Omega)$. Indeed, since $p < 2^*$ we may choose an $\theta > 0$ with $p \leq 2^* - \frac{\theta}{(2^*)'}$, where $(2^*)' = \frac{2n}{n+2}$. This guarantees that $(2^*)' \leq \frac{2^*-\theta}{p-1}$. Next the embedding of $H_0^1(\Omega)$ into $L^{2^*-\theta}(\Omega)$ being compact, we have that M is relatively compact in $L^{2^*-\theta}(\Omega)$. Then the set $M_p := \{|u|^{p-2}u : u \in M\}$ is relatively compact in $L^{\frac{2^*-\theta}{p-1}}(\Omega)$ and using the continuous embeddings

$$L^{\frac{2^*-\theta}{p-1}}(\Omega) \subset L^{(2^*)'}(\Omega) \subset H^{-1}(\Omega)$$

we find that M_p is relatively compact in $H^{-1}(\Omega)$. Thus, $s(M) = -\sigma\Delta_x^{-1}(M_p)$ is relatively compact in $H_0^1(\Omega)$ as desired.

From Theorem 4.3 one obtains the following result.

Theorem 5.1. *Let $h : [0,T] \times H_0^1(\Omega) \times H_0^1(\Omega) \to P_{kc}(H_0^1(\Omega))$ be such that $h(.,u,v)$ has a strongly measurable selection on $[0,T]$ for every $u, v \in H_0^1(\Omega)$, $h(t,.)$ is upper semicontinuous for a.e. $t \in [0,T]$, and for each $\tau > 0$ there exists $\nu \in L^2(0,T)$ such that $|h(t,u,v)|_{H_0^1(\Omega)} \leq \nu(t)$ for a.e. $t \in [0,T]$ and all $u, v \in H_0^1(\Omega)$ with $|u|_{H_0^1(\Omega)} \leq \tau$. Assume there are $a, b, a_0 \in \mathbb{R}_+$ and $\alpha, \gamma \in [1,2[$ and $\beta \in [0,\infty[$ such that*

$$-(w,v)_{H_0^1(\Omega)} \leq a|v|^\alpha_{H_0^1(\Omega)} + b|u|^\beta_{H_0^1(\Omega)}|v|^\gamma_{H_0^1(\Omega)}$$

for all $u, v \in H_0^1(\Omega)$, $w \in h(t,u,v)$ and for a.e. $t \in [0,T]$, and that for each bounded $M \subset H_0^1(\Omega)$,

$$\beta_{H_0^1(\Omega)}(h(t,M,H_0^1(\Omega))) \leq a_0\beta_{H_0^1(\Omega)}(M).$$

In addition assume that there exists $R > 0$ with

$$\varepsilon R^2 \geq aT^{\frac{2-\alpha}{2}}R^\alpha + b\frac{1}{2^\beta}T^{\frac{2+\beta-\gamma}{2}}R^{\beta+\gamma}.$$

Then for $a_0 < \frac{1}{mT}$, (5.1) has at least one solution $u \in H^2(0,T;H_0^1(\Omega))$ with

$$|u'|_2 = (\int_0^T |u'(t)|^2_{H_0^1(\Omega)} dt)^{\frac{1}{2}} \leq R.$$

Proof. For any bounded M, since $\beta_{H_0^1(\Omega)}(s(M)) = 0$, one has

$$\beta_{H_0^1(\Omega)}(g(t, M, H_0^1(\Omega))) \le a_0 \beta_{H_0^1(\Omega)}(M).$$

Recall that the space $H_0^1(\Omega)$ is separable. It follows that the unique solution $\varphi \in L^\infty(0, T; \mathbb{R}_+)$ of (4.5) with $\omega(t, \tau) = a_0 \tau$ is $\varphi = 0$ provided that $a_0 m T < 1$. Thus Theorem 4.3 applies. □

Corollary 5.2. *For every $f \in L^\infty(0, T; H_0^1(\Omega))$ the problem*

$$\begin{gathered} -u_{tt} - \varepsilon u_t + \sigma \Delta_x^{-1}(|u|^{p-2} u) + u = f(t, x) \quad \textit{a.e. on } [0, T] \times \Omega \\ u(t, .) \in H_0^1(\Omega) \quad \textit{for a.e. } t \in [0, T] \\ u(0, x) = -u(T, x), \quad u_t(0, x) = -u_t(T, x) \quad \textit{a.e. on } \Omega. \end{gathered}$$

has at least one solution $u \in H^2(0, T; H_0^1(\Omega))$ with

$$|u'|_2 \le \frac{|f|_\infty \sqrt{T}}{\varepsilon}.$$

Here $|f|_\infty = \operatorname{ess\,sup}_{t \in [0,T]} |f(t)|_{H_0^1(\Omega)}$.

Proof. In this case $h(t, u, v) = f(t) := f(t, .)$. Consequently all the assumptions of Theorem 5.1 are satisfied for $a = 0$, $b = |f|_\infty$, $\alpha = 1$, $\beta = 0$, $\gamma = 1$, $a_0 = 0$, $\nu(t) = |f(t)|_{H_0^1(\Omega)}$ and $R = \frac{|f|_\infty \sqrt{T}}{\varepsilon}$. □

(II) For the next application we look for a function $u = u(t, x)$ solving the problem

$$\begin{gathered} -u_{tt} - \varepsilon u_t + \sigma |u|_{L^2(\Omega)}^{p-2} u - \Delta_x u \in h(t, u, u_t) \quad \text{a.e. on } [0, T] \times \Omega \\ u(t, .) \in H_0^1(\Omega) \quad \text{for a.e. } t \in [0, T] \\ u(0, x) = -u(T, x), \quad u_t(0, x) = -u_t(T, x) \quad \text{a.e. on } \Omega. \end{gathered} \tag{5.2}$$

