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EXISTENCE AND NONEXISTENCE FOR SINGULAR SUBLINEAR PROBLEMS ON EXTERIOR DOMAINS

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Abstract. In this article we study the existence of radial solutions of $\Delta u + K(|x|)f(u) = 0$ on the exterior of the ball of radius $R > 0$ centered at the origin in $\mathbb{R}^N$ with $u = 0$ on ∂B_R, and $\lim_{|x|\to\infty} u(x) = 0$ where $N > 2$, $f(u) \sim \frac{-1}{|u|^{q-1}u}$ for u near 0 with $0 < q < 1$, and $f(u) \sim |u|^{p-1}u$ for large $|u|$ with $0 < p < 1$. Also, $K(|x|) \sim |x|^{-\alpha}$ with $N + q(N-2) < \alpha < 2(N-1)$ for large $|x|$.

1. Introduction

In this article we study the radial solutions of:

$$\Delta u + K(|x|)f(u) = 0, \quad x \in \mathbb{R}^N \backslash B_R \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\left(\mathbb{R}^N \backslash B_R\right) \tag{1.2}$$

$$u \to 0 \quad \text{as } |x| \to \infty \tag{1.3}$$

where B_R is the ball of radius $R > 0$ centered at the origin in $\mathbb{R}^N$, $K(x) > 0$ and $u : \mathbb{R}^N \to \mathbb{R}$ with $N > 2$. In addition, we suppose $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is locally Lipschitz and

- (H1) f is odd, there exists $\beta > 0$ such that $f < 0$ on $(0, \beta)$, $f > 0$ on (β, ∞).
- (H2) $g_1 : \mathbb{R} \to \mathbb{R}$ is continuous and
$$f(u) = \frac{-1}{|u|^{q-1}u} + g_1(u)$$
where $0 < q < 1$ and $g_1(0) = 0$.
- (H3) $g_2 : \mathbb{R} \to \mathbb{R}$ is continuous and $f(u) = |u|^{p-1}u + g_2(u)$, where $0 < p < 1$ and $\lim_{u\to+\infty} g_2(u)/|u|^p = 0$.

We let $F(u) = \int_0^u f(s)\,ds$. Since f is odd it follows that F is even and from (H2) it follows that f is integrable near $u = 0$. Thus F is continuous and $F(0) = 0$. It also follows that F is bounded below by $-F_0$ with $F_0 > 0$ and from (H3) we see there exists γ with $0 < \beta < \gamma$ such that

- (H4) $F < 0$ on $(0, \gamma)$, $F > 0$ on (γ, ∞), and $F > -F_0$ on $\mathbb{R}$.
- (H5) K and K' are continuous on $[R, \infty)$ with $K(r) > 0$, $2(N-1) + \frac{rK'}{K} > 0$, $N + q(N-2) < \alpha < 2(N-1)$ and $\lim_{r\to\infty} rK'/K = -\alpha$.
- (H6) There exists $K_1 > 0$ such that $\lim_{r\to\infty} r^\alpha K(r) = K_1 > 0$.

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Interest in the topic for this article comes from recent papers [2, 7, 9, 10] about solutions of differential equation problems on exterior domains. In [1] we studied (1.1)–(1.3) with $K(r) \sim r^{-\alpha}$, where f is singular at 0 and grows superlinearly at ∞, with various values of α. We proved existence of an infinite number of solutions. In this article we consider the case when f is singular at 0 and grows sublinearly at ∞. In this article we prove the following results.

Theorem 1.1. *Let $N > 2$, $R > 0$, $0 < p, q < 1$, $N + q(N-2) < \alpha < 2(N-1)$, and suppose* (H1)–(H6) *hold. Then given a non-negative integer, n_0, then there are solutions $u_0, u_1, \dots, u_{n_0}$ of* (1.1)–(1.3) *where u_k has exactly k zeros on (R, ∞) and $\lim_{r\to\infty} u_k(r) = 0$ if R is sufficiently small.*

Theorem 1.2. *Let $N > 2$, $R > 0$, $0 < p, q < 1$, $N + q(N-2) < \alpha < 2(N-1)$, and suppose* (H1)–(H6) *hold. Then there are no radial solutions of* (1.1)–(1.3) *if $R > 0$ is sufficiently large.*

2. Preliminaries

Since we are interested in studying radial solutions of (1.1)–(1.3), we assume that $r = |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}$, $u(r) = u(|x|)$ where $x \in \mathbb{R}^N$ and u satisfies

$$u''(r) + \frac{N-1}{r}u'(r) + K(r)f(u(r)) = 0 \quad \text{on } (R, \infty), \tag{2.1}$$

$$u(R) = 0, \quad \lim_{r\to\infty} u(r) = 0. \tag{2.2}$$

To prove existence we make the change of variables

$$u(r) = v(r^{2-N}). \tag{2.3}$$

Then

$$u'(r) = (2-N)r^{1-N}v'(r^{2-N}),$$
$$u''(r) = (2-N)(1-N)r^{-N}v'(r^{2-N}) + (2-N)^2 r^{2(1-N)}v''(r^{2-N}).$$

Letting $t = r^{2-N}$ and $r = t^{\frac{1}{2-N}}$ in (2.1)–(2.2) gives

$$v''(t) + h(t)f(v(t)) = 0 \quad \text{for } 0 < t < R^{2-N} \tag{2.4}$$

where from (H1)–(H6),

$$h(t) = \frac{1}{(N-2)^2} t^{\frac{2(N-1)}{2-N}} K(t^{\frac{1}{2-N}}) \sim \frac{t^{-\tilde{\alpha}}}{(N-2)^2} \quad \text{with } \tilde{\alpha} = \frac{2(N-1)-\alpha}{N-2} > 0. \tag{2.5}$$

Note that $2 - \tilde{\alpha} = \frac{\alpha-2}{N-2} > 0$. Also from (H5) and (H6) it follows that there is a constant $h_1 > 0$ with

$$\lim_{t\to 0^+} t^{\tilde{\alpha}} h(t) = h_1, \quad h'(t) < 0 \text{ on } (0, R^{2-N}], \quad 0 < \tilde{\alpha} + q < 1. \tag{2.6}$$

Then there are $h_0 > 0$ and $h_2 > 0$ such that

$$h_0 \le t^{\tilde{\alpha}} h(t) \le h_2 \quad \text{on } (0, R^{2-N}]. \tag{2.7}$$

We now consider (2.4) with

$$v(0) = 0, \quad v'(0) = a \ge 0 \tag{2.8}$$

and we try to find $a \ge 0$ such that $v(R^{2-N}) = 0$. We write v_a to emphasize the dependence of v on a. Let $a \ge 0$. We first show that there is a solution v_a of equation (2.4) on $(0, \epsilon)$ for small ϵ along with (2.8) and v_a, v_a' continuous on $[0, \epsilon)$.

This is a bit lengthy so we postpone this proof to the Appendix. We now assume v_a solves (2.4) on $(0,\epsilon)$ and v_a, v_a' continuous on $[0,\epsilon)$.

Next let $(0,B)\subset(0,R^{2-N})$ be the maximal open interval where the solution of (2.4) exists along with (2.8). We will show $B=R^{2-N}$. First, from the proof in the appendix we have that there exists $\epsilon>0$ such that $0<\epsilon\le B\le R^{2-N}$.

Now we define the energy of solution (2.4), (2.8) as

$$E_a(t)=\frac{1}{2}\frac{v_a'^2(t)}{h(t)}+F(v_a(t))\quad\text{for } 0<t<B. \tag{2.9}$$

Differentiating E_a, using (2.4) and since we know from (2.6) that $h'(t)<0$, then

$$E_a'(t)=-\frac{v_a'^2(t)h'(t)}{2h^2(t)}\ge 0\quad\text{on } (0,B). \tag{2.10}$$

Thus E_a is nondecreasing on $(0,B)$. Therefore,

$$0=\lim_{t\to 0^+}E_a(t)\le E_a(t)=\frac{1}{2}\frac{v_a'^2(t)}{h(t)}+F(v_a(t)) \tag{2.11}$$

so it follows that

$$E_a(t)>0\quad\text{for } 0<t<B. \tag{2.12}$$

Next we see that

$$\Big(\frac{1}{2}v_a'^2(t)+h(t)F(v_a(t))\Big)'=h'(t)F(v_a(t)). \tag{2.13}$$

Now let us show for fixed $a\ge 0$ that v_a and v_a' are continuous on $[0,R^{2-N}]$.

Lemma 2.1. *Assume* (H1)–(H6) *hold, $N>2$, and $a\ge 0$. Suppose v_a solves* (2.4). *Then $|v_a(t)|\le C$ and $|v_a'(t)|\le C$ for some constant C on* $[0,R^{2-N}]$ *and v_a, v_a' are continuous on* $[0,R^{2-N}]$.

