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ON SINGULAR SOLUTIONS OF A MAGNETOHYDRODYNAMIC NONLINEAR BOUNDARY LAYER EQUATION

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Abstract. This paper concerns the singular solutions of the equation

$$f''' + \kappa f f'' - \beta f'^2 = 0,$$

where $\beta < 0$ and $\kappa = 0$ or 1. This equation arises when modelling heat transfer past a vertical flat plate embedded in a saturated porous medium with an applied magnetic field. After suitable normalization, f' represents the velocity parallel to the surface or the non–dimensional fluid temperature. Our interest is in solutions which develop a singularity at some point (the blow-up point). In particular, we shall examine in detail the behavior of f near the blow-up point.

1. Introduction

We investigate a one layer model of magnetohydrodynamic (MHD) flow and heat transfer problems, which are of considerable practical interest. Such a system is important in understanding a variety of geophysical, astrophysical, chemical engineering and metallurgical processes (cooling of continuous strips or filaments, purification of molten metals, etc.).

Much progress has been made during the previous years in the development of MHD nonlinear boundary layer equations. Pavlov [22] was the first who examined the MHD flow over a stretching wall in an electrically conducting fluid, with an uniform magnetic field. Further studies are those of Chakrabarti and Gupta [9], Vajravelu [30], Takhar et al. [28, 27], Kumari et al. [19], Andersson et al. [1], Watanabe and Pop [31] and Sobha and Ramakrishna [26]. In particular, paper [26] focused mainly on the effect of magnetic field on temperature distribution.

Following the work by Sobha and Ramakrishna, the study of similarity solutions of Prandtl's equation for the steady two-dimensional heat transfer past a vertical plate embedded in a porous medium with an applied magnetic field leads to the differential equation (see Appendix)

$$f''' + \frac{1+m}{2} f f'' - m f'^2 = 0, \tag{1.1}$$

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subject to the boundary wall conditions

$$f(0) = 0, \quad f'(0) = \omega, \tag{1.2}$$

and the boundary (free) condition

$$\lim_{\eta\to\infty} f'(\eta) = f'(\infty) = 0. \tag{1.3}$$

The parameter m is related to the temperature distribution prescribed on the plate and ω is a magnetic parameter. For a physical consideration m and ω satisfy $-\frac{1}{3} \leq m \leq 1$ and $0 < \omega < 1$. In the absence of the magnetic field the parameter ω is equal to 1 (see for example [6], [11]). However, for the mathematical analysis we will be concerned with $-1 \leq m < 0$ and with every value of ω.

Problem (1.1)-(1.3) also arises in physically different contexts in fluid mechanics, as boundary layer flow on permeable surface with mass transfer parameter $a \neq 0$ [10], [20]. In this case initial conditions (1.2) take the form

$$f(0) = a, \quad f'(0) = 1. \tag{1.4}$$

The real number a is also referred to as the suction/injection parameter. The case $a > 0$ corresponds to suction and $a < 0$ to injection of the fluid. With $m = 0$ equation (1.1) is called the Blasius equation [7].

Problem (1.1), (1.3), (1.4) has been the subject of intensive study. Results concerning problem (1.1)–(1.3) can be found in [11] by Cheng and Minkowycz, for a different physical problem, in which the numerical solution has been performed in the case where $-\frac{1}{3} < m < 0$. For $m > -\frac{1}{2}$ numerical investigations are given in the works [17] by Ingham and Brown and [2] by Banks. The mathematical analysis is also considered in [17]. Some analytical results have been obtained by Belhachmi et al. [6]. The authors showed non-existence of solutions to (1.1)–(1.3) for $m \leq -\frac{1}{2}$. They also proved that this problem has an infinite number of solutions when $m = -\frac{1}{3}$ and uniqueness holds for $0 \leq m \leq \frac{1}{3}$.

Recently, multiple solutions of (1.1), (1.3), (1.4) were obtained by Guedda [14], for different values $-\frac{1}{3} < m < 0$. In particular, it is proved that for any $\tau > -\frac{m+1}{2}a$ the local solution to (1.1), (1.3) such that $f''(0) = \tau$ is global and satisfies

$$f'(\infty) = 0, \quad f(\eta) \sim L\eta^{\frac{1+m}{1-m}}, \tag{1.5}$$

as $\eta \to \infty$, for some $L > 0$. The case $-\frac{1}{2} < m < 0$ is also studied provided that $a \geq \sqrt{\frac{1}{m+1}}$.

In a recent paper [8] Brighi and Sari have conducted a discussion of the existence and the non-existence of solutions to problem (1.1), (1.3), (1.4), where the parameters a and m are taken on the whole range $(-\infty, \infty)$. Using dynamical system theories, the authors proved, among other results, that for $0 \leq m \leq 1$ and for any $a \in \mathbb{R}$ problem (1.1), (1.3), (1.4) has one and only one solution while for $m > 1$ multiple solutions exist included one and only one concave solution. For $-1 < m < -1/2$ the authors proved that there exists $a_\star^+ > 0$ such that the problem has no solution for any $a < a_\star^+$, while for $m < -1$ there exists $a_\star^- < 0$ such that a solution exists if and only if $a < a_\star^-$.

