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Analytic solutions of n-th order differential equations at a singular point *

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Abstract

Necessary and sufficient conditions are be given for the existence of analytic solutions of the nonhomogeneous n-th order differential equation at a singular point. Let L be a linear differential operator with coefficients analytic at zero. If L^* denotes the operator conjugate to L, then we will show that the dimension of the kernel of L is equal to the dimension of the kernel of L^*. Certain representation theorems from functional analysis will be used to describe the space of linear functionals that contain the kernel of L^*. These results will be used to derive a form of the Fredholm Alternative that will establish a link between the solvability of $Ly = g$ at a singular point and the kernel of L^*. The relationship between the roots of the indicial equation associated with $Ly = 0$ and the kernel of L^* will allow us to show that the kernel of L^* is spanned by a set of polynomials.

1 Introduction

In 1969, Harris, Sibuya and Weinberg [6] proved a theorem, based on certain techniques of functional analysis, that contains as corollaries the existence theorems of O. Perron and F. Lettenmeyer. Harris, Sibuya and Weinberg were able to show, under certain conditions, that there exists a polynomial $f(z)$ such that the linear differential system

$$z^D \frac{dy}{dz} - A(z)y = f(z),$$

where $A(z)$ is analytic at $z = 0$, and $D = diag\{d_1, \dots, d_n\}$ with nonnegative integers d_i, has a solution analytic at $z = 0$. Two years later, H.L. Turrittin [9] posed the following problem: Given the equation

$$\frac{dW}{dz} = \sum_{j=0}^{\infty} z^{-j} A_j W + \sum_{j=1}^{\infty} z^{-j} B_j$$

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where both series converge if $|z| \geq z_0 > 0$, with a formal solution

$$W(z) = \sum_{j=0}^{\infty} z^{-j} C_j,$$

what are the necessary and sufficient conditions that the formal solution converges? At the same time Turrittin speculated that the afore mentioned paper of Harris, Sibuya and Weinberg might lead to an answer to this question.

L.J. Grimm and L.M. Hall answered the question posed by Turrittin in [2]. Although their initial work was inspired by the Harris, Sibuya and Weinberg paper, Grimm and Hall actually employed the theory of normally solvable and Noetherian operators, much as Korobeĭnik did, to derive an analogue of the Fredholm alternative for certain systems of functional differential operators. Then they used A.E. Taylor's representation of Banach spaces of analytic functions [8] to obtain conditions for the solvability of certain nonhomogeneous equations they had studied previously in [3].

The primary difficulty encountered when applying the results of [2] comes when attempting to describe the cokernel (see [2], [4], [5]) of the operator.

In this paper we will completely describe the cokernel of the operator L, defined by

$$Ly(z) = z^n y^{(n)}(z) + z^{n-1} a_1(z) y^{(n-1)}(z) + \cdots + a_n(z) y(z) \tag{1.1}$$

where $a_1(z), \ldots, a_n(z)$ are analytic at the point $z = 0$.

We will produce solvability conditions for the nonhomogeneous Bessel equation, that are independent of the order, but match those of [5] if it is an equation of integer order.

First, we will define a collection of Banach spaces, each containing functions analytic at the origin, and such that L is a continuous linear operator on this collection. We will show that the dimension of the kernel of L is equal to the dimension of the cokernel of L. Then we will use Taylor's representation of Banach spaces of analytic functions [8] to essentially set up a relationship between the roots of the indicial equation (from the Frobenius method) and the necessary form of the nonhomogeneous term g that allows us to find a solution to $Ly = g$ analytic at the origin.

2 Definitions and Preliminaries

Let B_1 and B_2 be Banach spaces and let $T : B_1 \to B_2$ be a continuous linear operator with domain B_1. Denote the conjugate of T by T^*.

Definition The operator T is normally solvable if $T(B_1)$ is closed in B_2.

The following lemma, due to Korobeĭnik [7], gives another characterization of normal solvability. The annihilator of the set W will be denoted by $W^{\perp}$.

Lemma 2.1 *Let B_1 and B_2 be Banach spaces, and let $T : B_1 \to B_2$ be a continuous linear operator. Then the following two properties are equivalent.*

(a) T is normally solvable.