Proof. We first assume that there is a $t_{a,\gamma}\in[0,B)$ such that $v_a(t_{a,\gamma})=\gamma$ and $0\le v_a<\gamma$ on $[0,t_{a,\gamma})$.

We know from (H4) that $F(v_a)\le 0$ when $t\in[0,t_{a,\gamma}]$ so we have

$$0<\frac{1}{2}\frac{v_a'^2(t)}{h(t)}+F(v_a(t))\le\frac{1}{2}\frac{v_a'^2(t)}{h(t)}\ \text{on } (0,t_{a,\gamma}].$$

Thus $v_a'>0$ on $[0,t_{a,\gamma}]$. Also if we multiply (2.4) by v_a^q, use (H2), and integrate by parts on $(0,t)$ this gives

$$v_a^qv_a'-\int_0^t qv_a^{q-1}(s)v_a'^2(s)\,ds+\int_0^t h(s)v_a^q(s)g_1(v_a(s))\,ds=\int_0^t h(s)\,ds. \tag{2.14}$$

Thus

$$v_a^qv_a'+\int_0^t h(s)v_a^q(s)g_1(v_a(s))\,ds\ge\int_0^t h(s)\,ds. \tag{2.15}$$

Integrating (2.15) again and using (2.7) gives

$$\begin{aligned}\frac{v_a^{q+1}(t)}{q+1}+\int_0^t\int_0^s h(x)v_a^q(x)g_1(v_a(x))\,dx\,ds&=\int_0^t\int_0^s h(x)\,dx\,ds\\&\ge\frac{h_0t^{2-\tilde\alpha}}{(2-\tilde\alpha)(1-\tilde\alpha)}.\end{aligned} \tag{2.16}$$

Let L_1 be the Lipschitz constant for g_1 on $[0,\gamma]$ so then $|g_1(v_a)| \le L_1 v_a$ on $[0,t_{a,\gamma}]$. using this and since $v_a' > 0$ on $[0,t_{a,\gamma}]$ then:

$$\begin{aligned}\int_0^t \int_0^s h(x) v_a^q(x) g_1(v_a(x))\,dx\,ds &\le L_1 \int_0^t \int_0^s h(x) v_a^{q+1}(x)\,dx\,ds \\ &\le L_1 v_a^{q+1}(t) \int_0^t \int_0^s h(x)\,dx\,ds.\end{aligned}$$

using this in (2.16) and using (2.7) again we see that

$$\begin{aligned}\frac{h_0 t^{2-\tilde\alpha}}{(2-\tilde\alpha)(1-\tilde\alpha)} &\le v_a^{q+1}(t)\Big[\frac{1}{q+1} + \frac{L_1 h_2 t^{2-\tilde\alpha}}{(2-\tilde\alpha)(1-\tilde\alpha)}\Big] \\ &\le v_a^{q+1}(t)\Big[\frac{1}{q+1} + \frac{L_1 h_2 R^{(2-N)(2-\tilde\alpha)}}{(2-\tilde\alpha)(1-\tilde\alpha)}\Big].\end{aligned}$$

Therefore

$$v_a(t) \ge C_1 t^{\frac{2-\tilde\alpha}{1+q}} \quad \text{on } [0,t_{a,\gamma}] \tag{2.17}$$

where

$$C_1 = \Big[\frac{h_0(q+1)}{(2-\tilde\alpha)(1-\tilde\alpha) + L_1 h_2 (q+1) R^{(2-N)(2-\tilde\alpha)}}\Big]^{\frac{1}{q+1}} > 0.$$

Evaluating (2.17) at $t = t_{a,\gamma}$ gives

$$t_{a,\gamma} \le \Big(\frac{\gamma}{C_1}\Big)^{\frac{1+q}{2-\tilde\alpha}}. \tag{2.18}$$

Then from (2.17) and (2.7) we see that

$$\frac{h(t)}{v_a^q(t)} \le \frac{h_2}{{C_1}^q} t^{\frac{-\tilde\alpha-2q}{1+q}} \quad \text{on } (0,t_{a,\gamma}].$$

Rewriting (2.4) and substituting gives

$$v_a''(t) = \frac{h(t)}{v_a^q(t)} - h(t) g_1(v_a(t)) \le \frac{h_2}{{C_1}^q} t^{\frac{-\tilde\alpha-2q}{1+q}} + h_2 L_1 t^{-\tilde\alpha}\gamma \quad \text{on } (0,t_{a,\gamma}]. \tag{2.19}$$

Integrating on $(0,t)$ gives

$$v_a'(t) \le a + C_2 t^{\frac{1-\tilde\alpha-q}{1+q}} + C_3 t^{1-\tilde\alpha} \quad \text{on } [0,t_{a,\gamma}] \tag{2.20}$$

where $C_2 = \frac{h_2(1+q)}{C_1^q(1-\tilde\alpha-q)}$, $C_3 = \frac{h_2 L_1 \gamma}{1-\tilde\alpha}$. Integrating (2.20) on $(0,t)$ we have

$$v_a(t) \le at + C_4 t^{\frac{2-\tilde\alpha}{1+q}} + \frac{C_3}{2-\tilde\alpha} t^{2-\tilde\alpha} \quad \text{on } [0,t_{a,\gamma}] \tag{2.21}$$

where

$$C_4 = \frac{h_2(1+q)^2}{C_1^q(1-\tilde\alpha-q)(2-\tilde\alpha)}.$$

Evaluating (2.21) at $t = t_{a,\gamma}$ and using (2.18) we obtain

$$\gamma \le t_{a,\gamma}\Big(a + C_4\Big(\frac{\gamma}{C_1}\Big)^{\frac{1-\tilde\alpha-q}{2-\tilde\alpha}} + \frac{C_3}{2-\tilde\alpha}\Big(\frac{\gamma}{C_1}\Big)^{\frac{(1-\tilde\alpha)(1+q)}{2-\tilde\alpha}}\Big) = t_{a,\gamma}(a + C_5) \tag{2.22}$$

where

$$C_5 = C_4\Big(\frac{\gamma}{C_1}\Big)^{\frac{1-\tilde\alpha-q}{2-\tilde\alpha}} + \frac{C_3}{2-\tilde\alpha}\Big(\frac{\gamma}{C_1}\Big)^{\frac{(1-\tilde\alpha)(1+q)}{2-\tilde\alpha}}.$$

From (2.22) we have

$$\frac{1}{t_{a,\gamma}} \le \frac{a+C_5}{\gamma}. \tag{2.23}$$

Now from (2.20) and for $t \in [0, t_{a,\gamma}]$ we obtain

$$0 \le v_a'(t) \le a + C_2 t_{a,\gamma}{}^{\frac{1-\tilde{\alpha}-q}{1+q}} + C_3 t_{a,\gamma}{}^{1-\tilde{\alpha}} \le a + C_6 \quad \text{on } [0, t_{a,\gamma}] \tag{2.24}$$

where $C_6 = C_2 R^{\frac{(2-N)(1-\tilde{\alpha}-q)}{1+q}} + C_3 R^{(2-N)(1-\tilde{\alpha})}$. Thus $|v_a'|$ is bounded on $[0, t_{a,\gamma}]$ if $t_{a,\gamma} \le B$.

Now continuing to assume $t_{a,\gamma} \le B$ we integrate (2.13) on $(t_{a,\gamma}, t)$, using (2.24), $h' < 0$, and $-F_0 \le F(v_a)$ (by (H4)) then we obtain

$$\begin{aligned}
\frac{1}{2}v_a'^2(t) - h(t)F_0 &\le \frac{1}{2}v_a'^2(t) + h(t)F(v_a) \\
&= \frac{1}{2}v_a'^2(t_{a,\gamma}) + \int_{t_{a,\gamma}}^{t} h'(s)F(v_a(s))\,ds \\
&\le \frac{1}{2}(a+C_6)^2 - \int_{t_{a,\gamma}}^{t} h'(s)F_0\,ds \\
&= \frac{1}{2}(a+C_6)^2 - h(t)F_0 + h(t_{a,\gamma})F_0.
\end{aligned}$$

using (2.23) in the above we have

$$\frac{1}{2}v_a'^2(t) \le \frac{1}{2}(a+C_6)^2 + h(t_{a,\gamma})F_0 \le \frac{1}{2}(a+C_6)^2 + h_2 F_0 \Big(\frac{a+C_5}{\gamma}\Big)^{\tilde{\alpha}}. \tag{2.25}$$

Thus it follows from (2.25) and standard inequalities that $|v_a'|$ is bounded as

$$|v_a'| \le a + C_7 \quad \text{on } [0, B) \tag{2.26}$$

for some C_7 that does not depend on a if $0 < t_{a,\gamma} \le B$. Then

$$|v_a| = \Big|\int_0^t v_a'\,ds\Big| \le (a+C_7)t \le (a+C_7)B \quad \text{on } [0, B) \tag{2.27}$$

so $|v_a|$ is also bounded on $[0, B)$ if $t_{a,\gamma} \le B$.

