Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 102, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

UNIQUENESS OF TRAVELING WAVE SOLUTIONS FOR NON-MONOTONE CELLULAR NEURAL NETWORKS WITH DISTRIBUTED DELAYS

HUI-LING ZHOU, ZHIXIAN YU

Abstract. In this article, we study the uniqueness of traveling wave solutions for non-monotone cellular neural networks with distributed delays. First we establish a priori asymptotic behavior of the traveling wave solutions at infinity. Then, based on Ikehara's theorem, we prove the uniqueness of the solution $\psi(n-ct)$ with $c \le c_*$, where $c_* < 0$ is the critical wave speed.

1. Introduction

In this article, we study the uniqueness of traveling wave solution for the non-monotone cellular neural networks with distributed delays

$$\begin{aligned} x_n'(t) = -x_n(t) &+ \sum_{i=1}^{m} \int_0^{\tau} a_i J_i(y) f(x_{n-i}(t-y))dy \\ &+ \alpha \int_0^{\tau} J_{m+1}(y) f(x_n(t-y))dy + \sum_{j=1}^{l} \beta_j \int_0^{\tau} J_{m+1+j}(y) f(x_{n+j}(t-y))dy, \end{aligned} \tag{1.1}$$

where the constants $n \in Z$, $m, l \in N$, $\tau \ge 0$, and the varible $t \in \mathbb{R}$. We use the following assumptions:

(H0) (i) $\alpha > 0$, $a_1 > 0$, $a_i \ge 0$ $(i = 2, \dots, m)$, $\beta_1 > 0$ and $\beta_j \ge 0$ $(j = 2, \dots, l)$. $a = \sum_{i=1}^{m} a_i$ and $\beta = \sum_{j=1}^{l} \beta_j$.
(ii) $J_i : [0, \tau] \to (0, +\infty)$ is the piecewise continuous function satisfying $\int_0^{\tau} J_i(y)dy = 1$, where $0 < \tau < \infty$.

(H1) $f \in C([0, b], [0, \frac{b}{a+\alpha+\beta}])$, $f(0) = 0$, $\alpha f'(0) \ge 1$ and there exists $K > 0$ with $K \le b$ such that

$$(a + \alpha + \beta)f(K) = K, \quad |f(u) - f(v)| \le f'(0)|u - v| \quad \text{for } u, v \in [0, b].$$

(H2) $(a + \alpha + \beta)f(u) > u$ for $u \in (0, K)$ and $(a + \alpha + \beta)f(u) < u$ for $u \in (K, b]$.

(H3) There exist $\sigma > 0$, $\delta > 0$ and $M > 0$ such that

$$f(u) \ge f'(0)u - Mu^{1+\sigma} \quad \text{for } u \in [0, \delta].$$

2010 *Mathematics Subject Classification.* 35C07, 92D25, 35B35.
Key words and phrases. Cellular neural network; uniqueness; non-monotone; asymptotic behavior; distributed delays.

Submitted March 1, 2017. Published April 11, 2017.

A traveling wave solution (1.1) with speed c is a nonnegative bounded solution of the form $u_n(t) = \psi(n-ct)$ satisfying $\psi(-\infty) = 0$ and $\liminf_{\xi\to+\infty}\psi(\xi) > 0$. Substituting $u_n(t) = \psi(n-ct)$ in (1.1), we have the wave profile equation

$$\begin{aligned} -c\psi'(\xi) &= -\psi(\xi) + \sum_{i=1}^{m}\int_0^\tau a_i J_i(y) f(\psi(\xi - i + cy))dy \\ &\quad + \alpha\int_0^\tau J_{m+1}(y) f(\psi(\xi + cy))dy \\ &\quad + \sum_{j=1}^{l}\beta_j\int_0^\tau J_{m+1+j}(y) f(\psi(\xi + j + cy))dy. \end{aligned} \tag{1.2}$$

When the output function f is monotone, the existence of traveling wave solutions for many versions of CNNs (1.1) with delays or without delays has been widely investigated. see for example [10, 11, 13, 14, 15, 16, 17, 18, 21, 23, 24, 28, 30, 26]. The existence of entire solutions for (1.1) has been investigated by Wu and Hsu [23, 24]. Letting $J_i = \delta(y-\tau_i)$, $i = 1,\dots,m+l+1$, (1.1) reduces to the multiple discrete delays equation

$$\begin{aligned} w_n'(t) &= -w_n(t) + \sum_{i=1}^{m} a_i f(w_{n-i}(t-\tau_i)) + \alpha f(w_n(t-\tau_{m+1})) \\ &\quad + \sum_{j=1}^{l}\beta_j f(w_{n+j}(t-\tau_{m+1+j})). \end{aligned} \tag{1.3}$$

Yu and Mei [28] investigated uniqueness and stability of traveling wave solutions for (1.3) with the monotone output function. In [28] the authors used the technique in [3] to study uniqueness of travelling wave soluitons for (1.3) with discrete delays. We will extend this method to (1.1) with distributed delays.

For the non-monotone output function f, Yu et al. [27] only established the existence of non-critical traveling wave solutions. Yu and Zhao [31] further established the existence of the spreading speed, its coincidence with the minimal wave speed and the existence of critical waves for the non-monotone DCNNs (1.1). We summarize the existence of traveling wave solutions of (1.1) with the non-monotone output function in [27, 31] as follows.