Very recently, asymptotic properties of global unbounded solutions to a class of degenerate nonlinear differential boundary layer equations are obtained by Guedda and Kersner [15]. In particular, it is proved that any global solution to (1.1), where $-1 < m < 0$, such that $f(\infty) = \infty$, satisfies (1.5).

Based on these previous results, we may conclude (see below) that for $-1 < m < -1/2$ and $a < a_\star^+$ any local solution to (1.1), (1.4) blows up at a finite point.

The problem of the blowing–up solutions to boundary layer equations was first mentioned by Coppel [12]. The author classified all solutions of the Falkner-Scan differential equation [13]

$$f''' + ff'' + \beta(1 - f'^2) = 0, \tag{1.6}$$

where $0 \le \beta < 2$. In particular, it is shown that for $0 \le \beta < 1/2$, any blowing-up solution satisfies $f'(\eta) \sim -(2-\beta)f(\eta)^2/6$ as $\eta \to \eta_c$, where $0 < \eta_c < \infty$ is the blow–up point of f.

The initial value problem, with $m = 0$ or $\beta = 0$,

$$\begin{gathered} f''' + \frac{1}{2}ff'' = 0, \\ f(0) = a, \quad f'(0) = b, \quad f''(0) = \tau, \end{gathered} \tag{1.7}$$

where $a \in \mathbb{R}$, $b > 0$ and $\tau \le 0$, has been considered by Belhachmi et al. [5]. Among other results, it is shown that there exists $\tau^\star \le 0$ such that the unique solution to the Blasius problem (1.7) is not global, for any $\tau < \tau^\star$.

Recently, the absence of global solutions to

$$\begin{gathered} f''' + ff'' = 0, \\ f(0) = 1, \quad f'(0) = 0, \quad f''(0) = \tau, \end{gathered} \tag{1.8}$$

has been reconsidered in detail by Ishimura and Matsui [18]. By introducing the function v such that $v(-f) = f'^2$, the authors proved that for any $\tau < 0$, the solution f to (1.8) blows up at a some point $\eta_c = \eta_c(\tau)$, and that the blow-up coordinate f'/f^2 tends to $-1/3$ as $\eta \to \eta_c$. Then they deduced that

$$\lim_{\eta \uparrow \eta_c} (\eta_c - \eta) f(\eta) = -3.$$

In this work we extend the results of [18] to equation (1.1). Since the case $m = 0$ were investigated, we will suppose $-1 \le m < 0$. Let us note that if $m > -1$ the new function

$$\eta \mapsto \sqrt{\frac{m+1}{2}}\, f\Big(\sqrt{\frac{2}{m+1}}\eta\Big)$$

satisfies

$$f''' + ff'' - \beta f'^2 = 0, \tag{1.9}$$

where $\beta = \frac{2m}{m+1}$. If $m = -1$ or $\kappa = 0$ equation (1.1) reads

$$f''' + f'^2 = 0. \tag{1.10}$$

So, we shall be concerned with the ordinary differential equation

$$f''' + \kappa ff'' - \beta f'^2 = 0, \tag{1.11}$$

where $\beta < 0$ and $\kappa = 1$ or $\beta = -1$ and $\kappa = 0$ ($m = -1$). The initial conditions which we wish to consider are

$$f(0) = a, \quad f'(0) = \omega, \quad f''(0) = \tau, \tag{1.12}$$

where a, ω are real numbers, $\tau < 0$ and $\beta < 0$.

2. Existence of singular solutions for $\beta < 0$

In this section we are interested in conditions for non-global existence of solutions to (1.11), (1.12). First we note, according to a standard theory of ODE, that problem (1.11), (1.12) has a unique local solution f_τ defined on the maximal interval $[0, \eta_c), \eta_c \le \infty$. This solution is of class C^∞ on $[0, \eta_c)$ and satisfies

$$f_\tau''(\eta) + \kappa f_\tau(\eta) f_\tau'(\eta) = \tau + \kappa a\omega + (\kappa + \beta) \int_0^\eta f_\tau'(s)^2 ds, \tag{2.1}$$

for all $\eta < \eta_c$. If, in addition, η_c is finite $\lim_{\eta \uparrow \eta_c} |f_\tau(\eta)| + |f_\tau'(\eta)| + |f_\tau''(\eta)| = \infty$. In fact, following the work [12], the existence time η_c is characterized by the following result.