(b) The equation $Ty = u$, $u \in B_2$, is solvable in B_1 if and only if $u \in (\ker T^*)^\perp$.

Definition The d-characteristic of T is the ordered pair $(\alpha(T), \beta(T))$, where $\alpha(T) = \dim(\ker T)$ and $\beta(T) = \dim(B_2/T(B_1)$. $\beta(T)$ is called the defect number of T.

Definition The index of the operator T, denoted by *ind*(T), is the number $\beta(T) - \alpha(T)$.

Definition The operator T is called a Noetherian operator if T is normally solvable and if both $\alpha(T)$ and $\beta(T)$ are finite.

Remark If $\beta(T) < \infty$, then $\beta(T) = \dim(\operatorname{coker} T) = \dim(\ker T^*)$.

The following theorem is due to I.C. Gohberg and M.G. Kreĭn [1].

Theorem 2.2 *Let $T : B_1 \to B_2$ be a continuous Noetherian operator, and let $T_1 : B_1 \to B_2$ be a compact operator. Then $T_2 = T + T_1$ is a Noetherian operator from B_1 into B_2 with* $\operatorname{ind}(T_2) = \operatorname{ind}(T)$.

For an arbitrary Noetherian operator the following theorem holds.

Theorem 2.3 (Fredholm Alternative) *If B_1 and B_2 are Banach spaces and $T : B_1 \to B_2$ is a Noetherian operator with conjugate T^*, then*

(i) The equation $T\,y = u$ has a solution (in B_1) if and only if $u \in (\ker T^*)^\perp$.

(ii) The equation $T^*\,f = g$ has a solution (in B_2^*) if and only if $g \in (\ker T^{**})^\perp$.

The proof of this theorem can be found in Korobeĭnik [7, Lemma 2.1].

3 Properties of the Operator

Let $G = \{z : |z| < 1\}$ and define A_p to be the Banach space of functions $\upsilon(z)$ analytic in G and p times continuously differentiable on $\overline{G}$, with norm

$$\|\upsilon(z)\|_p = \{\max |\upsilon^{(i)}(z)|,\ 0 \le i \le p,\ |z| = 1\}.$$

Consider the operator $L : A_n \to A_0$, defined in (1.1), where the a_i are now in A_0. In order to apply Theorem 2.2 to the operator L, we need the following two results. A proof for each of these two lemmas can be easily obtained by following the techniques used by Korobeĭnik [7].

Lemma 3.1 *The operator $l_0 : A_n \to A_0$ defined by $l_0 y(z) \equiv z^n y^{(n)}(z)$ is Noetherian with $\alpha(l_0) = \beta(l_0)$.*

Lemma 3.2 *The operator $l_i : A_n \to A_0$, $1 \le i \le n$, defined by $l_i y(z) = z^{n-i} a_i(z) y^{(n-i)}(z)$, $a_i \in A_0$, is compact.*

Now we are able to apply Theorem 2.2 to the operator L to obtain the following theorem.

Theorem 3.3 *L is a Noetherian operator with $\alpha(L) = \beta(L)$.*

4 Description of the Banach Spaces

Let $\tilde{A}$ be the space of all functions that are analytic in G. If $f \in \tilde{A}$, $f(z)$ can be expanded in a power series in z,

$$f(z) = \sum_{k=0}^{\infty} f_k z^k, \quad \text{convergent in } G.$$

Definition If f and g are elements of $\tilde{A}$ such that

$$f(z) = \sum_{k=0}^{\infty} f_k z^k \quad \text{and} \quad g(z) = \sum_{k=0}^{\infty} g_k z^k\,,$$

we define the Hadamard product of f and g by

$$B(f, g; z) = \sum_{k=0}^{\infty} f_k g_k z^k\,, \quad z \in G.$$

The series $B(f, g; z)$, thus defined, is an element of $\tilde{A}$ and defines a continuous linear functional of f over A_0.

Definition Define A_0^0 as the class of all $F \in \tilde{A}$ such that

$$\lim_{r \to 1^-} B(u, F; r)$$

exists for each $u \in A_0$.