On the other hand if $0 \le v_a < \gamma$ on $[0, B)$ then a similar argument shows that (2.17) and (2.20) hold on $[0, B)$ and so again we see that $|v_a|, |v_a'|$ are bounded on $[0, B)$.

Thus $\lim_{t\to B^-} v_a(t) = D \in \mathbb{R}$. Also since $h(t)F(v_a(t))$ and $h'(t)F(v_a(t))$ are continuous on $[\epsilon, B)$ it follows by integrating (2.13) on $[\epsilon, B)$ that $\lim_{t\to B^-} v_a'(t) = D_1 \in \mathbb{R}$. From (2.12) we know $0 < E_a(t) \le \frac{1}{2}\frac{D_1^2}{h(B)} + F(D)$ on $[0, B)$ so D and D_1 cannot both be zero. If $B < R^{2-N}$ then the solution v_a can be extended to $[0, B+\epsilon)$ for some $\epsilon > 0$ by using the fact that D, D_1 are not both zero for if $D \ne 0$ then we can just use the standard existence theorem from differential equations and if $D = 0$ then $D_1 \ne 0$ and we can use the contraction mapping principle as we did in the appendix which contradicts the definition of B. Thus we see $B = R^{2-N}$. Also since v_a, v_a' are bounded on $[0, R^{2-N})$ then we see $\lim_{t\to (R^{2-N})^-} v_a$ exists and $\lim_{t\to (R^{2-N})^-} v_a'$ exists. Thus v_a, v_a' are continuous on $[0, R^{2-N}]$. This completes the proof. □

Lemma 2.2. *Let $N > 2$, $a \ge 0$. Assume* (H1)–(H6) *hold, and suppose $v_a(t)$ solves* (2.4), (2.8). *Then the solutions $v_a(t)$ continuously depend on the parameter $a \ge 0$ on $[0, R^{2-N}]$.*

Proof. Let $0 \le a_1 < a_2$. Since v_a, v_a' are continuous on $[0, R^{2-N}]$ it follows from (2.26) and (2.27) that v_a, v_a' are bounded on $[0, R^{2-N}]$. Then notice from (2.26) and (2.27) we have

$$|v_a'(t)| \le a_2 + C_7 \quad \text{on } [0, R^{2-N}]\ \forall a \text{ with } 0 \le a_1 \le a \le a_2, \tag{2.28}$$

$$|v_a(t)| \le (a_2 + C_7)R^{2-N} \quad \text{on } [0, R^{2-N}]\ \forall a \text{ with } 0 \le a_1 \le a \le a_2. \tag{2.29}$$

Thus we see that $|v_a'|$ and $|v_a|$ are uniformly bounded on $[0, R^{2-N}]$ for all a with $0 \le a_1 \le a \le a_2$.

Next, let $a^* \ge 0$ with $0 \le a_1 \le a^* \le a_2$. We will now show that $v_a \to v_{a^*}$ uniformly on $[0, R^{2-N}]$ as $a \to a^*$. We prove this by contradiction so suppose not. Then there exist A_j with $a_1 \le A_j \le a_2$ such that $A_j \to a^*$ as $j \to \infty$, $t_j \in [0, R^{2-N}]$ and there is an $\epsilon_2 > 0$ such that

$$|v_{A_j}(t_j) - v_{a^*}(t_j)| \ge \epsilon_2 \quad \forall j. \tag{2.30}$$

Since $A_j \to a^*$ as $j \to \infty$ and $0 \le a_1 \le A_j \le a_2$, by (2.28), (2.29) we see that v_{A_j} and v_{A_j}' are uniformly bounded on $[0, R^{2-N}]$ and therefore the v_{A_j} are equicontinuous on $[0, R^{2-N}]$. Then by the Arzela-Ascoli theorem there is a subsequence $v_{A_{j_l}}$ of v_{A_j} such that $v_{A_{j_l}} \to v_{a^*}$ uniformly on $[0, R^{2-N}]$. So as $l \to \infty$,

$$0 \leftarrow |v_{A_{j_l}}(t_{j_l}) - v_{a^*}(t_{j_l})| \ge \epsilon_2 > 0 \quad \text{which is impossible.}$$

Thus v_a varies continuously with a on $[0, R^{2-N}]$ for all a with $0 \le a_1 \le a \le a_2$. This completes the proof. □

Lemma 2.3. *Let $v_a(t)$ satisfy* (2.4), (2.8) *and assume that* (H1)–(H6) *hold. Then $\lim_{a\to\infty} \max_{[0,R^{2-N}]} v_a(t) = \infty$. In addition, if $v_a(t)$ has a first local maximum, M_a, with $0 < M_a \le R^{2-N}$, then $v_a(M_a) \to \infty$ as $a \to \infty$. Further, if a is sufficiently large, then v_a is increasing on $[0, R^{2-N}]$ and $v_a(R^{2-N}) \to \infty$ as $a \to \infty$.*

Proof. We assume by the way of contradiction that $\max_{[0,R^{2-N}]} v_a(t) \le C_8$ for some constant $C_8 > 0$ which does not depend on a for a large. Since $f(v_a) = -\frac{1}{|v|^{q-1}v_a} + g_1(v_a)$ and $g_1(v_a)$ is continuous on $[0, C_8]$ then there is a $C_9 > 0$ such that $|g_1(v_a)| \le C_9$ on $[0, R^{2-N}]$. Now either $v_a' > 0$ or v_a has a local maximum M_a and $v_a' > 0$ on $[0, M_a)$. We show that v_a cannot have a local maximum M_a for large a.

Integrating (2.4) over $(0, t)$ and estimating gives

$$v_a'(t) = a + \int_0^t h(s)\frac{1}{|v|_a^{q-1}v_a}\,ds - \int_0^t h(s)g_1(v_a)\,ds \ge a - C_9\int_0^t h(s)\,ds. \tag{2.31}$$

Recalling from (2.6) that $\tilde\alpha + q < 1$ and $q > 0$ it follows that $\tilde\alpha < 1$. Also from (2.7) we have $-h(t) \ge -h_2 t^{-\tilde\alpha}$. Then using this in (2.31) implies

$$v_a'(t) \ge a - \frac{C_9 h_2}{1-\tilde\alpha}t^{1-\tilde\alpha}. \tag{2.32}$$

Now if v_a has a local maximum then evaluating (2.32) at M_a gives

$$\frac{C_9 h_2}{1-\tilde\alpha}R^{(2-N)(1-\tilde\alpha)} \ge \frac{C_9 h_2}{1-\tilde\alpha}M_a^{1-\tilde\alpha} \ge a \tag{2.33}$$

but the right-hand side goes to infinity as $a \to \infty$ while the left-hand side is fixed and thus we obtain a contradiction. Thus we see if $a > 0$ is sufficiently large and v_a is bounded above by a constant that it is independent of a then $v_a' > 0$ on $[0, R^{2-N}]$. Next integrating (2.32) on $(0, t)$ we obtain:

$$C_8 \geq v_a(t) \geq at - \frac{C_9 h_2}{(1-\tilde{\alpha})(2-\tilde{\alpha})} t^{2-\tilde{\alpha}}. \tag{2.34}$$

Thus

$$C_8 \geq v_a(R^{2-N}) \geq aR^{2-N} - \frac{C_9 h_2}{(1-\tilde{\alpha})(2-\tilde{\alpha})} (R^{2-N})^{2-\tilde{\alpha}} \tag{2.35}$$

therefore the right-hand side of (2.35) approaches infinity as a approaches infinity, but the left-hand side is bounded by C_8. so we have a contradiction. Thus $\lim_{a\to\infty} \max_{[0,R^{2-N}]} v_a(t) = \infty$.