Proposition 2.1. *Let f_τ be the unique local solution to (1.11), (1.12), where $\kappa \in \{0, 1\}, \beta < 0$ and $\tau \in \mathbb{R}$. Assume $\eta_c < \infty$. Then*

$$\lim_{\eta \uparrow \eta_c} f_\tau(\eta) = -\infty.$$

Proof. First we show that $\sup_{[0,\eta_c)} |f_\tau(\eta)| = \infty$. We adapt an idea due to [12]. Suppose that this is not the case. Assume that $\kappa \neq -\beta$. From (2.1) we deduce

$$(\kappa+\beta)\big[f_\tau'(\eta) + \frac{\kappa}{2} f_\tau^2(\eta) - (\tau + \kappa a\omega)\eta - \omega - \frac{\kappa}{2} a^2\big] = (\kappa+\beta)^2 \int_0^\eta \int_0^t f_\tau'(s)^2 \, ds \, dt, \tag{2.2}$$

for all $\eta < \eta_c$. Because the right–hand side of (2.2) is positive and monotonic the left–hand side of (2.2), and (therefore) $(\kappa+\beta) f_\tau'(\eta)$ tends to ∞ as $\eta \to \eta_c$. Otherwise, f_τ' is bounded and by (2.1) f_τ'' is also bounded, which is absurd. Consequently, the function

$$v(\eta) = \int_0^\eta \int_0^t {f_\tau'}^2(s) \, ds \, dt$$

goes to ∞ as $\eta \to \eta_c$ and satisfies

$$\lim_{\eta \to \eta_c} v''(\eta) = \infty$$

and

$$v'' \le 2(\kappa + \beta)^2 v^2$$

on $(\eta_c - \varepsilon, \eta_c), \varepsilon > 0$ small. The last differential inequality yields, for some constant $C_1 > 0$,

$$v(\eta) \ge C_1 (\eta_c - \eta)^{-2}$$

as $\eta \to \eta_c$. Returning to (2.2) we deduce

$$(\kappa + \beta) f_\tau'(\eta) \ge C_2 (\eta_c - \eta)^{-2}, \quad C_2 = \text{const.} > 0,$$

and this implies, after integration, that $(\kappa + \beta) f_\tau(\eta)$ is not bounded as $\eta \to \eta_c$, a contradiction. Next we use the equation of f_τ to deduce

$$(f_\tau'' e^{\kappa F})' = \beta e^{\kappa F} {f_\tau'}^2, \tag{2.3}$$

where $F(\eta) = \int_0^\eta f(s) ds$, and (then) f_τ'' has at most one zero. Therefore, f_τ is monotonic on $(\eta_c - \varepsilon, \eta_c)$, $\varepsilon > 0$ small enough, and then $|f_\tau(\eta)| \to \infty$ as $\eta \to \eta_c$.

For $\kappa = -\beta$ and then $\kappa = 1$, we infer

$$f_\tau'' + f_\tau f_\tau' = \tau + a\omega,$$

$$f_\tau' + \frac{1}{2} f_\tau^2 = (\tau + a\omega)\eta + \omega + \frac{1}{2} a^2.$$

Hence, if f_τ is bounded we deduce that f'_τ and f''_τ are bounded, a contradiction.

It remains to prove that $f_\tau(\eta)$ approaches $-\infty$ as η approaches η_c. Because f_τ is monotonic on some (η_0, η_c) we assume that $f'_\tau(\eta)$ is nonnegative for all $\eta_0 < \eta < \eta_c$ and $f_\tau(\eta) \to \infty$ as $\eta \to \eta_c$. Define the energy-type function

$$E = \frac{1}{2} {f''_\tau}^2 - \frac{\beta}{3} {f'_\tau}^3, \tag{2.4}$$

which satisfies

$$E' = -\kappa f_\tau {f''_\tau}^2.$$

Thus f'_τ is bounded and then f_τ is also bounded on (η_0, η_c), which is impossible. Consequently, $f_\tau(\eta)$ goes to $-\infty$ as $\eta \to \eta_c$. □

The following result indicates that f_τ has a singularity for any $\tau < 0$.

Theorem 2.2. *Let $\omega \le 0, a \in \mathbb{R}$. Assume that $\kappa \in \{0, 1\}, \beta < 0$. For any $\tau < 0$ η_c is finite and the function f_τ satisfies*

$$\lim_{\eta \uparrow \eta_c} f_\tau(\eta) = -\infty.$$

Proof. First we assume that $\omega < 0$. We suppose for the sake of contradiction that f_τ is global; that is $\eta_c = \infty$. Because $f'_\gamma(\eta) < \omega$, for all $\eta > 0$, $f_\tau(\eta) < a + \omega\eta$ and tends to $-\infty$ as $\eta \to \infty$. Together with (1.11) the energy-type function E defined by (2.4) is monotonic increasing on (η_0, ∞), where $\eta_0 = \max\{0, -a/\omega\}$ and this infers

$$f''_\tau(t)^2 \ge \frac{2\beta}{3}\big(f'_\tau(\eta)^3 - f'(\eta_0)^3\big) + f''_\tau(\eta_0)^2,$$

for all $\eta \ge \eta_0$. One readily verifies that $f'_\gamma(\eta)$ tends to $-\infty$. Now, the function $g = -f_\tau$ is positive on (η_0, ∞), monotonic increasing, goes to ∞ with η and satisfies

$$g''(\eta) \ge \sqrt{\frac{|\beta|}{3}} g'(\eta)^{3/2},$$

for large η. A simple analysis of this inequality implies that g' is not global. A contradiction. Next assume that $\omega = 0$. Because $\tau < 0$ there exists a (small) real number $\eta_0 > 0$ such that $f'_\tau(\eta_0)$ and $f''_\tau(\eta_0)$ are negative. The new function $\overline{f}(\eta) = f_\tau(\eta + \eta_0)$ is a solution to (1.11) which satisfies $\overline{f}'(0) < 0$ and $\overline{f}''(0) < 0$. Hence f_τ is not global. □