Theorem 4.1 *A_0^0 is a Banach space if the norm is defined by*

$$\|F\| = \lim_{r \to 1^-} \sup_{\|u\|_0 = 1} |B(u, F; r)|, \quad u \in A_0.$$

Let A_0^* denote the space of all linear functionals defined on A_0. Recall that, as u varies over A_0, $B(u,F;r)$ defines a linear functional with norm $\sup_{\|u\|_0=1}|B(u,F;r)|$, $u\in A_0$. Then, if $F\in A_0^0$,

$$\gamma(u)=\lim_{r\to 1}B(u,F;r),\quad u\in A_0 \tag{4.1}$$

defines an element γ of A_0^*.

Define the mapping $\Lambda: A_0^0\to A_0^*$ by $\Lambda(F)=\gamma$. Let $\phi\in A_0^*$ and define the function

$$H(z)=\sum_{k=0}^{\infty}\phi(z^k)z^k,\quad z\in G.$$

Theorem 4.2 $H(z)\in A_0^0$.

Now, we may also define the mapping $\Gamma: A_0^*\to A_0^0$ by $\Gamma(\phi)=H$.

Theorem 4.3 *The mapping Λ is an isometric isomorphism and $\Lambda^{-1}=\Gamma$.*

Therefore, as a result of theorems 4.2 and 4.3, each linear functional in A_0^* can be uniquely represented (or determined) by an element in A_0^0. We restate this result in the following theorem.

Theorem 4.4 *Every linear functional $\gamma\in A_0^*$ is representable in the form*

$$\gamma(u)=\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}u(\rho e^{i\theta})F\big(\frac{r}{\rho}e^{-i\theta}\big)d\theta,\quad u\in A_0, \tag{4.2}$$

where $r<\rho<1$ and $F\in A_0^0$. F uniquely determines and is uniquely determined by γ, and $\|\gamma\|=\|F\|$.

5 Description of the Cokernel

Using the results of Theorem 4.4, we can apply the Fredholm Alternative to obtain the following theorem which gives necessary and sufficient conditions for the solvability of $Ly=g$, $g\in A_0$.

Theorem 5.1 *For $g\in A_0$, the equation $Ly=g$ has a solution in A_n if and only if*

$$\lim_{r\to 1}B(g,f;r)=0 \tag{5.1}$$

for all $f\in A_0^0$ such that

$$\lim_{r\to 1}B(Ly,f;r)=0 \tag{5.2}$$

for all $y\in A_n$.

Proof. Define the space $\mathcal{F}(L)$ by $\mathcal{F}(L) = \Gamma(\ker(L^*))$. Hence $\mathcal{F}(L)$ is a subspace of A_0^0 and can be represented by

$$\mathcal{F}(L) = \{f \in A_0^0 : f \text{ satisfies (5.2) for all } y \in A_n\}. \tag{5.3}$$

Now, from Theorem 2.3, the equation $Ly = g$ has a solution in A_n if and only if u is in $ker(L^*)^\perp$, and u is in $ker(L^*)^\perp$ if and only if $\gamma(u) = 0$ for all $\gamma \in ker(L^*)$. But by Theorem 4.4 this means

$$\gamma(u) = \lim_{r \to 1} B(g, f; r) = 0$$

for every $f \in \Gamma(\ker(L^*)) = \mathcal{F}(L)$. $\diamondsuit$

Equation (5.2) characterizes the cokernel of L ($\ker(L^*)$) and equation (5.1) characterizes the annihilator of the cokernel of L ($\ker(L^*)^\perp$). Next, we will use theorems 3.3 and 5.1 to describe completely the structure of the cokernel of L. We will do this by developing a rather intriguing relationship between the coefficients of a Frobenius series solution for $Ly = 0$ and the coefficients of a series for any member of A_0^0 that satisfies (5.2).

Definition The "factorial function" $r^{(k)}$ is defined as follows, according to the value of k:

(a) if $k = 1, 2, 3, \ldots$, then $r^{(k)} = r(r-1)(r-2)\cdots(r-k+1)$,

(b) if $k = 0$, then $r^{(0)} = 1$,

(c) if $k = -1, -2, -3, \ldots$, then $r^{(k)} = \frac{1}{(r+1)(r+2)\cdots(r-k)}$,

(d) if k is not an integer, then $r^{(k)} = \frac{\Gamma(r+1)}{\Gamma(r-k+1)}$.