Proposition 1.1. *Assume that* (H0)-(H3) *hold. Then there exists $c^* < 0$ (which is given in Lemma 2.1) such that for any $c \le c^*$,* (1.1) *admits a non-negative traveling wave solution $\psi(n-ct)$ with the wave speed $c^* < 0$ and satisfying*

$$\psi(-\infty) = 0 \quad \textit{and} \quad 0 < \liminf_{\xi\to+\infty}\psi(\xi) \le \limsup_{\xi\to+\infty}\psi(\xi) \le b. \tag{1.4}$$

The uniqueness of monotone travelling wave solutions for various evolution systems has been established; see for example [1, 2, 4, 5, 19, 20] and the references therein. The proof of uniqueness strongly relies on the monotonicity of travelling waves. It seems very difficult to extend the techniques in those literatures to the non-monotone evolution systems because the wave profile may lose the monotonicity and the study of the corresponding uniqueness is very limited, see, e.g., [6, 8, 9]. Recently, the authors in [25, 29] extend the technique in [3] to non-monotone lattice equations with discrete delays. In this article, we extend the technique in [3] to non-monotone CNNs with distributed delays.

The rest of this article is organized as follows. Section 2 is devoted to studying the asymptotic behavior of the traveling wave solutions. In Section 3, we prove the uniqueness of the solution.

2. Asymptotic behavior of traveling wave solutions

In this section, we consider the asymptotic behavior at negative infinity of any traveling wave solutions of (1.1). The characteristic equation of (1.2) at 0 is

$$\begin{aligned}\Delta(c,\lambda) = -c\lambda + 1 - f'(0)\Big[\sum_{i=1}^{m} a_i \int_0^{\tau} J_i(y) e^{\lambda(-i+cy)}dy \\ + \alpha\int_0^{\tau} J_{m+1}(y)e^{\lambda cy}dy + \sum_{j=1}^{l}\beta_j\int_0^{\tau} J_{m+1+j}(y)e^{\lambda(j+cy)}dy\Big].\end{aligned} \tag{2.1}$$

Lemma 2.1 ([26, Lemma 2.1])**.** *Assume that* (H0) *and* $\alpha f'(0) \geq 1$ *hold. Then there exist a unique pair of* $c_* < 0$ *and* $\lambda_* > 0$ *such that*

- (i) $\Delta(c_*,\lambda_*) = 0$, $\frac{\partial\Delta(c,\lambda)}{\partial\lambda}|_{c=c_*,\lambda=\lambda_*} = 0$;
- (ii) *For any* $c > c_*$ *and* $\lambda \in [0,+\infty)$, $\triangle(c,\lambda) < 0$;
- (iii) *For any* $c < c_*$, $\triangle(c,\lambda) = 0$ *has two positive roots* $\lambda_2 \geq \lambda_1 > 0$. *Moreover, if* $c < c_*$, $\triangle(c,\lambda) > 0$ *for any* $\lambda \in (\lambda_1,\lambda_2)$; *if* $c = c_*$, *then* $\lambda_1 = \lambda_2 = \lambda_*$.

Now we give a different version of Ikehara's Theorem, which can be found in [3].

Proposition 2.2. *Let* $F(\lambda) := \int_0^{+\infty} u(x)e^{-\lambda x}dx$, *where* $u(x)$ *is a positive decreasing function. Assume* $F(\lambda)$ *can be written as*

$$F(\lambda) = \frac{h(\lambda)}{(\lambda+\mu)^{k+1}},$$

where $k > -1$ *and* $h(\lambda)$ *is analytic in the strip* $-\mu \leq \Re\lambda < 0$. *Then*

$$\lim_{x\to+\infty}\frac{u(x)}{x^k e^{-\mu x}} = \frac{h(-\mu)}{\Gamma(\mu+1)}.$$

Remark 2.3. Changing the variable $t = -x$, and modifying the proof for Ikehara's Theorem given in [7], we can show the following version of Proposition 2.2. Let $F(\lambda) := \int_{-\infty}^{0} u(t)e^{-\lambda t}dt$, where $u(t)$ is a positive increasing function. Assume $F(\lambda)$ can be written as

$$F(\lambda) = \frac{h(\lambda)}{(\mu-\lambda)^{k+1}},$$

where $k > -1$ and $h(\lambda)$ is analytic in the strip $\mu - \epsilon < \Re\lambda \leq \mu$ for some $0 < \epsilon < \mu$. Then

$$\lim_{x\to-\infty}\frac{u(x)}{|x|^k e^{\mu x}} = \frac{h(\mu)}{\Gamma(\mu+1)}.$$

To apply Ikehara's Theorem, we need to assure that traveling wave solutions are positive.

Lemma 2.4. *Assume that* (H0)–(H3) *hold and let* $\psi(n-ct)$ *be a non-negative traveling wave of* (1.1) *with* $c \leq c_*$ *satisfying* (1.4). *Then* $\psi(\xi) > 0$ *for* $\xi \in \mathbb{R}$.

