Electronic Journal of Differential Equations, Vol. 2022 (2022), No. 63, pp. 1–25.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

SOLVABILITY OF INCLUSIONS INVOLVING PERTURBATIONS OF POSITIVELY HOMOGENEOUS MAXIMAL MONOTONE OPERATORS

DHRUBA R. ADHIKARI, ASHOK ARYAL, GHANSHYAM BHATT,
ISHWARI J. KUNWAR, RAJAN PURI, MIN RANABHAT

Abstract. Let X be a real reflexive Banach space and X^* be its dual space. Let G_1 and G_2 be open subsets of X such that $\overline{G}_2 \subset G_1$, $0 \in G_2$, and G_1 is bounded. Let $L : X \supset D(L) \to X^*$ be a densely defined linear maximal monotone operator, $A : X \supset D(A) \to 2^{X^*}$ be a maximal monotone and positively homogeneous operator of degree $\gamma > 0$, $C : X \supset D(C) \to X^*$ be a bounded demicontinuous operator of type (S_+) with respect to $D(L)$, and $T : \overline{G}_1 \to 2^{X^*}$ be a compact and upper-semicontinuous operator whose values are closed and convex sets in X^*. We first take $L = 0$ and establish the existence of nonzero solutions of $Ax + Cx + Tx \ni 0$ in the set $G_1 \setminus G_2$. Secondly, we assume that A is bounded and establish the existence of nonzero solutions of $Lx+Ax+Cx \ni 0$ in $G_1 \setminus G_2$. We remove the restrictions $\gamma \in (0,1]$ for $Ax+Cx+Tx \ni 0$ and $\gamma = 1$ for $Lx+Ax+Cx \ni 0$ from such existing results in the literature. We also present applications to elliptic and parabolic partial differential equations in general divergence form satisfying Dirichlet boundary conditions.

1. Introduction and preliminaries

Let X be a real reflexive Banach space and X^* be its topological dual space. The symbol 2^{X^*} denotes the collection of all subsets of X^*. The norm on X is denoted by $\|\cdot\|_X$. When there is no risk of misunderstanding, the norms on X and X^* are both denoted by $\|\cdot\|$. The pairing $\langle x^*, x\rangle$ denotes the value of the functional $x^* \in X^*$ at $x \in X$. The symbols $\partial Z, \operatorname{Int} Z, \overline{Z}$ and $\operatorname{co} Z$ denote the boundary, interior, closure, and convex hull of the set $Z \subset X$, respectively. The symbol $B_X(0,R)$ denotes the open ball of radius $R > 0$ with center at 0 in X. The symbols $\mathbb{R}$ and $\mathbb{R}_+$ denote $(-\infty,\infty)$ and $[0,\infty)$, respectively. For a sequence $\{x_n\}$ in X and $x_0 \in X$, we denote by $x_n \to x_0$ and $x_n \rightharpoonup x_0$ the strong convergence and weak convergence, respectively. Given another real Banach Y, an operator $T : X \supset D(T) \to Y$ is said to be *bounded* if it maps bounded subsets of the domain $D(T)$ onto bounded subsets of Y. The operator T is said to be *compact* if it maps bounded subsets of $D(T)$ onto relatively compact subsets in Y. The operator T is said to be *demicontinuous* if it is strong-to-weak continuous on $D(T)$.

2020 *Mathematics Subject Classification.* 47H14, 47H05, 47H11.

Key words and phrases. Topological degree theory; operators of type (S_+);
monotone operator; duality mapping; Yosida approximant.

Submitted April 1, 2022. Published August 30, 2022.

A multivalued operator A from X to X^* is written as $A : X \supset D(A) \to 2^{X^*}$, where $D(A) = \{x \in X : Ax \neq \emptyset\}$ is the effective domain of A. Here, Ax means $A(x)$, and these notations are used interchangeably in the sequel. We denote the graph of A by $\mathrm{Gr}(A)$, i.e., $\mathrm{Gr}(A) = \{(x, y) : x \in D(A), y \in Ax\}$.

Definition 1.1. An operator $A : X \supset D(A) \to 2^{X^*}$ is said to be *positively homogeneous of degree* $\gamma > 0$ if $(x, y) \in \mathrm{Gr}(A)$ implies $sx \in D(A)$ for all $s \geq 0$ and $(sx, s^\gamma y) \in \mathrm{Gr}(A)$.

Remark 1.2. An equivalent condition for an operator $A : X \supset D(A) \to 2^{X^*}$ to be positively homogeneous of degree $\gamma > 0$ is that $x \in D(A)$ implies $sx \in D(A)$ for all $s \geq 0$ and $s^\gamma Ax \subset A(sx)$. It follows that a positively homogeneous operator A of degree $\gamma > 0$ satisfies $0 \in A(0)$. When A is positively homogeneous of degree $\gamma > 0$, it can be verified that $x \in D(A)$ implies $sx \in D(A)$ for all $s > 0$ and $s^\gamma Ax = A(sx)$. However, in general, the property $s^\gamma Ax = A(sx)$ may not be true for $s = 0$. For example, let $A : \mathbb{R} \supset [0, \infty) \to 2^{\mathbb{R}}$ be given by

$$Ax = \begin{cases} (-\infty, 0] & \text{for } x = 0 \\ x^\gamma & \text{for } x > 0. \end{cases}$$

Clearly, $A(0) = (-\infty, 0] \neq \{0\}$.

A *gauge* function is a strictly increasing continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0) = 0$ and $\varphi(r) \to \infty$ as $r \to \infty$. The *duality mapping* of X corresponding to a gauge function φ is the mapping $J_\varphi : X \supset D(J_\varphi) \to 2^{X^*}$ defined by

$$J_\varphi x = \{x^* \in X^* : \langle x^*, x\rangle = \varphi(\|x\|)\|x\|,\ \|x^*\| = \varphi(\|x\|)\}, \quad x \in X.$$

The Hahn-Banach theorem ensures that $D(J_\varphi) = X$, and therefore $J_\varphi : X \to 2^{X^*}$ is, in general, a multivalued mapping. The duality mapping corresponding to the gauge function $\varphi(r) = r$ is called the *normalized duality* mapping and denoted by J. It is well-known that the duality mapping J_φ satisfies

$$J_\varphi x = \frac{\varphi(\|x\|)}{\|x\|} Jx, \quad x \in X \setminus \{0\}.$$

Since J is homogeneous of degree 1, we have

$$J_\varphi(sx) = \frac{\varphi(s\|x\|)}{\|x\|} Jx, \quad (s, x) \in \mathbb{R}_+ \times (X \setminus \{0\}).$$

In particular, when $\varphi(r) = r^{p-1}$, $1 < p < \infty$, we obtain $J_\varphi x = \|x\|^{p-2} Jx$, $x \in X \setminus \{0\}$, which implies

$$J_\varphi(sx) = s^{p-1} J_\varphi x, \quad (s, x) \in \mathbb{R}_+ \times X,$$

i.e., J_φ is positively homogeneous of degree $p - 1$.

When X is reflexive and both X and X^* are strictly convex, the inverse J_φ^{-1} of J_φ is the duality mapping of X^* with the gauge function $\varphi^{-1}(r) = r^{q-1}$, where q is given by $1/p + 1/q = 1$. It is easy to verify that

$$J_\varphi^{-1}(sx^*) = s^{q-1} J_\varphi^{-1} x^*, \quad (s, x^*) \in \mathbb{R}_+ \times X^*. \tag{1.1}$$

It is clear that J_φ is positively homogeneous of degree $\gamma > 0$ if and only if φ is positively homogeneous of degree $\gamma > 0$. Additional properties of duality mappings in connection with Banach space geometry can be found in Alber and Ryazantseva [7] and Cioranescu [19].

Definition 1.3. An operator $A : X \supset D(A) \to 2^{X^*}$ is said to be *monotone* if for all $(x, u), (y, v) \in \mathrm{Gr}(A)$ we have $\langle u - v, x - y\rangle \geq 0$. A monotone operator $A : X \supset D(A) \to 2^{X^*}$ is said to be *maximal monotone* if $\mathrm{Gr}(A)$ is maximal in $X \times X^*$, when $X \times X^*$ is partially ordered by set inclusion.

In what follows, we assume that X is reflexive and both X and X^* are strictly convex. It is well-known that the duality mapping J_φ is maximal monotone. A monotone operator A is maximal if and only if $R(A + \lambda J_\varphi) = X^*$ for all $\lambda \in (0, \infty)$ and all gauge functions φ. For a proof of this result for $\varphi(r) = r^{p-1}, 1 < p < \infty$, the reader is referred, for example, to Barbu [10, Theorem 2.3].

Definition 1.4. Let $L : X \supset D(L) \to X^*$ be a densely defined linear maximal monotone operator. An operator $C : X \supset D(C) \to X^*$ is said to be of *type* (S_+) *with respect to* $D(L)$ if for every sequence $\{x_n\} \subset D(L) \cap D(C)$ with $x_n \rightharpoonup x_0$ in X, $Lx_n \rightharpoonup Lx_0$ in X^* and
$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0\rangle \leq 0,$$
we have $x_n \to x_0$ in X. In this case, if $L = 0$, then C is said to be of *type* (S_+).

Definition 1.5. A family of operators $C(s) : X \supset G \to X^*, s \in [0, 1]$, is said to be a *homotopy of type* (S_+) *with respect to* $D(L)$ if for every sequence $\{x_n\} \subset D(L)\cap G$ with $x_n \rightharpoonup x_0$ in X and $Lx_n \rightharpoonup Lx_0$ in X^*, $\{s_n\} \subset [0, 1]$ with $s_n \to s_0$ and
$$\limsup_{n\to\infty} \langle C(s_n)x_n, x_n - x_0\rangle \leq 0,$$
we have $x_n \to x_0$ in X, $x_0 \in G$ and $C(s_n)x_n \rightharpoonup C(s_0)x_0$ in X^*. In this case, if $L = 0$, then $C(s)$ is said to be a *homotopy of type* (S_+). A homotopy $C(s)$ of type (S_+) with respect to $D(L)$ is *bounded* if the set $\{C(s)x : s \in [0, 1], x \in G\}$ is bounded.

Definition 1.6. An operator $T : X \supset D(T) \to 2^{X^*}$ is said to be of *class* (P) if
(i) it maps bounded sets to relatively compact sets;
(ii) for every $x \in D(T)$, Tx is a closed and convex subset of X^*; and
(iii) $T(\cdot)$ is upper-semicontinuous, i.e., for every closed set $F \subset X^*$, the set $T^-(F) = \{x \in D(T) : Tx \cap F \neq \emptyset\}$ is closed in X.

Hu and Papageorgiou introduced the operators of class (P) in [21]. We recall a compact-set valued upper-semicontinuous operator T is closed. Furthermore, given an operator T of class P and a sequence $\{(x_n, y_n)\} \subset \mathrm{Gr}(T)$ such that $x_n \to x \in D(T)$, the sequence $\{y_n\}$ has a cluster point in Tx.

This paper is organized as follows. In Section 2, we study variants of the standard Yosida approximants introduced in Brézis, Crandall, and Pazy [14] and their fundamental properties. Since the topological degree theory for $(S+)$-operators is employed to establish the main existence results in the later sections, we provide several results involving variants of Yosida approximants related to the Browder degree theory [16].

In Section 3, we first prove the existence of nonzero solutions of $Ax + Cx + Tx \ni 0$ by utilizing the topological degree theories developed by Browder [18] and Skrypnik [32]. In this case, A is maximal monotone with $A(0) = \{0\}$ and positively homogeneous of degree $\gamma > 0$, C is bounded demicontinuous of type (S_+), and T is of class (P). This result extends an analogous result for $\gamma \in (0, 1]$ established in [2] to an arbitrary degree of homogeneity $\gamma > 0$. Another main result established in this section is the existence of nonzero solutions of $Lx + Ax + Cx \ni 0$, where L, C are as above, and A is a bounded maximal monotone and positively homogeneous of degree $\gamma > 0$. This result extends an analogous result for $\gamma = 1$ established in [2] to an arbitrary degree of homogeneity $\gamma > 0$.

In Section 4, we present some applications of the theories developed in Section 3 to elliptic and parabolic partial differential equations, in general, divergence form that include p-Laplacian with $1 < p < \infty$ and satisfy Dirichlet boundary conditions.

For additional facts and various topological degree theories related to the subject of this paper, the reader is referred to Adhikari and Kartsatos [4, 5], Kartsatos and Lin [22], and Kartsatos and Skrypnik [24, 26]. For further information on functional analytic tools used herein, the reader is referred to Barbu [10], Browder [17], Pascali and Sburlan [28], Simons [30], Skrypnik [31, 32], and Zeidler [34].

2. Variants of Yosida approximants and related properties

Let X be a strictly convex and reflexive Banach space with strictly convex X^*. By using the duality mapping J_φ corresponding to an arbitrary gauge function φ, we study variants of the Yosida approximants in Brézis et al. [14] and resolvents of a maximal monotone operator $A : X \supset D(A) \to 2^{X^*}$. For each $\lambda > 0$ and each $x \in X$, the inclusion

$$0 \in J_\varphi(x_\lambda - x) + \lambda A x_\lambda \tag{2.1}$$

has a unique solution $x_\lambda \in D(A)$ (see Proposition 2.1 (i)). We define $J^\varphi_\lambda : X \to D(A) \subset X$ and $A^\varphi_\lambda : X \to X^*$ by

$$J^\varphi_\lambda x := x_\lambda \quad \text{and} \quad A^\varphi_\lambda x := \frac{1}{\lambda} J_\varphi(x - J^\varphi_\lambda x), \quad x \in X. \tag{2.2}$$

The operators A^φ_λ and J^φ_λ are variants of the standard Yosida approximant A_λ and resolvent J_λ of A. For each $x \in X$, we have

$$A^\varphi_\lambda x \in A(J^\varphi_\lambda x) \quad \text{and} \quad x = J^\varphi_\lambda x + J^{-1}_\varphi(\lambda A^\varphi_\lambda x).$$

When $\varphi(r) = r^{p-1}$, a splitting of x in terms of A^φ_λ and J^φ_λ is

$$x = J^\varphi_\lambda x + \lambda^{q-1} J^{-1}_\varphi(A^\varphi_\lambda x), \tag{2.3}$$

and therefore

$$A^\varphi_\lambda x = \left(A^{-1} + \lambda^{q-1} J^{-1}_\varphi\right)^{-1} x, \quad x \in X. \tag{2.4}$$

It is easy to verify that $A = A^\varphi_\lambda$ if and only if $A = 0$. In fact, if $A = 0$, then $J^\varphi_\lambda = I$, the identity operator on X. Moreover, if $0 \in D(A)$ and $0 \in A(0)$, then $A^\varphi_\lambda 0 = 0$.

