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OSCILLATION CRITERIA FOR SECOND-ORDER NEUTRAL DIFFERENTIAL EQUATIONS WITH DISTRIBUTED DEVIATING ARGUMENTS

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Abstract. Using a class of test functions $\Phi(t, s, T)$ defined by Sun [13] and a generalized Riccati technique, we establish some new oscillation criteria for the second-order neutral differential equation with distributed deviating argument

$$(r(t)\psi(x(t))Z'(t))' + \int_a^b q(t,\xi)f[x(g(t,\xi))]d\sigma(\xi) = 0, \quad t \geq t_0,$$

where $Z(t) = x(t) + p(t)x(t - \tau)$. The obtained results are different from most known ones and can be applied to many cases which are not covered by existing results.

1. Introduction and Preliminaries

Consider the second-order neutral differential equation with distributed deviating argument

$$(r(t)\psi(x(t))Z'(t))' + \int_a^b q(t,\xi)f[x(g(t,\xi))]d\sigma(\xi) = 0, \quad t \geq t_0, \tag{1.1}$$

where $Z(t) = x(t) + p(t)x(t - \tau)$, $\tau \geq 0$, and the following conditions are assumed to hold without further mentioning:

- (A1) $r, p \in C(I, \mathbb{R})$ and $0 \leq p(t) \leq 1$, $r(t) > 0$ for $t \in I$, $\int^\infty 1/r(s)ds = \infty$, $I = [t_0, \infty)$;
- (A2) $\psi \in C^1(\mathbb{R}, \mathbb{R})$, $\psi(x) > 0$ for $x \neq 0$;
- (A3) $f \in C(\mathbb{R}, \mathbb{R})$, $xf(x) > 0$ for $x \neq 0$;
- (A4) $q \in C(I \times [a, b], [0, \infty))$ and $q(t, \xi)$ is not eventually zero on any half-line $[t_u, \infty) \times [a, b]$, $t_u \geq t_0$;
- (A5) $g \in C(I \times [a, b], [0, \infty))$, $g(t, \xi) \leq t$ for $t \geq t_0$ and $\xi \in [a, b]$, $g(t, \xi)$ has a continuous and positive partial derivative on $I \times [a, b]$ with respect to the first variable t and nondecreasing with respect to the second variable ξ, respectively, and $\liminf_{t\to\infty} g(t, \xi) = \infty$ for $\xi \in [a, b]$;
- (A6) $\sigma \in C([a, b], \mathbb{R})$ is nondecreasing, and the integral of (1.1) is in the sense of Riemann-Stieltjes.

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Let $\tau^*(t) = \min\{\tau(t) = t - \tau, \delta(t) = \min g(t, \xi)$ for $\xi \in [a, b]\}$ and let $T^* = \min\{\tau^*(t) : t \geq 0\}$ and $\tau^*_{-1}(t) = \sup\{s \geq 0 : \tau^*(s)] \leq t\}$ for $t \geq T^*$. Clearly $\tau^*_{-1}(t) \geq t$ for $t \geq T^*$, $\tau^*_{-1}(t)$ is nondecreasing and coincides with the inverse of $\tau^*(t)$ when latter exists. By a solution of (1.1) we means a nontrivial real-valued function $x(t)$ which has the properties $Z(t) \in C^1([\tau^*_{-1}(t_0), \infty), \mathbb{R})$, and $r(t)\psi(x(t)Z'(t) \in C^1([\tau^*_{-1}(t_0), \infty), \mathbb{R})$. Our attention is restricted to those solutions $x(t)$ of (1.1) which exist on some half-line $[t_x, \infty)$ with $\sup\{|x(t)| : t \geq T\} > 0$ for any $T \geq t_x$, and satisfy (1.1). As usual, a solution $x(t)$ of (1.1) is called oscillatory if the set of its zeros is unbounded from above, otherwise, it is called nonoscillatory. (1.1) is called oscillatory if all solutions are oscillatory.

We note that second order neutral delay differential equations have various applications in problems dealing with vibrating masses attached to an elastic bar and some variational problems, etc. For further applications and questions concerning existence and uniqueness of solutions of neutral delay differential equations, see [8].

In the last decades, there has been an increasing interest in obtaining sufficient conditions for the oscillation and/or nonoscillation of solutions of second order linear and nonlinear neutral delay differential equations with distributed deviating arguments (see, for example, [4, 10, 14, 16, 17] and the references therein). Very recently, in [16], the results in [5, 11, 15] for second order differential equations have been extended to (1.1).

For other oscillation results of various neutral functional differential equations we refer the reader to the monographs [1, 2, 3, 6, 7].