If we seek a Frobenius series solution for $Ly = 0$ of the form $w(z) = \sum_{k=0}^{\infty} c_k z^{k+r}$, the indicial equation will be

$$F(r) = r^{(n)} + r^{(n-1)} a_{1_0} + \cdots + r^{(2)} a_{(n-2)_0} + r^{(1)} a_{(n-1)_0} + a_{n_0} = 0.$$

The following theorem is an immediate consequence of theorem 3.3 and 5.1.

Theorem 5.2 *If the equation $F(r) = 0$ does not have a nonnegative integer root, then the equation $Ly = g$ has a solution in A_n for any $g \in A_0$.*

Proof. We know that $dim(\ker(L^*)) = 0$ from Theorem 3.3. So $f \equiv 0$ is the only solution to (5.2) and (5.1) is satisfied for any $g \in A_0$. Therefore, by Theorem 5.1, equation $Ly = g$ has a solution in A_n for any $g \in A_0$. $\diamondsuit$

Now assume that the indicial equation, $F(r) = 0$, has at least one nonnegative integer root. We will show that the cokernel of L is always spanned by a set of polynomials and therefore the existence of a solution to $Ly = g$ in A_n

depends only on a finite number of the coefficients in the series expansion of $g(z)$. First, we need to rewrite equation (5.2). Let

$$f(z) = \sum_{k=0}^{\infty} f_k z^k \quad \text{and} \quad a_i(z) = \sum_{k=0}^{\infty} a_{i_k} z^k, \quad i = 1, \dots, n.$$

Then another form of (5.2) is:

$$\begin{gathered}
F(0)f_0 + \textstyle\sum_{k=1}^{\infty} a_{n_k} f_k = 0 \\
F(1)f_1 + \textstyle\sum_{k=1}^{\infty} \left[a_{(n-1)_k} + a_{n_k}\right] f_{k+1} = 0 \\
F(2)f_2 + \textstyle\sum_{k=1}^{\infty} \left[2a_{(n-2)_k} + 2a_{(n-1)_k} + a_{n_k}\right] f_{k+2} = 0 \\
F(3)f_3 + \textstyle\sum_{k=1}^{\infty} \left[6a_{(n-3)_k} + 6a_{(n-2)_k} + 3a_{(n-1)_k} + a_{n_k}\right] f_{k+3} = 0 \\
\vdots \\
F(n)f_n + \textstyle\sum_{k=1}^{\infty} \left[n^{(n-1)} a_{1_k} + n^{(n-2)} a_{2_k} + \dots + n a_{(n-1)_k} + a_{n_k}\right] f_{k+n} = 0 \\
\vdots \\
F(r)f_r + \textstyle\sum_{k=1}^{\infty} \left[r^{(n-1)} a_{1_k} + r^{(n-2)} a_{2_k} + \dots + r a_{(n-1)_k} + a_{n_k}\right] f_{k+r} = 0 \\
\vdots
\end{gathered} \tag{5.4}$$

Theorem 5.3 *If the equation $F(r) = 0$ has exactly one nonnegative integer root, r, then the cokernel of L is spanned by a polynomial of degree r.*

Proof. Since there is only one nonnegative integer root, r, we may reduce (5.4) to

$$\begin{gathered}
F(0)f_0 + \textstyle\sum_{k=1}^{r} a_{n_k} f_k = 0 \\
F(1)f_1 + \textstyle\sum_{k=1}^{r-1} \left[a_{(n-1)_k} + a_{n_k}\right] f_{k+1} = 0 \\
F(2)f_2 + \textstyle\sum_{k=1}^{r-2} \left[2a_{(n-2)_k} + 2a_{(n-1)_k} + a_{n_k}\right] f_{k+2} = 0 \\
F(3)f_3 + \textstyle\sum_{k=1}^{r-3} \left[6a_{(n-3)_k} + 6a_{(n-2)_k} + 3a_{(n-1)_k} + a_{n_k}\right] f_{k+3} = 0 \\
\vdots \\
F(r-1)f_{r-1} + \left[(r-1)^{(n-1)} a_{1_1} + \dots + (r-1)^{(1)} a_{(n-1)_1} + a_{n_1}\right] f_r = 0 \\
F(r)f_r = 0
\end{gathered} \tag{5.5}$$

by setting $f_k = 0$ for $k > r$. Back substitution for $f_k,\ k < r$, yields a polynomial of degree r in the cokernel of L. We also know that $dim(\ker(L)) = 1$, which implies, from Theorem 3.3, that $\dim(\ker(L^*)) = 1$. Therefore the cokernel of L is spanned by a polynomial of degree r. ♢