Proof. Assume that there exists ξ_0 such that $\psi(\xi_0) = 0$. Without loss of generality, we may assume ξ_0 is the left-most point. According to $\psi(\xi) \geq 0$ for $\xi \in \mathbb{R}$, we can easily see that $\psi(\xi)$ attains the minimum at ξ_0 and $\psi'(\xi_0) = 0$. According to (H0) and (H1), it follows from (1.2) that

$$\int_0^\tau J_{m+1}(y) f(\psi(\xi_0 + cy))dy = 0,$$

which implies that $f(\psi(\xi_0 + cy)) = 0$ for any $y \in [0, \tau]$. Thus, choosing some sufficiently small number $y_0 > 0$, we can obtain $\psi(\xi_0 + cy_0) = 0$ according to the continuity of $\psi(\xi)$ and $c < 0$. This contradicts to the choice of ξ_0, and completes the proof. □

Lemma 2.5. *Assume that* (H1)–(H3) *hold and let $\psi(n - ct)$ be any non-negative traveling wave of* (1.1) *with $c \leq c_*$ and satisfy* (1.4). *Then there exists a positive number $\rho > 0$ such that $\psi(\xi) = O(e^{\rho\xi})$ as $\xi \to -\infty$.*

Proof. Since $f'(0)(a + \alpha + \beta) > 1$, there exists $\epsilon_0 > 0$ such that

$$A := (1 - \epsilon_0) f'(0)(a + \alpha + \beta) - 1 > 0.$$

For such $\epsilon_0 > 0$, there exist $\delta_1 > 0$ such that $f(u) \geq (1 - \epsilon_0) f'(0) u$ for any $u \in [0, \delta_1]$. Since $\psi(-\infty) = 0$, there exists $M > 0$ and $\forall \xi \leq -M$ such that $\psi(\xi) < \delta_1$. Integrating (1.2) from η to ξ with $\xi \leq -l - M$, it follows that

$$\begin{aligned}
& - c[\psi(\xi) - \psi(\eta)] \\
&= - \int_\eta^\xi \psi(x)dx + \sum_{i=1}^m a_i \int_\eta^\xi \int_0^\tau J_i(y) f(\psi(x - i + cy))\, dy\, dx \\
&\quad + \alpha \int_\eta^\xi \int_0^\tau J_{m+1}(y) f(\psi(x + cy))\, dy\, dx \\
&\quad + \sum_{j=1}^l \beta_j \int_\eta^\xi \int_0^\tau J_{m+1+j}(y) f(\psi(x + j + cy))\, dy\, dx \\
&\geq - \int_\eta^\xi \psi(x)dx + f'(0)(1 - \epsilon_0) \Big[\sum_{i=1}^m a_i \int_\eta^\xi \int_0^\tau J_i(y) \psi(x - i + cy)\, dy\, dx \\
&\quad + \alpha \int_\eta^\xi \int_0^\tau J_{m+1}(y) \psi(x + cy)\, dy\, dx \\
&\quad + \sum_{j=1}^l \beta_j \int_\eta^\xi \int_0^\tau J_{m+1+j}(y) \psi(x + j + cy)\, dy\, dx \Big] \\
&= A \int_\eta^\xi \psi(x)dx + f'(0)(1 - \epsilon_0) \Big[\alpha \int_\eta^\xi \int_0^\tau J_{m+1}(y)(\psi(x + cy) - \psi(x))\, dy\, dx \\
&\quad + \sum_{i=1}^m a_i \int_\eta^\xi \int_0^\tau J_i(y)(\psi(x - i + cy) - \psi(x))\, dy\, dx \\
&\quad + \sum_{j=1}^l \beta_j \int_\eta^\xi \int_0^\tau J_{m+1+j}(y)(\psi(x + j + cy) - \psi(x))\, dy\, dx \Big].
\end{aligned} \tag{2.2}$$

Since $\psi(x)$ is differentiable, we have

$$\begin{aligned}\int_\eta^\xi (\psi(x-i+cy)-\psi(x))dx &= \int_\eta^\xi \int_0^{-i+cy} \psi'(x+s)dsdx \\ &= \int_0^{-i+cy} (\psi(\xi+s)-\psi(\eta+s))ds.\end{aligned}$$

Similarly,

$$\int_\eta^\xi (\psi(x+cy)-\psi(x))dx = \int_0^{cy} (\psi(\xi+s)-\psi(\eta+s))ds,$$

$$\int_\eta^\xi (\psi(x+j+cy)-\psi(x))dx = \int_0^{j+cy} (\psi(\xi+s)-\psi(\eta+s))ds.$$

Letting $\eta \to -\infty$ in (2.2), we obtain

$$\begin{aligned}&A\int_{-\infty}^\xi \psi(x)dx \\ &\le -c\psi(\xi) - f'(0)(1-\epsilon_0)\Big[\sum_{i=1}^m a_i \int_0^\tau \int_0^{-i+cy} J_i(y)\psi(\xi+s)\,ds\,dy \\ &\quad + \alpha \int_0^\tau \int_0^{cy} J_{m+1}(y)\psi(\xi+s)\,ds\,dy \\ &\quad + \sum_{j=1}^l \beta_j \int_0^\tau \int_0^{j+cy} J_{m+1+j}(y)\psi(\xi+s)\,ds\,dy\Big].\end{aligned} \tag{2.3}$$

From (2.3), we know that $\int_{-\infty}^\xi \psi(x)dx < +\infty$. Letting $\Phi(\xi) = \int_{-\infty}^\xi \psi(x)dx$ and integrating (2.3) from $-\infty$ to ξ, we have

$$\begin{aligned}&A\int_{-\infty}^\xi \Phi(x)dx \\ &\le -c\Phi(\xi) - f'(0)(1-\epsilon_0)\Big[\sum_{i=1}^m a_i \int_0^\tau \int_0^{-i+cy} J_i(y)\Phi(\xi+s)\,ds\,dy \\ &\quad + \alpha \int_0^\tau \int_0^{cy} J_{m+1}(y)\Phi(\xi+s)\,ds\,dy \\ &\quad + \sum_{j=1}^l \beta_j \int_0^\tau \int_0^{j+cy} J_{m+1+j}(y)\Phi(\xi+s)\,ds\,dy\Big] \\ &\le \varrho\Phi(\xi+\kappa)\end{aligned} \tag{2.4}$$

for some $\kappa > 0$ and $\varrho > 0$ according to the monotonicity of $\Phi(\xi)$, Letting $\varpi > 0$ such that $\varrho < A\varpi$, and for $\xi \le -l-M$, it follows that

$$\Phi(\xi-\varpi) \le \frac{1}{\varpi}\int_{\xi-\varpi}^\xi \Phi(x)dx \le \frac{1}{\varpi}\int_{-\infty}^\xi \Phi(x)dx \le \frac{\varrho}{A\varpi}\Phi(\xi+\kappa). \tag{2.5}$$