Now we show that if v_a has a first local maximum, M_a, on $[0, R^{2-N}]$, then $\lim_{a\to\infty} v_a(M_a) = \infty$. For if not we may again appeal to (2.33) as we did earlier to again get a contradiction. Thus the assumption that $v_a(M_a)$ is bounded is false. Therefore if $M_a \in [0, R^{2-N}]$ exists, then

$$\lim_{a\to\infty} v_a(M_a) = \infty. \tag{2.36}$$

Next we show that $v_a' > 0$ on $[0, R^{2-N}]$ if a is sufficiently large. So suppose not. Then there exists a first local maximum, M_a, of v_a, with $0 < M_a \leq R^{2-N}$. From (2.10)–(2.12) we have $E_a(t) > 0$ and $E_a'(t) \geq 0$. Thus for $0 \leq t \leq M_a$ we have

$$\frac{1}{2}\frac{v_a'^2(t)}{h(t)} + F(v_a(t)) \leq F(v_a(M_a)). \tag{2.37}$$

Rewriting and integrating (2.37) on $(0, M_a)$ gives

$$\begin{aligned} \int_0^{M_a} \frac{v_a'(t)\,dt}{\sqrt{2}\sqrt{F(v_a(M_a)) - F(v_a(t))}} &\leq \int_0^{M_a} \sqrt{h(t)}\,dt \\ &\leq \sqrt{h_2}\int_0^{R^{2-N}} t^{-\tilde{\alpha}/2}\,dt \\ &= \frac{2\sqrt{h_2}}{2-\tilde{\alpha}} \big(R^{2-N}\big)^{1-\frac{\tilde{\alpha}}{2}}. \end{aligned} \tag{2.38}$$

Since $v_a(M_a) \to \infty$ as $a \to \infty$ from (2.36) it follows from (H3) that $F(v_a(M_a)) - F(s) \leq C_{10} {v_a}^{p+1}(M_a)$ for some constant $C_{10} > 0$. Then after changing variables on the left-hand side of (2.38) and rewriting we obtain

$$\begin{aligned} \frac{{v_a}^{\frac{1-p}{2}}(M_a)}{\sqrt{2C_{10}}} &= \frac{v_a(M_a)}{\sqrt{2}\sqrt{C_{10}{v_a}^{p+1}(M_a)}} \\ &\leq \int_0^{v_a(M_a)} \frac{ds}{\sqrt{2}\sqrt{F(v_a(M_a)) - F(s)}} \\ &= \frac{2\sqrt{h_2}}{2-\tilde{\alpha}} \big(R^{2-N}\big)^{1-\frac{\tilde{\alpha}}{2}}. \end{aligned} \tag{2.39}$$

This yields a contradiction since the right-hand side of (2.39) is finite but $0 < p < 1$ and by (2.36) the left-hand side of (2.39) goes to infinity as $a \to \infty$. Thus the assumption that v_a has a local maximum on $[0, R^{2-N}]$ if a is sufficiently large is false. Therefore if a is sufficiently large then v_a is increasing on $[0, R^{2-N}]$ and so $v_a(R^{2-N}) = \max_{[0,R^{2-N}]} v_a(t)$. Since from the first part of the proof we know that $\lim_{a\to\infty} \max_{[0,R^{2-N}]} v_a(t) = \infty$ it follows that $\lim_{a\to\infty} v_a(R^{2-N}) = \infty$. This completes the proof. □

Lemma 2.4. *Let $v_a(t)$ satisfy (2.4), (2.8) and assume (H1)–(H6) hold. Let $R > 0$ be sufficiently small. Then $v_a(t)$ has a local maximum, M_a, and a zero, Z_a, with $0 < M_a < Z_a < R^{2-N}$ if a is sufficiently small. In addition, if $R > 0$ is sufficiently small then v_a has n zeros on $[0, R^{2-N}]$.*

Proof. Let us suppose instead that $v_a'(t) > 0$ on $[0, R^{2-N}]$ for all sufficiently small a and R sufficiently small. Then from (2.18) it follows that $t_{a,\gamma} \le C_{11}$ where C_{11} is independent of a. Thus $t_{a,\gamma} < R^{2-N}$ if R is sufficiently small. Since v_a is continuous and increasing then for $t > t_{a,\gamma}$ we have $\gamma = v_a(t_{a,\gamma}) < v_a(t)$. Since $v_a'(t) > 0$ and $f(v_a) > 0$ on $[\gamma, \infty)$ with $f(v_a) \to \infty$ as $v_a \to \infty$ by (H3) it follows that there exists $C_{12} > 0$ such that $f(v_a) \ge C_{12} > 0$ on $[t_{a,\gamma}, R^{2-N}]$. Then

$$v_a''(t) + C_{12}h(t) \le v_a''(t) + h(t)f(v_a(t)) = 0 \quad \text{on } [t_{a,\gamma}, R^{2-N}]. \tag{2.40}$$

Rewriting and integrating on $(t_{a,\gamma}, t)$ gives

$$0 < v_a'(t) \le v_a'(t_{a,\gamma}) - C_{12}\Big[\frac{t^{1-\tilde\alpha} - t_{a,\gamma}^{1-\tilde\alpha}}{1-\tilde\alpha}\Big]. \tag{2.41}$$

From (2.6) we know $0 < \tilde\alpha < 1$ and it follows from (2.26) that if $0 \le a \le a_0$ then

$$|v_a'(t)| \le a + C_7 \le a_0 + C_7. \tag{2.42}$$

Thus $v_a'(t_{a,\gamma})$ is bounded by a constant that is independent of a when a is sufficiently small and so it follows that the right-hand side of (2.41) becomes negative if R is sufficiently small which contradicts the assumption that $v_a'(t) > 0$ on $[0, R^{2-N}]$. Thus if a is sufficiently small and R is sufficiently small then there is an M_a with $0 < M_a < R^{2-N}$ such that $v_a' > 0$ on $(0, M_a)$ and $v_a'(M_a) = 0$.

Next, we want to show that v_a has a zero on $[0, R^{2-N}]$ if a and R are sufficiently small. In order to do this we will show that $v_a \to v_0$ uniformly on $[0, R^{2-N}]$ as $a \to 0^+$ where

$$\begin{gathered} v_0'' + h(t)f(v_0) = 0, \\ v_0(0) = 0 = v_0'(0). \end{gathered}$$

Then we will show v_0 has a zero and since $v_a \to v_0$ uniformly as $a \to 0^+$ it will follow that v_a has a zero if a is sufficiently small and R is sufficiently small.

It follows from Lemmas 2.1 and 2.2, and (2.28)–(2.29) that v_a, v_a' are uniformly bounded on $[0, R^{2-N}]$ for all $0 \le a \le a_0$ for some $a_0 > 0$. Therefore there is a subsequence of the v_a, say v_{a_j}, such that $v_{a_j} \to v_0$ uniformly on $[0, R^{2-N}]$ by the Arzela-Ascoli Theorem as $a_j \to 0$.

Now we assume there is a $t_{a,\beta}$ with $0 < t_{a,\beta} < R^{2-N}$ such that $v_a(t_{a,\beta}) = \beta$ and $0 \le v_a(t) < \alpha$ on $[0, t_{a,\beta})$. It follows from (2.21) and an argument similar to (2.22) that

$$\beta \le t_{a,\beta}(a + C_5) \tag{2.43}$$

and as in (2.19) we have

$$0 \le v_a'' \le \frac{h_2}{C_1^q} t^{\frac{-\tilde\alpha-2q}{1+q}} + h_2L_1\beta t^{-\tilde\alpha} \le C_{13}t^{\frac{-\tilde\alpha-2q}{1+q}} \quad \text{on } [0, t_{a,\beta}] \tag{2.44}$$

where $C_{13} = \frac{h_2}{C_1^q} + h_2L_1\beta R^{\frac{(2-N)(2-\tilde\alpha)q}{1+q}}$.

Thus for $0 < x < y < t_{a,\beta}$ and since $0 < \frac{1-\tilde{\alpha}-q}{1+q} < 1$ we have

$$\begin{aligned} 0 \le v_a'(y) - v_a'(x) &= \int_x^y v_a''(t)\,dt \\ &\le C_{13} \int_x^y t^{\frac{-\tilde{\alpha}-2q}{1+q}}\,dt \\ &= C_{14} |y^{\frac{1-\tilde{\alpha}-q}{1+q}} - x^{\frac{1-\tilde{\alpha}-q}{1+q}}| \\ &\le C_{14} |y-x|^{\frac{1-\tilde{\alpha}-q}{1+q}} \end{aligned} \tag{2.45}$$

where $C_{14} = \frac{1+q}{1-\tilde{\alpha}-q} C_{13}$. And since $0 < \frac{\beta}{a_0+C_5} \le t_{a,\beta}$ from (2.43) it follows from this that the v_a' are equicontinuous on $[0, \frac{\beta}{a_0+C_5}]$ for $0 \le a \le a_0$ and so $v_{a_j}' \to v_0'$ uniformly on $[0, \frac{\beta}{a_0+C_5}]$ by the Arzela-Ascoli Theorem.

Now if $0 < v_a < \beta$ on $[0, R^{2-N}]$ then we see (2.44) and (2.45) hold $[0, R^{2-N}]$. Next we choose t_0 with $0 < t_0 < \frac{\beta}{a_0+C_5}$. Then integrating (2.13) on (t_0, t) gives:

$$\frac{1}{2} v_{a_j}'^2(t) + h(t) F(v_{a_j}(t)) = \frac{1}{2} v_{a_j}'^2(t_0) + \int_{t_0}^t h'(s) F(v_{a_j}(s))\,ds. \tag{2.46}$$

Now since $v_{a_j} \to v_0$ uniformly and since $v_{a_j}'(t_0) \to v_0'(t_0)$ it then follows that $v_{a_j}' \to v_0'$ uniformly on $[t_0, R^{2-N}]$, and so combined with the earlier fact $v_{a_j}' \to v_0'$ uniformly on $[0, \frac{\beta}{a_0+C_5}]$ we see that $v_{a_j}' \to v_0'$ uniformly on $[0, R^{2-N}]$.