The purpose of this paper is to establish some new oscillation criteria for (1.1) by introducing a class of functions $\Phi(t, s, T)$ defined in the recent paper [13] and a generalized Riccati technique. Our results are different from most known ones in the sense that they are given in the form that $\limsup_{t\to\infty}[\cdot]$ is greater than a constant, rather than in the form $\limsup_{t\to\infty}[\cdot] = +\infty$. Thus, our results can be applied to many cases, which are not covered by existing ones.

Following the idea of Sun [13], we say that a function $\Phi = \Phi(t, s, T)$ belongs to a function class X, denoted by $\Phi \in$ X, if $\Phi \in C(E, \mathbb{R})$, where $E = \{(t, s, T) : t_0 \leq T \leq s \leq t < \infty\}$, which satisfies $\Phi(t, t, T) = \Phi(t, T, T) = 0$, $\Phi(t, s, T) \neq 0$ for $T < s < t$, and has the partial derivative $\partial\Phi/\partial s$ on E such that $\partial\Phi/\partial s$ is locally integrable with respect to s on E.

We now recall to introduce another class of functions defined by Philos [11] which is used extensively. Let $D_0 = \{(t, s) : t > s > t_0\}$ and $D = \{(t, s) : t \geq s \geq t_0\}$. Say that $H \in C(D, \mathbb{R})$ belongs to a function class Y, denoted by $H \in$ Y, if $H(t, t) = 0$ for $t \geq t_0$, $H(t, s) > 0$ on D_0, H has continuous partial derivatives on D_0 with respect to t and s.

Let us state three sets of conditions commonly used as in literature; see for example [12, 16], which we rely on:

- (S1) $f'(x)$ exists, $f'(x) \geq k_1$ and $\psi(x) \leq L^{-1}$ for $x \neq 0$;
- (S2) $f'(x)$ exists, $f'(x)/\psi(x) \geq k_2$ for $x \neq 0$;
- (S3) $f(x)/x \geq k_3$ and $\psi(x) \leq L^{-1}$ for $x \neq 0$,
  where k_1, k_2, k_3 and L are positive real numbers.

In addition, we will use the following conditions as in [12, 16]:

- (N1) There exists a positive real number M such that $\pm f(\pm uv) \geq Mf(u)f(v)$ for $uv \geq 0$;
- (N2) $u\psi'(u) > 0$ for $u \neq 0$.

In order to prove our theorems we need the following three Lemmas whose proof can be found in [16].

Lemma 1.1 ([16]). *Suppose that (S1) and (N1) are satisfied. Let $x(t)$ be an eventually positive solution of* (1.1). *Then there exists a $T_0 \ge t_0$ such that*

$$Z(t) > 0, \quad Z'(t) > 0 \quad (r(t)\psi(x(t))Z'(t))' \le 0, \quad \text{for } t \ge T_0. \tag{1.2}$$

Moreover, for $t \ge T_0$,

$$(r(t)\psi(x(t))Z'(t))' + Mf[Z(g(t,a))]\int_a^b q(t,\xi)f[1-p(g(t,\xi))]d\sigma(\xi) \le 0. \tag{1.3}$$

Lemma 1.2 ([16]). *Suppose that (S2), (N1) and (N2) are satisfied. Let $x(t)$ be an eventually positive solution of* (1.1). *Then there exists a $T_0 \ge t_0$ such that* (1.2) *and* (1.3) *hold.*

Lemma 1.3 ([16]). *Suppose that (S3) is satisfied. Let $x(t)$ be an eventually positive solution of* (1.1). *Then there exists a $T_0 \ge t_0$ such that* (1.2) *holds. Moreover,*

$$(r(t)\psi(x(t))Z'(t))' + k_3 Z[g(t,a)]\int_a^b q(t,\xi)[1-p(g(t,\xi))]d\sigma(\xi) \le 0, \quad t \ge T_0. \tag{1.4}$$

2. Kamenev-type oscillation criteria

Theorem 2.1. *Let (S1) and (N1) hold. If there exist functions $\varphi \in C^1(I,\mathbb{R}^+)$, $\phi \in C^1(I,\mathbb{R})$ and $\Phi \in \mathrm{X}$ such that for each $T \ge t_0$,*

$$\limsup_{t\to\infty}\int_T^t \Big\{\Phi^2(t,s,T)\Theta_1(s) - \gamma_1(s)\big[\Phi'_s(t,s,T) + \frac{1}{2}l_1(s)\Phi(t,s,T)\big]^2\Big\}ds > 0, \tag{2.1}$$

where

$$l_1(t) = \frac{\varphi'(t)}{\varphi(t)} + \frac{2k_1Lg'(t,a)\phi(t)}{r[g(t,a)]}, \quad \gamma_1(t) = \frac{r[g(t,a)]\varphi(t)}{k_1Lg'(t,a)},$$