The choice of an appropriate gauge function is essential for the main existence results in this paper. The following proposition summarizes some important properties of A^φ_λ and J^φ_λ along the lines of analogous properties of A_λ and J_λ. A complete proof is provided here for the reader's convenience.

Proposition 2.1. *Let X be a strictly convex and reflexive Banach space with strictly convex dual X^* and $A : X \supset D(A) \to 2^{X^*}$ be a maximal monotone operator. Then the following statements hold.*

(i) *The operator A^φ_λ is single-valued, monotone, bounded on bounded subsets of X, and demicontinuous from X to X^*.*
(ii) *For every $x \in D(A)$ and $\lambda > 0$, we have*

$$\|A^\varphi_\lambda x\| \le |Ax| := \inf\{\|x^*\| : x^* \in Ax\}.$$

(iii) *The operator* J_λ^φ *is bounded on bounded subsets of* X*, demicontinuous from* X *to* $D(A)$*, and*
$$\lim_{\lambda\to 0} J_\lambda^\varphi x = x \quad \text{for all } x \in \overline{\operatorname{co} D(A)}.$$
(iv) *If* $\lambda_n \to 0$*,* $x_n \rightharpoonup x$ *in* X*,* $A_{\lambda_n}^\varphi x_n \rightharpoonup y$ *and*
$$\limsup_{n,m\to\infty} \langle A_{\lambda_n}^\varphi x_n - A_{\lambda_m}^\varphi x_m, x_n - x_m\rangle \le 0,$$
then $(x,y)\in \operatorname{Gr}(A)$ *and*
$$\lim_{n,m\to\infty} \langle A_{\lambda_n}^\varphi x_n - A_{\lambda_m}^\varphi x_m, x_n - x_m\rangle = 0.$$
(v) *For every sequence* $\{\lambda_n\}$ *with* $\lambda_n \to 0$*,* $A_{\lambda_n}^\varphi x \rightharpoonup A^{\{0\}}x$ *for all* $x\in D(A)$*. In addition, if* X^* *is uniformly convex, then* $A_{\lambda_n}^\varphi x \to A^{\{0\}}x$ *for all* $x \in D(A)$*.*

(vi) *If* $\lambda_n \to 0$ *and* $x \notin \overline{D(A)}$*, then*
$$\lim_{n\to\infty} \|A_{\lambda_n}^\varphi x\| = \infty.$$

Proof. (i) We first show that J_λ^φ is single-valued. Given $x\in X$ and $\lambda>0$, let x_λ and $\tilde x_\lambda$ be solutions of (2.1). Take $y\in Ax_\lambda$ and $\tilde y \in A\tilde x_\lambda$ such that
$$J_\varphi(x_\lambda - x) + \lambda y = 0 \quad\text{and}\quad J_\varphi(\tilde x_\lambda - x) + \lambda \tilde y = 0.$$
This along with the monotonicity of A and J_φ implies
$$\langle J_\varphi(x_\lambda - x) - J_\varphi(\tilde x_\lambda - x), (x_\lambda - x) - (\tilde x_\lambda - x)\rangle = 0. \tag{2.5}$$
Since X is strictly convex, it follows that J_φ is strictly monotone, i.e., for $u_1,u_2\in X$, we have
$$\langle J_\varphi u_1 - J_\varphi u_2, u_1 - u_2\rangle > 0 \text{ if and only if } u_1 \ne u_2.$$
It follows from (2.5) that $x_\lambda = \tilde x_\lambda$. Thus, J_λ^φ is single-valued, and therefore A_λ^φ is also single-valued. It is easy to verify the monotonicity of A_λ^φ.

To show A_λ^φ is bounded, let $B\subset X$ be bounded. For each $x\in B$, let $x_\lambda = J_\lambda^\varphi x$. Let $(u,v)\in \operatorname{Gr}(A)$. Using (2.1), it follows that
$$\langle J_\varphi(x_\lambda - x) + \lambda y_\lambda, x_\lambda - u\rangle = 0,$$
where $y_\lambda \in Ax_\lambda$. This implies
$$\langle J_\varphi(x_\lambda - x), x_\lambda - u\rangle = -\lambda\langle y_\lambda, x_\lambda - u\rangle \le \lambda\langle v, u - x_\lambda\rangle.$$
The last inequality follows from the monotonicity of A. It then follows that
$$\begin{aligned}\langle J_\varphi(x_\lambda - x), x_\lambda - x\rangle &= \langle J_\varphi(x_\lambda - x), x_\lambda - u\rangle + \langle J_\varphi(x_\lambda - x), u - x\rangle\\ &\le \lambda\langle v, u - x_\lambda\rangle + \langle J_\varphi(x_\lambda - x), u - x\rangle\\ &= \lambda\langle v, u - x\rangle + \lambda\langle v, x - x_\lambda\rangle + \langle J_\varphi(x_\lambda - x), u - x\rangle.\end{aligned} \tag{2.6}$$
This implies
$$\varphi(\|x_\lambda - x\|)\|x_\lambda - x\| \le \lambda\|v\|\left(\|u - x\| + \|x_\lambda - x\|\right) + \varphi(\|x_\lambda - x\|)\|u - x\|. \tag{2.7}$$
If $\{x_\lambda : x\in B\}$ is unbounded, the inequality (2.7) yields a contradiction. Thus, J_λ^φ is bounded on B. Since J_φ is bounded on B, it follows from (2.2) that A_λ^φ is also bounded on B.

Let $\{x_n\}\subset X$ be such that $x_n \to x_0 \in X$ as $n\to\infty$. Denote $u_n = J_\lambda^\varphi x_n$ and $v_n = A_\lambda^\varphi x_n$, so that
$$J_\varphi(u_n - x_n) + \lambda v_n = 0. \tag{2.8}$$

Since J_λ^φ and A_λ^φ are bounded on bounded sets, both $\{u_n\}$ and $\{v_n\}$ are bounded. Since J_φ and A are monotone, it follows from

$$\begin{aligned}&\langle J_\varphi(u_n - x_n) - J_\varphi(u_m - x_m), (u_n - x_n) - (u_m - x_m)\rangle\\ &= -\lambda\langle v_n - v_m, (u_n - x_n) - (u_m - x_m)\rangle\end{aligned}$$

that

$$\lim_{n,m\to\infty}\langle v_n - v_m, u_n - u_m\rangle = 0,$$
$$\lim_{n,m\to\infty}\langle J_\varphi(u_n - x_n) - J_\varphi(u_m - x_m), (u_n - x_n) - (u_m - x_m)\rangle = 0.$$

Passing to subsequences, we may assume that $u_n \rightharpoonup u_0$ in X, $v_n \rightharpoonup v_0$ in X^*, and $J_\varphi(u_n - x_n) \rightharpoonup w_0$ in X^* for some $u_0 \in X$ and some $v_0, w_0 \in X^*$. By [10, Lemma 2.3], it follows that $(u_0, v_0) \in \mathrm{Gr}(A)$ and $(u_0 - x_0, w_0) \in \mathrm{Gr}(J_\varphi)$. Using all these in (2.8), we obtain $J_\varphi(u_0 - u_0) + \lambda v_0 = 0$, which implies $u_0 = J_\lambda^\varphi x_0$ and $v_0 = A_\lambda^\varphi x_0$, i.e., $J_\lambda^\varphi x_n \rightharpoonup J_\lambda^\varphi x_0$ and $A_\lambda^\varphi x_n \rightharpoonup A_\lambda^\varphi x_0$ as $n \to \infty$. This proves the demicontinuity of J_λ and A_λ.

(ii) Let $x \in D(A)$ and $\lambda > 0$. Let $y \in Ax$ and $x_\lambda = J_\lambda^\varphi x$. Then

$$\begin{aligned}0 &\le \langle y - A_\lambda x, x - x_\lambda\rangle\\ &= \langle y, x - x_\lambda\rangle - \frac{1}{\lambda}\varphi(\|x - x_\lambda\|)\|x - x_\lambda\|\\ &\le \|y\|\|x - x_\lambda\| - \frac{1}{\lambda}\varphi(\|x - x_\lambda\|)\|x - x_\lambda\|,\end{aligned}$$

which implies $\varphi(\|x - x_\lambda\|) \le \lambda\|y\|$, and therefore

$$\|A_\lambda^\varphi x\| = \frac{1}{\lambda}\|J_\varphi(x - x_\lambda)\| \le \|y\|.$$

Consequently, $\|A_\lambda^\varphi x\| \le |Ax| := \inf\{\|y\| : y \in Ax\}$.

(iii) The boundedness of J_λ^φ on bounded subsets of X and its demicontinuity are already proved in (i). Let $x \in \overline{\mathrm{co}D(A)}$ and $(u, v) \in \mathrm{Gr}(A)$. Following the arguments that lead to (2.7), we find that $\{x_\lambda - x : \lambda > 0\}$ is bounded, and therefore $\{J_\varphi(x_\lambda - x) : \lambda > 0\}$ is bounded. Let $\{\lambda_n\} \subset (0, \infty)$ be such that $\lambda_n \to 0$. Let $y \in X^*$ be such that $J_\varphi(x_{\lambda_n} - x) \rightharpoonup y$ in X^*. Then (2.6) yields

$$\limsup_{n\to\infty}\varphi(\|x_{\lambda_n} - x\|)\|x_{\lambda_n} - x\| \le \langle y, u - x\rangle.$$

It is clear that this argument applies to all $u \in \overline{\mathrm{co}D(A)}$. Taking $u = x$, we obtain

$$\lim_{n\to\infty}\varphi(\|x_{\lambda_n} - x\|)\|x_{\lambda_n} - x\| = 0.$$

By the homeomorphic property of the gauge function φ, it follows that we must have $x_{\lambda_n} \to x$ as $n \to \infty$. This completes the proof of (iii).

(iv) Let $u_n = J_{\lambda_n}^\varphi x_n$ for all n. Since $\{A_{\lambda_n}^\varphi x_n\}$ is bounded, it follows that

$$\varphi(\|x_n - u_n\|) = \varphi(\|x_n - J_{\lambda_n}^\varphi x_n\|) = \|J_\varphi(x_n - J_{\lambda_n}^\varphi x_n)\| = \lambda_n\|A_{\lambda_n}^\varphi x_n\| \to 0$$

as $n \to \infty$. This implies $\|x_n - u_n\| \to 0$ as $n \to \infty$. Since

$$\begin{aligned}&\langle A_{\lambda_n}^\varphi x_n - A_{\lambda_m}^\varphi x_m, x_n - x_m\rangle\\ &= \langle A_{\lambda_n}^\varphi x_n - A_{\lambda_m}^\varphi x_m, u_n - u_m\rangle + \langle A_{\lambda_n}^\varphi x_n - A_{\lambda_m}^\varphi x_m, (x_n - u_n) - (x_m - u_m)\rangle\end{aligned}$$

and A is monotone, it follows as in Brézis et al. [14] that

$$\lim_{n,m\to\infty}\langle A^{\varphi}_{\lambda_n}x_n - A^{\varphi}_{\lambda_m}x_m, x_n - x_m\rangle = 0 \text{ and } \lim_{n,m\to\infty}\langle A^{\varphi}_{\lambda_n}x_n - A^{\varphi}_{\lambda_m}x_m, u_n - u_m\rangle = 0.$$

The conclusion of (iv) now follows from [10, Lemma 2.3].

(v) Let $x \in D(A)$. Since X^* is reflexive and strictly convex and Ax is a closed and convex subset of X^*, it follows that there exists a unique element of Ax, denoted by $A^{\{0\}}x$, such that $\|A^{\{0\}}x\| = \inf\{\|x^*\| : x^* \in Ax\}$. Let $\{\lambda_n\} \subset (0,\infty)$ be such that $\lambda_n \to 0$ and $A^{\varphi}_{\lambda_n}x \rightharpoonup y$ in X^* as $n \to \infty$. As in the proof of (iv), with $x_n = x$, we have $y \in Ax$. In view of part (ii), it follows that

$$\|y\| \le \liminf_{n\to\infty}\|A^{\varphi}_{\lambda_n}x\| \le \limsup_{n\to\infty}\|A^{\varphi}_{\lambda_n}x\| \le \|A^{\{0\}}x\|,$$

and therefore we must have $y = A^{\{0\}}x$ and $A^{\varphi}_{\lambda_n}x \rightharpoonup A^{\{0\}}x$ in X^*. Moreover, if X^* is uniformly convex, then, by [10, Lemma 1.1], we obtain $A^{\varphi}_{\lambda_n}x \to A^{\{0\}}x$ in X^*.

