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BOUNDEDNESS OF SOLUTIONS TO FOURTH ORDER DIFFERENTIAL EQUATIONS WITH OSCILLATORY RESTORING AND FORCING TERMS

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Abstract. This article concerns the fourth order differential equation

$$x^{(iv)} + ax''' + bx'' + g(x') + h(x) = p(t).$$

Using the Cauchy formula for the particular solution of non-homogeneous linear differential equations with constant coefficients, we prove that the solution and its derivatives up to order three are bounded.

1. Introduction

In this article, we study the boundedness of solutions to the fourth-order nonlinear differential equation

$$x^{(iv)} + ax''' + bx'' + g(x') + h(x) = p(t) \tag{1.1}$$

where $a > 0$ and $b > 0$ are positive constants with $a^2 > 4b$, g, h, and p and their first derivatives are continuous functions depending on the arguments shown. In addition, h and p are oscillatory.

Several authors have investigated the boundedness of solutions of certain differential equations of the fourth order. We can mention in this direction, the works of Afuwape and Adesina [1] where the frequency-domain approach was used. Other articles in this connection include Tiryaki and Tunc [10], Tunc [11, 12, 13], Tunc and Tiryaki [14] where the second Lyapunov method was used. All these results generalize in one way or another some results on third order nonlinear differential equations see for instance [2, 3, 4, 5, 6, 7, 8, 9].

The present work was motivated by a relatively recent paper of Andres [2], where the existence of a bounded solution for a third order non-linear differential equation with oscillatory restoring and forcing terms was proved. We shall use the Cauchy formula for the particular solution of non homogeneous linear part of the equation (1.1), to prove that the solution $x(t)$ and its derivatives $x'(t)$, $x''(t)$ and $x'''(t)$ are bounded.

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2. Assumptions and Main Result

The basic assumptions on the functions which appear in (1.1) are the following:

- (i) h and p are oscillatory in the following sense: For each argument there exist such numbers $\beta_1 > \alpha_1 > u > \beta_{-1} > \alpha_{-1}$ that for $f(\alpha_1) < 0, f(\beta_1) > 0, f(\alpha_{-1}) < 0, f(\beta_{-1}) > 0$ where f is either $h(x)$ or $p(t)$, u is either x or t and all the roots of the restoring term $h(x)$ are isolated.
- (ii) (a) $|h(x)| \le H$, (b) $|h'(x)| \le H'$;
- (iii) (a) $|g(x')| < cx' \le G$, (b) $|g'(x')| \le G'$;
- (iv) (a) $|p(t)| \le P$, (b) $|p'(t)| \le P'$, (c) $|\int_0^t P(\tau)d\tau| \le P_0$, (d) $\limsup_{t\to\infty}|P(t)| > 0$.

The main result of this paper is as follows.

Theorem 2.1. *Assume there exist positive constants $H, H', G, G', P, P', P_0, R$ such that for $|x| > R$ and $t > 0$ the conditions (ii) and (iii) hold. If in addition,*

$$\min[d(\overline{x}_k, \overline{x}_{k+1}), d(\overline{x}_k, \overline{x}_{k-1})] > 2\frac{G+H+P}{b}\Big(\frac{2}{a} + \frac{a}{b}\Big) + \frac{P_0}{b},$$

where $\overline{x}_k$ are roots of $h(x)$, $h'(x_k) > 0$ and $\overline{x}_{k-1}, \overline{x}_{k+1}$ denote the couple adjacent roots of $\overline{x}_k (k = 0, \pm 2, \pm 4, \dots)$; then all solutions $x(t)$ of equation (1.1) are bounded and for each of them there exists a root $\overline{x}$ of $h(x)$ such that $(x(t) - \overline{x})$ oscillates.

3. Preliminary Results

To prove our main result, we shall need the following result.