We now assume that the indicial equation for $Ly = 0$ has $q > 1$ nonnegative integer roots, $r_1, \dots, r_q$. Assume $r_1 > r_2 > \dots > r_q$ and let $N = r_1 - r_q$. Under these assumptions we are guaranteed, following the standard Frobenius series solution technique, at least one solution in A_n. If there is to be more than one linearly independent solution in A_n, it can be determined by using the smaller root, r_q, of the indicial equation and calculating the first $N+1$ coefficients of the Frobenius series. If $c_0, c_1, \dots, c_N$ denote these coefficients and $H = (h_{i,j})$ is the upper-triangular matrix where

$$h_{i,j} = \begin{cases} 0, & \text{if } i > j \\ F(r_1 - i + 1), & \text{if } i = j \\ (r_1 - j + 1)^{(n-1)} a_{1_{j-i}} + \cdots & \\ +(r_1 - j + 1)^{(1)} a_{(n-1)_{j-i}} + a_{n_{j-i}}, & \text{if } i < j \end{cases}$$

then the system of equations, $H\vec{c} = \vec{0}$,

$$\begin{bmatrix} F(r_1) & h_{1,2} & h_{1,3} & \cdots & h_{1,N} & h_{1,N+1} \\ 0 & F(r_1-1) & h_{2,3} & \cdots & h_{2,N} & h_{2,N+1} \\ 0 & 0 & F(r_1-2) & \cdots & h_{3,N} & h_{3,N+1} \\ \vdots & \cdots & 0 & \ddots & \cdots & \vdots \\ 0 & \cdots & \vdots & \ddots & F(r_q+1) & h_{N,N+1} \\ 0 & 0 & 0 & \cdots & 0 & F(r_q) \end{bmatrix} \begin{bmatrix} c_N \\ c_{N-1} \\ c_{N-2} \\ \vdots \\ c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{5.6}$$

determines these coefficients. In other words, the number of linearly independent solutions to (5.6) coincides with the number of linearly independent solutions in A_n of $Ly = 0$.

Definition Let

$$B = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{bmatrix},$$

then define the backward transpose of B as

$$B^t = \begin{bmatrix} b_{n,n} & \cdots & b_{2,n} & b_{1,n} \\ \vdots & \ddots & \vdots & \vdots \\ b_{n,2} & \cdots & b_{2,2} & b_{1,2} \\ b_{n,1} & \cdots & b_{2,1} & b_{1,1} \end{bmatrix}.$$

Return to the infinite set of equations (5.4) which determine the cokernel of L. We want to consider the system of equations

$$\begin{bmatrix} F(r_q) & h_{N,N+1} & h_{N-1,N+1} & \cdots & h_{2,N+1} & h_{1,N+1} \\ 0 & F(r_q+1) & h_{N-1,N} & \cdots & h_{2,N} & h_{1,N} \\ 0 & 0 & F(r_q+2) & \cdots & h_{2,N-1} & h_{1,N-1} \\ \vdots & \cdots & 0 & \ddots & \cdots & \vdots \\ 0 & \cdots & \vdots & \ddots & F(r_1-1) & h_{1,2} \\ 0 & 0 & 0 & \cdots & 0 & F(r_1) \end{bmatrix} \begin{bmatrix} f_{r_q} \\ f_{r_q+1} \\ f_{r_q+2} \\ \vdots \\ f_{r_1-1} \\ f_{r_1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{5.7}$$

given specifically by the $r_q th$ through the $r_1 th$ equations of (5.4). The coefficient matrix for this system of equations will be denoted by H^t since it is the backward transpose of the matrix H in (5.6).

If we set $f_k = 0$, $k > r_1$, then the number of linearly independent solutions to (5.7) coincides with the number of linearly independent polynomials in the cokernel of L. The following lemma is the basis of a very lucrative relationship between the number of linearly independent solutions in A_n of $Ly = 0$ and the number of linearly independent polynomials in the cokernel of L.