(vi) Suppose, on the contrary, that there is a sequence $\{\lambda_n\}$ with $\lambda_n \to 0$ and an element $x \notin \overline{D(A)}$ such that $\{\|A^{\varphi}_{\lambda_n}x\|\}$ is bounded. Let $R > 0$ be such that $\|A^{\varphi}_{\lambda_n}x\| \le R$ for all n. Then, by (2.2), we have

$$\varphi(\|x - J^{\varphi}_{\lambda_n}x\|) = \|J_\varphi(x - J^{\varphi}_{\lambda_n}x)\| \le R\lambda_n.$$

Since φ^{-1} is also a gauge function, we obtain $J^{\varphi}_{\lambda_n}x \to x$ as $n \to \infty$. This implies $x \in \overline{D(A)}$, a contradiction. □

A proof of the following lemma for $\varphi(r) = r$ can be found in Boubakari and Kartsatos [13]. Since we are dealing here with an arbitrary gauge function φ, we provide a complete proof.

Lemma 2.2. *Let $A : X \supset D(A) \to 2^{X^*}$ be maximal monotone and $G \subset X$ be bounded. Let $0 < \lambda_1 < \lambda_2$. Then there exists a constant K, independent of λ, such that*

$$\|A^{\varphi}_{\lambda}x\| \le K$$

for all $x \in \overline{G}$ and $\lambda \in [\lambda_1, \lambda_2]$.

Proof. For every $x \in X$, we have

$$A^{\varphi}_{\lambda}x = \frac{1}{\lambda}J_\varphi(x - x_\lambda),$$

where $x_\lambda = J^{\varphi}_{\lambda}x$. Let $(u,v) \in \mathrm{Gr}(A)$. In view of (2.7) in the proof of (i) in Proposition 2.1, we have

$$\begin{aligned}\varphi(\|x_\lambda - x\|)\|x_\lambda - x\| &\le \lambda\|v\|\left(\|u - x\| + \|x_\lambda - x\|\right) + \varphi(\|x_\lambda - x\|)\|u - x\|\\ &\le \lambda_2\|v\|\left(\|u - x\| + \|x_\lambda - x\|\right) + \varphi(\|x_\lambda - x\|)\|u - x\|.\end{aligned}$$

By the properties of the gauge function φ, it follows that $\varphi(\|x_\lambda - x\|)$ must be bounded, i.e., there exists a constant $K_0 > 0$ such that

$$\varphi(\|x_\lambda - x\|) \le K_0$$

for all $x \in \overline{G}$ and all $\lambda \in [\lambda_1, \lambda_2]$. Consequently, we have

$$\|A^{\varphi}_{\lambda}x\| = \frac{1}{\lambda}\varphi(\|x_\lambda - x\|) \le \frac{1}{\lambda_1}K_0 =: K$$

for all $x \in \overline{G}$ and all $\lambda \in [\lambda_1, \lambda_2]$. □

By a well-known renorming theorem due to Troyanski [33], a reflexive Banach space X can be renormed with an equivalent norm with respect to which both X and X^* become locally uniformly convex (therefore strictly convex). With such a renorming, the duality mapping J_φ is a homeomorphism from X onto X^*. Henceforth, we assume that both X and X^* are reflexive and locally uniformly convex.

The following lemma involving A_λ^φ and J_λ^φ plays an important role in the sequel. Its proof is omitted here because of its similarity to [6, Lemma 1], except that, for the general φ here, we must make use of

$$x = J_\lambda^\varphi x + J_\varphi^{-1}(\lambda A_\lambda^\varphi x) \text{ and } \langle A_\lambda^\varphi x, J_\varphi^{-1}(\lambda A_\lambda^\varphi x)\rangle = \varphi^{-1}(\lambda \|A_\lambda^\varphi x\|)\|A_\lambda^\varphi x\|, \quad x \in X.$$

The lemma for A_λ and J_λ is essentially due to Brézis et al. [14].

Lemma 2.3. *Let $A : X \supset D(A) \to 2^{X^*}$ and $S : X \supset D(S) \to 2^{X^*}$ be maximal monotone operators such that $0 \in D(A) \cap D(S)$ and $0 \in S(0) \cap A(0)$. Assume that $A + S$ is maximal monotone and that there is a sequence $\{\lambda_n\} \subset (0, \infty)$ such that $\lambda_n \to 0$, and a sequence $\{x_n\} \subset D(S)$ such that $x_n \rightharpoonup x_0 \in X$ and $A_{\lambda_n}^\varphi x_n + w_n^* \rightharpoonup y_0^* \in X^*$, where $w_n^* \in S x_n$. Then the following statements are true.*

(i) *The inequality*
$$\lim_{n\to\infty} \langle A_{\lambda_n}^\varphi x_n + w_n^*, x_n - x_0\rangle < 0 \tag{2.9}$$
is impossible.

(ii) *If*
$$\lim_{n\to\infty} \langle A_{\lambda_n}^\varphi x_n + w_n^*, x_n - x_0\rangle = 0, \tag{2.10}$$
then $x_0 \in D(A + S)$ and $y_0^* \in (A + S)x_0$.

Definition 2.4. An operator $A : X \supset D(A) \to 2^{X^*}$ is said to be *strongly quasibounded* if for every $S > 0$ there exists $K(S) > 0$ such that $\|x\| \le S$ and $\langle x^*, x\rangle \le S$ for some $x^* \in Ax$ imply $\|x^*\| \le K(S)$.

It is obvious that a bounded operator is strongly quasibounded. With regard to possibly unbounded operators, Browder and Hess [18] and Pascali and Sburlan [28] have shown that a monotone operator A with $0 \in \operatorname{Int} D(A)$ is strongly quasibounded. The following lemma with the particular case $\varphi(r) = r$ addressed in Kartsatos and Quarcoo[23, Lemma D] is needed in the sequel.

Lemma 2.5. *Let $A : X \supset D(A) \to 2^{X^*}$ be a strongly quasibounded maximal monotone operator such that $0 \in A(0)$. Let $\{\lambda_n\} \subset (0, \infty)$ and $\{x_n\} \subset X$ be such that*
$$\|x_n\| \le S \quad \text{and} \quad \langle A_{\lambda_n}^\varphi x_n, x_n\rangle \le S_1 \quad \text{for all } n,$$
where S, S_1 are positive constants. Then there exists a number $K > 0$ such that $\|A_{\lambda_n}^\varphi x_n\| \le K$ for all n.

Proof. Denote $w_n = A_{\lambda_n}^\varphi x_n$ and $u_n = J_{\lambda_n}^\varphi x_n$ for all n. Then we have
$$w_n \in A u_n \quad \text{and} \quad x_n = u_n + J_\varphi^{-1}(\lambda_n w_n).$$
In view of $0 \in A(0)$, we obtain
$$\begin{aligned} 0 \le \langle w_n, u_n\rangle &= \langle w_n, x_n - J_\varphi^{-1}(\lambda_n w_n)\rangle \\ &= \langle w_n, x_n\rangle - \langle w_n, J_\varphi^{-1}(\lambda_n w_n)\rangle \\ &= \langle w_n, x_n\rangle - \varphi^{-1}(\lambda_n \|w_n\|)\|w_n\| \end{aligned}$$

$$\leq S_1 - \varphi^{-1}(\lambda_n\|w_n\|)\|w_n\|.$$

This yields $\langle w_n, u_n\rangle \leq S_1$ and $\varphi^{-1}(\lambda_n\|w_n\|)\|w_n\| \leq S_1$ for all n. Suppose $\{w_n\}$ is unbounded. Then there exists a subsequence, denoted again by $\{w_n\}$, such that $\|w_n\| \to \infty$ and $1 \leq \|w_n\|$ for all n. Consequently, $\varphi^{-1}(\lambda_n\|w_n\|) \leq S_1$ for all n, and since $x_n = u_n + J_\varphi^{-1}(\lambda_n w_n)$, it follows that

$$\lambda_n\|w_n\| = \|J_\varphi(x_n - u_n)\| = \varphi(\|x_n - u_n\|).$$

This implies $\|x_n - u_n\| = \varphi^{-1}(\lambda_n\|w_n\|) \leq S_1$ for all n. Since $\{x_n\}$ is bounded, we obtain the boundedness of $\{u_n\}$ and $\{\langle w_n, u_n\rangle\}$, which contradicts the strong quasiboundedness of A. Consequently, $\{w_n\}$ is bounded. □

For the rest of this paper, we take the gauge function $\varphi(r) = r^{p-1}$, $p > 1$. For the special case $\varphi(r) = r$, the reader can find proofs of Lemma 2.6 in Kartsatos and Skrypnik [25] when $0 \in A(0)$ and in Asfaw and Kartsatos [8], without the condition $0 \in A(0)$. We note that Zhang and Chen in [35, Lemma 2.7] proved the continuity of $x \mapsto A_\lambda x$ on $D(A)$ for each $\lambda > 0$, also without the condition $0 \in A(0)$. In [8, Lemma 6], however, the continuity of $x \mapsto A_\lambda x$ on X is used with no mention of its validity. We next provide a detailed proof of the continuity of the mapping $(\lambda, x) \mapsto A_\lambda^\varphi x$ on $(0,\infty) \times X$.

Lemma 2.6. *Let $A : X \supset D(A) \to 2^{X^*}$ be a maximal monotone operator. Then the mapping $(\lambda, x) \mapsto A_\lambda^\varphi x$ is continuous on $(0,\infty) \times X$.*

Proof. We first prove the continuity of $x \mapsto A_{\lambda_0}^\varphi x$ on X for each fixed $\lambda_0 > 0$. To this end, let $\{x_n\} \subset X$ be such that $x_n \to x_0 \in X$. By Lemma 2.2, we have the boundedness of $\{A_{\lambda_0}^\varphi x_n\}$, and therefore

$$\lim_{n\to\infty}\langle A_{\lambda_0}^\varphi x_n - A_{\lambda_0}^\varphi x_0, x_n - x_0\rangle = 0. \tag{2.11}$$

We know that

$$x_n = J_{\lambda_0}^\varphi x_n + \lambda_0^{q-1} J_\varphi^{-1}(A_{\lambda_0}^\varphi x_n) \quad \text{and} \quad x_0 = J_{\lambda_0}^\varphi x_0 + \lambda_0^{q-1} J_\varphi^{-1}(A_{\lambda_0}^\varphi x_0). \tag{2.12}$$

Since $A_{\lambda_0}^\varphi x_n \in A(J_{\lambda_0}^\varphi x_n)$ and $A_{\lambda_0}^\varphi x_0 \in A(J_{\lambda_0}^\varphi x_0)$, the monotonicity of A together with (2.11) and (2.12) yields

$$\lim_{n\to\infty}\langle A_{\lambda_0}^\varphi x_n - A_{\lambda_0}^\varphi x_0, J_\varphi^{-1}(A_{\lambda_0}^\varphi x_n) - J_\varphi^{-1}(A_{\lambda_0}^\varphi x_0)\rangle = 0. \tag{2.13}$$

Since J_φ^{-1} is a duality mapping from X^* to X, it follows, in view of [19, Proposition 2.17], that

$$A_{\lambda_0}^\varphi x_n \to A_{\lambda_0}^\varphi x_0 \quad \text{as} \quad n \to \infty.$$

This proves the continuity of $A_{\lambda_0}^\varphi$ on X.

We now proceed to prove the continuity of $(\lambda, x) \mapsto A_\lambda^\varphi x$ on $(0,\infty) \times X$. Let $\{\lambda_n\} \subset (0,\infty)$ and $\{x_n\} \subset X$ be such that $\lambda_n \to \lambda_0 \in (0,\infty)$ and $x_n \to x_0 \in X$ as $n \to \infty$. Let $G \subset X$ be a bounded set that contains x_n for all n. Rename $\lambda_1, \lambda_2 > 0$ such that $\lambda_n \in [\lambda_1, \lambda_2]$ for all n. Since

$$J_{\lambda_n}^\varphi x_n \in A^{-1}(A_{\lambda_n}^\varphi x_n) \quad \text{and} \quad x_n = J_{\lambda_n}^\varphi x_n + \lambda_n^{q-1} J_\varphi^{-1}(A_{\lambda_n}^\varphi x_n),$$

it follows that

$$\begin{aligned} J_{\lambda_n}^\varphi x_n + \lambda_0^{q-1} J_\varphi^{-1}(A_{\lambda_n}^\varphi x_n) &\in A^{-1}(A_{\lambda_n}^\varphi x_n) + \lambda_0^{q-1} J_\varphi^{-1}(A_{\lambda_n}^\varphi x_n) \\ &= \left(A^{-1} + \lambda_0^{q-1} J_\varphi^{-1}\right)(A_{\lambda_n}^\varphi x_n). \end{aligned}$$

This implies

$$\begin{aligned} A^{\varphi}_{\lambda_n} x_n &= \left(A^{-1} + \lambda_0^{q-1} J_{\varphi}^{-1}\right)^{-1} \left(J^{\varphi}_{\lambda_n} x_n + \lambda_0^{q-1} J_{\varphi}^{-1}(A^{\varphi}_{\lambda_n} x_n)\right) \\ &= A^{\varphi}_{\lambda_0} \left(J^{\varphi}_{\lambda_n} x_n + \lambda_0^{q-1} J_{\varphi}^{-1}(A^{\varphi}_{\lambda_n} x_n)\right) \\ &= A^{\varphi}_{\lambda_0} \left(x_n + (\lambda_0^{q-1} - \lambda_n^{q-1}) J_{\varphi}^{-1}(A^{\varphi}_{\lambda_n} x_n)\right). \end{aligned}$$

By Lemma 2.2, $\{A^{\varphi}_{\lambda_n} x_n\}$ is bounded, and so is $\{J_{\varphi}^{-1}(A^{\varphi}_{\lambda_n} x_n)\}$. Since $\lambda_n \to \lambda_0$, we have $(\lambda_0^{q-1} - \lambda_n^{q-1}) J_{\varphi}^{-1}(A^{\varphi}_{\lambda_n} x_n) \to 0$ as $n \to \infty$. The continuity of $A^{\varphi}_{\lambda_0}$ implies $A^{\varphi}_{\lambda_n} x_n \to A^{\varphi}_{\lambda_0} x_0$ as $n \to \infty$. This completes the proof. □

Remark 2.7. We anticipate that Lemma 2.6 holds for any gauge function φ. Since the formula (2.4) may not hold for A^{φ}_{λ} with a general φ, the above proof does not go through and this subject may be of independent research interest.