Now taking limits in (2.46) gives

$$\frac{1}{2} v_0'^2(t) + h(t) F(v_0(t)) = \frac{1}{2} v_0'^2(t_0) + \int_{t_0}^t h'(s) F(v_0(s))\,ds \text{ on } (0, R^{2-N}].$$

Letting $t_0 \to 0^+$ gives

$$\frac{1}{2} v_0'^2(t) + h(t) F(v_0(t)) = \int_0^t h'(s) F(v_0(s))\,ds.$$

Then from (2.4) and (H3) we see that $v_{a_j}'' \to v_0''$ at all points where $v_0(t) \neq 0$ and at these points we have

$$\begin{gathered} v_0'' + h(t) f(v_0) = 0, \\ v_0(0) = v_0'(0) = 0. \end{gathered}$$

As at the beginning of the proof of this lemma it follows that v_0 has a local maximum, M_0, and $v_0(M_0) > \gamma$ if $R > 0$ is sufficiently small. Now we assume by way of contradiction $v_0 > \gamma$ on $[M_0, R^{2-N}]$. Then we have $\frac{f(v_0)}{v_0} > 0$ on $[M_0, R^{2-N}]$ so there is a $C_{15} > 0$ such that $\frac{f(v_0)}{v_0} \ge C_{15} > 0$ when $\gamma \le v_0 \le v_0(M_0)$. Thus substituting in (2.4) and using (2.7) we obtain

$$v_0''(t) + \frac{h_0 C_{15}}{t^{\tilde{\alpha}}} v_0(t) \le 0.$$

So $v_0'' < 0$ while $\gamma \le v_0 \le v_0(M_0)$. Integrating $v_0'' < 0$ twice on $(M_0 + \epsilon, t)$ we have

$$v_0(t) \le v_0(M_0 + \epsilon) + v_0'(M_0 + \epsilon)(t - (M_0 + \epsilon)). \tag{2.47}$$

Now if R is sufficiently small then R^{2-N} will be very large and thus we may choose t sufficiently large so that the right-hand side of (2.47) becomes negative contradicting that $v_0 \geq \gamma$. So there exists $t_{\gamma_0} > M_0$ such that $v_0(t_{\gamma_0}) = \gamma$ and $v_0' < 0$ on (M_0, t_{γ_0}) if R is sufficiently small.

Next while $\beta < \frac{\gamma+\beta}{2} \leq v_0 \leq \gamma$ then $f(v_0) > 0$ so $v_0'' < 0$. Integrating $v_0'' < 0$ twice on (t_{γ_0}, t) gives

$$v_0(t) < \gamma + v_0'(t_{\gamma_0})(t - t_{\gamma_0}) \quad \text{with } v_0'(t_{\gamma_0}) < 0.$$

Now again if R is sufficiently small then R^{2-N} is very large and so we can choose t sufficiently large from which it would follow that $v_0(t) < \frac{\gamma+\beta}{2}$ contradicting that $v_0(t) \geq \frac{\gamma+\beta}{2}$. So there is a $t_{\gamma_1} > t_{\gamma_0}$ such that $v_0(t_{\gamma_1}) = \frac{\gamma+\beta}{2}$.

Now assume $v_0(t) > 0$ on (M_0, R^{2-N}). Then recall that $\frac{1}{2}\frac{v_0'^2}{h(t)} + F(v_0) > 0$ and there exists $C_{16} > 0$ so $-F(v_0) \geq C_{16} v_0^{1-q}$ for $t > t_{\gamma_1}$. Therefore,

$$-\frac{v_0'}{v_0^{\frac{1-q}{2}}} \geq \sqrt{2C_{16}h_0}\, t^{-\tilde{\alpha}/2} \quad \text{on } (t_{\gamma_1}, t).$$

Integrating on (t_{γ_1}, t) gives

$$0 < v_0^{\frac{1+q}{2}}(t) \leq \Big(\frac{\gamma+\beta}{2}\Big)^{\frac{1+q}{2}} - \frac{(1+q)\sqrt{2C_{16}h_0}}{2-\tilde{\alpha}}\big[t^{\frac{2-\tilde{\alpha}}{2}} - t_{\gamma_1}^{\frac{2-\tilde{\alpha}}{2}}\big]. \tag{2.48}$$

And again if R is sufficiently small then we can choose t sufficiently large so that the right-hand side of (2.48) becomes negative contradicting that $v_0 > 0$. Thus v_0 has a first positive zero, Z_1, on $[0, R^{2-N}]$ if $R > 0$ is sufficiently small. Also $0 < \frac{1}{2}\frac{v_0'^2}{h(t)} + F(v_0)$ for $t > 0$ so $0 < \frac{1}{2}\frac{v_0'^2(Z_1)}{h(Z_1)}$ and therefore $v_0'(Z_1) < 0$. Thus $v_0(Z_1 + \epsilon) < 0$ for $\epsilon > 0$ sufficiently small. Then since $v_a \to v_0$ uniformly on $[0, Z_1 + \epsilon]$ it follows that $v_a(Z_1 + \epsilon) < 0$ if a is sufficiently small and therefore if $a > 0$ and R are sufficiently small we see that v_a has a zero $0 < Z_{1,a} < R^{2-N}$. Then as at the beginning of the proof where we showed that v_a has a local maximum, a similar argument shows v_a has a local minimum, m_a, with $Z_{1,a} < m_a$ and then v_a has a second zero, $Z_{2,a}$, with $Z_{2,a} > m_a$, if $a > 0$ and R are sufficiently small. Continuing in this way we can find n zeros on $[0, R^{2-N}]$ if R is small enough. This completes the proof. □

3. Proof of main Results

Proof of Theorem 1.1. Consider the set

$$S_0 = \{a > 0 : v_a(t) > 0 \text{ on } (0, R^{2-N})\}.$$

If a is sufficiently large then $v_a(t) > 0$ on $(0, R^{2-N})$ by Lemma 2.3 and therefore $v_a \in S_0$ if a is sufficiently large. Thus $S_0 \neq \emptyset$. Also if a and R are sufficiently small then v_a has a zero on $(0, R^{2-N})$ by Lemma 2.4. Thus S_0 is bounded from below by a positive constant if R is sufficiently small. Now let

$$a_0 = \inf S_0.$$

We now show that $v_{a_0} > 0$ on $(0, R^{2-N})$ and $v_{a_0}(R^{2-N}) = 0$. Suppose on the contrary that there exists a zero, $Z_{a_0} \in (0, R^{2-N})$, and $v_{a_0} > 0$ on $(0, Z_{a_0})$ with $v_{a_0}(Z_{a_0}) = 0$. Then $0 < E_a(Z_{a_0}) = \frac{1}{2}\frac{v_{a_0}'^2(Z_{a_0})}{h(Z_{a_0})}$ so $v_{a_0}'(Z_{a_0}) < 0$.

Thus for $Z_{a_0} < t_1 < R^{2-N}$ and t_1 close to Z_{a_0} we have $v_{a_0}(t_1) < 0$. Then for a close to a_0 with $a < a_0$ then $v_a(t_1) < 0$ by continuous dependence (Lemma

2.2) but this contradicts the definition of a_0. Thus $v_{a_0} > 0$ on $(0, R^{2-N})$ and so $v_{a_0}(R^{2-N}) \geq 0$.

Next suppose that $v_{a_0}(R^{2-N}) > 0$. Then $v_{a_0} > 0$ on $(0, R^{2-N}]$ and for a close to a_0 with $a < a_0$ then $v_a > 0$ on $(0, R^{2-N}]$. But since $a < a_0$, it follows that $a \notin S_0$ so v_a must have a zero on $(0, R^{2-N}]$ which contradicts that $v_a > 0$ on $(0, R^{2-N}]$. Thus $v_{a_0}(R^{2-N}) = 0$. Also since E_a non-decreasing it follows that $0 < E_a(R^{2-N}) = \frac{1}{2}\frac{v_{a_0}'^2(R^{2-N})}{h(R^{2-N})}$ so $v_{a_0}'(R^{2-N}) < 0$.