Lemma 5.4 *Let B be an $n \times n$ matrix and B^t be its backward transpose, then $B^t \sim B$.*

Proof. Denote the $n \times n$ backward identity by P, i.e.,

$$P = \begin{bmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ \vdots & & \cdots & & \vdots \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then $B^t = PB^TP$ where B^T is the usual transpose of B. It is known that $B^T \sim B$ which implies that there exists a nonsingular matrix R such that $B^T = RBR^{-1}$. So, if we let $Q = PR$ then Q is a nonsingular matrix such that $B^t = QBQ^{-1}$. $\diamondsuit$

Now, from Lemma 5.4, we know that $H \sim H^t$. In other words, there exists a nonsingular matrix, Q, such that $QH = H^tQ$. Therefore, equations (5.6) and (5.7) will have the same number of linearly independent solutions directly related to each other by the matrix Q. We state this result in the following lemma.

Lemma 5.5 *If Q is a nonsingular matrix such that $QH = H^tQ$ and $[c_N \cdots c_0]^T$ is a solution of (5.6) then $Q[c_N \cdots c_0]^T$ is a solution of (5.7).*

Together, lemmas 5.4 and 5.5 describe the unique relationship between the kernel and cokernel of the operator L. Thus, we can completely describe the structure of the cokernel of L when $F(r) = 0$ has two or more nonnegative integer roots. These results are presented in the following theorem.

Theorem 5.6 *Assume the indicial equation, $F(r) = 0$, has q ($q \leq n$) nonnegative integer roots, $r_1, \dots, r_q$, with $r_1 > r_2 > \cdots > r_q$. Let $N_i = r_i - r_{i+1}$ and $\eta(p) = \sum_{i=1}^{p} N_i$, $p \in \{1, \dots, q-1\}$. Then, if $Ly = 0$ has $m+1$ ($m+1 \leq q$) linearly independent solutions in A_n, there exist integers $p_1 < p_2 < \cdots < p_m \leq q-1$ such that the cokernel of L is spanned by the functions $f_0(z) \cup \{f_j(z)\}_{j=1}^m$ where $f_0(z)$ is a polynomial of degree r_q and $f_j(z)$ is a polynomial of degree $r_q + \eta(p_j)$.*

Proof. Following the Frobenius method for solving $Ly = 0$ we find a solution in A_n associated with the largest root, r_1. Then we check for a solution in A_n associated with each of the smaller roots. With the hypotheses above we will get solutions associated with the following $m+1$ roots,

$$\{r_1, r_1 - \eta(p_1), r_1 - \eta(p_2), \dots, r_1 - \eta(p_m)\},$$

where $p_1, p_2, \dots, p_m$ are integers such that $1 \le p_1 < p_2 < \cdots < p_m \le q-1$, and these will be linearly independent. Thus (5.6) will have $m+1$ linearly independent solutions, where each of the coefficients

$$\{c_N, c_{N-\eta(p_1)}, c_{N-\eta(p_2)}, \dots, c_{N-\eta(p_m)}\}$$

are arbitrary. From lemmas 5.4 and 5.5, we will get $m+1$ linearly independent solutions to (5.7), where each of the coefficients

$$\{f_{r_q}, f_{r_q+\eta(p_1)}, f_{r_q+\eta(p_2)}, \dots, f_{r_q+\eta(p_m)}\}$$

are arbitrary.

Therefore, each of the polynomials

$$\begin{aligned} f_0(z) &= f_{0_0} + f_{0_1} z + \cdots + f_{0_{r_q}} z^{r_q} \\ f_1(z) &= f_{1_0} + f_{1_1} z + \cdots + f_{1_{r_q+\eta(p_1)}} z^{r_q+\eta(p_1)} \\ f_2(z) &= f_{2_0} + f_{2_1} z + \cdots + f_{2_{r_q+\eta(p_2)}} z^{r_q+\eta(p_2)} \\ &\vdots \\ f_m(z) &= f_{m_0} + f_{m_1} z + \cdots + f_{m_{r_q+\eta(p_m)}} z^{r_q+\eta(p_m)} \end{aligned} \tag{5.8}$$

is an element of $\ker(L^*)$. But, from Theorem 3.3 we know that

$$\dim(\ker L^*) = \dim(\ker L) = m+1.$$

Hence, the cokernel of L is spanned by the polynomials in (5.8). $\diamondsuit$

Theorems 5.1, 5.2, 5.3, and 5.6 describe completely the cokernel of the operator L, which is either zero dimensional or is spanned by polynomials. Therefore, if the nonhomogeneous term in $Ly = g$ is represented by the series

$$g(z) = \sum_{k=0}^{\infty} g_k z^k, \tag{5.9}$$

the solvability in A_n of $Ly = g$ is determined by a finite number of the coefficients in (5.9).