Let G be an open and bounded subset of X. Let $L : X \supset D(L) \to X^*$ be densely defined linear maximal monotone, $A : X \supset D(A) \to 2^{X^*}$ maximal monotone, and $C(s) : X \supset \overline{G} \to X^*$, $s \in [0, 1]$, a bounded homotopy of type (S_+) with respect to $D(L)$. Since $\mathrm{Gr}(L)$ is closed in $X \times X^*$, the space $Y = D(L)$ associated with the graph norm $\|x\|_Y = \|x\|_X + \|Lx\|_{X^*}$, $x \in Y$, becomes a real reflexive Banach space. We may assume that Y and its dual Y^* are locally uniformly convex.

Let $j : Y \to X$ be the natural embedding and $j^* : X^* \to Y^*$ its adjoint. Since $j : Y \to X$ is continuous, we have $D(j^*) = X^*$. This implies that j^* is also continuous. Since j^{-1} is not necessarily bounded, we have, in general, $j^*(X^*) \neq Y^*$. Moreover, $j^{-1}(\overline{G}) = \overline{G} \cap D(L)$ is closed and $j^{-1}(G) = G \cap D(L)$ is open, $\overline{j^{-1}(G)} \subset j^{-1}(\overline{G})$, and $\partial(j^{-1}(G)) \subset j^{-1}(\partial G)$.

We define $M : Y \to Y^*$ by $(Mx, y) := \langle Ly, J^{-1}(Lx)\rangle$, $x, y \in Y$, where the duality pairing $(\cdot, \cdot)$ is in $Y^* \times Y$, and J^{-1} is the inverse of the duality map $J : X \to X^*$ and is identified with the duality map from X^* to $X^{**} = X$. Also, for every $x \in Y$ such that $Mx \in j^*(X^*)$, we have $J^{-1}(Lx) \in D(L^*)$, $Mx = j^* \circ L^* \circ J^{-1}(Lx)$, and

$$(Mx - My, x - y) = \langle Lx - Ly, J^{-1}(Lx) - J^{-1}(Ly)\rangle \geq 0$$

for all $y \in Y$ such that $My \in j^*(X^*)$. Moreover, it is easy to see that M is continuous on Y, and therefore M is maximal monotone.

We now define $\hat{L} : Y \to Y^*$ and $\hat{C}(s) : j^{-1}(\overline{G}) \to Y^*$ by $\hat{L} = j^* \circ L \circ j$ and $\hat{C}(s) = j^* \circ C(s) \circ j$, respectively, and for each $t > 0$, we also define $\hat{A}^{\varphi}_t : Y \to Y^*$ by $\hat{A}^{\varphi}_t = j^* \circ A^{\varphi}_t \circ j$, where A^{φ}_t is the Yosida approximant of A corresponding to the gauge function φ.

The next lemma employs Lemma 2.5 and follows as in [5, Lemma 5], and therefore its proof is omitted.

Lemma 2.8. *Let $G \subset X$ be open and bounded. Assume the following:*

- (i) *$L : X \supset D(L) \to X^*$ is linear, maximal monotone with $\overline{D(L)} = X$;*
- (ii) *$A : X \supset D(A) \to 2^{X^*}$ is strongly quasibounded, maximal monotone with $0 \in A(0)$; and*
- (iii) *$C(t) : X \supset \overline{G} \to X^*$ is a bounded homotopy of type (S_+) with respect to $D(L)$.*

Then, for a continuous curve $f(s), 0 \leq s \leq 1$, in X^*, the set

$$F = \left\{x \in j^{-1}(\overline{G}) : \hat{L} + \hat{A}^{\varphi}_t + \hat{C}(s) + tMx = j^* f(s) \text{ for some } t > 0,\ s \in [0, 1]\right\}$$

is bounded in Y.

The next two propositions are essential for the existence results in Section 2 and Section 3.

Proposition 2.9. *Let $A : X \supset D(A) \to 2^{X^*}$ be maximal monotone and $C : X \supset D(C) \to X^*$ be bounded, demicontinuous and of type (S_+). Suppose that $G \subset X$ is open and bounded such that $0 \in A(0)$, $p \in X^*$, and*

$$p \notin (A + C)x$$

for all $x \in \partial G \cap D(A) \cap D(C)$. Then the following statements hold.

(i) *There exists $t_0 > 0$ such that*

$$A^{\varphi}_t x + Cx \neq p$$

for all $x \in \partial G \cap D(C)$ and $t < t_0$.

(ii) *For fixed $t_1, t_2 > 0$, define $q(t) := tt_1 + (1 - t)t_2$, $t \in [0, 1]$. Then the operator*

$$H(t, x) = A^{\varphi}_{q(t)}x + Cx, \quad (t, x) \in [0, 1] \times \overline{G}$$

is a homotopy of type (S_+).

(iii) *For every sequence $\{t_n\} \subset (0, \infty)$ such that $t_n \to 0$, $\lim_{n\to\infty} \mathrm{d}_{S_+}(A^{\varphi}_{t_n} + C, G, p)$ exists and does not depend on the choice of $\{t_n\}$.*

Proof. (i) Without loss of generality, we assume that $p = 0$. In fact, if $p \neq 0$, then we replace C with $C - p$. Suppose that (iii) is false. Then there exist $\{t_n\} \subset (0, \infty)$ and $\{x_n\} \subset \partial G$ such that $t_n \to 0$ and

$$A^{\varphi}_{t_n}x_n + Cx_n = 0 \tag{2.14}$$

for all n. Since C is bounded, $\{Cx_n\}$ is bounded. This implies that $\{A^{\varphi}_{t_n}x_n\}$ is also bounded. We may assume that there exist $x_0 \in X$ and $w_0 \in X^*$ such that $x_n \rightharpoonup x_0$ in X and $A^{\varphi}_{t_n}x_n \rightharpoonup w_0$ in X^*. If

$$\limsup_{n\to\infty}\langle Cx_n, x_n - x_0\rangle > 0,$$

we find a subsequence of $\{x_n\}$, denoted again by itself, such that

$$\lim_{n\to\infty}\langle Cx_n, x_n - x_0\rangle > 0.$$

In view of (2.14), we obtain

$$\lim_{n\to\infty}\langle A^{\varphi}_{t_n}x_n, x_n - x_0\rangle < 0;$$

however, this is impossible by (i) of Lemma 2.3. We then must have

$$\limsup_{n\to\infty}\langle Cx_n, x_n - x_0\rangle \leq 0.$$

By the (S_+)−property of C, we have $x_n \to x_0$, and consequently

$$\lim_{n\to\infty}\langle A^{\varphi}_{t_n}x_n, x_n - x_0\rangle = 0.$$

By (ii) of Lemma 2.3, we obtain $x_0 \in D(A)$ and $w_0 \in Ax_0$. Since C is demicontinuous, $Cx_n \rightharpoonup Cx_0$ in X^*. This implies $w_0 = -Cx_0$, i.e., $0 \in (A + C)(\partial G)$, contradicting $0 \notin (A + C)(\partial G)$.

(ii) Let $t_1, t_2 \in (0, t_0]$ be such that $t_1 < t_2$. Consider the following one-parameter family of operators:

$$H(t,x) := A^{\varphi}_{q(t)}x + Cx, \quad (t,x) \in [0,1] \times \overline{G}.$$

We prove that $H(t,\cdot)$ is a bounded homotopy of type (S_+). The boundedness of $H(\cdot,\cdot)$ follows from Lemma 2.2 and the boundedness of C. Let $\{t_n\} \subset [0,1]$ and $\{x_n\} \subset \overline{G}$ satisfy $t_n \to t_0$ and $x_n \rightharpoonup x_0$ in X, and

$$\limsup_{n\to\infty} \langle A^{\varphi}_{q(t_n)}x_n + Cx_n, x_n - x_0\rangle \leq 0. \tag{2.15}$$

Using the monotonicity of $A^{\varphi}_{q(t)}$ in (2.15), we obtain

$$\limsup_{n\to\infty} \langle A^{\varphi}_{q(t_n)}x_0 + Cx_n, x_n - x_0\rangle \leq 0. \tag{2.16}$$

By Lemma 2.6, we have $A^{\varphi}_{q(t_n)}x_0 \to A^{\varphi}_{q(t_0)}x_0$, and therefore (2.16) yields

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0\rangle \leq 0.$$

Since C is demicontinuous and of type S_+, it follows that $x_n \to x_0$ in X and $Cx_n \rightharpoonup Cx_0$ in X^*. Consequently, we have

$$A^{\varphi}_{q(t_n)}x_n + Cx_n \rightharpoonup A^{\varphi}_{q(t_0)}x_0 + Cx_0$$

as $n \to \infty$. This proves that $H(t,\cdot)$, $t \in [0,1]$, is a homotopy of type (S_+).

(iii) By the invariance of the degree, d_{S_+}, for (S_+)-mappings under the homotopies of type (S_+), we have

$$\mathrm{d}_{S_+}(A^{\varphi}_{t_1}, G, 0) = \mathrm{d}_{S_+}(H(0,\cdot), G, 0) = \mathrm{d}_{S_+}(H(1,\cdot), G, 0) = \mathrm{d}_{S_+}(A^{\varphi}_{t_2}, G, 0).$$

It follows that $\mathrm{d}_{S_+}(A^{\varphi}_t, G, 0)$ exists and is independent of $t \in (0, t_0]$. □

Remark 2.10. Let A, C, G, and p be the same as in Proposition 2.9. When we define a degree mapping of $A + C$, denoted by $\mathrm{D}(A + C, G, p)$, by

$$\mathrm{D}(A + C, G, p) = \lim_{t\to 0^+} \mathrm{d}_{S_+}(A^{\varphi}_t, G, p),$$

we can verify that the degree mapping D has the same four basic properties as the Browder degree in [16]. By the uniqueness of the Browder degree established by Berkovits and Miettunen [12], the degree D coincides with the Browder degree for $A + C$.

By replacing $\hat{T}_t$ everywhere in [5, Lemma 5, Lemma 6, and Lemma 8] with $\hat{A}^{\varphi}_t$ with the gauge function $\varphi(r) = r^{p-1}$ and by following the methodology used in [5] in conjunction with Lemmas2.3, 2.5,2.6, and 2.8, we obtain Proposition 2.11 below. Its proof is omitted here because the method of proof is similar to that in [5] and Proposition 2.9, except for having to deal with $\hat{A}^{\varphi}_t$. For further properties of $L + A + C$ in relation to the following proposition for $\varphi(r) = r$, the reader is referred to Addou and Mermri [1] and Adhikari and Kartsatos[5].

Proposition 2.11. *Let $G \subset X$ be open and bounded. Assume that $L : X \supset D(L) \to X^*$ is linear, maximal monotone with $\overline{D(L)} = X$; $A : X \supset D(A) \to 2^{X^*}$ is strongly quasibounded, maximal monotone with $0 \in A(0)$; and $C(t) : X \supset \overline{G} \to X^*$, $t \in [0,1]$, is a bounded homotopy of type (S_+) with respect to $D(L)$. Suppose that*

$$0 \notin (L + A + C(t))x$$

for all $x \in \partial G \cap D(L) \cap D(A)$. Then the following statements hold.

(i) *There exists* $t_0 > 0$ *such that*
$$\hat{L}x + \hat{A}_t^{\varphi}x + \hat{C}(t)x + tMx \neq 0$$
for all $(t, x) \in [0, 1] \times (\partial G \cap D(L))$ *and* $t < t_0$.

(ii) *For fixed numbers* $t_1, t_2 > 0$, *define* $q(t) := tt_1 + (1-t)t_2$, $t \in [0, 1]$. *Then the operator*
$$\hat{H}(t, x) = \hat{L}x + \hat{A}_{q(t)}^{\varphi}x + \hat{C}(t)x + s(t)Mx,$$
with $(t, x) \in [0, 1] \times (\overline{G} \cap D(L))$, *is a homotopy of type* (S_+).

(iii) *For every sequence* $\{t_n\} \subset (0, \infty)$ *such that* $t_n \to 0$,
$$\lim_{n\to\infty} \mathrm{d}_{S_+}(\hat{L} + \hat{A}_{t_n}^{\varphi} + \hat{C}(t) + t_nM, G, 0)$$
exists and does not depend on the choice of $\{t_n\}$.

3. Existence of nontrivial solutions

Hu and Papageorgiou generalized in [21] the Browder degree theory [16] to the mappings of the form $A+C+T$, where $A : X \supset D(A) \to 2^{X^*}$ is maximal monotone with $0 \in A(0)$, $C : X \supset D(C) \to X^*$ is bounded demicontinuous of type (S_+), and T is of class (P). With an application of the (S_+)-degree developed by Browder [16] and Skrypnik [32], we prove in Theorem 3.3 the existence of nonzero solutions of $Ax + Cx + Tx \ni 0$ when $A + C + T$ satisfies certain boundary conditions, and the operator A, in addition, is positively homogeneous of degree $\gamma > 0$. This result extends the existence result for $\gamma \in (0, 1]$ in [2] to $\gamma > 0$ (see also [6, Theorem 6] for $\gamma = 1$).

The following lemma, which is crucial to the existence results in this section, shows that positively homogeneous maximal monotone operators transmit the homogeneity into their Yosida approximants corresponding to J_φ with $\varphi(r) = r^{p-1}$, $p > 1$, and a suitable value of p.