Lemma 3.1. *If there exist positive constants, H, G, P such that for all $x \in \Re^1$ and $t \ge 0$ the assumptions (ii)(a), (iii)(a) and (iv)(a) hold, then each solution $x(t)$ of (1.1) satisfies the inequalities*

$$\limsup_{t\to\infty}|x'(t)| \le \frac{G}{c} := D', \tag{3.1}$$

$$\limsup_{t\to\infty}|x''(t)| \le \frac{H+G+P}{b} := D'', \tag{3.2}$$

$$\limsup_{t\to\infty}|x'''(t)| \le \frac{2(H+G+P)}{a} := D'''. \tag{3.3}$$

Proof. Let $z = x''$ then the equation (1.1) reduces to

$$z'' + az' + bz = P(t) - g(x'(t)) - h(x). \tag{3.4}$$

Equation (3.4) can also be rewritten as

$$z'' + az' + bz = B, \tag{3.5}$$

where $B = P(t) - g(x'(t)) - h(x)$. Thus the general solution of the equation (3.5) satisfies

$$|x''(t)| = |z(t)| = C_1 e^{a_1 t} + C_2 e^{a_2 t} + \int_0^t \frac{e^{a_1\tau} - e^{a_2\tau}}{a_1 - a_2}(B)d\tau,$$

where $a_{1,2} = \big(-a \pm \sqrt{a^2 - 4b}\big)/2$ and constants C_1 and C_2 are arbitrary. Hence by the virtue of assumptions (i)-(iv) for $t \ge 0$, we have not only

$$\Big|\int_0^t \frac{e^{a_1\tau} - e^{a_2\tau}}{a_1 - a_2}[P(t) - g(x'(t)) - h(x)]d\tau\Big| \le \frac{H+G+P}{b}\Big(1 + \frac{a_2 e^{a_1 t} - a_1 e^{a_2 t}}{a_1 - a_2}\Big),$$

but also

$$\limsup_{t\to\infty} |x''(t)| \le \frac{H+G+P}{b} =: D''.$$

Furthermore on substituting $w = z'$ in (3.4), we have

$$w' + aw = P(t) - bx''(t) - g(x'(t)) - h(x).$$

Following the same argument used in obtaining the general solution for the equation (3.5), we have

$$|x'''(t)| = |w(t)| = Ce^{-at} + \int_0^t e^{-a\tau}[P(t) - bx''(t) - g(x'(t)) - h(x)]d\tau,$$

and by assumptions (i)-(iii) for $t \ge T_x$, we have not only

$$\begin{aligned} |\int_0^t e^{-a\tau}[P(t) - bx''(t) - g(x'(t)) - h(x)]d\tau| &\le 2\frac{H+G+P}{a}\int_0^t e^{-a\tau}d\tau \\ &\le 2\frac{H+G+P}{a}(1 - e^{-a(t-T_x)}), \end{aligned}$$

but also

$$\limsup_{t\to\infty} |x'''(t)| \le 2\frac{H+G+P}{a} =: D'''.$$

To establish the inequality (3.1), we use the assumption (iii)(a); i.e., given that $|g(x')| < cx' \le G$, we have

$$|cx'(t)| \le c|x'(t)| \le G;$$

i.e., $|x'(t)| \le G/c$. Hence

$$\limsup_{t\to\infty} |x'(t)| \le \frac{G}{c} := D'.$$

This completes the proof of the lemma 3.1. $\square$

Lemma 3.2. *Under the assumptions of Lemma 3.1. If (ii)(b) and (v)(d) hold for $x \in \Re^1$, then every solution $x(t)$ of* (1.1) *either satisfies the relation*

$$\lim_{t\to\infty} x(t) = \overline{x}, \quad \lim_{t\to\infty} x'(t) = \lim_{t\to\infty} x''(t) = \lim_{t\to\infty} x'''(t) = 0 \quad (h(\overline{x}) = 0,) \tag{3.6}$$

or there exists a root $\overline{x}$ of $h(x)$ such that $(x(t) - \overline{x})$ oscillates.