6 Examples

In this section we will illustrate some of the preceding results. Consider the nonhomogeneous Bessel equation of order ν.

Example 6.1 Let $L_1 : A_2 \to A_0$ be defined by

$$L_1\, y(z) = z^2\, y''(z) + zy'(z) + (z^2 - \nu^2)y(z).$$

If we use the Frobenius method to solve $L_1 y = 0$ we obtain the indicial function $F(r) = (r-\nu)(r+\nu)$, so ν and $-\nu$ are the roots of the indicial equation. Without loss of generality assume that $\nu \geq 0$. If ν is not an integer, $F(r) = 0$ will not have a nonnegative integer root. Thus, by Theorem 5.2, $Ly = g$ will have a solution in A_2 for any function $g \in A_0$. If ν is an integer, then $F(r) = 0$ has exactly one nonnegative integer root, ν. Thus, by Theorem 5.3, the cokernel of L_1 is spanned by a polynomial of degree ν. Following the proof of Theorem 5.3 we are able to produce this polynomial. First, we derive the finite system of equations

$$\begin{gathered}
F(0)f_0 + \textstyle\sum_{k=1}^{\nu} a_{n_k}\, f_k = 0 \\
F(1)f_1 + \textstyle\sum_{k=1}^{\nu-1} \left[a_{(n-1)_k} + a_{n_k}\right] f_{k+1} = 0 \\
F(2)f_2 + \textstyle\sum_{k=1}^{\nu-2} \left[2a_{(n-2)_k} + 2a_{(n-1)_k} + a_{n_k}\right] f_{k+2} = 0 \\
F(3)f_3 + \textstyle\sum_{k=1}^{\nu-3} \left[6a_{(n-3)_k} + 6a_{(n-2)_k} + 3a_{(n-1)_k} + a_{n_k}\right] f_{k+3} = 0 \\
\vdots \\
F(\nu-1)f_{\nu-1} + \left[(\nu-1)^{(n-1)}a_{1_1} + \cdots + (\nu-1)^{(1)}a_{(n-1)_1} + a_{n_1}\right] f_\nu = 0 \\
F(\nu)f_\nu = 0,
\end{gathered}$$

similar to those in (5.5); then, back substitution yields the polynomial

$$f(z) = z^\nu + \sum_{m=1}^{\lfloor \frac{\nu}{2} \rfloor} \frac{(\nu-m-1)!}{2^{2m}\, m!(\nu-1)!} z^{\nu-2m},$$

which spans the cokernel of L_1. So, by Theorem 5.1, $L_1 y = g$ will have a solution in A_2 if and only if the coefficients in the series expansion of $g(z) = \sum_{k=0}^{\infty} g_k z^k$ satisfy

$$g_\nu + \sum_{m=1}^{\lfloor \frac{\nu}{2} \rfloor} \frac{(\nu-m-1)!}{2^{2m}\, m!(\nu-1)!} g_{\nu-2m} = 0.$$

Remark This example was given previously by Hall [5]. However, by applying the techniques developed in this article, we have obtained solvability conditions for $L_1 y = g$, the nonhomogeneous Bessel equation of order ν, independent of the value of ν. We obtained this condition, which matches that of [5] if ν is an integer, without converting the equation to a system.

Example 6.2 Let $L_2 : A_2 \to A_0$ be defined by

$$L_2\, y(z) = z^2\, y''(z) - 2z(z-2)y'(z) + 2(2-3z)y(z).$$

Here, $F(r) = 0$ has roots $r_1 = 4$, $r_2 = 1$ and $N = r_1 - r_2 = 3$. We will use Theorem 5.6 to describe the cokernel of the operator L_2. First, we obtain the coefficient matrices of (5.6) and (5.7).