Define $h(\xi) = \Phi(\xi)e^{-\rho\xi}$, where $\rho = \frac{1}{\rho+\varpi}\ln\frac{A\varpi}{\varrho} > 0$. Hence,

$$h(\xi-\varpi) = \Phi(\xi-\varpi)e^{-\rho(\xi-\varpi)} \le \frac{\varrho}{A\varpi}e^{\rho(\kappa+\varpi)}h(\xi+\kappa) = h(\xi+\kappa),$$

which implies h is bounded. Therefore, $\Phi(\xi) = O(e^{\rho\xi})$ when $\xi \to -\infty$. Integrating (1.2) from $-\infty$ to ξ, it follows from (H2) that

$$
\begin{aligned}
-c\psi(\xi) &= \sum_{i=1}^{m} a_i \int_0^\tau \int_{-\infty}^{\xi} J_i(y) f(\psi(x-i+cy))\,dx\,dy \\
&\quad + \alpha \int_0^\tau \int_{-\infty}^{\xi} J_{m+1}(y) f(\psi(x+cy))\,dx\,dy \\
&\quad + \sum_{j=1}^{l} \beta_j \int_0^\tau \int_{-\infty}^{\xi} J_{m+1+j}(y) f(\psi(x+j+cy))\,dx\,dy - \Phi(\xi) \\
&\le f'(0) \sum_{i=1}^{m} a_i \int_0^\tau \int_{-\infty}^{\xi} J_i(y) \psi(x-i+cy)\,dx\,dy \\
&\quad + \alpha f'(0) \int_0^\tau \int_{-\infty}^{\xi} J_{m+1}(y) \psi(x+cy)\,dx\,dy \\
&\quad + f'(0) \sum_{j=1}^{l} \beta_j \int_0^\tau \int_{-\infty}^{\xi} J_{m+1+j}(y) \psi(x+j+cy)\,dx\,dy - \Phi(\xi) \\
&= -\Phi(\xi) + f'(0) \sum_{i=1}^{m} a_i \int_0^\tau J_i(y) \Phi(\xi-i+cy) dy \\
&\quad + \alpha f'(0) \int_0^\tau J_{m+1}(y) \Phi(\xi+cy) dy \\
&\quad + f'(0) \sum_{j=1}^{l} \beta_j \int_0^\tau J_{m+1+j}(y) \Phi(\xi+j+cy) dy.
\end{aligned} \tag{2.6}
$$

Thus, we have $\psi(\xi) = O(e^{\rho\xi})$ when $\xi \to -\infty$. With the help of Ikehara's theorem, we obtain the asymptotic behavior of traveling wave solutions at $-\infty$. $\square$

Proposition 2.6. *Assume that* (H1)–(H3) *hold and let $\psi(n-ct)$ be any non-negative traveling wave of* (1.1) *with the wave speed $c \le c_*$ and satisfy* (1.4). *Then*

$$
\lim_{\xi \to -\infty} \frac{\psi(\xi)}{e^{\lambda_1 \xi}} \text{ exists for } c < c_*, \quad \lim_{\xi \to -\infty} \frac{\psi(\xi)}{|\xi| e^{\lambda_* \xi}} \text{ exists for } c = c_*. \tag{2.7}
$$

Proof. According to Lemma 2.5, we define the two-sided Laplace transform of ψ for $0 < \Re\lambda < \rho$,

$$
L(\lambda) \equiv \int_{-\infty}^{+\infty} \psi(x) e^{-\lambda x} dx.
$$

We claim that $L(\lambda)$ is analytic for $0 < \Re\lambda < \lambda_1$ and has a singularity at $\lambda = \lambda_1$. Note that

$$
\begin{aligned}
&- c\psi'(\xi) + \psi(\xi) - f'(0) \sum_{i=1}^{m} a_i \int_0^\tau J_i(y) \psi(\xi-i+cy) dy \\
&- \alpha f'(0) \int_0^\tau J_{m+1}(y) \psi(\xi+cy) dy
\end{aligned}
$$

$$
\begin{aligned}
&- f'(0)\sum_{j=1}^{l}\beta_j\int_0^\tau J_im+1+j\psi(\xi+j+cy)dy\\
&=\sum_{i=1}^{m}a_i\int_0^\tau J_i(y)[f(\psi(\xi-i+cy))-f'(0)\psi(\xi-i+cy)]dy\\
&\quad+\alpha\int_0^\tau J_{m+1}(y)[f(\psi(\xi+cy))-f'(0)\psi(\xi-i+cy)]dy\\
&\quad+\sum_{j=1}^{l}\beta_j\int_0^\tau J_{m+1+j}[f(\psi(\xi+j+cy))-f'(0)\psi(\xi-i+cy)]dy\\
&=: Q(\psi)(\xi).
\end{aligned}
$$

Multiplying the two sides of the above equality by $e^{-\lambda\xi}$ and integrating ξ on $\mathbb{R}$, we obtain

$$
\Delta(c,\lambda)L(\lambda)=\int_{-\infty}^{+\infty}e^{-\lambda x}Q(\psi)(x)dx. \tag{2.8}
$$