The next result considers the case $\omega > 0$ and $\beta < -2$ for $\kappa = 1$. The case $\kappa = 0$ will be treated in detail in the next section. The condition $\beta < -\frac{1}{2}$ is plainly satisfied for $-1 < m < -\frac{1}{2}$. According to [8] there exists $a^+_\star > 0$ such that the problem

$$\begin{gathered} f''' + ff'' - \beta {f'}^2 = 0, \\ f(0) = a, \quad f'(0) = 1, \quad f'(\infty) = 0, \end{gathered} \tag{2.5}$$

has no solution for any $a < a^+_\star$. On the other hand, it should be noticed that if f is a solution to (1.11) then it is for the function $\eta \mapsto \gamma f(\gamma\eta)$, for any $\gamma > 0$. Consequently, problem (2.5), with $f'(0) = \omega$ instead of $f'(0) = 1$ has no solution for any $a < \sqrt{\omega} a^+_\star$. Clearly, this deduction and the results of [15] lead to the following result.

Theorem 2.3. *Let $\omega > 0$. Assume that $\beta < -2$ and $a < \sqrt{\omega} a_\star^+$. For any τ the local solution f_τ is not global ($\eta_c < \infty$) and satisfies*

$$\lim_{\eta \uparrow \eta_c} f_\tau(\eta) = -\infty.$$

Having proved that f_τ blows up at a finite point (under favorable conditions), we determine its precise asymptotic behavior, closely following the analysis of [18].

3. Asymptotic behavior of blowing-up solutions

The purpose of this section is to study the asymptotic behavior of any possible blowing-up solution to (1.11), where $\kappa \in \{0,1\}$ and $\beta < 0$ are mainly assumed.

3.1. **The limit case $\beta = -\infty$ ($\kappa = 0$).** In this short subsection we examine the structure of solutions to problem (1.10), (1.12) for different τ and ω. Solving this problem is equivalent to finding a solution $g\,(= f_\tau')$ to the following ODE

$$g'' + g^2 = 0, \tag{3.1}$$

accompanied with the initial conditions

$$g(0) = \omega, \quad g'(0) = \tau. \tag{3.2}$$

In this subsection, we assume that the real numbers ω and τ take place on the whole $\mathbb{R}$. In the phase plane (g, g') the curve of the above problem are given by

$$g'^2 + \frac{2}{3} g^3 = \gamma(\tau, \omega), \tag{3.3}$$

where $\gamma(\tau, \omega) = \tau^2 + \frac{2}{3}\omega^3$, or

$$g' = \pm \sqrt{\gamma(\tau, \omega) - \frac{2}{3} g^3}, \tag{3.4}$$

as soon as $\gamma(\tau, \omega) \geq \frac{2}{3} g^3$. If $\gamma(\tau, \omega) = 0$ the problem can be solved explicitly. In this case $\omega \leq 0$. If $\omega = 0$ we get $g \equiv 0$ and for $\omega < 0$ we deduce from (3.4) that

$$g(\eta) = -\frac{6}{(\eta_c - \eta)^2}, \tag{3.5}$$

where

$$\eta_c^2 = \frac{6}{|\omega|}.$$

Consequently, if $\eta_c < 0$ ($\tau > 0$) the solution g is global and tends to 0 as η goes to infinity and for $\eta_c > 0$ ($\tau < 0$) the solution g is not global and tends to $-\infty$ as η approaches η_c. For $\gamma(\tau, \omega) \neq 0$, we assume first that g is global and that g' is positive for large η. Recall that g' is monotonic decreasing (see (3.1)). Since g is monotonic increasing we conclude that g and g' are bounded, there exists a real number g_∞ such that $g(\eta)$ tends to g_∞ as η tends to infinity and $g'(\eta)$ tends to zero as η tends to infinity and then $g_\infty = 0$ by using again equation (3.1). This leads to $\gamma(\tau, \omega) = 0$, thanks to (3.3). A contradiction. Therefore, there exists $\eta_0 \geq 0$ such that $g'(\eta) < 0$ on (η_0, ∞). So, we may assume without lost of generality that $g'(\eta) < 0$ for all $\eta \geq 0$ ($\tau < 0$), and consider equation (3.4) with minus instead of $\pm$, which gives

$$\int_{g(\eta)}^{\omega} \frac{ds}{\sqrt{\gamma(\tau, \omega) - \frac{2}{3} s^3}} = \eta.$$

A simple analysis of the above integral shows that g cannot be global. This means that there exists a real number $\eta_c = \eta_c(\tau, \omega)$, such that $\lim_{\eta\to\eta_c} g(\eta) = -\infty$. Moreover, the blow-up point η_c is given by

$$\int_{-\infty}^{\omega} \frac{ds}{\sqrt{\gamma(\tau,\omega) - \frac{2}{3}s^3}} = \eta_c. \tag{3.6}$$

Next, one sees from (3.3) that

$$\lim_{\eta\to\eta_c} g'(\eta)^2 g^{-3}(\eta) = -\frac{2}{3},$$

and then

$$\lim_{\eta\to\eta_c} (\eta_c - \eta)^2 g(\eta) = -6.$$

Returning to the original function f_τ we summarize the main result of the present subsection in the following.