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \textit{and} \quad H^t = \begin{bmatrix} 0 & -4 & 0 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, from Lemma 5.4, we know that $H \sim H^t$ and so there exists a nonsingular matrix

$$Q = \begin{bmatrix} 1 & -2 & 2 & 8 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

such that $QH = H^t Q$.

Now, $\begin{bmatrix} 1 & 2 & -2 & 1 \end{bmatrix}^T$ is a solution to $H\vec{(c)} = \vec{0}$ which implies, by Lemma 5.5, that $Q \begin{bmatrix} 1 & 2 & -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T$ is a solution to $H^t \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix}^T = \vec{0}$. Thus, by setting $f_k = 0$, $k > 4$, in (5.4) we can use back substitution to get $f_0 = \dfrac{3}{2} f_1$. So, we get two linearly independent polynomials, $f_1(z) = z^4$ and $f_2(z) = z + \frac{3}{2}$ that span the cokernel of L_2.

Thus, by Theorem 5.1, $L_2 y = g$ will have a solution in A_2 if and only if the coefficients in the series expansion of $g(z)$ satisfy both $g_4 = 0$ and $g_1 = -\frac{3}{2} g_0$.

Remark Back substitution is necessary to find the f_k, $k < r_q = 1$, since the equation $H^t \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix}^T = \vec{0}$ is derived from the first through the fourth equations of (5.4).

Example 6.3 Let $L_3 : A_4 \to A_0$ be defined by

$$L_3\, y(z) = z^4\, y^{(4)}(z) + z^3 a_1(z) y^{(3)}(z) + z^2 a_2(z) y''(z) + z a_3(z) y' + a_4(z) y(z),$$

where $a_1(z) = z^4 - 4$, $a_2(z) = 12z^2 - \frac{1}{2}z + 6$, $a_3(z) = -3z^3 + 13z^2 - z - 1$, and $a_4(z) = 6z^4 + z^2 - 11z + 1$.

In this case, $F(r) = 0$ has roots $r_1 = 5$, $r_2 = 4$, $r_3 = 1$, and $r_4 = 0$. So, by Theorem 5.6, we know that the cokernel of L_3 is spanned by the function $f_0(z) = 1$ and, possibly as many as three more polynomials, $f_j(z)$, of degree $\eta(p_j)$, $1 \le p_j \le 3$. To apply Theorem 5.6, i.e., determine the polynomials in the cokernel of L_3, it is enough to find the matrix H^t of (5.7) and solve

$H^t \begin{bmatrix} f_0 & f_1 & f_2 & f_3 & f_4 & f_5 \end{bmatrix}^T = \vec{0}.$

$$H^t = \begin{bmatrix} 0 & -11 & 1 & 0 & 6 & 0 \\ 0 & 0 & -12 & 14 & -3 & 6 \\ 0 & 0 & 12 & -14 & 3 & -6 \\ 0 & 0 & 0 & 12 & -17 & 112 \\ 0 & 0 & 0 & 0 & 0 & -21 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $\begin{bmatrix} f_0 & f_1 & f_2 & f_3 & f_4 & f_5 \end{bmatrix}^T = \begin{bmatrix} \alpha & \frac{533}{11}\beta & 101\beta & 102\beta & 72\beta & 0 \end{bmatrix}^T$ is a solution to $H^t \begin{bmatrix} f_0 & f_1 & f_2 & f_3 & f_4 & f_5 \end{bmatrix}^T = \vec{0}$, where α and β are arbitrary constants.

Therefore, by setting $f_k = 0$, $k > 5$, we get two linearly independent polynomials, $f_0(z) = 1$ and $f_1(z) = \frac{533}{11}z + 101z^2 + 102z^3 + 72z^4$ that span the cokernel of L_3. So there exists only one integer, $p_1 = 2$, such that the cokernel of L_3 is spanned by $f_0(z)$, a polynomial of degree $r_4 = 0$, and $f_1(z)$, a polynomial of degree $r_4 + \eta(p_1) = 0 + (1+3)$. So, by Theorem 5.1, $L_3 y = g$ will have a solution in A_4 if and only if the coefficients in the series expansions of $g(z)$ satisfy the conditions $g_0 = 0$ and $\frac{533}{11}g_1 + 101g_2 + 102g_3 + 72g_4 = 0$.

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