Next let us define

$$S_1 = \{a > 0 : v_a(t) \text{ solves (2.4), (2.8) and has exactly one zero on } (0, R^{2-N})\}.$$

If we choose a slightly smaller than a_0 and R sufficiently small then it follows from Lemma 2.4 that v_a has at least one zero, Z_{a_1}, on $(0, R^{2-N})$ and Z_{a_1} is close to R^{2-N}. Also we know $v_{a_0}'(R^{2-N}) < 0$ so if a is sufficiently close to a_0 then $v_a' < 0$ on (Z_{a_1}, R^{2-N}). Thus v_a has at most one zero on $(0, R^{2-N})$ if a is sufficiently close to a_0. Therefore S_1 is nonempty. We also know from Lemma 2.4 that if R is sufficiently small then v_a has a second zero on $(0, R^{2-N})$. Therefore S_1 is bounded from below. So let

$$a_1 = \inf S_1.$$

In a similar way we can show that v_{a_1} has exactly one zero on $(0, R^{2-N})$ and $v_{a_1}(R^{2-N}) = 0$. In a similar fashion we can show that if n_0 is a given nonnegative integer then if $R > 0$ is sufficiently small then there exists $a_0, a_1, \dots, a_{n_0}$ such that v_{a_k} has k zeros on $(0, R^{2-N})$ and $v_{a_k}(R^{2-N}) = 0$. Finally, let $u_k(r) = v_{a_k}(r^{2-N})$. Then $u_k(r)$ satisfies (1.1)–(1.3) and u_k has k zeros on (R, ∞). This completes the proof. □

Proof of Theorem 1.2. Suppose there is a solution, v_a, of (2.4) with $v_a(0) = v_a(R^{2-N}) = 0$. This then implies that v_a has a local maximum, M_a, with $0 < M_a < R^{2-N}$ and $v_a'(M_a) = 0$. Since E_a is non-decreasing (by (2.10)) then for $0 < t < M_a$,

$$0 < \frac{1}{2}\frac{v_a'^2}{h(t)} + F(v_a(t)) = E_a(t) \leq E_a(M_a) = F(v_a(M_a)). \tag{3.1}$$

Thus $v_a(M_a) > \gamma$. Rewriting and integrating (3.1) on $(0, M_a)$ gives

$$\begin{aligned} \int_0^{M_a} \frac{v_a'(t)\,dt}{\sqrt{2}\sqrt{F(v_a(M_a)) - F(v_a(t))}} &\leq \int_0^{M_a} \sqrt{h_2}\ t^{-\tilde{\alpha}/2}\,dt \\ &= \frac{2\sqrt{h_2}}{2-\tilde{\alpha}}\ M_a^{\frac{2-\tilde{\alpha}}{2}} \\ &\leq \frac{2\sqrt{h_2}}{2-\tilde{\alpha}}(R^{2-N})^{\frac{2-\tilde{\alpha}}{2}}. \end{aligned} \tag{3.2}$$

Since $\tilde{\alpha} < 1$ and from (H4) we have $-F(v_a(t)) \leq F_0$ so it follows that $F(v_a(M_a)) - F(v_a(t)) \leq F(v_a(M_a)) + F_0$ which we apply to (3.2) to obtain

$$\int_0^{M_a} \frac{v_a'(t)\,dt}{\sqrt{2}\sqrt{F(v_a(M_a)) - F(v_a(t))}} \geq \frac{v_a(M_a)}{\sqrt{2}\sqrt{F(v_a(M_a)) + F_0}}. \tag{3.3}$$

Next from (H3) it follows that there is a constant $F_1 > 0$ such that $F(x) \le F_1|x|^{p+1}$ for all x and therefore it follows from (3.2)-(3.3) and that $v_a(M_a) > \gamma$ that

$$\frac{\gamma^{\frac{1-p}{2}}}{\sqrt{2}\sqrt{F_1 + \frac{F_0}{\gamma^{p+1}}}} \le \frac{v_a^{\frac{1-p}{2}}(M_a)}{\sqrt{2}\sqrt{F_1 + \frac{F_0}{v_a^{p+1}(M_a)}}} \le \frac{2\sqrt{h_2}}{2-\tilde{\alpha}}\, (R^{2-N})^{\frac{2-\tilde{\alpha}}{2}}. \tag{3.4}$$

Thus the right-hand side of (3.4) goes to zero if R sufficiently large but the left-hand side of (3.4) is positive and independent of R. Thus (1.1)–(1.3) has no solutions if R is sufficiently large. This completes the proof. □

4. Appendix

Lemma 4.1. *Let $a > 0$ and (H1)–(H6) hold. Then there exists a solution v_a of (2.4), (2.8) on $(0, \epsilon]$ for some $\epsilon > 0$.*

Proof. This is similar to the proof of existence in [1] which we include here for completeness. First integrate (2.4) over $(0, t)$ and use (2.8). This gives

$$v_a'(t) = a - \int_0^t h(s)f(v_a(s))\, ds \quad \text{for } t > 0. \tag{4.1}$$

Integrate again over $(0, t)$ and using (2.8) gives

$$v_a(t) = at - \int_0^t \int_0^s h(x)f(v_a(x))\, dx\, ds \qquad \text{for } t > 0. \tag{4.2}$$

Now let $W(t) = \frac{v_a(t)}{t}$ so $v_a(t) = tW(t)$ and $W(0) = \lim_{t\to 0^+} \frac{v_a(t)}{t} = v_a'(0) = a$. Rewriting (4.2) we obtain

$$W(t) = a - \frac{1}{t}\int_0^t \int_0^s h(x)f\left(xW(x)\right)\, dx\, ds \quad \text{for } t > 0. \tag{4.3}$$

We now we solve equation (4.3) on $(0, \epsilon]$ by a fixed point method as follows. Let us define

$$\begin{aligned} S = \Big\{ &W : [0, \epsilon] \to \mathbb{R} \text{ with } W(0) = a > 0, W \in C[0, \epsilon] \text{ and} \\ &|W(t) - a| \le \frac{a}{2} \text{ on } [0, \epsilon] \Big\} \end{aligned} \tag{4.4}$$

where $C[0, \epsilon]$ is the set of continuous functions on $[0, \epsilon]$ and $\epsilon > 0$. Let

$$\|W\| = \sup_{x\in[0,\epsilon]} |W(x)|.$$

Then $(S, \|\cdot\|)$ is a Banach space. Let us define a map T on S by

$$TW(t) = \begin{cases} a & \text{for } t = 0 \\ a - \frac{1}{t}\int_0^t\int_0^s h(x)f\left(xW(x)\right)\, d\, ds & \text{for } 0 < t \le \epsilon. \end{cases}$$

From (4.4) we see $0 < \frac{a}{2} \le W(x) \le \frac{3a}{2}$ on $[0, \epsilon]$ so it follows that $|\frac{-1}{x^q W^q(x)}| \le \frac{2^q x^{-q}}{a^q}$ on $(0, \epsilon]$ and since we know from (H1)–(H2) that $g_1(x)$ is locally Lipschitz this then implies that there exists $L_1 > 0$ such that

$$|g_1(x)| \le L_1|x| \quad \text{on } [0, \gamma]. \tag{4.5}$$

Now let $W \in S$ and suppose $0 < \epsilon < \frac{2\gamma}{3a}$. Then on $[0, \epsilon]$ we have

$$0 \le xW(x) < \epsilon\frac{3a}{2} < \frac{2\gamma}{3a}\frac{3a}{2} = \gamma.$$

using (H2), (2.6), and (4.5) we estimate

$$|h(x)f(xW(x))| = \big|h(x)\big(\frac{-1}{x^q W^q(x)} + g_1(xW(x))\big)\big| \le \frac{h_2 2^q}{a^q} x^{-(\tilde{\alpha}+q)} + \frac{3ah_2L_1}{2} x^{1-\tilde{\alpha}}.$$

Recalling from (2.6) that $\tilde{\alpha} + q < 1$ then integrating once over $[0,t]$ gives

$$\int_0^t |h(x)f\left(xW(x)\right)| \, dx \le \frac{A_1}{a^q} t^{1-\tilde{\alpha}-q} + A_2 a t^{2-\tilde{\alpha}} \tag{4.6}$$

where $A_1 = \frac{h_2 2^q}{(1-\tilde{\alpha}-q)}$ and $A_2 = \frac{3h_2L_1}{2(2-\tilde{\alpha})}$. Thus from (4.6) we have

$$\lim_{t\to 0^+} \int_0^t |h(x)f\left(xW(x)\right)| \, dx = 0. \tag{4.7}$$

Integrating (4.6) again gives

$$\int_0^t \int_0^s |h(x)f\left(xW(x)\right)| \, dx\, ds \le \frac{A_3 t^{2-\tilde{\alpha}-q}}{a^q} + aA_4 t^{3-\tilde{\alpha}} \tag{4.8}$$

where $A_3 = \frac{h_2 2^q}{(2-\tilde{\alpha}-q)(1-\tilde{\alpha}-q)}$ and $A_4 = \frac{3h_2L_1}{2(2-\tilde{\alpha})(3-\tilde{\alpha})}$. So we see

$$\lim_{t\to 0^+} \int_0^t \int_0^s |h(x)f\left(xW(x)\right)| \, dx\, ds = 0. \tag{4.9}$$

We now show that $T(W) \in S$ for each $W \in S$ if $\epsilon > 0$ is sufficiently small so we first let $W \in S$. It follows then from (4.9) that $T(W)$ is continuous on $[0,\epsilon]$. Thus we see $\lim_{t\to 0^+} TW(t) = a$ and so $|TW(t) - a| \le \frac{a}{2}$ on $[0,\epsilon]$ if $\epsilon > 0$ is sufficiently small. Therefore $T : S \to S$ if ϵ is sufficiently small.