Proof. Substituting a fixed bounded solution $x(t)$ of (1.1) into itself and integrating the result from T_x to t, we have

$$\begin{aligned} &\int_{T_x}^t h(x(\tau)d\tau \\ &= -\{b[x'(t) - x'(T_x)] + a[x''(t) - x''(T_x)] + \int_{T_x}^t g(x'(\tau))d\tau\} + \int_{T_x}^t P(\tau)d\tau. \end{aligned} \tag{3.7}$$

By condition (iii)(a), we have that

$$\begin{aligned} &\int_{T_x}^t h(x(\tau)d\tau \\ &< -\{b[x'(t) - x'(T_x)] + a[x''(t) - x''(T_x)] + c[x(t) - x(T_x)]\} + \int_{T_x}^t P(\tau)d\tau. \end{aligned} \tag{3.8}$$

Define $I(t) \equiv \int_{T_x}^t P(\tau))d\tau$. By the virtue of the above condition, the assertion of the Lemma 3.1 and the boundedness of $x(t)$, there exists a constant M_x such that for $t \geq T_x$,

$$|I(t)| \leq M_x;$$

i.e.,

$$\Big|\int_{T_x}^t h(x(\tau))d\tau\Big| \leq |I(t)| \leq M_x. \tag{3.9}$$

Now let us assume that $x(t)$ does not converge to any root $\overline{x}$ of $h(x)$; i.e.,

$$\limsup_{t\to\infty} |x(t) - \overline{x}| > 0 \tag{3.10}$$

and simultaneously, for $t \geq T_x$,

$$h(x(t)) \geq 0 \quad \text{or} \quad h(t) \leq 0. \tag{3.11}$$

Then

$$H(t) :\equiv \int_{T_x}^t h(x(\tau))d\tau$$

for $t \geq T_x$ which is a composed monotone function with a finite or infinite limit for $t \to \infty$. Since (3.9) implies that divergent case can be disregarded, it follows from (3.10) that not only

$$\lim_{t\to\infty} \int_{T_x}^t |h(x(\tau))|d\tau = \lim_{t\to\infty} \Big|\int_{T_x}^t |h(x(\tau))|d\tau\Big| \leq M_x, \tag{3.12}$$

but also

$$\lim_{t\to\infty} |x(t) - \overline{x}| = 0\,. \tag{3.13}$$

Otherwise (i.e., if $\limsup_{t\to\infty}|x(t) - \overline{x}| > 0$) the inequality (3.10) together with the fact that the roots of $h(x)$ are isolated yields

$$\liminf_{t\to\infty} |h(x(t))| = \liminf_{t\to\infty}|h(x(t)) - h(\overline{x})| > 0$$

which is a contradiction to (3.12). Thus (3.9) and (3.11) imply

$$\limsup_{t\to\infty}|h(x(t))| = \limsup_{t\to\infty}|h(x(t)) - h(\overline{x})| > 0 = \liminf_{t\to\infty}|h(x(t))|.$$

In what follows, $d(x, y)$ denotes the distance between x and y. Consequently, there exists such a sequence $t_i \geq T_x$ and a constant $\widetilde{H} > 0$, such that

$$\liminf_{t\to\infty\Rightarrow t_i\to\infty} d(t_i, t_{i-1}) > 0, \quad |h(x(t_i)| \geq \widetilde{H},$$

and such that

$$M_x \geq \lim_{t\to\infty} \int_{t_{i-1}}^{t_i} |h(x(\tau))|d\tau = \sum_{i=2}^{\infty} \int_{t_{i-1}}^{t_i} |h(x(\tau))|d\tau$$

implies

$$\lim_{t\to\infty\Rightarrow t_i\to\infty} \int_{t_{i-1}}^{t_i} |h(x(\tau))|d\tau = 0,$$

or

$$H' \limsup_{t\to\infty} |x'(t)| \geq \lim_{t\to\infty} \Big|\frac{dh(x(t))}{dx(t)} x'(t)\Big| = \limsup_{t\to\infty} \Big|\frac{dh(t)}{dt}\Big| = \infty.$$