Lemma 3.1. *Let* $A : X \supset D(A) \to 2^{X^*}$ *be maximal monotone and positively homogeneous of degree* $\gamma > 0$. *Then, for each* $t > 0$, *the Yosida approximant* A_t^{φ} *corresponding to the gauge function* $\varphi(r) = r^{p-1}$, $p > 1$, *satisfies*
$$A_t^{\varphi}(sx) = \begin{cases} s^{\gamma}A_{ts^{\gamma+1-p}}^{\varphi}x & \text{for } (s, x) \in (\mathbb{R}_+ \setminus \{0\}) \times X \\ 0 & \text{for } (s, x) \in \{0\} \times X. \end{cases} \tag{3.1}$$
Consequently, if $p = \gamma + 1$, *then* A_t^{φ} *is positively homogeneous of degree* γ, *i.e.,*
$$A_t^{\varphi}(sx) = s^{\gamma}A_t^{\varphi}x \quad \text{for all } (s, x) \in \mathbb{R}_+ \times X.$$

Proof. Let $t > 0$ be fixed. The case $s = 0$ is trivial. Assume $s > 0$, and let
$$y = A_t^{\varphi}(sx) = (A^{-1} + t^{q-1}J_{\varphi}^{-1})^{-1}(sx), \quad x \in X,$$
where q satisfies $1/p + 1/q = 1$. Then
$$y \in A(-t^{q-1}J_{\varphi}^{-1}y + sx) = A\left(s\left(-t^{q-1}s^{-1}J_{\varphi}^{-1}y + x\right)\right).$$
This means
$$\left(s\left(-t^{q-1}s^{-1}J_{\varphi}^{-1}y + x\right),\ y\right) \in \mathrm{Gr}(A).$$
Since A is positively homogeneous of degree $\gamma > 0$, we obtain
$$\left(-t^{q-1}s^{-1}J_{\varphi}^{-1}y + x,\ s^{-\gamma}y\right) \in \mathrm{Gr}(A),$$

i.e.,

$$s^{-\gamma}y \in A\left(-t^{q-1}s^{-1}J_\varphi^{-1}y + x\right).$$

In view of (1.1), we have

$$s^{-\gamma(1-q)}J_\varphi^{-1}(s^{-\gamma}y) = J_\varphi^{-1}y,$$

and therefore

$$s^{-\gamma}y \in A\left(-t^{q-1}s^{\gamma(q-1)-1}J_\varphi^{-1}(s^{-\gamma}y) + x\right).$$

This implies

$$x \in \left(A^{-1} + t^{q-1}s^{\gamma(q-1)-1}J_\varphi^{-1}\right)(s^{-\gamma}y).$$

Using

$$t^{q-1}s^{\gamma(q-1)-1} = (ts^\gamma)^{q-1}\left(s^{1-p}\right)^{q-1} = \left(ts^{\gamma+1-p}\right)^{q-1},$$

we obtain

$$y = s^\gamma\left(A^{-1} + \left(ts^{\gamma+1-p}\right)^{q-1}J_\varphi^{-1}\right)^{-1}x = s^\gamma A^\varphi_{ts^{\gamma+1-p}}x.$$

Thus, we have

$$A_t^\varphi(sx) = s^\gamma A^\varphi_{ts^{\gamma+1-p}}x.$$

Clearly, A_t^φ is positively homogeneous of degree γ if $p = \gamma + 1$. □

Remark 3.2. In the settings of Lemma 3.1 with $p = \gamma + 1$, it follows from (2.3) that the resolvent J_t^φ is positively homogeneous of degree 1 in the following sense: for each $t > 0$, we have $J_t^\varphi(sx) = sJ_t^\varphi x$ for all $x \in X$ and all $s \geq 0$.

Theorem 3.3. *Assume that $G_1, G_2 \subset X$ are open, bounded with $0 \in G_2$ and $\overline{G_2} \subset G_1$. Let $A : X \supset D(A) \to 2^{X^*}$ be maximal monotone and positively homogeneous of degree $\gamma > 0$ with $A(0) = \{0\}$; $C : \overline{G}_1 \to X^*$ bounded, demicontinuous and of type (S_+); and $T : \overline{G}_1 \to 2^{X^*}$ of class (P). Assume, further, that*

- (H1) *there exists $v_0^* \in X^* \setminus \{0\}$ such that $Ax + Cx + Tx \not\ni \lambda v_0^*$ for all $(\lambda, x) \in \mathbb{R}_+ \times (D(A) \cap \partial G_1)$, and*
- (H2) *$Ax + Cx + Tx + \lambda Jx \not\ni 0$ for all $(\lambda, x) \in \mathbb{R}_+ \times (D(A) \cap \partial G_2)$.*

Then the inclusion $Ax + Cx + Tx \ni 0$ has a nonzero solution $x \in D(A) \cap (G_1 \setminus G_2)$.

Proof. To study the solvability of the inclusion

$$Ax + Cx + Tx \ni 0,\ x \in \overline{G}_1,$$

we consider the associated approximate equation

$$A_t^\varphi x + Cx + q_\epsilon x = 0,\ t > 0, x \in \overline{G}_1, \epsilon > 0. \tag{3.2}$$

Here, the gauge function is taken to be $\varphi(r) = r^{p-1}$, $1 < p < \infty$ so that $\gamma = p - 1$, and $q_\epsilon : \overline{G_1} \to X^*$ is an approximate continuous Cellina-selection as in [9, Lemma 6] and [21] satisfying $q_\epsilon x \in T(B_\epsilon(x) \cap \overline{G_1}) + B_\epsilon(0)$ for all $x \in \overline{G_1}$ and $q_\epsilon(\overline{G_1}) \subset \overline{\operatorname{co} T(\overline{G_1})}$.

We show that the equation (3.2) has a solution $x_{t,\epsilon}$ in $G_1 \setminus G_2$ for all sufficiently small t and ϵ. To this end, we first show that there exist $\tau_0 > 0$, $t_0 > 0$ and $\epsilon_0 > 0$ such that the equation

$$A_t^\varphi x + Cx + q_\epsilon x = \tau v_0^* \tag{3.3}$$

has no solution in G_1 for every $\tau \geq \tau_0$, $t \in (0, t_0]$ and $\epsilon \in (0, \epsilon_0]$.

Assuming the contrary, let $\{\tau_n\} \subset (0, \infty)$, $\{t_n\} \subset (0, \infty)$, $\{\epsilon_n\} \subset (0, \infty)$ and $\{x_n\} \subset G_1$ be such that $\tau_n \to \infty$, $t_n \to 0$, $\epsilon_n \to 0$ and

$$A_{t_n}^\varphi x_n + Cx_n + q_{\epsilon_n}x_n = \tau_n v_0^*. \tag{3.4}$$

We can assume that $q_{\epsilon_n} x_n \to g^* \in X^*$ in view of the properties of T. Then $\|A^\varphi_{t_n} x_n\| \to \infty$ as $\|\tau_n v_0^*\| \to \infty$ and $\{Cx_n\}$ is bounded. Thus, from (3.4), we obtain

$$\frac{A^\varphi_{t_n} x_n}{\|A^\varphi_{t_n} x_n\|} + \frac{Cx_n}{\|A^\varphi_{t_n} x_n\|} + \frac{q_{\epsilon_n} x_n}{\|A^\varphi_{t_n} x_n\|} = \frac{\tau_n}{\|A^\varphi_{t_n} x_n\|} v_0^*. \tag{3.5}$$

This implies

$$\frac{\tau_n \|v_0^*\|}{\|A^\varphi_{t_n} x_n\|} \to 1 \quad \text{so that} \quad \frac{\tau_n}{\|A^\varphi_{t_n} x_n\|} \to \frac{1}{\|v_0^*\|} \quad \text{as } n \to \infty. \tag{3.6}$$

Since $p - 1 = \gamma$, by Lemma 3.1, A^φ_t is also homogeneous of degree $\gamma = p - 1$, and therefore we obtain

$$\frac{A^\varphi_{t_n} x_n}{\|A^\varphi_{t_n} x_n\|} = A^\varphi_{t_n}\Big(\frac{x_n}{\|A^\varphi_{t_n} x_n\|^{1/\gamma}}\Big). \tag{3.7}$$

Let $u_n = x_n / \|A^\varphi_{t_n} x_n\|^{1/\gamma}$. It is clear that $u_n \to 0$. In view of (3.5), (3.6), and (3.7), we obtain $A^\varphi_{t_n} u_n \to h$ with $h = v_0^*/\|v_0^*\|$. This implies

$$\lim_{n\to\infty} \langle A^\varphi_{t_n} u_n, u_n\rangle = \langle h, 0\rangle = 0.$$

Since $t_n \to 0$, by (ii) of Lemma 2.3 with $S = 0$ we obtain $0 \in D(A)$ and $h \in A(0) = \{0\}$, a contradiction to $\|h\| = 1$.

We now consider the homotopy mapping

$$H_1(s, x, t, \epsilon) = A^\varphi_t x + Cx + q_\epsilon x - s\tau_0 v_0^*, \quad s \in [0, 1],\ x \in \overline{G_1}, \tag{3.8}$$

where $t \in (0, t_0]$ and $\epsilon \in (0, \epsilon_0]$ are fixed. By following the arguments as in [2, Theorem 3.1], we can show that, for every $s \in [0, 1]$ the operator $x \mapsto Cx - s\tau_0 v_0^*$ is bounded, demicontinuous and of type (S_+) on $\overline{G_1}$, and that the equation $H_1(s, x, t, \epsilon) = 0$ has no solution in ∂G_1 for all sufficiently small $t \in (0, t_0]$, $\epsilon \in (0, \epsilon_0]$ and all $s \in [0, 1]$. In doing this, we need to use Lemma 2.3. The details are omitted.

It follows from Proposition 2.9 that the mapping $H_1(s, x, t, \epsilon)$ is an admissible homotopy for the degree, d_{S_+}, of (S_+)-mappings, and $\mathrm{d}_{S_+}(H_1(s, \cdot, t, \epsilon), G_1, 0)$ is well-defined and is a constant for all $s \in [0, 1]$ and for all $t \in (0, t_0]$, $\epsilon \in (0, \epsilon_0]$.

Assume that

$$\mathrm{d}_{S_+}(H_1(1, \cdot, t_1, \epsilon_1), G_1, 0) \neq 0,$$

for some sufficiently small $t_1 \in (0, t_0]$ and $\epsilon_1 \in (0, \epsilon_0]$. Then the equation

$$A^\varphi_{t_1} x + Cx + g_{\epsilon_1} x = \tau_0 v_0^*$$

has a solution in G_1. However, this contradicts our choice of the number τ_0 in (3.3). Consequently,

$$\mathrm{d}_{S_+}(A^\varphi_t + C + q_\epsilon, G_1, 0) = \mathrm{d}_{S_+}(H_1(0, \cdot, t, \epsilon), G_1, 0) = 0, \quad t \in (0, t_0],\ \epsilon \in (0, \epsilon_0].$$

We next consider the homotopy mapping

$$H_2(s, x, t, \epsilon) = s(A^\varphi_t x + Cx + q_\epsilon x) + (1 - s)Jx, \quad (s, x) \in [0, 1] \times \overline{G_2}. \tag{3.9}$$

We claim that there exist $t_1 \in (0, t_0]$ and $\epsilon_1 \in (0, \epsilon_0]$ such that $H_2(s, x, t, \epsilon) = 0$ has no solution on ∂G_2 for any $s \in [0, 1]$, any $t \in (0, t_1]$ and any $\epsilon \in (0, \epsilon_1]$. To prove the claim, we assume the contrary and then follow the argument used in [2, Theorem 3.1] along with the properties of A^φ_t established in Lemma 2.3 to arrive at a contradiction to (H2). For the sake of convenience, we assume that t_0 and ϵ_0 are sufficiently small so that we may take $t_1 = t_0$ and $\epsilon_1 = \epsilon_0$.

It follows from Proposition 2.9 that $H_2(s,x,t,\epsilon)$ is an admissible homotopy for the degree of (S_+)-mappings and $\mathrm{d}_{S_+}(H_2(s,\cdot,t,\epsilon),G_2,0)$ is well-defined and constant for all $s\in[0,1]$, all $t\in(0,t_0]$ and all $\epsilon\in(0,\epsilon_0]$. By the invariance of the (S_+)-degree, for all $t\in(0,t_0]$ and $\epsilon\in(0,\epsilon_0]$, we have

$$\begin{aligned}\mathrm{d}_{S_+}(H_2(1,\cdot,t,\epsilon),G_2,0)&=\mathrm{d}_{S_+}(A_t^\varphi+C+q_\epsilon,G_2,0)\\&=\mathrm{d}_{S_+}(H_2(0,\cdot,t,\epsilon),G_2,0)\\&=\mathrm{d}_{S_+}(J,G_2,0)=1.\end{aligned}$$

Thus, for all $t\in(0,t_0]$, $\epsilon\in(0,\epsilon_0]$, we have

$$\mathrm{d}_{S_+}(A_t^\varphi+C+q_\epsilon,G_1,0)\neq\mathrm{d}_{S_+}(A_t^\varphi+C+q_\epsilon,G_2,0).$$

Using the excision property of the (S_+)-degree, which is an easy consequence of its finite-dimensional approximations, for every $t\in(0,t_0]$ and every $\epsilon\in(0,\epsilon_0]$, there exists a solution $x_{t,\epsilon}\in G_1\setminus G_2$ of $A_t^\varphi x+Cx+q_\epsilon x=0$. Let $t_n\in(0,t_0]$ and $\epsilon_n\in(0,\epsilon_0]$ be such that $t_n\to0$, $\epsilon_n\to0$ and let $x_n\in G_1\setminus G_2$ be the corresponding solutions of $A_t^\varphi x+Cx+q_\epsilon x=0$, i.e.,

$$A_{t_n}^\varphi x_n+Cx_n+q_{\epsilon_n}x_n=0.$$

We may assume that $x_n\rightharpoonup x_0$ in X and $q_{\epsilon_n}x_n\to g^*\in X^*$. We observe that

$$\langle A_{t_n}^\varphi x_n,x_n-x_0\rangle=-\langle Cx_n+q_{\epsilon_n}x_n,x_n-x_0\rangle.$$

If

$$\limsup_{n\to\infty}\langle Cx_n+q_{\epsilon_n}x_n,x_n-x_0\rangle>0,$$

then we obtain a contradiction from (i) of Lemma 2.3 with $S=0$ there. Consequently,

$$\limsup_{n\to\infty}\langle Cx_n+q_{\epsilon_n}x_n,x_n-x_0\rangle\le0,$$

and hence

$$\limsup_{n\to\infty}\langle Cx_n,x_n-x_0\rangle\le0.$$

By the (S_+)-property of C, we obtain $x_n\to x_0\in\overline{G_1\setminus G_2}$. Then $Cx_n\rightharpoonup Cx_0$ and $A_{t_n}^\varphi x_n\rightharpoonup -Cx_0-g^*$. Using this in (ii) of Lemma 2.3 with $S=0$ there, we obtain $x_0\in D(A)$ and $-Cx_0-g^*\in Ax_0$. By a property of the selection $q_{\epsilon_n}x_n$ as in Hu and Papageorgiou [21], we have $g^*\in Tx_0$, and therefore $Ax_0+Cx_0+Tx_0\ni0$. We also have

$$x_0\in\overline{G_1\setminus G_2}=(G_1\setminus G_2)\cup\partial(G_1\setminus G_2)\subset(G_1\setminus G_2)\cup\partial G_1\cup\partial G_2.$$

By (H1) and (H2), we have $x_0\notin\partial G_1\cup\partial G_2$, and hence $x_0\in D(A)\cap(G_1\setminus G_2)$. □

Remark 3.4. We point out that the condition $A(0)=\{0\}$ on the homogeneous maximal monotone operator A used in Theorem 3.3 is rather mild in view of Rockafellar's result [29] which says that a monotone map is locally bounded at every point in the interior of its domain.