Here again Ω is a bounded domain of $\mathbb{R}^n$, $p > 2$, $\varepsilon > 0$ and $\sigma \in \mathbb{R}$, but we need no upper bound for p. Now we let $E = L^2(\Omega)$, $A = \Delta_x$ be the Laplace operator with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and $s(u) = -\sigma |u|_{L^2(\Omega)}^{p-2} u$. We note that the conditions (i) and (ii) in Theorem 4.3 hold with

$$\phi(u) = \begin{cases} \frac{1}{2} \int_\Omega |\nabla u|^2 dx, & u \in H^1(\Omega) \\ +\infty, & \text{otherwise.} \end{cases}$$

and $\psi(u) = -\frac{\sigma}{p} |u|_{L^2(\Omega)}^p$. From Theorem 4.3 one obtains the following result.

Theorem 5.3. *Let $h : [0, T] \times L^2(\Omega) \times L^2(\Omega) \to P_{kc}(L^2(\Omega))$ be such that $h(., u, v)$ has a strongly measurable selection on $[0, T]$ for every $u, v \in L^2(\Omega)$, $h(t, .)$ is upper semicontinuous for a.e. $t \in [0, T]$, and for every $\tau > 0$ there exists $\nu \in L^2(0, T)$ such that $|h(t, u, v)|_{L^2(\Omega)} \le \nu(t)$ for a.e. $t \in [0, T]$ and all $u, v \in L^2(\Omega)$ with $|u|_{L^2(\Omega)} \le \tau$. Assume there are $a, b, a_0 \in \mathbb{R}_+$ and $\alpha, \gamma \in [1, 2[$ and $\beta \in [0, \infty[$ such that*

$$-(w, v)_{L^2(\Omega)} \le a |v|_{L^2(\Omega)}^\alpha + b |u|_{L^2(\Omega)}^\beta |v|_{L^2(\Omega)}^\gamma$$

for all $u, v \in L^2(\Omega)$, $w \in h(t, u, v)$ and for a.e. $t \in [0, T]$, and that for each bounded $M \subset L^2(\Omega)$,

$$\beta_{L^2(\Omega)}(h(t, M, L^2(\Omega))) \le a_0 \beta_{L^2(\Omega)}(M).$$

In addition assume that there exists $R > 0$ with

$$\varepsilon R^2 \ge a T^{\frac{2-\alpha}{2}} R^\alpha + b \frac{1}{2^\beta} T^{\frac{2+\beta-\gamma}{2}} R^{\beta+\gamma}.$$

Then for sufficiently small $|\sigma|$ *and* a_0 (5.2) *has a solution* $u \in H^2(0,T;L^2(\Omega))$ *with*

$$|u'|_2 = \Big(\int_0^T |u'(t)|^2_{L^2(\Omega)} dt\Big)^{1/2} \le R.$$

Proof. For any $u, v \in L^2(\Omega)$ with $|u|_{L^2(\Omega)}, |v|_{L^2(\Omega)} \le \eta$, we have

$$\begin{aligned}
|s(u)-s(v)|_{L^2(\Omega)} &= |\sigma| ||u|^{p-2}_{L^2(\Omega)} u - |v|^{p-2}_{L^2(\Omega)} v|_{L^2(\Omega)} \\
&\le |\sigma| (||u|^{p-2}_{L^2(\Omega)} (u-v)|_{L^2(\Omega)} + |(|u|^{p-2}_{L^2(\Omega)} - |v|^{p-2}_{L^2(\Omega)}) v|_{L^2(\Omega)}) \\
&\le |\sigma| (\eta^{p-2} |u-v|_{L^2(\Omega)} + (p-2)\eta^{p-2} |u-v|_{L^2(\Omega)}) \\
&= |\sigma|(p-1)\eta^{p-2} |u-v|_{L^2(\Omega)}.
\end{aligned}$$

Hence for any bounded $M \subset L^2(\Omega)$ one has

$$\beta_{L^2(\Omega)}(g(t,M,L^2(\Omega))) \le [|\sigma|(p-1)|M|^{p-2} + a_0] \beta_{L^2(\Omega)}(M)$$

where, as above, $g(t,u,v) = s(u) + h(t,u,v)$, and $|M| = \sup_{u,v\in M} |u-v|_{L^2(\Omega)}$. It is easily seen that the unique solution $\varphi \in L^\infty(0,T;\mathbb{R}_+)$ of (4.5) with

$$\omega(t,\tau) = [|\sigma|(p-1)\eta^{p-2} + a_0]\tau$$

where $\eta = R\max\{1, \sqrt{T}/2\}$, is $\varphi = 0$ provided that $|\sigma|$ and a_0 are small enough. Thus Theorem 4.3 applies. □

Corollary 5.4. *For every* $f \in L^\infty(0,T;L^2(\Omega))$*, if* $|\sigma|$ *is sufficiently small the problem*

$$\begin{gathered}
-u_{tt} - \varepsilon u_t + \sigma |u|^{p-2}_{L^2(\Omega)} u - \Delta_x u = f(t,x) \quad \text{a.e. on } [0,T]\times\Omega \\
u(t,.) \in H^1_0(\Omega) \quad \text{for a.e. } t \in [0,T] \\
u(0,x) = -u(T,x), \quad u_t(0,x) = -u_t(T,x) \quad \text{a.e. on } \Omega.
\end{gathered}$$

has at least one solution $u \in H^2(0,T;L^2(\Omega))$ *with* $|u'|_2 \le \frac{|f|_\infty \sqrt{T}}{\varepsilon}$. *Here* $|f|_\infty = \operatorname{ess\,sup}_{t\in[0,T]} |f(t)|_{L^2(\Omega)}$.

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