(1) If $\tau^2 = -\frac{2}{3}\omega^3$ and $\tau > 0$ the solution f_τ is global and given by

$$f_\tau(\eta) = 6\frac{1}{\eta_c + \eta} + a - \frac{6}{\eta_c},$$

where $\eta_c = \sqrt{6/|\omega|}$.

(2) If $\tau^2 = -\frac{2}{3}\omega^3$ and $\tau < 0$ the solution f_τ is not global and given by

$$f_\tau(\eta) = -6\frac{1}{\eta_c - \eta} + a + \frac{6}{\eta_c}, \quad \eta_c = \sqrt{6/|\omega|}. \tag{3.7}$$

(3) If $\tau^2 \neq -\frac{2}{3}\omega^3$ the solution f_τ is not global and satisfies

$$\lim_{\eta\to\eta_c} (\eta_c - \eta) f_\tau(t) = -6, \tag{3.8}$$

where $\eta_c > 0$ is the blow-up point, which is given by (3.6), for $\tau < 0$.

The above results show, in particular, that problem (1.1)-(1.3)) has no non trivial (similarity) solution for any $\omega \geq 0$, if $m = -1$ even if $f(0) \neq 0$ [24, pp. 244–246], [21].

3.2. **The case** $\kappa = 1$**.** We shall be concerned with problem (1.11), (1.12), where $\kappa = 1$ and $\beta < 0$. Our interest is in solutions which develop a singularity. In fact, the aim of the present subsection is to establish the asymptotic behavior of any possible singular solution at its blow-up point. Let us note that if we look for a singular solution to (1.11), where $\kappa \in \{0, 1\}$ and $\beta < 0$, under the form

$$f^\star(\eta) = A(\eta_c - \eta)^{-\gamma},$$

where $A \neq 0, \eta_c > 0$ and $\gamma > 0$, we find that $\gamma = 1$ and $A = 6/(\beta - 2\kappa)$,

The main result is as follows.

Theorem 3.1. *Let f be a solution to* (1.11)*, where $\kappa = 1$ and $\beta < 0$. Assume that there exists a real number η_c such that* $\lim_{\eta\uparrow\eta_c} f(\eta) = -\infty$*. Then*

$$f(\eta) \sim \frac{6}{\beta - 2}\frac{1}{\eta_c - \eta}, \quad \text{as } \eta \uparrow \eta_c. \tag{3.9}$$

The case $\beta = -1$ is easy to analyze. Setting $h = f' + \frac{1}{2}f^2$ one sees, from (1.11), $h'' = (1+\beta)f'^2$. In such situation the solution f satisfies the Riccati equation

$$f'(\eta) + \frac{1}{2}f(\eta)^2 = \lambda\eta + \delta, \tag{3.10}$$

where $\lambda = f''(0) + f(0)f'(0)$ and $\delta = f'(0) + \frac{1}{2}f^2(0)$. Since f is not global we infer

$$\lim_{\eta\uparrow\eta_c} \frac{f'(\eta)}{f(\eta)^2} = -\frac{1}{2}. \tag{3.11}$$

Finally, a simple integration of the above yields (3.9) with $\beta = -1$.

To prove Theorem 3.1 we use some modification and adaptation of an idea used in [18] for the Blasius equation and introduced by Toland [29]. To obtain (3.9), equation (1.11) will be reduced to a second order equation in which f is regarded as an independent variable. Since f and f' are monotonic decreasing and tend to $-\infty$ as η approaches η_c, there exists a real number $0 \le \eta_0 < \eta_c$ such that f and f' are negative on (η_0, η_c). Without loss of generality we may assume that $f(\eta_0) = 0, f''(\eta_0) < 0$. Defining

$$x = -f, \quad v(x) = f'(\eta(x))^2.$$

and using (1.11) we arrive at the second order differential equation

$$v''(x) = -2\beta\sqrt{v(x)} + x\frac{v'(x)}{\sqrt{v(x)}}, \quad x > 0. \tag{3.12}$$

The initial condition is given by

$$v(0) = f'(\eta_0)^2 > 0,\ v'(0) = -2f''(\eta_0) > 0. \tag{3.13}$$

Setting

$$w(s) = \frac{v(x)}{x^4}, \quad x = e^s,\ x \ge x_0,$$

for x_0 large, equation (3.12) becomes

$$w'' + 7w' + 12w - 2(2-\beta)\sqrt{w} - w'w^{-1/2} = 0. \tag{3.14}$$

Therefore, in the remainder of this section we study equations (3.12) and (3.14). We shall see that the solution v to (3.12),(3.13) is global, equation (3.14) is satisfied on some (s_0, ∞) and $w(s)$ goes to $\frac{(2-\beta)^2}{36}$ as s tends to infinity, which leads to

$$\lim_{\eta\to\eta_c} \frac{f'(\eta)}{f(\eta)^2} = \frac{2-\beta}{6}, \tag{3.15}$$

and then (3.9) is satisfied. We start with simple results which are crucial for the proof. We distinguish between the cases $1+\beta \ge 0$ and $1+\beta < 0$.