The existence of nonzero solutions of $Lx+Ax+Cx\ni0$, where the maximal monotone operator A is strongly quasibounded and positively homogeneous of degree $\gamma=1$, is established in [2]. In the following theorem, we extend this result to an arbitrary degree $\gamma>0$ for the same combination of operators in the spirit of the Berkovits-Mustonen theory in [11] and the theories developed in [6]. We recall that the maximal monotone operator A investigated in [6] is strongly quasibounded. However, by a result of Hess [20], a strongly quasibounded and positively homogeneous operator of degree $\gamma > 0$ is necessarily bounded. Therefore, in the following theorem, we assume that the maximal monotone operator A is bounded.

Theorem 3.5. *Assume that $G_1, G_2 \subset X$ are open, bounded with $0 \in G_2$ and $\overline{G_2} \subset G_1$. Let $L : X \supset D(L) \to X^*$ be linear maximal monotone with $\overline{D(L)} = X$, and $A : X \supset D(A) \to 2^{X^*}$ bounded, maximal monotone and positively homogeneous of degree $\gamma > 0$. Also, let $C : \overline{G_1} \to X^*$ be bounded, demicontinuous and of type (S_+) with respect to $D(L)$. Moreover, assume that*

- (H3) *there exists $v^* \in X^* \setminus \{0\}$ such that $Lx + Ax + Cx \not\ni \lambda v^*$ for all $(\lambda, x) \in \mathbb{R}_+ \times (D(L) \cap D(A) \cap \partial G_1)$, and*
- (H4) *$Lx + Ax + Cx + \lambda Jx \not\ni 0$ for all $(\lambda, x) \in \mathbb{R}_+ \times (D(L) \cap D(A) \cap \partial G_2)$.*

Then the inclusion $Lx + Ax + Cx \ni 0$ has a solution $x \in D(L) \cap D(A) \cap (G_1 \setminus G_2)$.

Proof. We begin by observing that a positively homogeneous and bounded maximal monotone operator A of degree $\gamma > 0$ satisfies $0 \in D(A)$ and $A(0) = \{0\}$. To solve the inclusion

$$Lx + Ax + Cx \ni 0, \quad x \in \overline{G_1}, \tag{3.10}$$

let us consider the associated equation

$$\hat{L}x + \hat{A}_t^{\varphi}x + \hat{C}x + tMx = 0, \quad t \in (0, \infty),\ x \in j^{-1}(\overline{G_1}). \tag{3.11}$$

Here, the gauge function is $\varphi(r) = r^{p-1}$, $1 < p < \infty$, and $\gamma = p-1$. We can show as in [5, Lemma 5] that there exists $R > 0$ such that the open ball $B_Y(0, R)$ contains all the solutions of (3.11). We recall that $Y = D(L)$.

We shall prove that (3.11) has a solution $x_t \in j^{-1}(G_1 \setminus G_2)$ for all sufficiently small $t > 0$. We first claim that there exist $\tau_0 > 0$, $t_0 > 0$ such that

$$\hat{L}x + \hat{A}_t^{\varphi}x + \hat{C}x + tMx = \tau j^* v^* \tag{3.12}$$

has no solution in $G_R^1(Y) := j^{-1}(G_1) \cap B_Y(0, R)$ for all $t \in (0, t_0]$ and all $\tau \in [\tau_0, \infty)$. Assume the contrary and let $\{\tau_n\} \subset (0, \infty)$, $\{t_n\} \subset (0, 1)$ and $\{x_n\} \subset G_R^1(Y)$ such that $\tau_n \to \infty$, $t_n \to 0$ and

$$\hat{L}x_n + \hat{A}_{t_n}^{\varphi}x_n + \hat{C}x_n + t_n Mx_n = \tau_n j^* v^*. \tag{3.13}$$

We note that j^* is one-to-one because $j(Y) = Y$, which is dense in X. This implies that j^*v^* is nonzero, and therefore $\|\tau_n j^* v^*\|_{Y^*} \to +\infty$. Also, the sequence $\{x_n\}$ is bounded in Y and so we may assume that $x_n \rightharpoonup x_0$ in X and $Lx_n \rightharpoonup Lx_0$ in X^*. In particular, $\{Lx_n\}$ is bounded in X^*. Since $Mx_n \in j^*(X^*)$, we have $J^{-1}(Lu) \in D(L^*)$ and

$$Mx_n = j^* L^* J^{-1}(Lx_n).$$

Since j^*, L^*, J^{-1} are bounded, we have the boundedness of $\{Mx_n\}$. It is clear that $\hat{L}x_n$ and $\hat{C}x_n$ are bounded in Y^*, and therefore (3.13) implies that $\|\hat{A}_{t_n}^{\varphi}x_n\|_{Y^*} \to \infty$. Since A is positively homogeneous of degree γ, applying Lemma 3.1 for $\gamma = p-1$ shows that each $A_{t_n}^{\varphi}$ is also positively homogeneous of $\gamma = p-1$. Consequently,

$$\frac{\hat{A}_{t_n}^{\varphi}x_n}{\|\hat{A}_{t_n}^{\varphi}x_n\|_{Y^*}} = \hat{A}_{t_n}^{\varphi}\Big(\frac{x_n}{\|\hat{A}_{t_n}^{\varphi}x_n\|_{Y^*}^{1/\gamma}}\Big) \tag{3.14}$$

for all n. Define $\beta_n := 1/\|\hat{A}^{\varphi}_{t_n} x_n\|_{Y^*}$ and $\delta_n := \beta_n^{1/\gamma}$. Since $\|\hat{A}^{\varphi}_{t_n} x_n\|_{Y^*} \to \infty$, it follows that $\beta_n x_n \to 0$ and $\delta_n x_n \to 0$ in X as $n \to \infty$. From (3.13) and (3.14), we find

$$\hat{L}(\beta_n x_n) + \hat{A}^{\varphi}_{t_n}(\delta_n x_n) + \beta_n \hat{C} x_n + t_n \beta_n M x_n = \tau_n \beta_n j^* v^*. \tag{3.15}$$

Because $\|\hat{A}^{\varphi}_{t_n}(\delta_n x_n)\|_{Y^*} = 1$ and the remaining terms on the left in (3.15) converge to 0 in X^* as $n \to \infty$, we obtain $\tau_n \beta_n \to 1/\|j^* v^*\|_{Y^*}$, and therefore $\hat{A}^{\varphi}_{t_n}(\delta_n x_n) \to y_0$, where $y_0 = j^* v_* / \|j^* v^*\|_{Y^*}$. Since $u_n := \delta_n x_n \to 0$ as $n \to \infty$, we have $\langle \hat{A}^{\varphi}_{t_n} u_n, u_n \rangle \to \langle y_0, 0 \rangle = 0$ as $n \to \infty$. By Lemma 2.3, (ii), we have $y_0 \in A(0) = \{0\}$, which is a contradiction to $\|y_0\|_{Y^*} = 1$.

We now consider the homotopy $H : [0,1] \times Y \to Y^*$ defined by

$$H(s,x) = \hat{L}x + \hat{A}^{\varphi}_t x + \hat{C}x + tMx - s\tau_0 j^* v^*, \quad s \in [0,1],\ x \in j^{-1}(\overline{G_1}), \tag{3.16}$$

where $t \in (0, t_0]$ is fixed. It can be easily seen that $C - s\tau_0 v^*$ is bounded demicontinuous on $\overline{G_1}$ and of type (S_+) with respect to $D(L)$.

We now show that the equation $H(s,x) = 0$ has no solution on the boundary $\partial G^1_R(Y)$. Here, the number $R > 0$ is increased, if necessary, so that the ball $B_Y(0,R)$ now also contains all the solutions x of $H(s,x) = 0$. To this end, assume the contrary so that there exist $\{t_n\} \subset (0, t_0]$, $\{s_n\} \subset [0,1]$, and $\{x_n\} \subset \partial G^1_R(Y)$ such that $t_n \to 0$, $s_n \to s_0$, $x_n \rightharpoonup x_0$ in Y, $A^{\varphi}_{t_n} x_n \rightharpoonup w^*$ in X^*, $Cx_n \rightharpoonup c^*$ and

$$\hat{L}x_n + \hat{A}^{\varphi}_{t_n} x_n + \hat{C}x_n + t_n M x_n = s_n \tau_0 j^* v^*. \tag{3.17}$$

Here, the boundedness of $\{A^{\varphi}_{t_n} x_n\}$ follows as in Step I of [3, Proposition 1], except that we now use $A^{\varphi}_{t_n}$ in place of the operators T_{s_n} used in [3]. Since $x_n \rightharpoonup x_0$ in Y, we have $x_n \rightharpoonup x_0$ in X and $Lx_n \rightharpoonup Lx_0$ in X^*. Also, since $x_n \in B_Y(0,R)$ and

$$\partial(j^{-1}(G_1) \cap B_Y(0,R)) \subset \partial(j^{-1}(G_1)) \cup \partial B_Y(0,R) \subset j^{-1}(\partial G_1) \cup \partial B_Y(0,R),$$

we have $x_n \in j^{-1}(\partial G_1) = \partial G_1 \cap Y \subset \partial G_1$. We now follow the arguments as in [2, Theorem 2.2] in conjunction with Lemma 2.3 to arrive at

$$\langle Lx_0 + w^* + Cx_0 - s_0 \tau_0 v^*, u \rangle = 0$$

for all $u \in Y$, where $x_0 \in D(A)$ and $w^* \in Ax_0$. Since Y is dense in X, we have $Lx_0 + Tx_0 + Cx_0 \ni s_0 \tau_0 v^*$, which contradicts the hypothesis (H3) because $x_0 \in D(L) \cap D(T) \cap \partial G_1$.

We shrink t_0, if necessary, so that

$$H(s,x) = 0, \quad s \in [0,1],\ x \in \overline{G^1_R(Y)}$$

has no solution on the boundary $\partial G^1_R(Y)$ for all $t \in (0, t_0]$ and all $s \in [0,1]$. It now follows from Proposition 2.11 that $H(s,x)$ is an admissible homotopy for the (S_+)-degree, d_{S_+}, and therefore $\mathrm{d}_{S_+}(H(s,\cdot), G^1_R(Y), 0)$, is well-defined and remains constant for all $s \in [0,1]$. Also, by Proposition 2.11, the limit

$$\lim_{t \to 0+} \mathrm{d}_{S_+}(H(1,\cdot), G^1_R(Y), 0)$$

exists. By shrinking t_0 further, if necessary, we find that $\mathrm{d}_{S_+}(H(1,\cdot), G^1_R(Y), 0) =$ a constant for all $t \in (0, t_0]$. Suppose, if possible, that

$$\mathrm{d}_{S_+}(H(1,\cdot), G^1_R(Y), 0) \neq 0$$

for some $t_1 \in (0, t_0]$. Then there exists $x_0 \in G^1_R(Y)$ such that

$$\hat{L}x + \hat{A}^{\varphi}_{t_1} x + \hat{C}x + t_1 Mx = \tau_0 j^* v^*.$$

This contradicts the choice of τ_0 as stated in (3.12). Since

$$\mathrm{d}_{S_+}(H(0,\cdot), G_R^1(Y), 0) = \mathrm{d}_{S_+}(H(1,\cdot), G_R^1(Y), 0),$$

we have

$$\mathrm{d}_{S_+}(\hat{L} + \hat{A}_t^{\varphi} + \hat{C} + tM, G_R^1(Y), 0) = \mathrm{d}_{S_+}(H(0,\cdot), G_R^1(Y), 0) = 0 \tag{3.18}$$

for all $t \in (0, t_0]$.