We know that the left-hand side of (2.8) is analytic for $0<\Re\lambda<\rho$. According to (H3), for any $\overline{u}>0$, there exists $\overline{d}>0$ such that $f(u)\geq f'(0)u-\overline{d}u^{\sigma+1}$, for all $u\in[0,\overline{u}]$, where $\overline{d}:=\max\{d,\gamma^{-(\sigma+1)}\max_{u\in[\gamma,\overline{u}]}\{f'(0)u-f(u)\}\}$. Thus,

$$
\begin{aligned}
&-\overline{d}\Big[\sum_{i=1}^{m}a_i\int_0^\tau J_i(y)\psi^{\sigma+1}(\xi-i+cy)dy+\alpha\int_0^\tau J_{m+1}(y)\psi^{\sigma+1}(\xi+cy)dy\\
&+\sum_{j=1}^{l}\beta_j\int_0^\tau J_{m+1+j}\psi^{\sigma+1}(\xi+j+cy)dy\Big]\\
&\leq Q(\psi)(\xi)\leq 0.
\end{aligned} \tag{2.9}
$$

Choose $\upsilon>0$ such that $\frac{\upsilon}{\sigma}<\rho$. Then for any $\Re\lambda\in(0,\rho+\upsilon)$, we have

$$
\begin{aligned}
&\Big|\int_{-\infty}^{+\infty}e^{-\lambda x}Q(\psi)(x)dx\Big|\\
&\leq\overline{d}\int_{-\infty}^{+\infty}e^{-\lambda\xi}\Big[\sum_{i=1}^{m}a_i\int_0^\tau J_i(y)\psi^{\sigma+1}(\xi-i+cy)dy\\
&\quad+\alpha\int_0^\tau J_{m+1}(y)\psi^{\sigma+1}(\xi+cy)dy\\
&\quad+\sum_{j=1}^{l}\beta_j\int_0^\tau J_{m+1+j}\psi^{\sigma+1}(\xi+j+cy)dy\Big]d\xi\\
&=\overline{d}[\sum_{i=1}^{m}a_i\int_0^\tau J_i(y)e^{\lambda(-i+cy)}dy+\alpha\int_0^\tau J_{m+1}(y)e^{\lambda cy}dy\\
&\quad+\sum_{j=1}^{l}\beta_j\int_0^\tau J_{m+1+j}e^{\lambda(j+cy)}dy]\int_{-\infty}^{+\infty}e^{-\lambda x}\psi^{\sigma+1}(x)dx\\
&\leq\overline{d}\Big[\sum_{i=1}^{m}a_i\int_0^\tau J_i(y)e^{\lambda(-i+cy)}dy+\alpha\int_0^\tau J_{m+1}(y)e^{\lambda cy}dy
\end{aligned}
$$

$$+\sum_{j=1}^{l}\beta_j\int_0^\tau J_{m+1+j}e^{\lambda(j+cy)}dy\Big]L(\lambda-\upsilon)\Big(\sup_{\xi\in\mathbb{R}}\psi(\xi)e^{\frac{-\upsilon\xi}{\sigma}}\Big)^\sigma<+\infty.$$

We use properties of Laplace transform [22, p. 58]. Since $\psi>0$ according to Lemma 2.4, there exists a real number D such that $L(\lambda)$ is analytic for $0<\Re\lambda<D$ and $L(\lambda)$ has a singularity at $\lambda=D$. Thus, when $c\le c_*$, $L(\lambda)$ is analytic for $\Re\lambda\in(0,\lambda_1)$ and $L(\lambda)$ has a singularity at $\lambda=\lambda_1$.

According to (2.8), we have

$$F(\lambda):=\int_{-\infty}^0\psi(x)e^{-\lambda x}dx=\frac{\int_{-\infty}^{+\infty}e^{-\lambda x}Q(\psi)(x)dx}{\Delta(c,\lambda)}-\int_0^{+\infty}\psi(x)e^{-\lambda x}dx.$$

Define $H(\lambda)=F(\lambda)(\lambda_1-\lambda)^{k+1}$, where $k=0$ if $c<c_*$ and $k=1$ if $c=c_*$.

We claim that $H(\lambda)$ is analytic in the strip $S:=\{\lambda\in C|0<\Re\lambda\le\lambda_1\}$. Indeed, define

$$G(\lambda)=\frac{\int_{-\infty}^{+\infty}e^{-\lambda x}Q(\psi)(x)dx}{\Delta(c,\lambda)/(\lambda_1-\lambda)^{k+1}}=L(\lambda)(\lambda_1-\lambda)^{k+1}.$$

It is easily seen that $G(\lambda)$ is analytic in the strip $\{\lambda\in C|0<\Re\lambda<\lambda_1\}$.