Lemma 3.2. *Let v be the solution to* (3.12), (3.13) *where $-1 \le \beta < 0$. Then v is global, increasing and tends to infinity with x. Moreover, there exists $x_1 > 0$ (large) such that the following*

$$\sqrt{v(x)} \le \frac{3}{2}x^2, \tag{3.16}$$

holds for all x in (x_1, ∞).

Proof. Since $v(0)$ and $v'(0)$ are positive (see (3.13)), there exists an $\varepsilon > 0$ such that $v(x) > 0, v'(x) > 0$ for all $0 \le x \le \varepsilon$. Assume that there exists $\varepsilon_0 > \varepsilon$ such that $v' > 0$ on $[0, \varepsilon_0)$ and $v'(\varepsilon_0) = 0$. Hence v is positive on $[0, \varepsilon_0]$ and, using (3.12), $v''(x) > 0$, for all $x \in [0, \varepsilon_0]$. Therefore, v' is monotonic increasing on $[0, \varepsilon_0]$ and positive on $[0, \varepsilon_0)$, which contradicts the hypothesis $v'(\varepsilon_0) = 0$. Hence,

$$v > 0, \quad v' > 0, \quad v'' > 0,$$

as long as v exists. Note that if v is global, necessarily $v(x)$ tends to infinity with x.

To show that v is global and satisfies (3.16) we put

$$H(x) = v'(x) - 2x\sqrt{v(x)}, \quad \forall\ x \in [0, x_c).$$

Hence

$$H'(x) = -2(1+\beta)\sqrt{v(x)} \le 0,$$

and then

$$\begin{gathered} v'(x) \le 2x\sqrt{v(x)} + v'(0), \\ (\sqrt{v(x)})' \le x + \frac{v'(0)}{2\sqrt{v(0)}}, \end{gathered} \tag{3.17}$$

since $v(x) \ge v(0)$. Integrating the last inequality over $(0, x)$ leads to

$$\sqrt{v(x)} \le \frac{1}{2}x^2 + \frac{xv'(0)}{2\sqrt{v(0)}} + \sqrt{v(0)}.$$

Using this and (3.17) we deduce that v is global and estimate (3.16) is satisfied. $\square$

The following result gives the lower bound of $\sqrt{v}/x^2$ for x large.

Lemma 3.3. *Let v be the solution to* (3.12), (3.13) *where $-1 \le \beta < 0$. Then there exists $x_2 > 0$, large, such that*

$$\sqrt{v(x)} \ge \frac{1}{12}(2 - 3\beta)x^2, \tag{3.18}$$

holds for all x in (x_2, ∞).

Proof. Let $G(x) = 5v(x) - 3xv'(x)$, for $x \ge x_1$, where x_1 is given by Lemma 3.2. We have

$$G'(x) = 2v'(x)\big[1 - \frac{3}{2}\frac{x^2}{\sqrt{v(x)}}\big] + 6\beta x\sqrt{v(x)},$$

thanks to (3.12). It follows from Lemma 3.2 that $G'(x) \le 0$ for all $x \ge x_1$, and then

$$\begin{gathered} 5v(x) - 3xv'(x) \le 5v(x_1) - 3x_1v'(x_1), \quad \forall x \ge x_1, \\ 4v(x) - 3xv'(x) \le 5v(x_1) - 3x_1v'(x_1) - v(x), \quad \forall x \ge x_1. \end{gathered}$$

Since $v(x)$ tends to infinity with x we deduce that there exists x_3, large such that

$$4v(x) \le 3xv'(x), \quad \forall x \ge x_3. \tag{3.19}$$

On the other hand, the new function

$$V(x) = v'(x) - \frac{2}{5}(2 - 3\beta)x\sqrt{v(x)}$$

satisfies

$$V'(x) = 2\frac{1+\beta}{5}\big[\frac{3}{2}x\frac{v'(x)}{\sqrt{v(x)}} - 2\sqrt{v(x)}\big].$$

Due to (3.19) we have $V'(x) \ge 0$ for all $x \ge x_3$. Hence

$$v'(x) - \frac{2}{5}(2-3\beta)x\sqrt{v(x)} \ge v'(x_3) - \frac{2}{5}(2-3\beta)x_2\sqrt{v(x_3)},$$

which leads to (3.18). □

Next we consider the case $1+\beta < 0$.

Lemma 3.4. *Let v be the solution to* (3.12)*),* (3.13) *where $1+\beta<0$. Then v is global, increasing and tends to infinity with x. Moreover, the following*

$$\sqrt{v(x)} \ge \frac{1}{2}x^2, \tag{3.20}$$

holds for all $x \ge 0$.