$$\Theta_1(t) = \varphi(t)\Big\{M\int_a^b q(t,\xi)f[1-p(g(t,\xi))]d\sigma(\xi) + \frac{k_1Lg'(t,a)\phi^2(t)}{r[g(t,a)]} - \phi'(t)\Big\},$$

then (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1) on I. Without loss of generality we assume that $x(t) > 0$ for $t \ge t_0$, (The case of $x(t) < 0$ can be considered similarly). By Lemma 1.1, there exists a $T_0 \ge t_0$ such that (1.2) and (1.3) hold. Define

$$v(t) = \varphi(t)\Big[\frac{r(t)\psi(x(t))Z'(t)}{f[Z(g(t,a))]} + \phi(t)\Big] \quad \text{for all } t \ge T_0. \tag{2.2}$$

Differentiating (2.2) and using (1.3), it follows that

$$\begin{aligned} v't) \le \frac{\varphi'(t)}{\varphi(t)}v(t) - \varphi(t)\Big[M\int_a^b q(t,\xi)f[1-p(g(t,\xi))]d\sigma(\xi) \\ + \frac{r(t)\psi(x(t))Z'(t)}{f^2[Z(g(t,a))]}f'[Z(g(t,a))]Z'[g(t,a)]g'(t,a) - \phi'(t)\Big]. \end{aligned}$$

Since $g(t,a) \le t$ and $(r(t)\psi(x(t))Z'(t))' \le 0$, we have

$$r(t)\psi(x(t))Z'(t) \le r[g(t,a)]\psi[x(g(t,a))]Z'[g(t,a)].$$

Thus,

$$\begin{aligned} v'(t) &\le \frac{\varphi'(t)}{\varphi(t)}v(t) - \varphi(t)\Big[M\int_a^b q(t,\xi)f[1-p(g(t,\xi))]d\sigma(\xi) - \phi'(t)\Big] \\ &\quad - \frac{k_1\varphi(t)g'(t,a)}{r[g(t,a)]\psi[x(g(t,a))]}\Big(\frac{r(t)\psi(x(t))Z'(t)}{f[Z(g(t,a))]}\Big)^2 \\ &\le \frac{\varphi'(t)}{\varphi(t)}v(t) - \varphi(t)\Big[M\int_a^b q(t,\xi)f[1-p(g(t,\xi))]d\sigma(\xi) - \phi'(t)\Big] \\ &\quad - \frac{k_1 L\,\varphi(t)g'(t,a)}{r[g(t,a)]}\Big(\frac{v(t)}{\varphi(t)} - \phi(t)\Big)^2. \end{aligned}$$

So that

$$v'(s) \le -\Theta_1(s) + l_1(s)v(s) - \frac{1}{\gamma_1(s)}v^2(s). \tag{2.3}$$

Multiplying (2.3) by $\Phi^2(t,s,T_0)$, and integrating from T_0 to t, we get

$$\begin{aligned} &\int_{T_0}^t \Phi^2(t,s,T_0)\Theta_1(s)ds \\ &\le \int_{T_0}^t \Phi^2(t,s,T_0)[-v'(s)+l_1(s)v(s)]ds - \int_{T_0}^t \frac{1}{\gamma_1(s)}\Phi^2(t,s,T_0)v^2(s)ds. \end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned} &\int_{T_0}^t \Phi^2(t,s,T_0)\Theta_1(s)ds \\ &\le 2\int_{T_0}^t \Phi(t,s,T_0)\big[\Phi'_s(t,s,T_0) + \frac{1}{2}l_1(s)\Phi(t,s,T_0)\big]v(s)ds \\ &\quad - \int_{T_0}^t \frac{1}{\gamma_1(s)}\Phi^2(t,s,T_0)v^2(s)ds \\ &= \int_{T_0}^t \gamma_1(s)\big[\Phi'_s(t,s,T_0) + \frac{1}{2}l_1(s)\Phi(t,s,T_0)\big]^2 ds \\ &\quad - \int_{T_0}^t \frac{1}{\gamma_1(s)}\big\{\Phi(t,s,T_0)v(s) - \gamma_1(s)\big[\Phi'_s(t,s,T_0) + \frac{1}{2}l_1(s)\Phi(t,s,T_0)\big]\big\}^2 ds, \end{aligned}$$

which implies

$$\int_{T_0}^t \big\{\Phi^2(t,s,T_0)\Theta_1(s) - \gamma_1(s)\big[\Phi'_s(t,s,T_0) + \frac{1}{2}l_1(s)\Phi(t,s,T_0)\big]^2\big\}ds \le 0.$$