To prove that $G(\lambda)$ is analytic for $\Re\lambda=\lambda_1$, we only need to prove that $\Delta(c,\lambda)=0$ does not have any zero with $\Re\lambda=\lambda_1$ other than $\lambda=\lambda_1$. Indeed, letting $\lambda=\lambda_1+i\widetilde{\lambda}$, we have

$$\begin{aligned}0=-c\widetilde{\lambda}+1-f'(0)\Big[&\sum_{k=1}^{m}a_k\int_0^\tau J_k(y)e^{\lambda_1(-k+cy)}\cos(-k+cy)\widetilde{\lambda}dy\\&+\alpha\int_0^\tau J_{m+1}(y)e^{\lambda_1cy}\cos cy\widetilde{\lambda}dy\\&+\sum_{j=1}^{l}\beta_j\int_0^\tau J_{m+1+j}(y)e^{\lambda_1(j+cy)}\cos(j+cy)\widetilde{\lambda}dy\Big]\end{aligned}\tag{2.10}$$

and

$$\begin{aligned}0=-c\widetilde{\lambda}-f'(0)\Big[&\sum_{k=1}^{m}a_k\int_0^\tau J_k(y)e^{\lambda_1(-k+cy)}\sin(-k+cy)\widetilde{\lambda}dy\\&+\alpha\int_0^\tau J_{m+1}(y)e^{\lambda_1cy}\sin cy\widetilde{\lambda}dy\\&+\sum_{j=1}^{l}\beta_j\int_0^\tau J_{m+1+j}(y)e^{\lambda_1(j+cy)}\sin(j+cy)\widetilde{\lambda}dy\Big].\end{aligned}\tag{2.11}$$

It follows from (2.10) and (2.11) that $\widetilde{\lambda}=0$.

According to the above argument, $G(\lambda)$ is analytic in S, and $H(\lambda)$ is also analytic in S. Moreover, we claim that $H(\lambda_1)>0$. Indeed, notice that $H(\lambda_1)=G(\lambda_1)$. On the other hand, $\int_{-\infty}^{+\infty}e^{-\lambda_1x}Q(\psi)(x)dx<0$ according to (2.9) and $\lim_{\lambda\to\lambda_1^-}\Delta(c,\lambda)/(\lambda_1-\lambda)^{k+1}<0$ according to Lemma 2.1.

Since $\psi(\xi)$ may be non-monotone, Ikehara's Theorem could be directly used. Thus, we need to make a function transformation, i.e., $\widehat{\psi}(\xi)=\psi(\xi)e^{p\xi}$, where

$p = \frac{-1}{c} > 0$. It follows from (1.2) that

$$\begin{aligned}\widehat{\psi}'(\xi) = \frac{-1}{c}\Big[\sum_{i=1}^{m}\int_0^\tau a_i J_i(y) f(\psi(\xi - i + cy))dy + \alpha\int_0^\tau J_{m+1}(y) f(\psi(\xi + cy))dy \\ + \sum_{j=1}^{l}\beta_j\int_0^\tau J_{m+1+j}(y) f(\psi(\xi + j + cy))dy\Big]e^{p\xi} > 0.\end{aligned}$$

Therefore, $\widehat{\psi}(\xi)$ is increasing and $\widehat{\psi}(\xi) > 0$. Now we apply the Ikehara's Theorem to $\widehat{\psi}(\xi)$. Let

$$\widehat{F}(\lambda) := \int_{-\infty}^0 \widehat{\psi}(x)e^{-\lambda x}dx = F(\lambda - p).$$

Then

$$\widehat{F}(\lambda) = \frac{\widehat{H}(\lambda)}{((\lambda_1 + p) - \lambda)^{k+1}},$$

where $\widehat{H}(\lambda) = H(\lambda - p)$ is analytic for $p < \Re\lambda \le \lambda_1 + p$ and $\widehat{H}(\lambda_1 + p) = H(\lambda_1) > 0$. According to Remark 2.3, the limits exists and

$$\lim_{\xi\to-\infty}\frac{\widehat{\psi}(\xi)}{|\xi|^k e^{(\lambda_1+p)\xi}} = \lim_{\xi\to-\infty}\frac{\psi(\xi)}{|\xi|^k e^{\lambda_1\xi}},$$

i.e.,

$$\lim_{\xi\to-\infty}\frac{\psi(\xi)}{e^{\lambda_1\xi}} \text{ exists for } c < c_*, \quad \lim_{\xi\to-\infty}\frac{\psi(\xi)}{|\xi|e^{\lambda_*\xi}} \text{ exists for } c = c_*.$$

This completes the proof. □

3. Uniqueness of traveling wave solutions

In this section, we show the following unique result of traveling wave solutions of (1.1).

Theorem 3.1. *Assume that* (H1)–(H3) *hold. Let $\psi(n - ct)$ be a traveling wave of* (1.1) *with the wave speed $c \le c_*$, which is given in Proposition 1.1. If $\phi(n - ct)$ is any non-negative traveling wave of* (1.1) *with the same wave speed c satisfying* (1.4)*, then ϕ is a translation of ψ; more precisely, there exists $\bar{\xi} \in \mathbb{R}$ such that $\phi(n - ct) = \psi(n - ct + \bar{\xi})$.*

Proof. From Proposition 2.6, there exist two positive numbers ϑ_1 and ϑ_2 such that

$$\lim_{\xi\to-\infty}\frac{\phi(\xi)}{|\xi|^k e^{\lambda_1\xi}} = \vartheta_1, \quad \lim_{\xi\to-\infty}\frac{\psi(\xi)}{|\xi|^k e^{\lambda_1\xi}} = \vartheta_2$$

where $k = 0$ for $c < c_*$, and $k = 1$ for $c = c_*$. For $\epsilon > 0$, define

$$\omega(\xi) := \frac{\phi(\xi) - \psi(\xi + \bar{\xi})}{e^{\lambda_1\xi}} \text{ for } c < c_*, \quad \omega_\epsilon(\xi) := \frac{\phi(\xi) - \psi(\xi + \bar{\xi})}{(\epsilon|\xi| + 1)e^{\lambda_*\xi}} \text{ for } c = c_*, \tag{3.1}$$

where $\bar{\xi} = \frac{1}{\lambda_1}\ln\frac{\vartheta_1}{\vartheta_2}$. Then $\omega(\pm\infty) = 0$ and $\omega_\epsilon(\pm\infty) = 0$.