Proof. Arguing as in the proof of Lemma 3.3 the function v is increasing and tends to infinity with x if $x_c = \infty$. To prove that v is global we show first that estimate (3.20) holds on $(0, x_c)$. Using again the function $H(x) = v'(x) - 2x\sqrt{v(x)}$ and the ODE satisfied by v to deduce that $H'(x) > 0$ for all $0 \le x < x_c$. Hence

$$v'(x) - 2x\sqrt{v(x)} \ge v'(0) \ge 0, \tag{3.21}$$

and then $(\sqrt{v(x)})' \ge x$, which leads to (3.20). It remains to prove that v is global. To this end we use the equation of v and estimates (3.20), (3.21) to get

$$\begin{gathered}\frac{v''(x)}{v'(x)} = -2\beta\frac{\sqrt{v(x)}}{v'(x)} + \frac{x}{\sqrt{v(x)}},\\ \frac{v''(x)}{v'(x)} \le (|\beta|+2)\frac{1}{x},\end{gathered}$$

for all $x \in (0, x_c)$. Integrating the above inequality over (x_0, x), $x_0 > 0$, yields

$$v'(x) \le v'(x_0)\left(\frac{x}{x_0}\right)^{|\beta|+2}.$$

Hence v is global. □

Remark 3.5. Lemmas 3.2 and 3.3 indicate, in particular, that if $1+\beta \ge 0$ the function $s \to w(s)$ is uniformly bounded on (s_1, ∞) for some s_1 large.

Corollary 3.6. *Set $\Gamma = \inf\{\frac{1}{2}, \frac{2-3\beta}{12}\}$. Let v be the global solution to* (3.12), (3.13) *where $\beta < 0$. Then there exists $x_0 > 0$ large such that*

$$\frac{\sqrt{v(x)}}{x^2} \ge \Gamma, \tag{3.22}$$

for all $x \ge x_0$.

Remark 3.7. Estimate (3.22) can also be used for proving that the existence interval $(0, \eta_c)$ of (1.11) is bounded. In view of $\eta'(x) = \frac{1}{\sqrt{v(x)}}$ we have

$$\eta_c = \eta(x_0) + \int_{x_0}^{\infty} \frac{dx}{\sqrt{v(x)}},$$
$$\eta_c \leq \eta(x_0) + \frac{1}{\Gamma} \int_{x_0}^{\infty} \frac{dx}{x^2};$$

therefore, η_c is bounded.

Next, we examine the limit of $w(s)$ as $s \to \infty$. We note first that

$$w(s) \geq \Gamma^2, \quad s \geq s_0. \tag{3.23}$$

The proof of Theorem 3.1 is an immediate consequence of the following lemma which is our final result.

Lemma 3.8. *Let v be the global solution to* (3.12)), (3.13), *where $\beta < 0$. Then*

$$\lim_{x\to\infty} \frac{\sqrt{v(x)}}{x^2} = \frac{2-\beta}{6}.$$

Proof. The proof of this lemma will amount to proving that

$$\lim_{s\to\infty} w(s) = \frac{(2-\beta)^2}{36}. \tag{3.24}$$

The proof of (3.24) is short and different from the one given in [18]. By (3.14) the function

$$I(s) = \frac{1}{2} w'(s)^2 + 6w(s)^2 - \frac{4(2-\beta)}{3} w(s)^{3/2}$$

satisfies

$$I'(s) = -7w'(s)^2 w(s)^{-1/2} \big[\sqrt{w(s)} - \frac{1}{7}\big].$$

Therefore, $I'(s) \leq 0$, for all $s \geq s_0$, thanks to (3.23) and the definition of Γ. It follows from this that w and then w' are bounded. Because

$$0 \leq M w'(s)^2 \leq -I'(s),$$

where $M = 7[1 - \frac{1}{\Gamma}\frac{1}{7}] > 0$, we deduce that w' is square integrable. Using again equation (3.14) one sees that w'' is also bounded. Now, we use the identity

$$w'(s)^3 = w'(s_0)^3 + 3\int_{s_0}^{s} w'(\tau)^2 w''(\tau) d\tau,$$

to show that $w'(s)$ has a finite limit as $s \to \infty$ and this limit is zero, since w' is square integrable. Next we get, by differentiating (3.14),

$$w''' + 7w'' + 12w' - (2-\beta)w'w^{-1/2} - w''w^{-1/2} + \frac{1}{2}w'^2 w^{-3/2} = 0.$$

Hence w''' is bounded and we have

$$\begin{aligned}\int_{s_0}^{s} w'''(r)w'(r)dr &= -\frac{7}{2}\left(w'(s)^2 - w'(s_0)^2\right) - \int_{s_0}^{s} w'(r)^2 \left(12 - (2-\beta)w(r)^{-1/2}\right) dr \\ &\quad + \int_{s_0}^{s} w''(r)w'(r)w(r)^{-1/2} dr - \frac{1}{2}\int_{s_0}^{s} w'(r)^3 w(r)^{-3/2} dr.\end{aligned}$$