This contradicts (2.1) and completes the proof. $\square$

Theorem 2.2. *Let (S2), (N1), (N2) hold. If there exist functions $\varphi \in C^1(I,\mathbb{R}^+)$, $\phi \in C^1(I,\mathbb{R})$ and $\Phi \in \mathrm{X}$ such that for each $T \ge t_0$,*

$$\limsup_{t\to\infty}\int_T^t \Big\{\Phi^2(t,s,T)\Theta_2(s) - \gamma_2(s)\big[\Phi'_s(t,s,T) + \frac{1}{2}l_2(s)\Phi(t,s,T)\big]^2\Big\}ds > 0, \tag{2.4}$$

where

$$l_2(t) = \frac{\varphi'(t)}{\varphi(t)} + \frac{2k_2 g'(t,a)\phi(t)}{r[g(t,a)]}, \quad \gamma_2(t) = \frac{r[g(t,a)]\varphi(t)}{k_2 g'(t,a)},$$

$$\Theta_2(t) = \varphi(t)\Big\{M\int_a^b q(t,\xi)f[1-p(g(t,\xi))]d\sigma(\xi) + \frac{k_2 g'(t,a)\phi^2(t)}{r[g(t,a)]} - \phi'(t)\Big\},$$

then (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1) on I, say $x(t) > 0$ for $t \geq t_0$. Then, by Lemma 1.2, there exists a $T_0 \geq t_0$ such that (1.2) and (1.3) hold. We consider the function $v(t)$ defined by (2.2), and proceeding as in the proof of Theorem 2.1 to get

$$\begin{aligned} v'(t) &\leq \frac{\varphi'(t)}{\varphi(t)}v(t) - \varphi(t)\Big[M\int_a^b q(t,\xi)f[1-p(g(t,\xi))]d\sigma(\xi) - \phi'(t)\Big] \\ &\quad - \frac{\varphi(t)g'(t,a)}{r[g(t,a)]\psi[x(g(t,a))]}\frac{f'[Z(g(t,a))]}{\psi[x(g(t,a))]}\Big(\frac{r(t)\psi(x(t))Z'(t)}{f[Z(g(t,a))]}\Big)^2. \end{aligned}$$

Now, we use $x[g(t,a)] \leq Z[g(t,a)]$ and $(N2)$ to obtain

$$\frac{f'[Z(g(t,a))]}{\psi[x(g(t,a))]} \geq \frac{f'[Z(g(t,a))]}{\psi[Z(g(t,a))]} \geq k_2.$$

Therefore,

$$\begin{aligned} v'(t) &\leq \frac{\varphi'(t)}{\varphi(t)}v(t) - \varphi(t)\Big[M\int_a^b q(t,\xi)f[1-p(g(t,\xi))]d\sigma(\xi) - \phi'(t)\Big] \\ &\quad - \frac{k_2\varphi(t)g'(t,a)}{r[g(t,a)]}\Big(\frac{r(t)\psi(x(t))Z'(t)}{f[Z(g(t,a))]}\Big)^2 \\ &= -\Theta_2(t) + l_2(t)v(t) - \frac{1}{\gamma_2(t)}v^2(t). \end{aligned}$$

The rest of the proof is as in Theorem 2.1. □

Theorem 2.3. *Let (S3) holds. If there exist functions $\varphi \in C^1(I,\mathbb{R}^+)$, $\phi \in C^1(I,\mathbb{R})$ and $\Phi \in \mathrm{X}$ such that for each $T \geq t_0$,*

$$\limsup_{t\to\infty}\int_T^t \Big\{\Phi^2(t,s,T)\Theta_3(s) - \gamma_3(s)\big[\Phi_s'(t,s,T) + \frac{1}{2}l_3(s)\Phi(t,s,T)\big]^2\Big\}ds > 0, \quad (2.5)$$

where

$$l_3(t) = \frac{\varphi'(t)}{\varphi(t)} + \frac{2Lg'(t,a)\phi(t)}{r[g(t,a)]}, \quad \gamma_3(t) = \frac{r[g(t,a)]\varphi(t)}{Lg'(t,a)},$$

$$\Theta_3(t) = \varphi(t)\Big\{k_3\int_a^b q(t,\xi)[1-p(g(t,\xi))]d\sigma(\xi) + \frac{Lg'(t,a)\phi^2(t)}{r[g(t,a)]} - \phi'(t)\Big\},$$

then (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1) on I, say $x(t) > 0$ for $t \geq t_0$. Then, by Lemma 1.3, there exists a $T_0 \geq t_0$ such that (1.2) and (1.4) hold. We define the function

$$v(t) = \varphi(t)\Big[\frac{r(t)\psi(x(t))Z'(t)}{Z[g(t,a)]} + \phi(t)\Big] \quad \text{for } t \geq T_0. \quad (2.6)$$

