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# REGULARITY FOR ANISOTROPIC QUASI-LINEAR PARABOLIC EQUATIONS WITH VARIABLE GROWTH

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Abstract. In this article, we study a class of anisotropic quasi-linear parabolic equations with variable exponents. Following DiBenedetto's intrinsic scaling method, we prove local continuity of solutions under the condition for which only local boundedness was known.

## 1. Introduction

The study of nonlinear partial differential equations gained a significant importance in the recent past years, not only for their physical and biological relevance but, and no less important, also for the mathematical novelties intrinsically related to the subject. The development of the regularity theory for degenerate and/or singular parabolic PDEs is one example of the contemporary analysis of nonlinear PDEs. One has to go back to the late fifties to encounter the now standard procedure that allows one to get a regularity result for the solutions of nonlinear PDEs: regularity for elliptic PDEs was established by De Giorgi [[17]]; while Moser [23, 24, 25, 26], Nash [27] and DiBenedetto [12] dealt with parabolic PDEs.

In this article we are concerned with the anisotropic parabolic equation

$$u_t - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big[\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x,t)-2}\frac{\partial u}{\partial x_i}\Big] = 0 \quad \text{in } \Omega_T, \tag{1.1}$$

where $\Omega_T = \Omega \times (0,T]$, $\Omega$ is a bounded simple-connected domain in $\mathbb{R}^N$ and $0 < T < +\infty$. Throughout the paper we assume that the exponents $p_i(x,t)$ are given measurable functions in $\Omega_T$ such that for all $i = 1,\dots,N$

$$p_i(x,t) \subset (p_i^-, p_i^+) \subseteq [p^-, p^+] \subset (2,\infty).$$

with finite constants $p^{\pm}, p_i^{\pm} > 2$. Moreover, we assume that $p_i$ satisfies the following log-continuity condition:

$$\begin{gathered} |p_i(x,t) - p_i(y,\tau)| \le \frac{c_0}{\ln \frac{1}{|t-\tau|+|x-y|}} \quad \forall (x,t),\ (y,\tau) \in \Omega_T, \\ |t-\tau| + |x-y| \le \frac{1}{2}. \end{gathered} \tag{1.2}$$

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In this framework, a particularly relevant class of interest is given by functionals with anisotropic structures, i.e. those whose energy sees each derivatives being penalized with a different exponent. Yet firstly studied by Marcellini [22], further contributions have been given by Leonetti [19, 20], Acerbi and Fusco [1], Fusco and Sbordone [15, 16].

Anisotropic equations like (1.1) have strong physical background. They emerge, for instance, from the mathematical description of the dynamics of fluids with different conductivities in different directions. We refer to the extensive books by Antontsev-Díaz-Shmarev [3] and Bear [8] for discussions in this direction. They also appear in biology, see Bendahmane-Karlsen [9] and Bendahmane-Langlais-Saad [10], as a model describing the spread of an epidemic disease in heterogeneous environments.

Our aim here is to obtain a local regularity result for local weak solutions of (1.1). In order to achieve this goal, and since the equation is degenerate (the diffusion coefficient vanishes when $|\frac{\partial u}{\partial x_i}| = 0$), the idea is to study the equation within a geometry that takes this feature into consideration. The building blocks of DiBenedetto's intrinsic scaling method is to show that the continuity of the solution at a point follows from measuring its oscillation in a sequence of nested and shrinking cylinders, with vertex at that point, and showing that the oscillation converge to zero as the cylinders shrink to the point. To fully understand the technical procedure, based on the study of an alternative argument which makes use of energy and logarithmic estimates, one has not only to be familiar with Dibenedetto's technique (see [12, 13, 30]) but also to overcome the difficulty of having an $(x_i, t)-$dependence on the exponents $p_i$ for $i = 1, \dots, N$.

The local continuity of the anisotropic elliptic equation have been studied, and the results are well documented. We refer to [14, 21] for the results and the references to the original papers. It is known that the local solutions of the isotropic parabolic case of equation (1.1) with variable growth are locally Hölder continuous [2, 7]. To the best of the author's knowledge, no regularity result is known for the anisotropic parabolic equations with variable growth.

## 2. Preliminary and main results

2.1. **Function spaces.** We recall in what follows some definitions and basic proprieties of the generalized Lebesgue-Sobolev spaces. We begin by defining the variable exponent Lebesgue space as follows

$$L^{p(x)}(\Omega) := \big\{u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u|^{p(x)}\,dx\,dt < +\infty\big\}.$$

This set equipped with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} := \inf\big\{\alpha > 0 : \int_\Omega |\frac{u}{\alpha}|^{p(x)}\,dx\,dt < 1\big\}$$

becomes a reflexive Banach space. Now, we define the Sobolev space

$$W^{1,p(x)}(\Omega) := \{u \in L^{p(x)}(\Omega), \nabla u \in L^{p(x)}(\Omega)\}$$

endowed with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} := \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

In addition, if $p(x)$ is log-Hölder continuous then $W_0^{1,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Now, in connection with the anisotropic operators that we are considering, we need to recall the definitions of the anisotropic Sobolev spaces:

$$\begin{gathered}W^{1,(p_i)}(\Omega)=\big\{u\in W^{1,1}(\Omega),\ \frac{\partial u}{\partial x_i}\in L^{p_i}(\Omega),\text{for } i=1,\dots,N\big\},\\ W_0^{1,(p_i)}(\Omega)=\big\{u\in W_0^{1,1}(\Omega),\ \frac{\partial u}{\partial x_i}\in L^{p_i}(\Omega),\text{for } i=1,\dots,N\big\}.\end{gathered}$$