Therefore, the integral $\int_{s_0}^{\infty} w'''(r)w'(r)dr$ is finite, and then an integration by parts shows that w'' is square integrable. As a consequence, the previous equality implies that w'' tend to 0 at infinity. Finally, we deduce from (3.14) and (3.23),

$$|1 - \frac{6}{2-\beta}\sqrt{w}| \leq \frac{1}{2(2-\beta)}\frac{|w'|}{\Gamma^2} + \frac{1}{2\Gamma(2-\beta)}|w'' + 7w'|,$$

and get (3.24). The proof is completed. □

4. Appendix: Mathematical modelling

The materials presented here are based on many references. For example the works [4] by Bejan and Nield, [33] by Wooding and [26] by Sobha and Ramakrishna. The starting point is the boundary layer system

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}, \tag{4.1}$$

$$\frac{\partial p}{\partial x} + g\rho + \mu k^{-1}u + \sigma B_0^2 u = 0, \quad \frac{\partial p}{\partial y} + \mu k^{-1}v + \sigma B_0^2 v = 0, \tag{4.2}$$

where u, v are the velocity components, describes the 2D stationary heat convection and T is the temperature of the fluid. The constants $\mu, k, \alpha, g, \sigma$ and B_0 are, respectively, viscosity, permeability, thermal diffusivity, gravitational acceleration, the electric conductivity and applied magnetic field. The unknown functions p and ρ are, respectively, is the pressure and ρ is the $T-$dependent density, defined from the Boussinesq approximation [33]

$$\rho = \rho_0(1 - \beta_1(T - T_0)), \tag{4.3}$$

where ρ_0 is the density at a reference temperature T_0, and β_1 is a constant. Usually, the reference temperature is T_∞; the temperature far from the plate and then $\rho_0 = \rho_\infty$ is the value of ρ far from the plate, the reference density [25].

The wall temperature distribution is assumed to be a power function of the distance from the origin;

$$T_w(x) = T_0 + Ax^m,$$

where $A > 0$ is a constant and m is a real number. The boundary conditions are

$$v(x,0) = 0, \quad u(x,0) = u_w x^m, \quad T(x,0) = T_w(x), \tag{4.4}$$

and

$$T(x,\infty) = T_\infty, \quad u(x,\infty) = 0. \tag{4.5}$$

The above model can be expressed in a simpler form by introducing the stream function $\psi(u = \frac{\partial\psi}{\partial y}, v = -\frac{\partial\psi}{\partial x})$ and applying boundary approximations. PDEs (4.1), (4.2) are reduced to

$$(\frac{\mu}{k} + \sigma B_0^2)\frac{\partial^2\psi}{\partial y^2} = \rho_0 g\beta_1\frac{\partial T}{\partial y}, \tag{4.6}$$

$$\alpha\frac{\partial^2 T}{\partial y^2} = \frac{\partial T}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial T}{\partial y}\frac{\partial\psi}{\partial x}. \tag{4.7}$$

We then perform the similarity transformations in the usual way,

$$\eta = \sqrt{R_{a_x}}\frac{y}{x}, \tag{4.8}$$

$$\psi(x,y) = \alpha\sqrt{R_{a_x}}f(\eta), \quad T(x,y) = T_0 + (T_w - T_0)\theta(\eta), \tag{4.9}$$

where

$$R_{a_x} = \frac{\beta_1 g \rho_0 k (T_w - T_0) x}{\alpha \mu}, \tag{4.10}$$

is the modified local Rayleigh number in a porous medium. Equations (4.6) and (4.7) now reduce to

$$f'' = \omega \theta', \quad \omega = \frac{M^2}{M^2 + N^2}, \tag{4.11}$$

$$\theta'' + \frac{1+m}{2} f \theta' - m f' \theta = 0, \tag{4.12}$$

where $M^2 = \frac{1}{k}$ and $N^2 = \frac{\sigma B_0^2}{\mu}$ is the magnetic parameter.

The boundary conditions read

$$f(0) = 0, \quad f'(0) = \omega, \quad \theta(0) = 1,$$
$$\theta(\eta) \to 0, \quad f'(\eta) \to 0,$$

as $\eta \to \infty$. Therefore we get

$$f' = \omega \theta, \tag{4.13}$$

$$\begin{gathered} f''' + \frac{1+m}{2} f f'' - m f'^2 = 0, \\ f(0) = 0, \quad f'(0) = \omega, \end{gathered} \tag{4.14}$$

$$f'(\eta) \to 0, \quad \text{as } \eta \to \infty. \tag{4.15}$$

Note that equation (4.13) can also be obtained from the wall condition

$$u_w = \omega \beta_1 g \rho_0 k A,$$

and condition (4.15) can be replaced by $f''(0) = \tau$, where the real number τ has a physical meaning, since the local Nusselt number, Nu_x, is given by (in the usual dimensionless form)

$$Nu_x = -\sqrt{Ra_x}\theta'(0) = -\sqrt{Ra_x}\frac{\tau}{\omega}.$$

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