According to the assertion of the Lemma 3.1, this is impossible and that is why $(x(t) - \overline{x})$ necessarily oscillates. The remaining part of the lemma follows from the assertion

$$x(t) \in C^n[0, \infty), \quad \lim_{t\to\infty} |x^n(t)| < \infty, \tag{3.14}$$

$\lim_{t\to\infty}|x(t)| < \infty$ implies $\lim_{t\to\infty} x^k(t) = 0$, where $n \geq 2$ is a natural numbers and $k = 1, \dots (n-1)$. This completes the proof. □

Lemma 3.3. *Under the assumptions of the Lemma 3.2, suppose that (iv)(b) holds for all $t \geq 0$, and $\limsup_{t\to\infty}|p(t)| > 0$ hold, where P' is a suitable constant, then for every bounded solution $x(t)$ of* (1.1) *there exists a root $\overline{x}$ of $h(x)$ such that $(x(t) - \overline{x})$ oscillates.*

Proof. If Lemma 3.3 does not hold, then according to Lemma 3.2, equations (3.6) hold and the fifth derivative of $x(t)$ satisfies

$$x^v(t) = p'(t) - ax^{iv}(t) - bx'''(t) - g'(x'(t))x''(t) - h'(x(t))x'(t).$$

But by the ultimate boundedness of $x'(t), x''(t), x'''(t)$ and $x^{iv}(t)$, there exists a constant D_5 such that

$$\limsup_{t\to\infty}|x^v(t)| \leq D_5$$

which according to (3.14) gives the relations

$$\lim_{t\to\infty} x(t) = \overline{x} \Rightarrow \lim_{t\to\infty} h(x(t)) = h(\overline{x}) = 0, \lim_{t\to\infty} x^j(t) = 0,$$

$j = 1, 2, 3$, or

$$\limsup_{t\to\infty}|p(t)| = \left|x^{iv}(t) + ax'''(t) + bx''(t) + g(x'(t)) + h(x(t))\right| = 0,$$

which is a contradiction to $\limsup_{t\to\infty} |p(t)| > 0$. □

Proof of Theorem 2.1. Let us assume on the contrary, that $x(t)$ is an unbounded solution of (1.1); i.e., $\limsup_{t\to\infty} x(t) = \infty$. It will follow from Lemma 3.1 that there exists a number $T_0 \geq 0$ large enough such that for $t \geq T_0$,

$$|x'(t)| \leq D' + \epsilon_1,$$
$$|x''(t)| \leq D'' + \epsilon_2$$
$$|x'''(t)| \leq D''' + \epsilon_3$$

with $\epsilon_i > 0$, $(i = 1, 2, 3)$ small enough constants. Let $T_1 \geq T_0$ be the last point with $x(T_1) = \overline{x}_k$, (k even) and $T_2 > T_1$ be the first point with $x(T_2) = \overline{x}_{k+1}$. Integrating (1.1) from T_1 to t, $T_1 \leq t \leq T_2$, we have

$$\begin{aligned} &[x'''(t) - x'''(T_1)] + a[x''(t) - x''(T_1)] + b[x'(t) - x'(T_1)] \\ &+ \int_{T_1}^t g(x'(\tau))d\tau + \int_{T_1}^t h(\tau)d\tau \\ &= \int_{T_1}^t P(\tau)d\tau. \end{aligned} \tag{3.15}$$

However, on replacing $\int_{T_1}^t g(x'(\tau))d\tau$ with $c[x(t) - x(T_1)]$, for $T_1 \leq t \leq T_2$, we have $h(x(t))sgnx(t) \geq 0$. Multiplying (3.15) by $\operatorname{sgn} x$, we obtain

$$|x(t)| \leq |x(T_1)| + \frac{1}{c}[D''' + aD'' + bD' + P_0] + \epsilon,$$

where $c > 0$ and $\epsilon > 0$ is arbitrary small constant, a contradiction to $x(T_2) = \overline{x}_{k+1}$ with respect to condition (ii) of Theorem 2.1. The remaining part of the proof follows from the Lemma 3.3. □

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