First, we consider $c < c_*$. Since $\omega(\pm\infty) = 0$, $\sup_{\xi\in\mathbb{R}}\{\omega(\xi)\}$ and $\inf_{\xi\in\mathbb{R}}\{\omega(\xi)\}$ are finite. Without loss of generality, we assume $\sup_{\xi\in\mathbb{R}}\{\omega(\xi)\} \ge |\inf_{\xi\in\mathbb{R}}\{\omega(\xi)\}|$. If $\omega(\xi) \not\equiv 0$, there exists ξ_0 such that

$$\omega(\xi_0) = \max_{\xi\in\mathbb{R}}\omega(\xi) = \sup_{\xi\in\mathbb{R}}\omega(\xi) > 0, \quad \omega'(\xi_0) = 0.$$

We claim that for all $i,j\in\mathbb{Z}$, we have

$$\omega(\xi_0-i+cy)=\omega(\xi_0+cy)=\omega(\xi_0+j+cy)=\omega(\xi_0)$$

for $y\in[0,\tau]$. Suppose on the contrary that one of three inequalities $\omega(\xi_0-i+cy)<\omega(\xi_0)$, $\omega(\xi_0+cy)<\omega(\xi_0)$ and $\omega(\xi_0+j+cy)<\omega(\xi_0)$ for some i_0, j_0 must hold. According to (1.2), (3.1) and (H2), we obtain

$$\begin{aligned}
0&=c\omega'(\xi_0)\\
&=-c\lambda_1\omega(\xi_0)+\omega(\xi_0)\\
&\quad-e^{-\lambda_1\xi_0}\sum_{i=1}^{m}a_i\int_0^\tau J_i(y)[f(\phi(\xi_0-i+cy))-f(\psi(\xi_0+\overline{\xi}-i+cy))]dy\\
&\quad-e^{-\lambda_1\xi_0}\alpha\int_0^\tau J_{m+1}(y)[f(\phi(\xi_0+cy))-f(\psi(\xi_0+\overline{\xi}+cy))]dy\\
&\quad-e^{-\lambda_1\xi_0}\sum_{j=1}^{l}\beta_j\int_0^\tau J_{m+1+j}(y)[f(\phi(\xi_0+j+cy))-f(\psi(\xi_0+\overline{\xi}+j+cy))]dy\\
&>-c\lambda_1\omega(\xi_0)+\omega(\xi_0)-f'(0)\omega(\xi_0)\Big[\sum_{i=1}^{m}a_i\int_0^\tau J_i(y)e^{\lambda_1(-i+cy)}dy\\
&\quad+\alpha\int_0^\tau J_{m+1}(y)e^{\lambda_1 cy}dy+\sum_{j=1}^{l}\beta_j\int_0^\tau J_{m+1+j}(y)e^{\lambda_1(cy+j)}dy\Big]\\
&=-\omega(\xi_0)\triangle(c,\lambda_1)=0,
\end{aligned}$$

which is a contradiction. Thus, $\omega(\xi_0+cy_0)=\omega(\xi_0)$ also holds for $y_0\in(0,\tau)$. Again by bootstrapping, $\omega(\xi_0+kcy_0)=\omega(\xi_0)$ for all $k\in\mathbb{Z}$ and $\omega(+\infty)=0$. Therefore, we have $\phi(\xi)\equiv\psi(\xi+\overline{\xi})$ for $\xi\in\mathbb{R}$, which contradicts to $\omega(\xi)\not\equiv 0$.

Next, we consider $c=c_*$. Assume $\sup_{\xi\in\mathbb{R}}\{\omega(\xi)\}\ge|\inf_{\xi\in\mathbb{R}}\{\omega(\xi)\}|$. If $\omega_\epsilon(\xi)\neq 0$, there exists ξ_0^ϵ such that

$$\omega_\epsilon(\xi_0^\epsilon)=\max_{\xi\in\mathbb{R}}\{\omega_\epsilon(\xi)\}=\sup_{\xi\in\mathbb{R}}\{\omega_\epsilon(\xi)\}>0,\quad \omega_\epsilon'(\xi_0^\epsilon)=0.$$

Now we divide this part into three cases:

Case 1: Suppose that $\xi_0^\epsilon\to+\infty$ as $\epsilon\to 0$. It follows from (3.1) and (1.2) that

$$\begin{aligned}
&c_*[\phi'(\xi_0^\epsilon)-\psi'(\xi_0^\epsilon+\overline{\xi})]\\
&=\phi(\xi_0^\epsilon)-\sum_{i=1}^{m}a_i\int_0^\tau J_i(y)[f(\phi(\xi_0^\epsilon-i+c_*y))-f(\psi(\xi_0^\epsilon+\overline{\xi}-i+c_*y))]dy\\
&\quad-\alpha\int_0^\tau J_{m+1}(y)[f(\phi(\xi_0^\epsilon+c_*y))-f(\psi(\xi_0^\epsilon+\overline{\xi}+c_*y))]dy-\psi(\xi_0^\epsilon+\overline{\xi})\\
&\quad-\sum_{j=1}^{l}\beta_j\int_0^\tau J_{m+1+j}(y)f(\phi(\xi_0^\epsilon+j+c_*y))-f(\psi(\xi_0^\epsilon+\overline{\xi}+j+c_*y))]dy.
\end{aligned}$$