We next prove that T is a contraction mapping if ϵ is sufficiently small. Let $W_1, W_2 \in S$ and suppose $0 < \epsilon < \frac{2\gamma}{3a}$. Then

$$TW_1(t) - TW_2(t) = -\frac{1}{t}\int_0^t \int_0^s h(x)[f(xW_1(x)) - f(xW_2(x))]\, d\, ds. \tag{4.10}$$

By (H2) we have $f(xW(x)) = -x^{-q}W^{-q}(x) + g_1(xW(x))$ where $0 < q < 1$. Then as earlier before (4.5) we see that $0 \le xW_i \le \epsilon\frac{3a}{2} < \gamma$ on $[0,\epsilon]$ for $i = 1,2$ therefore using (4.5) this gives

$$\begin{aligned} |f(xW_1(x)) - f(xW_2(x))| &= |\frac{-1}{x^q}[\frac{1}{W_1^q} - \frac{1}{W_2^q}] + g_1(xW_1(x)) - g_1(xW_2(x))| \\ &\le \frac{1}{x^q}\Big|\frac{1}{{W_1}^q} - \frac{1}{{W_2}^q}\Big| + L_1 x|W_1 - W_2|. \end{aligned} \tag{4.11}$$

Next applying the mean value theorem we see that the right-hand side of (4.11) is bounded by

$$\frac{1}{x^q}\big[\frac{q}{W_3^{q+1}}|W_1 - W_2|\big] + L_1 x|W_1 - W_2|$$

where W_3 is between W_1 and W_2. Since $W_i \in S$ for $i = 1,2,3$ and $|W_i - a| \le \frac{a}{2}$ then $\frac{a}{2} \le W_i \le \frac{3a}{2}$ on $[0,\epsilon]$. Therefore it follows that ${W_3}^{q+1} \ge \left(\frac{a}{2}\right)^{q+1}$ and so we have

$$|f(xW_1(x)) - f(xW_2(x))| \le |W_1 - W_2|\big[\frac{q}{x^q}\big(\frac{2}{a}\big)^{q+1} + L_1 x\big] \quad \text{on } (0,\epsilon]. \tag{4.12}$$

Recalling that $|h(t)| \le \frac{h_2}{t^{\tilde\alpha}}$ and $\tilde\alpha + q < 1$ from (2.6), and $t \in (0,\epsilon]$, then using (4.12) in (4.10) gives

$$\begin{aligned}|TW_1 - TW_2| &\le \frac{1}{t}\int_0^t \int_0^s \frac{h_2}{x^{\tilde\alpha}} |W_1 - W_2| \big[\frac{q}{x^q}\big(\frac{2}{a}\big)^{q+1} + L_1 x\big]\,dx\,ds \\ &\le \frac{1}{t}\|W_1 - W_2\| \int_0^t \int_0^s \frac{h_2}{x^{\tilde\alpha}} \big[\frac{q}{x^q}\big(\frac{2}{a}\big)^{q+1} + L_1 x\big]\,dx\,ds \\ &\le \|W_1 - W_2\| \big[\frac{A_5 \epsilon^{1-q-\tilde\alpha}}{a^{q+1}} + A_6 \epsilon^{2-\tilde\alpha}\big],\end{aligned}$$

where $A_5 = \frac{h_2\ q\ 2^{q+1}}{(2-q-\tilde\alpha)(1-q-\tilde\alpha)}$ and $A_6 = \frac{h_2 L_1}{(3-\tilde\alpha)(2-\tilde\alpha)}$. Since

$$\lim_{\epsilon\to 0^+} \big[\frac{A_5 \epsilon^{1-q-\tilde\alpha}}{a^{q+1}} + A_6 \epsilon^{2-\tilde\alpha}\big] = 0,$$

for $\epsilon > 0$ sufficiently small we see that

$$|TW_1 - TW_2| \le c\|W_1 - W_2\|,$$

where

$$c = \frac{A_5 \epsilon^{1-q-\tilde\alpha}}{a^{q+1}} + A_6 \epsilon^{2-\tilde\alpha}. \tag{4.13}$$

Thus for ϵ sufficiently small we see $0 < c < 1$ and therefore T is a contraction mapping on S.

Thus by the contraction mapping principle [5] there exists a unique solution $W \in S$ to $TW = W$ on $[0,\epsilon]$ for some $\epsilon > 0$. And then $v_a(t) = tW(t)$ is a solution of (2.4) on $(0,\epsilon]$ for some $\epsilon > 0$. This completest the proof. ☐

Lemma 4.2. *Let $a = 0$ and (H1)–(H6) hold. Then there exists a solution $v_0 > 0$ of equation (2.4) with $v_0(0) = v_0'(0) = 0$ on $(0,\epsilon]$ for some $\epsilon > 0$.*

Proof. Suppose first that v_0 is a solution to (2.4) on $(0,\epsilon]$ with

$$v_0(0) = 0, \quad v_0'(0) = 0. \tag{4.14}$$

Let us determine the behavior of $v_0(t)$ on $(0,\epsilon)$. using the fact that $f(v_a) = \frac{-1}{|v_a|^{q-1}v_a} + g_1(v_a)$ where $0 < q < 1$, $g_1(0) = 0$, and g_1 is continuous at 0, then integrating (2.4) on $(0,t)$ and using $v_0'(0) = 0$ gives:

$$v_0'(t) = -\int_0^t h(s) f\Big(v_0(s)\Big)\,ds.$$

Integrating again on $(0,t)$ and using $v_0(0) = 0$ gives

$$v_0(t) = -\int_0^t \int_0^s h(x) f\Big(v_0(x)\Big)\,dx\,ds. \tag{4.15}$$

Now let $v_0(t) = t^{\frac{2-\tilde\alpha}{1+q}} W(t)$ where $W(0) \ne 0$. Rewriting (4.15) we have

$$W(t) = \frac{1}{t^{\frac{2-\tilde\alpha}{1+q}}} \int_0^t \int_0^s h(x)\Big[\frac{1}{x^{\frac{(2-\tilde\alpha)q}{1+q}} W^q(x)} - g\Big(x^{\frac{2-\tilde\alpha}{1+q}} W(x)\Big)\Big]\,dx\,ds. \tag{4.16}$$

Assuming $W(t)$ is continuous at 0, taking the limit of (4.16) and using L'Hôpital's rule twice gives

$$\begin{aligned}
W(0) &= \lim_{t\to 0^+} W(t)\\
&= A_7 \lim_{t\to 0^+} \frac{t^{\tilde\alpha}h(t)\big[\frac{t^{-\tilde\alpha}}{t^{\frac{(2-\tilde\alpha)q}{1+q}}\ W^q(t)}\big] - g_1\big(t^{\frac{2-\tilde\alpha}{1+q}}W(t)\big)h(t)}{t^{\frac{-\tilde\alpha-2q}{1+q}}}\\
&= A_7\Big[\lim_{t\to 0^+}\frac{t^{\tilde\alpha}h(t)}{W^q(t)} - \lim_{t\to 0^+}\frac{h(t)g_1\big(t^{\frac{2-\tilde\alpha}{1+q}}W(t)\big)}{t^{\frac{-\tilde\alpha-2q}{1+q}}}\Big]\\
&= \frac{A_7h_1}{W^q(0)} - A_7\lim_{t\to 0^+}\frac{h(t)g_1\big(t^{\frac{2-\tilde\alpha}{1+q}}W(t)\big)}{t^{\frac{-\tilde\alpha-2q}{1+q}}},
\end{aligned} \tag{4.17}$$

where $A_7 = \big(\frac{1+q}{2-\tilde\alpha}\big)\big(\frac{1+q}{1-\tilde\alpha-q}\big)$. Since $t^{\tilde\alpha}h(t)\to h_1>0$, by (2.6) as $t\to 0^+$, $0<\tilde\alpha<1$ and $|g_1(v)|\le L_1|v|$ on $[0,\gamma]$ it follows that

$$\Big|\frac{h(t)g_1\big(t^{\frac{2-\tilde\alpha}{1+q}}W(t)\big)}{t^{\frac{-\tilde\alpha-2q}{1+q}}}\Big| \le \frac{h_2t^{-\tilde\alpha}L_1t^{\frac{2-\tilde\alpha}{1+q}}}{t^{\frac{-\tilde\alpha-2q}{1+q}}}|W(t)| = h_2L_1t^{2-\tilde\alpha}|W(t)|\to 0$$

as $t\to 0^+$. Then we have

$$W^{q+1}(0) = h_1A_7 = \frac{h_1(1+q)^2}{(2-\tilde\alpha)(1-\tilde\alpha-q)},$$

hence

$$W(0) = \Big[\frac{h_1(1+q)^2}{(2-\tilde\alpha)(1-\tilde\alpha-q)}\Big]^{\frac{1}{q+1}} \equiv b_0. \tag{4.18}$$