Differentiating (2.6) and using (1.4), we obtain

$$\begin{aligned} v'(t) &\le \frac{\varphi'(t)}{\varphi(t)}v(t) - \varphi(t)\Big\{k_3\int_a^b q(t,\xi)[1-p(g(t,\xi))]d\sigma(\xi) - \phi'(t)\Big\} \\ &\quad - \frac{\varphi(t)g'(t,a)}{r[g(t,a)]\psi[x(g(t,a))]}\left(\frac{r(t)\psi(x(t))Z'(t)}{Z(g(t,a))}\right)^2 \\ &\le \frac{\varphi'(t)}{\varphi(t)}v(t) - \varphi(t)\Big\{k_3\int_a^b q(t,\xi)[1-p(g(t,\xi))]d\sigma(\xi) - \phi'(t)\Big\} \\ &\quad - \frac{L\varphi(t)g'(t,a)}{r[g(t,a)]}\left(\frac{v(t)}{\varphi(t)} - \phi(t)\right)^2 \\ &= -\Theta_3(t) + l_3(t)v(t) - \frac{1}{\gamma_3(t)}v^2(t). \end{aligned}$$

The rest of the proof follows the same lines as that of Theorem 2.1. □

Let $\Phi(t,s,T) = \sqrt{H_1(t,s)H_2(s,T)}$, where $H_1, H_2 \in Y$. By Theorems 2.1-2.3, we have the following interesting results.

Corollary 2.4. *Let (S1), (N1) hold. If there exist functions $\varphi \in C^1(I,\mathbb{R}^+)$, $\phi \in C^1(I,\mathbb{R})$, $\Phi \in \mathrm{X}$ and $H_1, H_2 \in \mathrm{Y}$ such that for each $T \ge t_0$,*

$$\limsup_{t\to\infty}\int_T^t H_1(t,s)H_2(s,T)\Big\{\Theta_1(s) - \frac{1}{4}\gamma_1(s)[h_1(t,s)+h_2(s,T)+l_1(s)]^2\Big\}ds > 0, \tag{2.7}$$

where $h_1(t,s)$ and $h_2(s,T)$ are defined by

$$\frac{\partial H_1(t,s)}{\partial s} = h_1(t,s)H_1(t,s) \quad \text{and} \quad \frac{\partial H_2(s,T)}{\partial s} = h_2(s,T)H_2(s,T), \tag{2.8}$$

then (1.1) is oscillatory.

Corollary 2.5. *Let (S2), (N1), (N2) hold. If there exist functions $\varphi \in C^1(I,\mathbb{R}^+)$, $\phi \in C^1(I,\mathbb{R})$, $\Phi \in \mathrm{X}$ and $H_1, H_2 \in \mathrm{Y}$ such that for each $T \ge t_0$,*

$$\limsup_{t\to\infty}\int_T^t H_1(t,s)H_2(s,T)\Big\{\Theta_2(s) - \frac{1}{4}\gamma_2(s)[h_1(t,s)+h_2(s,T)+l_2(s)]^2\Big\}ds > 0, \tag{2.9}$$

where $h_1(t,s)$ and $h_2(s,T)$ are defined by (2.8), then (1.1) is oscillatory.

Corollary 2.6. *Let (S3) hold. If there are functions $\varphi \in C^1(I,\mathbb{R}^+)$, $\phi \in C^1(I,\mathbb{R})$, $\Phi \in \mathrm{X}$ and $H_1, H_2 \in \mathrm{Y}$ such that for each $T \ge t_0$,*

$$\limsup_{t\to\infty}\int_T^t H_1(t,s)H_2(s,T)\Big\{\Theta_3(s) - \frac{1}{4}\gamma_3(s)[h_1(t,s)+h_2(s,T)+l_3(s)]^2\Big\}ds > 0, \tag{2.10}$$

where $h_1(t,s)$ and $h_2(s,T)$ are defined by (2.8), then (1.1) is oscillatory.

Let $\Phi(t,s,T) = (t-s)(s-T)^\alpha$ for $\alpha > 1/2$. By Theorems 2.1-2.3, we can establish the following important results.

Corollary 2.7. *Let (S1) and (N1) hold. If there exist functions $\varphi \in C^2(I,\mathbb{R}^+)$, $\phi \in C^1(I,\mathbb{R})$ and constants $\alpha > 1/2$, $m_1 > 0$ such that $\gamma_1(t) \le m_1$ and for each*