We claim that for all $i,j\in\mathbb{Z}$,

$$\omega_\epsilon(\xi_0^\epsilon-i+c_*y)=\omega_\epsilon(\xi_0^\epsilon+c_*y)=\omega_\epsilon(\xi_0^\epsilon+j+c_*y)=\omega_\epsilon(\xi_0^\epsilon)$$

for $y\in[0,\tau]$. Suppose for the contrary that one of three inequalities $\omega_\epsilon(\xi_0^\epsilon-i+c_*y)<\omega_\epsilon(\xi_0^\epsilon)$, $\omega_\epsilon(\xi_0^\epsilon+c_*y)<\omega_\epsilon(\xi_0^\epsilon)$ and $\omega_\epsilon(\xi_0^\epsilon+j+c_*y)<\omega_\epsilon(\xi_0^\epsilon)$ for some i_0 and

j_0 must hold. Choose $\epsilon > 0$ sufficiently small such that $\xi_0^\epsilon - m + c_*\tau > 0$. Thus,

$$
\begin{aligned}
& - c_*\omega_\epsilon(\xi_0^\epsilon)\epsilon - c_*\lambda_*\omega_\epsilon(\xi_0^\epsilon)(\epsilon\xi_0^\epsilon + 1) \\
& \le -\omega_\epsilon(\xi_0^\epsilon)(\epsilon|\xi_0^\epsilon| + 1) \\
& \quad + f'(0)\sum_{i=1}^m a_i \int_0^\tau J_i(y)e^{\lambda_*(-i+c_*y)}[\epsilon|\xi_0^\epsilon - i + c_*y| + 1]\omega_\epsilon(\xi_0^\epsilon - i + c_*y)dy \\
& \quad + \alpha f'(0)\int_0^\tau J_{m+1}(y)e^{\lambda_*c_*y}[\epsilon|\xi_0^\epsilon + c_*y| + 1]\omega_\epsilon(\xi_0^\epsilon + c_*y)dy \\
& \quad + f'(0)\sum_{j=1}^l \beta_j \int_0^\tau J_{m+1+j}(y)e^{\lambda_*(j+c_*y)}[\epsilon|\xi_0^\epsilon + j + c_*y| + 1]\omega_\epsilon(\xi_0^\epsilon + j + c_*y)dy \\
& < -\omega_\epsilon(\xi_0^\epsilon)(\epsilon|\xi_0^\epsilon| + 1) + f'(0)\sum_{i=1}^m a_i \int_0^\tau J_i(y)e^{\lambda_*(-i+c_*y)}[\epsilon|\xi_0^\epsilon - i + c_*y| + 1]\omega_\epsilon(\xi_0^\epsilon)dy \\
& \quad + \alpha f'(0)\int_0^\tau J_{m+1}(y)e^{\lambda_*c_*y}[\epsilon|\xi_0^\epsilon + c_*y| + 1]\omega_\epsilon(\xi_0^\epsilon)dy \\
& \quad + f'(0)\sum_{j=1}^l \beta_j \int_0^\tau J_{m+1+j}(y)e^{\lambda_*(j+c_*y)}[\epsilon|\xi_0^\epsilon + j + c_*y| + 1]\omega_\epsilon(\xi_0^\epsilon)dy,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
& - c_*\omega_\epsilon(\xi_0^\epsilon)\epsilon + \omega_\epsilon(\xi_0^\epsilon)(\epsilon\xi_0^\epsilon + 1)\Delta(c_*;\lambda_*) \\
& < f'(0)\sum_{i=1}^m a_i \int_0^\tau J_i(y)e^{\lambda_*(-i+c_*y)}\epsilon(-i + c_*y)\omega_\epsilon(\xi_0^\epsilon)dy \\
& \quad + \alpha f'(0)\int_0^\tau J_{m+1}(y)e^{\lambda_*(c_*y)}\epsilon c_*y\omega_\epsilon(\xi_0^\epsilon)dy \\
& \quad + f'(0)\sum_{j=1}^l \beta_j \int_0^\tau J_{m+1+j}(y)e^{\lambda_*(j+c_*y)}[\epsilon(j + c_*y)]\omega_\epsilon(\xi_0^\epsilon)dy.
\end{aligned}
\tag{3.2}
$$

This contradicts $\frac{\partial\Delta(c,\lambda)}{\partial\lambda}|_{c=c_*,\lambda=\lambda_*} = 0$. Repeating the arguments, we have $\omega_\epsilon(\xi_0^\epsilon) = \omega_\epsilon(\xi_0^\epsilon + kc_*y_0)$ for all $k \in Z$ and some $y_0 \in (0,\tau)$. It follows from that $\omega_\epsilon(+\infty) = 0$, we can obtain $\phi(\xi) \equiv \psi(\xi + \overline{\xi})$ for $\xi \in \mathbb{R}$, which contradicts $\omega_\epsilon(\xi) \not\equiv 0$.

Similar to the process in [3], $\phi(\xi) \equiv \psi(\xi + \overline{\xi})$ for $\xi \in \mathbb{R}$ still holds for

Case 2: Suppose $\xi_0^\epsilon \to -\infty$ as $\epsilon \to 0$ and **Case 3:** Suppose ξ_0^ϵ is bounded. This completes the proof. □

Acknowledgements. Zhi-Xian Yu was supported by the National Natural Science Foundation of China, by the Shanghai Leading Academic Discipline Project(No. XTKX2012), by the Innovation Program of Shanghai Municipal Education Commission (No.14YZ096), and by the Hujiang Foundation of China (B14005).

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Hui-Ling Zhou
College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China
E-mail address: `hlzhou@163.com`

Zhixian Yu (corresponding author)
College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China
E-mail address: `zxyu0902@163.com`