Next, we consider the homotopy $\widetilde{H} : [0,1] \times Y \to Y^*$ defined by

$$\widetilde{H}(s,x) = s(\hat{L}x + \hat{A}_t^{\varphi}x + \hat{C}x) + tMx + (1-s)\hat{J}x, \quad s \in [0,1],\ x \in j^{-1}(\overline{G_2}).$$

As in [3, Step III, p. 29], it can be shown that there exists $t_0 > 0$ (shrink it to a smaller number if necessary) such that all the solutions of

$$\widetilde{H}(s,x) = 0, \quad t \in (0, t_0],\ s \in [0,1]$$

are bounded in Y. We enlarge the previous number $R > 0$, if necessary, so that all solutions of $\widetilde{H}(s,x) = 0$ as described above are contained in $B_Y(0,R)$ in Y.

Again, by following arguments similar to that in [2, Theorem 2.2], we can show the existence of $t_1 \in (0, t_0]$ such that the equation $\widetilde{H}(s,x) = 0$ has no solutions on $\partial G_R^2(Y)$ for any $t \in (0, t_1]$ and any $s \in [0,1]$. Here, $G_R^2(Y) := j^{-1}(G_2) \cap B_Y(0,R)$. In fact, if we assume the contrary, we can arrive at a situation that contradicts (H4). At this point, we replace the number t_0 chosen previously with t_1 and call it t_0 again. Let us fix $t \in (0, t_0]$ and consider the homotopy equation

$$\widetilde{H}(s,x) = s(\hat{L}x + \hat{A}_t^{\varphi}x + \hat{C}x) + tMx + (1-s)\hat{J}x = 0, \quad s \in [0,1],\ x \in \overline{G_R^2(Y)}. \tag{3.19}$$

It is already discussed that (3.19) has no solution on $\partial G_R^2(Y)$. We note that $\widetilde{H}$ is an affine homotopy of bounded demicontinuous operators of type (S_+) on $\overline{G_R^2(Y)}$; namely, $\hat{L} + \hat{A}_t^{\varphi} + \hat{C} + tM$ and $tM + \hat{J}$. We also note here that $tM + \hat{J}$ is strictly monotone. In view of Proposition 2.11, it follows that $\widetilde{H}(s,x)$ is an admissible homotopy for the (S_+)-degree, d_{S_+}, which satisfies

$$\mathrm{d}_{S_+}(\widetilde{H}(1,\cdot), G_R^2(Y), 0) = \mathrm{d}_{S_+}(\widetilde{H}(0,\cdot), G_R^2(Y), 0). \tag{3.20}$$

This implies

$$\mathrm{d}_{S_+}(\hat{L} + \hat{A}_t^{\varphi} + \hat{C} + tM, G_R^2(Y), 0) = \mathrm{d}_{S_+}(tM + \hat{J}, G_R^2(Y), 0) = 1 \tag{3.21}$$

for all $t \in (0, t_0]$. The last equality follows from [15, Theorem 3, (iv)]. From (3.18) and (3.21), we obtain

$$\mathrm{d}_{S_+}(\hat{L} + \hat{A}_t^{\varphi} + \hat{C} + tM, G_R^1(Y), 0) \neq \mathrm{d}_{S_+}(\hat{L} + \hat{A}_t^{\varphi} + \hat{C} + tM, G_R^2(Y), 0)$$

for all $t \in (0, t_0]$. By the excision property of the (S_+)-degree, for each $t \in (0, t_0]$, there exists a solution $x_t \in G_R^1(Y) \setminus G_R^2(Y)$ of the equation

$$\hat{L}x + \hat{A}_t^{\varphi}x + \hat{C}x + tMx = 0.$$

We now pick a sequence $\{t_n\} \subset (0, t_0]$ such that $t_n \to 0$ and denote the corresponding solution x_t by x_n, i.e.,

$$\hat{L}x_n + \hat{A}_{t_n}^{\varphi}x_n + \hat{C}x_n + t_nMx_n = 0.$$

Since Y is reflexive, we have $x_n \rightharpoonup x_0 \in Y$ by passing to a subsequence. This implies $x_n \rightharpoonup x_0$ in X and $Lx_n \rightharpoonup Lx_0$ in X^*. By the boundedness (therefore strong quasiboundedness) of A, we may assume, in view of Lemma 2.5, that $A_{t_n}^{\varphi}x_n \rightharpoonup w^* \in X^*$. By a standard argument in conjunction with Lemma 2.3 and the (S_+)-property of C with respect to $D(L)$, we obtain $x_n \to x_0 \in \overline{G_R^1(Y) \setminus G_R^2(Y)}$. By Lemma 2.3 and the demicontinuity of C, we have $x_0 \in D(A)$, $w^* \in Ax_0$, and $Cx_n \rightharpoonup Cx_0$ in X^*. Thus, $Lx_0 + Ax_0 + Cx_0 \ni 0$.

Finally, to show $x_0 \in G_1 \setminus G_2$, we note that

$$G_R^1(Y) \setminus G_R^2(Y) = (G_1 \setminus G_2) \cap Y \cap B_Y(0, R) \subset G_1 \setminus G_2.$$

Consequently, $x_n \in G_1 \setminus G_2$ for all n, and therefore

$$x_0 \in \overline{G_1 \setminus G_2} \subset (G_1 \setminus G_2) \cup \partial(G_1 \setminus G_2) \subset (G_1 \setminus G_2) \cup \partial G_1 \cup \partial G_2.$$

By (H3) and (H4), $x_0 \notin \partial G_1 \cup \partial G_2$ and hence $x_0 \in D(L) \cap D(T) \cap (G_1 \setminus G_2)$. □

3.1. **Open Problem.** Does Theorem 3.5 hold true if the boundedness of A is dropped? Since a positively homogeneous operator that is strongly quasibounded is necessarily bounded, it is desirable to determine whether Theorem 3.5 holds if A is assumed to be "quasibounded". An operator $A : X \supset D(A) \to 2^{X^*}$ is said to be *quasibounded* if for every $S > 0$ there exists $K(S) > 0$ such that $\|x\| \le S$ and $\langle x^*, x\rangle \le S\|x\|$ for some $x^* \in Ax$ imply $\|x^*\| \le K(S)$. The notions of quasibounded and strongly quasibounded operators were introduced in Hess [20].

4. Applications

In this section, we apply Theorems 3.3 and 3.5 to elliptic and parabolic boundary value problems in general divergence form which are obtained by modifying relevant examples from Berkovits and Mustonen [11], Kittilä [27], and Adhikari [2].

Application 4.1. We consider the space $X = W_0^{m,p}(\Omega)$ with the integer $m \ge 1$, the number $p \in (1, \infty)$, and the domain $\Omega \subset \mathbb{R}^N$ with smooth boundary. We let N_0 denote the number of all multi-indices $\alpha = (\alpha_1, \dots, \alpha_N)$ such that $|\alpha| = \alpha_1 + \dots + \alpha_N \le m$. For $\xi = (\xi_\alpha)_{|\alpha|\le m} \in \mathbb{R}^{N_0}$, we have a representation $\xi = (\eta, \zeta)$, where $\eta = (\eta_\alpha)_{|\alpha|\le m-1} \in \mathbb{R}^{N_1}$, $\zeta = (\zeta_\alpha)_{|\alpha|=m} \in \mathbb{R}^{N_2}$ and $N_0 = N_1 + N_2$. We let

$$\xi(u) = (D^\alpha u)_{|\alpha|\le m}, \quad \eta(u) = (D^\alpha u)_{|\alpha|\le m-1}, \quad \text{and} \quad \zeta(u) = (D^\alpha u)_{|\alpha|=m},$$

where $D^\alpha = \prod_{i=1}^N \left(\frac{\partial}{\partial x_i}\right)^{\alpha_i}$. We write $\nabla u := (D^\alpha u)_{|\alpha|=1}$, and when $|\alpha| = k \in \{1, 2, \dots, m\}$, we simply write $D^k u := (D^\alpha u)_{|\alpha|=k}$. Also, define $q := p/(p-1)$.

We now consider the partial differential expression in divergence form

$$\sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \xi(u)), \quad x \in \Omega.$$

The functions $A_\alpha : \Omega \times \mathbb{R}^{N_0} \to \mathbb{R}$ are assumed to be Carathéodory, i.e., each $A_\alpha(x, \xi)$ is measurable in x for fixed $\xi \in \mathbb{R}^{N_0}$ and continuous in ξ for almost all $x \in \Omega$. We assume the following conditions on A_α:

(H5) There exist $p \in (1, \infty)$, $c_1 > 0$, and $\kappa_1 \in L^q(\Omega)$ such that

$$|A_\alpha(x, \xi)| \le c_1|\xi|^{p-1} + \kappa_1(x), \quad x \in \Omega, \;\; \xi \in \mathbb{R}^{N_0}, \;\; |\alpha| \le m.$$

(H6) The Leray-Lions condition

$$\sum_{|\alpha|=m} [A_\alpha(x, \eta, \zeta_1) - A_\alpha(x, \eta, \zeta_2)](\zeta_{1_\alpha} - \zeta_{2_\alpha}) > 0$$

is satisfied for every $x \in \Omega$, $\eta \in \mathbb{R}^{N_1}$ and $\zeta_1, \zeta_2 \in \mathbb{R}^{N_2}$ with $\zeta_1 \ne \zeta_2$.

(H7)
$$\sum_{|\alpha|\le m}[A_\alpha(x,\xi_1)-A_\alpha(x,\xi_2)](\xi_{1_\alpha}-\xi_{2_\alpha})\ge 0$$
is satisfied for every $x\in\Omega$ and $\xi_1,\xi_2\in\mathbb{R}^{N_0}$.

(H8) There exist $c_2>0$, $\kappa_2\in L^1(\Omega)$ such that
$$\sum_{|\alpha|\le m}A_\alpha(x,\xi)\xi_\alpha\ge c_2|\xi|^p-\kappa_2(x),\quad x\in\Omega,\ \xi\in\mathbb{R}^{N_0}.$$

(H9) Each $A_\alpha(x,\xi)$ is homogeneous of degree $\gamma>0$ with respect to ξ.

If an operator $A:W_0^{m,p}(\Omega)\to W^{-m,q}(\Omega)$ is given by
$$\langle Au,v\rangle=\int_\Omega\sum_{|\alpha|\le m}A_\alpha(x,\xi(u))D^\alpha v,\quad u,\,v\in W_0^{m,p}(\Omega),\tag{4.1}$$
then the conditions (H5), (H7) imply that A is bounded, continuous, and monotone as discussed in Kittilä [27, pp. 25-26] and Pascali and Sburlan [28, pp. 274-275]. Since A is continuous, it is maximal monotone. Moreover, the condition (H9) implies that A is positively homogeneous of degree $\gamma>0$. For example, for $m=1$, we have $|\alpha|\le 1$, and when
$$A_\alpha(x,\eta,\zeta)=\begin{cases}|\zeta|^{p-2}\zeta_\alpha & \text{for } |\alpha|=1\\ 0 & \text{for } |\alpha|=0,\end{cases}$$
the operator A in (4.1) is given by $A:=-\Delta_p$, where Δ_p is the $p-$Laplacian from $W_0^{1,p}(\Omega)$ to $W^{-1,q}(\Omega)$ defined as
$$\Delta_p u:=\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right),\quad u\in W_0^{1,p}(\Omega).$$
It is clear that Δ_p is positively homogeneous of degree $p-1$.

Similarly, the condition (H5), with A_α replaced by C_α, implies that the operator
$$\langle Cu,v\rangle=\int_\Omega\sum_{|\alpha|\le m}C_\alpha(x,\xi(u))D^\alpha v,\qquad u,\,v\in W_0^{m,p}(\Omega),\tag{4.2}$$
is a bounded continuous mapping. We also know that conditions (H5), (H6), and (H8), with C_α in place of A_α everywhere, imply that the operator C is of type (S_+) (see Kittilä [27, p. 27]).

We also consider a multifunction $H:\Omega\times\mathbb{R}^{N_1}\to 2^{\mathbb{R}}$ such that

(H10) $H(x,r)=[\varphi(x,r),\psi(x,r)]$ is measurable in x and upper semicontinuous in r, where $\varphi,\psi:\Omega\times\mathbb{R}^{N_1}\to\mathbb{R}$ are measurable functions; and

(H11) $|H(x,r)|=\max[|\varphi(x,r)|,|\psi(x,r)|]\le a(x)+c_2|r|$ a.e. on $\Omega\times\mathbb{R}^{N_1}$, where $a(\cdot)\in L^q(\Omega)$, $c_2>0$.

Define $T:W_0^{m,p}\to 2^{W^{-m,q}(\Omega)}$ by
$$\begin{aligned}Tu=\Big\{h\in W^{-m,q}(\Omega):\exists w\in L^q(\Omega)\text{ such that } w(x)\in H(x,u(x))\\ \text{and }\langle h,v\rangle=\int_\Omega w(x)v(x)\text{ for all } v\in W_0^{m,p}(\Omega)\Big\}.\end{aligned}$$
It is well-known that T is upper-semicontinuous and compact with closed and convex values (see [21, p. 254]), and therefore T is of class (P).

We now state the following theorem as an application of Theorem 3.3.

Theorem 4.2. *Assume that the operators A, C, and T are defined as above. Assume, further, that the rest of the conditions of Theorem 3.3 are satisfied for two balls $G_1 = B_{\delta_1}(0)$ and $G_2 = B_{\delta_2}(0)$, where $0 < \delta_2 < \delta_1$. Then the Dirichlet boundary value problem*

$$\sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha \Big(A_\alpha(x, \xi(u)) + C_\alpha(x, \xi(u)) \Big) + H(x,u) \ni 0, \quad x \in \Omega,$$
$$D^\alpha u(x) = 0, \quad x \in \partial\Omega, \quad |\alpha| \le m-1,$$

has a "weak" nonzero solution $u \in B_{\delta_1}(0) \setminus B_{\delta_2}(0) \subset W_0^{m,p}(\Omega)$, which satisfies the inclusion $Au + Cu + Tu \ni 0$.