$T \geq t_0$,

$$\limsup_{t\to\infty} \frac{1}{t^{2\alpha+1}} \int_T^t (t-s)^2(s-T)^{2\alpha}\Big[\frac{1}{m_1}\Theta_1(s)+\frac{1}{2}l_1'(s)-\frac{1}{4}l_1^2(s)\Big]ds > \frac{\alpha}{(2\alpha-1)(2\alpha+1)}, \tag{2.11}$$

then (1.1) *is oscillatory.*

Proof. Suppose that (1.1) has a nonoscillatory solution $x(t) > 0$. By using the same arguments as in the proof of Theorem 2.1 and denoting $(t-s)(s-T_0)^\alpha$ by $\Phi(t,s,T_0)$, we have

$$\begin{aligned}
&\int_{T_0}^t \Phi^2(t,s,T_0)\Theta_1(s)ds\\
&\leq \int_{T_0}^t \gamma_1(s)\big[\Phi_s'(t,s,T_0)+\frac{1}{2}l_1(s)\Phi(t,s,T_0)\big]^2ds\\
&\leq m_1\int_{T_0}^t \Big\{\Phi_s'^{\,2}(t,s,T_0)+\Phi_s'(t,s,T_0)\Phi(t,s,T_0)l_1(s)+\frac{1}{4}\Phi^2(t,s,T_0)l_1^2(s)\Big\}ds.
\end{aligned}$$

Noting that

$$\int_{T_0}^t \Phi_s'(t,s,T_0)\Phi(t,s,T_0)l_1(s)ds = -\frac{1}{2}\int_{T_0}^t \Phi^2(t,s,T_0)l_1'(s)ds,$$

we get

$$\begin{aligned}
&\frac{1}{m_1}\int_{T_0}^t \Phi^2(t,s,T_0)\Theta_1(s)ds\\
&\leq \int_{T_0}^t \Big\{\Phi_s'^{\,2}(t,s,T_0)-\frac{1}{2}\Phi^2(t,s,T_0)l_1'(s)+\frac{1}{4}\Phi^2(t,s,T_0)l_1^2(s)\Big\}ds\\
&= \int_{T_0}^t \Phi^2(t,s,T_0)\big[-\frac{1}{2}l_1'(s)+\frac{1}{4}l_1^2(s)\big]ds+\int_{T_0}^t [\alpha(t-s)(s-T_0)^{\alpha-1}-(s-T_0)^\alpha]^2ds\\
&= \int_{T_0}^t \Phi^2(t,s,T_0)\big[-\frac{1}{2}l_1'(s)+\frac{1}{4}l_1^2(s)\big]ds+\frac{\alpha}{(2\alpha-1)(2\alpha+1)}(t-T_0)^{2\alpha+1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t^{2\alpha+1}}\int_{T_0}^t (t-s)^2(s-T_0)^{2\alpha}\Big[\frac{1}{m_1}\Theta_1(s)+\frac{1}{2}l_1'(s)-\frac{1}{4}l_1^2(s)\Big]ds\\
&\leq \frac{\alpha}{(2\alpha-1)(2\alpha+1)},
\end{aligned}$$

which contradicts (2.11) and completest the proof. □

Similar to the proof of Corollary 2.7, by Theorems 2.2 and 2.3, we can easily obtain the following results.

Corollary 2.8. *Let (S2), (N1), (N2) hold. If there exist functions* $\varphi \in C^2(I,\mathbb{R}^+)$, $\phi \in C^1(I,\mathbb{R})$ *and constants* $\alpha > 1/2$, $m_2 > 0$ *such that* $\gamma_2(t) \leq m_2$ *and for each* $T \geq t_0$,

$$\limsup_{t\to\infty} \frac{1}{t^{2\alpha+1}} \int_T^t (t-s)^2(s-T)^{2\alpha}\Big[\frac{1}{m_2}\Theta_2(s)+\frac{1}{2}l_2'(s)-\frac{1}{4}l_2^2(s)\Big]ds > \frac{\alpha}{(2\alpha-1)(2\alpha+1)}, \tag{2.12}$$

then (1.1) *is oscillatory.*

Corollary 2.9. *Let (S3) holds. If there exist functions $\varphi \in C^2(I,\mathbb{R}^+)$, $\phi \in C^1(I,\mathbb{R})$ and constants $\alpha > 1/2$, $m_3 > 0$ such that $\gamma_3(t) \le m_3$ and for each $T \ge t_0$,*

$$\limsup_{t\to\infty} \frac{1}{t^{2\alpha+1}} \int_T^t (t-s)^2(s-T)^{2\alpha}\Big[\frac{1}{m_3}\Theta_3(s)+\frac{1}{2}l_3'(s)-\frac{1}{4}l_3^2(s)\Big]ds > \frac{\alpha}{(2\alpha-1)(2\alpha+1)}, \tag{2.13}$$

then (1.1) *is oscillatory.*

3. Interval oscillation criteria

We can easily see that the results in Section 2 involve the integral of the coefficients p, q and r, and hence, requires the information of the coefficients on the entire half-line $[t_0,\infty)$. In this section, we will establish several interval oscillation criteria for (1.1).