Now let $W(t)=b_0Y(t)$. Then $Y(0)=1$ and (4.16) becomes

$$Y(t) = \frac{1}{t^{\frac{2-\tilde\alpha}{1+q}}}\int_0^t\int_0^s h(x)\Big[\frac{1}{x^{\frac{(2-\tilde\alpha)q}{1+q}}b_0{}^{q+1}Y^q(x)} - \frac{g_1\big(x^{\frac{2-\tilde\alpha}{1+q}}b_0Y(x)\big)}{b_0}\Big]\,dx\,ds. \tag{4.19}$$

Now we attempt to solve (4.19) by using the contraction mapping principle. Let us define

$$\begin{aligned}
J = \Big\{&Y\in C[0,\epsilon] : Y(0)=1 \text{ and } |Y-1|<\delta \text{ where}\\
&0<\delta<1 \text{ is sufficiently small}\Big\}.
\end{aligned} \tag{4.20}$$

Let $\|Y\| = \sup_{x\in[0,\epsilon]}|Y(x)|$. Then $(J,\|\cdot\|)$ is a Banach space. Now we define T on J by

$$\begin{aligned}
&TY(t)\\
&= \begin{cases} 1 & \text{for } t=0\\[4pt] \frac{1}{t^{\frac{2-\tilde\alpha}{1+q}}}\int_0^t\int_0^s h(x)\Big[\frac{1}{x^{\frac{(2-\tilde\alpha)q}{1+q}}b_0{}^{q+1}Y^q(x)} - \frac{g_1\big(x^{\frac{2-\tilde\alpha}{1+q}}b_0Y(x)\big)}{b_0}\Big]\,dx\,ds & \text{for } 0<t\le\epsilon. \end{cases}
\end{aligned}$$

It is straightforward to show $TY(t)$ is continuous and from (4.18) and L'Hôpital's rule $\lim_{t\to 0^+}TY(t)=1$. Thus it follows that $|TY(t)-1|\le\delta$ on $[0,\epsilon)$ if ϵ sufficiently small, and therefore $T:J\to J$ if ϵ and δ are sufficiently small.

Since $|Y-1|<\delta<1$ then $0<1-\delta<Y<1+\delta$ and this implies that $\frac{1}{Y^q}\le\frac{1}{(1-\delta)^q}$. Let us suppose $Y_1, Y_2\in J$ then

$$\begin{aligned}
&TY_1(t)-TY_2(t)\\
&=\frac{1}{t^{\frac{2-\tilde\alpha}{1+q}}}\int_0^t\int_0^s h(x)\Big[\frac{1}{x^{\frac{(2-\tilde\alpha)q}{1+q}}{b_0}^{q+1}}\Big(\frac{1}{Y_1^q(x)}-\frac{1}{Y_2^q(x)}\Big)\Big]\,dx\,ds\\
&\quad-\frac{1}{b_0t^{\frac{2-\tilde\alpha}{1+q}}}\int_0^t\int_0^s h(x)\Big[g_1\Big(x^{\frac{2-\tilde\alpha}{1+q}}b_0Y_1(x)\Big)+g_1\Big(x^{\frac{2-\tilde\alpha}{1+q}}b_0Y_2(x)\Big)\Big]\,dx\,ds.
\end{aligned}\tag{4.21}$$

For the integral in (4.21) since $Y_1, Y_2\in J$, then by the mean value theorem there is a Y_3 between Y_1, Y_2 where $|Y_i-1|<\delta$ for $i=1,2,3$ (and therefore $1-\delta<Y_3<1+\delta$) then

$$|\frac{1}{Y_1^q}-\frac{1}{Y_2^q}|=\frac{q}{{Y_3}^{q+1}}|Y_1-Y_2|\le\frac{q}{(1-\delta)^{q+1}}|Y_1-Y_2|.$$

Then using (2.6) the integral in (4.21) becomes

$$\begin{aligned}
&\Big|\frac{1}{t^{\frac{2-\tilde\alpha}{1+q}}}\int_0^t\int_0^s h(x)\Big[\frac{1}{x^{\frac{(2-\tilde\alpha)q}{1+q}}{b_0}^{q+1}}\Big(\frac{1}{{Y_1}^q}-\frac{1}{{Y_2}^q}\Big)\Big]dx\,ds|\\
&\le\frac{q}{(1-\delta)^{1+q}{b_0}^{1+q}}\frac{|Y_1-Y_2|}{t^{\frac{(2-\tilde\alpha)q}{1+q}}}\int_0^t\int_0^s\frac{h(x)}{x^{\frac{(2-\tilde\alpha)q}{1+q}}}dx\,ds\\
&\le\frac{h_2q}{(1-\delta)^{1+q}{b_0}^{1+q}}\frac{|Y_1-Y_2|}{t^{\frac{(2-\tilde\alpha)q}{1+q}}}\int_0^t\int_0^s x^{\frac{-(\tilde\alpha+2q)}{1+q}}\,dx\,ds\\
&\le\frac{(1+q)^2h_2q}{(2-\tilde\alpha)(1-\tilde\alpha-q)(1-\delta)^{1+q}}\frac{|Y_1-Y_2|}{{b_0}^{1+q}}t^{\frac{(2-\tilde\alpha)(1-q)}{1+q}}.
\end{aligned}$$

Recalling ${b_0}^{q+1}=h_1A_7=h_1\big(\frac{1+q}{2-\tilde\alpha}\big)\big(\frac{1+q}{1-\tilde\alpha-q}\big)$ we obtain the right-hand side of (4.21) is bounded by

$$\frac{h_2q}{h_1(1-\delta)^{1+q}}\epsilon^{\frac{(2-\tilde\alpha)(1-q)}{1+q}}\quad\|Y_1-Y_2\|.$$

Since $\delta>0$ and $0<q<1$ we see that for $\epsilon>0$ sufficiently small,

$$\frac{h_2q}{h_1(1-\delta)^{1+q}}\epsilon^{\frac{(2-\tilde\alpha)(1-q)}{1+q}}\le d<1.$$

For the integral in (4.21) since g_1 is locally Lipschitz at 0, it follows that

$$\Big|g_1\Big(x^{\frac{2-\tilde\alpha}{1+q}}b_0Y_1(x)\Big)-g_1\Big(x^{\frac{2-\tilde\alpha}{1+q}}b_0Y_2(x)\Big)\Big|\le L_1\ b_0\ x^{\frac{2-\tilde\alpha}{1+q}}\ \|Y_1-Y_2\|$$

so substituting this into (4.21) gives

$$\begin{aligned}
&\Big|\frac{-1}{b_0t^{\frac{2-\tilde\alpha}{1+q}}}\int_0^t\int_0^s h(x)\Big[g_1\Big(x^{\frac{2-\tilde\alpha}{1+q}}b_0Y_1(x)\Big)-g_1\Big(x^{\frac{2-\tilde\alpha}{1+q}}b_0Y_2(x)\Big)\Big]\,dx\,ds\Big|\\
&\le\frac{|Y_1-Y_2|h_2L_1}{t^{\frac{2-\tilde\alpha}{1+q}}}\int_0^t\int_0^s x^{-\tilde\alpha+\frac{2-\tilde\alpha}{1+q}}\,dx\,ds\\
&\le|Y_1-Y_2|h_2L_1A_8t^{\frac{2+q}{1+q}}
\end{aligned}\tag{4.22}$$

where $A_8=\big(\frac{1+q}{1+(2+q)(1-\tilde\alpha)}\big)\big(\frac{1+q}{(2+q)(2-\tilde\alpha)}\big)$. Since $\lim_{t\to0^+}h_2L_1A_8t^{\frac{2+q}{1+q}}=0$ we can choose ϵ small enough so that $h_2L_1A_8t^{\frac{2+q}{1+q}}<\frac{1-d}{2}$ and so combining (4.21) and

(4.22) we obtain

$$|TY_1(t) - TY_2(t)| \leq \frac{1+d}{2} \|Y_1 - Y_2\|$$

where $0 \leq d < 1$ and thus $\frac{1+d}{2} < 1$.

Thus T is a contraction mapping, so by the contraction mapping principle [5] there is a unique solution $Y \in J$ to $T(Y) = Y$ on $[0, \epsilon]$. Then $v_a(t) = t^{\frac{2-\tilde{\alpha}}{1+q}} W(t)$ is a solution of (2.4), (4.14) on $[0, \epsilon]$ for some $\epsilon > 0$. This completes the proof. □

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