Application 4.3. Let Ω be a bounded open set in $\mathbb{R}^N$ with smooth boundary, $m \ge 1$ an integer, and $a > 0$. Set $Q = \Omega \times [0,a]$. Consider differential operators of the form

$$\frac{\partial u}{\partial t}(x,t) + \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha \Big(A_\alpha(x,t,\xi(u(x,t)) + C_\alpha(x,t,\xi(u(x,t))\Big) \tag{4.3}$$

in Q. The functions $A_\alpha = A_\alpha(x,t,\xi)$ and $C_\alpha = C_\alpha(x,t,\xi)$ are defined for $(x,t) \in Q$, $\xi = (\xi_\alpha)_{|\alpha|\le m} = (\eta, \zeta) \in \mathbb{R}^{N_0}$ with $\eta = (\eta_\gamma)_{|\alpha|\le m-1} \in \mathbb{R}^{N_1}$, $\zeta = (\zeta_\alpha)_{|\alpha|=m} \in \mathbb{R}^{N_2}$, and $N_1 + N_2 = N_0$. We assume that each $A_\alpha(x,t,\xi)$ satisfies the usual Carathéodory condition. We consider the following conditions.

(H12) (Continuity) For some $p > 1$, $c_1 > 0$, $g \in L^q(Q)$ with $q = p/(p-1)$, we have
$$|A_\alpha(x,t,\eta,\zeta)| \le c_1(|\zeta|^{p-1} + |\eta|^{p-1} + g(x,t)),$$
for $(x,t) \in Q$, $\xi = (\eta,\zeta) \in \mathbb{R}^{N_0}$ and $|\alpha| \le m$.

(H13) (Monotonicity)
$$\sum_{|\alpha|\le m} (A_\alpha(x,t,\xi_1) - A_\alpha(x,t,\xi_2))(\xi_{1_\alpha} - \xi_{2_\alpha}) \ge 0 \text{ for } (x,t) \in Q \text{ and } \xi_1, \xi_2 \in \mathbb{R}^{N_0}.$$

(H14) (Leray-Lions)
$$\sum_{|\alpha|=m} (A_\alpha(x,t,\eta,\zeta) - A_\alpha(x,t,\eta,\zeta^*))(\zeta_\alpha - \zeta^*_\alpha) > 0,$$
for $(x,t) \in Q$, $\eta \in \mathbb{R}^{N_1}$ and $\zeta, \zeta^* \in \mathbb{R}^{N_2}$.

(H15) (Coercivity) There exist $c_0 > 0$ and $h \in L^1(Q)$ such that
$$\sum_{|\alpha|\le m} A_\alpha(x,t,\xi) \ge c_0|\xi|^p - h(x,t), \quad (x,t) \in Q \text{ and } \xi \in \mathbb{R}^{N_0}.$$

(H16) Each $A_\alpha(x,t,\xi)$ is homogeneous of degree $\gamma > 0$ with respect to ξ.

Under condition (H12), the second term of (4.3) with $C_\alpha = 0$ generates a continuous bounded operator $A : X \to X^*$ defined by

$$\langle Au, v\rangle = \sum_{|\alpha|\le m} \int_Q A_\alpha(x,t,\xi(u(x,t)))D^\alpha v, \quad u, v \in X,$$

where $X = L^p(0,a;V)$, $X^* = L^q(0,a;V^*)$, and $V = W_0^{m,p}(\Omega)$. With the additional conditions (H13) and (H16), the operator A is maximal monotone and positively homogeneous of degree γ. Under (H12), (H14), and (H15) with A_α replaced by

C_α and other obvious changes, the second term in (4.3) with $A_\alpha = 0$ generates a continuous, bounded operator C defined as

$$\langle Cu, v\rangle = \sum_{|\alpha|\le m} \int_Q C_\alpha(x,t,\xi(u(x,t)))D^\alpha v, \quad u,v \in X,$$

which satisfies the condition (S_+) with respect to $D(L)$, where the operator L is defined as follows. The operator $\partial/\partial t$ generates an operator $L : X \supset D(L) \to X^*$, where

$$D(L) = \{v \in X : v' \in X^*,\ v(0) = 0\},$$

via the relation

$$\langle Lu, v\rangle = \int_0^a \langle u'(t), v(t)\rangle_V \,\mathrm{d}t, \quad u \in D(L),\ v \in X,$$

where $\langle\cdot,\cdot\rangle_V$ is the duality pairing in $V^* \times V$. The symbol $u'(t)$ is the generalized derivative of $u(t)$, i.e.,

$$\int_0^a u'(t)\varphi(t)\mathrm{d}t = -\int_0^a \varphi'(t)u(t)\,\mathrm{d}t, \quad \varphi \in C_0^\infty(0,a).$$

We can verify, as in Zeidler [34], that L is densely defined, linear and maximal monotone.

Given $h \in L^q(Q)$, define $h^* \in X^*$ by

$$\langle h^*, v\rangle = \int_Q hv, \quad v \in X.$$

As an application of Theorem 3.5, we obtain the following theorem.

Theorem 4.4. *Assume that the operators L, A, and C are as above, with A_α satisfying* (H12), (H13), *and* (H16)*, and C_α in place of A_α satisfying* (H12), (H14), *and* (H15)*. Assume, for a given $h \in L^q(Q)$, that the rest of the conditions of Theorem 3.5 are satisfied when C is replaced with $C - h^*$ for two balls $G_1 = B_{\delta_1}(0)$ and $G_2 = B_{\delta_2}(0)$ in $X = L^p(0,a;V)$, where $0 < \delta_2 < \delta_1$ and $V = W_0^m(\Omega)$. Then the initial-boundary value problem*

$$\begin{gathered}\frac{\partial u}{\partial t} + \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha\Big(A_\alpha(x,t,\xi(u)) + C_\alpha(x,t,\xi(u))\Big) = h(x,t),\\ D^\alpha u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,a], \quad |\alpha| \le m-1,\\ u(x,0) = 0, \quad x \in \Omega,\end{gathered}$$

has a "weak" nonzero solution $u \in B_{\delta_1}(0) \setminus B_{\delta_2}(0) \subset L^p(0,a;V)$ satisfying

$$Lu + Au + Cu = h^*.$$

Acknowledgments. This research was carried out by members of the Analysis Group of the Association of Nepalese Mathematicians in America (ANMA) within the *Collaborative Research in Mathematical Sciences* program. Ghanshyam Bhatt acknowledges Tennessee State University for supporting his engagement in this research with a Non-Instructional Assignment (NIA) grant for the years 2021-2022. The authors express their gratitude to the anonymous referees for providing feedback which helped to improve the article.

References

[1] A. Addou, B. Mermri; Topological degree and application to a parabolic variational inequality problem, *Int. J. Math. . Sci.,* **25** (2001), no. 4, 273–287.

[2] D. R. Adhikari; Nontrivial solutions of inclusions involving perturbed maximal monotone operators, *Electron. J. Differential Equations,* **2017** (2017), no. 151, 1–21.

[3] D. R. Adhikari, A. G. Kartsatos; Invariance of domain and eigenvalues for perturbations of densely defined linear maximal monotone operators, *Appl. Anal.,* **95** (2016), no. 1, 24–43.

[4] D. R. Adhikari, A. G. Kartsatos; A new topological degree theory for perturbations of the sum of two maximal monotone operators, *Nonlinear Anal.,* **74** (2011), no. 14, 4622–4641.

[5] D. R. Adhikari, A. G. Kartsatos; Strongly quasibounded maximal monotone perturbations for the Berkovits-Mustonen topological degree theory, *J. Math. Anal. Appl.,* **348** (2008), no. 1, 12–136.

[6] D. R. Adhikari, A. G. Kartsatos; Topological degree theories and nonlinear operator equations in Banach spaces, *Nonlinear Anal.,* **69** (2008), no. 4, 1235–1255.

[7] Y. Alber, I. Ryazantseva; Nonlinear ill-posed problems of monotone type, *Springer, Dordrecht,* 2006.

[8] T. M. Asfaw, A. G. Kartsatos; A Browder topological degree theory for multi-valued pseudomonotone perturbations of maximal monotone operators, *Adv. Math. Sci. Appl.,* **22** (2012), no. 1, 91–148.

[9] J. P. Aubin, A. Cellina; Differential inclusions, *Springer-Verlag,* 1984.

[10] V. Barbu; Nonlinear semigroups and differential equations in Banach spaces, *Noordhoff Int. Publ.,* Leyden (The Netherlands), 1975.

[11] J. Berkovits, V. Mustonen; Topological degree for perturbations of linear maximal monotone mappings and applications to a class of parabolic problems, *Rend. Mat. Appl.,* **12** (1992), no. 3, 597–621.

[12] J. Berkovits, M. Miettunen; On the uniqueness of the Browder degree, *Proc. Amer. Math. Soc.,* **136** (2008), no. 10, 3467–3476.

[13] I. Boubakari, A. G. Kartsatos; The Leray-Schauder approach to the degree theory for (S_+)-perturbations of maximal monotone operators in separable reflexive Banach spaces, *Nonlinear Anal.,* **70** (2009), no. 12, 4350–4368.

[14] H. Brézis, M. G. Crandall, A. Pazy; Perturbations of nonlinear maximal monotone sets in Banach spaces, *Comm. Pure Appl. Math.,* **23** (1970), 123–144.

[15] F. E. Browder; The degree of mapping and its generalizations, *Contemp. Math.,* **21** (1983), 15–40.

[16] F. E. Browder; Fixed point theory and nonlinear problems, *Bull. Amer. Math. Soc.,* **9** (1983), no. 1, 1–39.

[17] F. E. Browder; Nonlinear operators and nonlinear equations of evolution in Banach spaces, nonlinear functional analysis, *Proc. Sympos. Pure Appl. Math.,* **18** (1976), 1–308.

[18] F. E. Browder, P. Hess; Nonlinear mappings of monotone type in Banach spaces, *J. Functional Analysis,* **11** (1972), 251–294.

[19] I. Cioranescu; Geometry of Banach spaces, duality mappings and nonlinear problems, *Kluwer Acad. Publ.,* Dordrecht, 1990.

[20] P. Hess; On nonlinear mappings of monotone type homotopic to odd operators, *J. Functional Analysis,* **11** (1972), 138–167.

[21] S. Hu, N. S. Papageorgiou; Generalizations of Browder's degree, *Trans. Amer. Math. Soc.,* **347** (1995), no.1, 233–259.

[22] A. G. Kartsatos, J. Lin; Homotopy invariance of parameter-dependent domains and perturbation theory for maximal monotone and m-accretive operators in Banach spaces, *Adv. Differential Equations,* **8** (2003), no.2, 129–160.

[23] A. G. Kartsatos, J. Quarcoo; A new topological degree theory for densely defined $(S_+)_L$-perturbations of multivalued maximal monotone operators in reflexive separable Banach spaces, *Nonlinear Anal.,* **69** (2008), no. 8, 2339–2354.

[24] A. G. Kartsatos, I. V. Skrypnik; Degree theories and invariance of domain for perturbed maximal monotone operators in Banach spaces, *Adv. Differential Equations,* **12** (2007), no. 11, 1275–1320.

[25] A. G. Kartsatos, I. V. Skrypnik; A new topological degree theory for densely defined quasi-bounded $(\widetilde{S}_+)$-perturbations of multivalued maximal monotone operators in reflexive Banach spaces, *Abstr. Appl. Anal.,* (2005), no. 2, 121–158.
[26] A. G. Kartsatos, I. V. Skrypnik; On the eigenvalue problem for perturbed nonlinear maximal monotone operators in reflexive Banach spaces, *Trans. Amer. Math. Soc.,* **358** (2006), no. 9, 3851–3881.
[27] A. Kittilä; On the topological degree for a class of mappings of monotone type and applications to strongly nonlinear elliptic problems, *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertations,* **91** (1994), 48pp.
[28] D. Pascali, S. Sburlan; Nonlinear mappings of monotone type, *Sijthoff and Noordhoof*, Bucharest, 1978.
[29] R. Rockafellar, Local boundedness of nonlinear monotone operators, *Michigan Math. J.*, **16** (1969) 397–407.
[30] S. Simons; Minimax and monotonicity, vol. 1693, *Springer-Verlag*, Berlin, 1998.
[31] I. V. Skrypnik; Nonlinear elliptic boundary value problems, *BG Teubner*, 1986.
[32] I. V. Skrypnik; Methods for analysis of nonlinear elliptic boundary value problems, vol. 139, *American Mathematical Society*, 1994.
[33] S. L. Troyanski; On locally uniformly convex and differentiable norms in certain non-separable Banach spaces, *Studia Math.*, **37** (1971), 173–180.
[34] E. Zeidler; Nonlinear functional analysis and its applications, **II/B**, *Springer-Verlag*, New York, 1990.
[35] S.-S. Zhang, Y.-Q. Chen; Degree theory for multivalued (S)-type mappings and fixed point theorems, *Appl. Math. Mech.*, **11** (1990), no. 5, 441–454.

Dhruba R. Adhikari
Department of Mathematics, Kennesaw State University, Marietta, GA 30060, USA
Email address: dadhikar@kennesaw.edu

Ashok Aryal
Mathematics Department, Minnesota State University Moorhead, Moorhead, MN 56563, USA
Email address: ashok.aryal@mnstate.edu

Ghanshyam Bhatt
Department of Mathematical Sciences, Tennessee State University, Nashville, TN 37209, USA
Email address: gbhatt@tnstate.edu

Ishwari J. Kunwar
Department of Mathematics and Computer Science, Fort Valley State University, Fort Valley, GA 31030, USA
Email address: kunwari@fvsu.edu

Rajan Puri
Department of Mathematics, Wake Forest University, Winston-Salem, NC 27109, USA
Email address: purir@wfu.edu

Min Ranabhat
Department of Mathematical Sciences, University of Delaware, EWG 315, Newark, DE 19716, USA
Email address: ranabhat@udel.edu