Theorem 3.1. *Let (S1) and (N1) hold. If for each $T \ge t_0$, there exist functions $\varphi \in C^1(I,\mathbb{R}^+)$, $\phi \in C^1(I,\mathbb{R})$, $\Phi \in \mathrm{X}$ and two constants $d > c \ge T$ such that*

$$\int_c^d \Big\{\Phi^2(d,s,c)\Theta_1(s) - \gamma_1(s)\big[\Phi_s'(d,s,c) + \frac{1}{2}l_1(s)\Phi(d,s,c)\big]^2\Big\}ds > 0, \tag{3.1}$$

where Θ_1, γ_1, l_1 are defined as in Theorem 2.1, then (1.1) *is oscillatory.*

Proof. With the proof of Theorem 2.1, where t and T are replaced by d and c, respectively, we can easily see that every solution of (1.1) has at least one zero in (c,d); i.e., every solution of (1.1) has arbitrarily large zeros on $[t_0,\infty)$. This completes the proof of Theorem 3.1. □

Similar to the proof of Theorem 3.1, we can establish the following theorems.

Theorem 3.2. *Let (S2), (N1) and (N2) hold. If for each $T \ge t_0$, there exist functions $\varphi \in C^1(I,\mathbb{R}^+)$, $\phi \in C^1(I,\mathbb{R})$, $\Phi \in \mathrm{X}$ and two constants $d > c \ge T$ such that*

$$\int_c^d \Big\{\Phi^2(d,s,c)\Theta_2(s) - \gamma_2(s)\big[\Phi_s'(d,s,c) + \frac{1}{2}l_2(s)\Phi(d,s,c)\big]^2\Big\}ds > 0, \tag{3.2}$$

where Θ_2, γ_2, l_2 are defined as in Theorem 2.2, then (1.1) *is oscillatory.*

Theorem 3.3. *Let (S3) holds. If for each $T \ge t_0$, there exist functions $\varphi \in C^1(I,\mathbb{R}^+)$, $\phi \in C^1(I,\mathbb{R})$, $\Phi \in \mathrm{X}$ and two constants $d > c \ge T$ such that*

$$\int_c^d \Big\{\Phi^2(d,s,c)\Theta_3(s) - \gamma_3(s)\big[\Phi_s'(d,s,c) + \frac{1}{2}l_3(s)\Phi(d,s,c)\big]^2\Big\}ds > 0, \tag{3.3}$$

where Θ_3, γ_3, l_3 are defined as in Theorem 2.3, then (1.1) *is oscillatory.*

Corollary 3.4. *Let (S1) and (N1) hold. If for each $T \ge t_0$, there exist functions $H_1, H_2 \in Y$ and two constants $d > c \ge T$ such that*

$$\int_c^d H_1(d,s)H_2(s,c)\Big\{\Theta_1(s) - \frac{1}{4}\gamma_1(s)\big[h_1(d,s) + h_2(s,c) + l_1(s)\big]^2\Big\}ds > 0, \tag{3.4}$$

where $h_1(d,s)$ and $h_2(s,c)$ are defined by (2.8)*, then* (1.1) *is oscillatory.*

Corollary 3.5. *Let (S2), (N1) and (N2) hold. If for each $T \geq t_0$, there exist functions $H_1, H_2 \in Y$ and two constants $d > c \geq T$ such that*

$$\int_c^d H_1(d,s)H_2(s,c)\Big\{\Theta_2(s) - \frac{1}{4}\gamma_2(s)[h_1(d,s) + h_2(s,c) + l_2(s)]^2\Big\}ds > 0, \quad (3.5)$$

where $h_1(d,s)$ and $h_2(s,c)$ are defined by (2.8), then (1.1) is oscillatory.

Corollary 3.6. *Let (S3) holds. If for each $T \geq t_0$, there exist functions $H_1, H_2 \in Y$ and two constants $d > c \geq T$ such that*

$$\int_c^d H_1(d,s)H_2(s,c)\Big\{\Theta_3(s) - \frac{1}{4}\gamma_3(s)[h_1(d,s) + h_2(s,c) + l_3(s)]^2\Big\}ds > 0, \quad (3.6)$$

where $h_1(d,s)$ and $h_2(s,c)$ are defined by (2.8), then (1.1) is oscillatory.

4. Examples

Example 4.1. Consider the equation

$$\Big(\frac{1}{e^t(1+x^2(t))}\big(x(t) + (1-e^{-t})x(t-1)\big)'\Big)' + \mu\int_{\frac{1}{2}}^1 \frac{\xi x(\ln t\xi)}{t}d\xi = 0, \quad t \geq 2, \ (4.1)$$

where $r(t) = e^{-t}$, $\psi(x) = 1/(1+x^2)$, $p(t) = 1 - e^{-t}$, $\mu > 1$, $g(t,\xi) = \ln t\xi$, $f(x) = x$ and $q(t,\xi) = \mu\xi/t$.

If we take $k_1 = L = M = 1$, $m_1 = 2$, $\phi(t) = t$ and $\varphi(t) = 1$, then

$$\Theta_1(t) = \frac{\mu}{2t^2} + \frac{1}{2}t^2 - 1, \quad l_1(t) = t, \quad \gamma_1(t) = 2.$$

Thus, the left-hand side of (2.11) takes the following from

$$\frac{\mu}{4}\limsup_{t\to\infty}\frac{1}{t^{2\alpha+1}}\int_T^t (t-s)^2(s-T)^{2\alpha}\frac{1}{s^2}ds = \frac{\mu}{4}\frac{1}{\alpha(2\alpha-1)(2\alpha+1)}.$$

For any $\mu > 1$, there exists a constant $\alpha > 1/2$ such that $\mu/4 > \alpha^2$, i.e.,

$$\frac{\mu}{4}\frac{1}{\alpha(2\alpha-1)(2\alpha+1)} > \frac{\alpha}{(2\alpha-1)(2\alpha+1)},$$

i.e., (2.11) holds for $\mu > 1$. By Corollary 2.7, (4.1) is oscillatory for $\mu > 1$.

Example 4.2. Consider the equation

$$\Big(x^2(t)\big(x(t) + (1 - \frac{1}{t})x(t-1)\big)'\Big)' + \int_{\frac{1}{2}}^1 t^4\xi^3x^3(t\xi)d\xi = 0, \quad t \geq 2, \quad (4.2)$$

where $r(t) = 1$, $\psi(x) = x^2$, $p(t) = 1 - 1/t$, $g(t,\xi) = t\xi$, $f(x) = x^3$ and $q(t,\xi) = t^4\xi^3$.

If we take $\varphi(t) = 1$, $\phi(t) = 0$, $m_2 = 2/3$, $k_2 = 3$ and $M = 1$, then

$$\Theta_2(t) = \frac{t}{2}, \quad \gamma_2(t) = \frac{2}{3}, \quad l_2(t) = 0.$$

For any constant $T \geq 2$, there exists $n \in \mathbb{N}_0 = \{1, 2, \cdots\}$ such that $2n\pi \geq T$. Let $d = (2n+1)\pi$, $c = 2n\pi \geq T$ and $H_1(t,s)H_2(s,T) = |\sin(t-s)\sin(s-T)|$, we have

$$H_1((2n+1)\pi, s)H_2(s, 2n\pi) = \sin^2 s \quad \text{for} \quad c \leq s \leq d.$$

Thus, the left-hand side of (3.5) takes the from

$$\int_{2n\pi}^{(2n+1)\pi} \sin^2 s\big(\frac{s}{2} - \frac{4}{3}\cot^2 s\big)ds = \frac{(4n+1)\pi^2}{8} - \frac{2\pi}{3} > 0.$$

Hence, by Corollary 3.5, (4.2) is oscillatory.

Example 4.3. Consider the equation

$$\Big(\frac{1}{t(1+x^2(t))}\big(x(t)+\frac{1}{2}x(t-1)\big)'\Big)'+\mu\int_0^1\frac{\xi}{t^2}\frac{x^3(\sqrt{t+\xi})+3x(\sqrt{t+\xi})}{1+x^2(\sqrt{t+\xi})}d\xi=0,\quad t\geq 1, \tag{4.3}$$

where $r(t)=1/t$, $\psi(x)=1/(1+x^2)$, $p(t)=1/2$, $g(t,\xi)=\sqrt{t+\xi}$, $f(x)=(x^3+3x)/(1+x^2)$, $q(t,\xi)=\mu\xi/t^2$ and $\mu>2$.

If we take $k_3=L=1$, $m_3=2$, $\varphi(t)=1$, $\phi(t)=1/t$, then

$$\Theta_3(t)=\frac{\mu+6}{4t^2},\quad l_3(t)=\frac{1}{t},\quad \gamma_3(t)=2.$$

Thus, the left-hand side of (2.13) takes the following from

$$\frac{\mu}{8}\limsup_{t\to\infty}\frac{1}{t^{2\alpha+1}}\int_T^t(t-s)^2(s-T)^{2\alpha}\frac{1}{s^2}ds=\frac{\mu}{8}\frac{1}{\alpha(2\alpha-1)(2\alpha+1)}.$$

So, for any $\mu>2$, there exists a constant $\alpha>1/2$ such that

$$\frac{\mu}{8}\frac{1}{\alpha(2\alpha-1)(2\alpha+1)}>\frac{\alpha}{(2\alpha-1)(2\alpha+1)}.$$

Therefore, by Corollary 2.9, Equation (4.3) is oscillatory for $\mu>2$.

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