The space $W_0^{1,(p_i)}(\Omega)$ also denotes the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{1,(p_i)}=\sum_{i=1}^N\|\frac{\partial u}{\partial x_i}\|_{L^{p_i}(\Omega)}.$$

The theory of anisotropic Sobolev spaces is developed in [18, 28, 29], and in particular, the corresponding Sobolev embedding theorems were studied there. We define

$$p^*=\frac{N\overline{p}}{N-\overline{p}},\ \text{for } \overline{p}<N \quad\text{and}\quad \frac{1}{\overline{p}}=\frac{1}{N}\sum_{i=1}^N\frac{1}{p_i^-}. \tag{2.1}$$

In [29] it is proved that if $\overline{p}<N$, then

$$W_0^{1,(p_i^-)}(\Omega)\hookrightarrow L^r(\Omega),\quad \forall r\in[1,p^*].$$

This embedding is continuous and also compact if $r<p^*$. If $\overline{p}\ge N$, then

$$W_0^{1,(p_i^-)}(\Omega)\hookrightarrow L^r(\Omega),\quad \forall r\in[1,+\infty).$$

The following Sobolev type inequality is also proved; if $\overline{p}<N$, then there exists a positive constant $C$, depending only on $\Omega$, $p_i^-$, and $N$, such that

$$\|u\|_{r,\Omega}\le C\prod_{i=1}^N\|\frac{\partial u}{\partial x_i}\|_{p_i^-,\Omega}^{1/N},\quad \forall r\in[1,p^*], \tag{2.2}$$

for any $u\in W_0^{1,(p_i^-)}(\Omega)$.

For a.e. $t\in(0,T)$ we introduce the anisotropic Banach space

$$\begin{gathered}V_t(\Omega)=\big\{u(x):u(x)\in L^2(\Omega)\cap W_0^{1,1}(\Omega),|\frac{\partial u(x)}{\partial x_i}|^{p_i(x,t)}\in L^1(\Omega)\big\},\\ \|u\|_{V_t(\Omega)}=\|u\|_{2,\Omega}+\sum_{i=1}^N\|\frac{\partial u}{\partial x_i}\|_{p_i(\cdot,t),\Omega}.\end{gathered}$$

The elements of the space $V_t(\Omega)$ depend on $t\in(0,T)$ as a parameter and the norms $\|u\|_{V_t(\Omega)}$ are functions of $t$. By $W(\Omega_T)$ we denote the Banach space

$$\begin{gathered}W(\Omega_T)=\big\{u:(0,T)\mapsto V_t(\Omega):u\in L^2(\Omega_T),\,|\frac{\partial u}{\partial x_i}|^{p_i(x,t)}\in L^1(\Omega_T),\,u=0\text{ on }\Gamma\big\},\\ \|u\|_{W(\Omega_T)}=\|u\|_{2,\Omega_T}+\sum_{i=1}^N\|\frac{\partial u}{\partial x_i}\|_{p_i(\cdot),\Omega_T}.\end{gathered}$$

2.2. **Mollification in time.** Since weak solutions of parabolic equations, respectively inequalities possess only weak regularity properties with respect to time, it is in principle not possible to use the solution itself as a test-function in the weak formulation of the problem. In order to be nevertheless able to test properly, there are several possibilities to smooth the solution with respect to the time direction. To overcome these faculties, we consider the Friedrichs mollifier as was done in [2]. Indeed, taking the kernel

$$\rho \geq 0, \quad \rho \in C_0^\infty(\mathbb{R}^N), \quad \rho(x) \equiv 0 \text{ for } |x| \geq 1, \quad \int_{\mathbb{R}^N} \rho(x)\, dx = 1,$$

we introduce regularization of $f \in L^{p(x,t)}_{\rm loc}(\Omega_T)$ by

$$\begin{gathered} I^h f = f_h(x,t) = h^{-1} \int_t^{t+h} \int_{|x-y|\leq h} f(y,\tau)\rho_h(x-y)\, dy\, d\tau, \\ \rho_h(x) = h^{-N}\rho(h^{-1}x), \end{gathered} \tag{2.3}$$

and consider these inside the cylinder $\Omega_T$, i.e., in cylinders $\Omega'_T = \Omega' \times (T_1, T_2)$, where $\Omega' \subset \Omega$, $0 < T_1 < T_2 < T$. The basic property of the mollification, which can be retrieved from [2, Lemma 2.1], is summarized in the following lemma.

**Lemma 2.1.** *If the exponent p satisfies condition* (1.2)*, then $f_h \to f$ in $L^{p(x,t)}_{\rm loc}(\Omega_T)$ as $h \to 0$, for any $f \in L^{p(x,t)}_{\rm loc}(\Omega_T)$.*

2.3. **Formulation of the problem.** We will consider here local weak solutions of equation (1.1), the existence of such solutions is guaranteed by [5, 6].

**Definition 2.2.** A local weak solution of (1.1) is a measurable function $u(x,t)$ defined in $\Omega_T$, such that

(i) $u \in W(\Omega_T) \cap C([0,T]; L^2(\Omega))$;
(ii) for every subset $K$ of $\Omega$ and for every subinterval $[t_1, t_2]$ of $(0,T]$, we have

$$\Big[\int_K u\phi\, dx\Big]_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K \Big\{ -u\phi_t + \sum_{i=1}^N |\frac{\partial u}{\partial x_i}|^{p_i(x,t)-2}\frac{\partial u}{\partial x_i}\cdot\frac{\partial \phi}{\partial x_i}\Big\}\, dx\, dt = 0, \tag{2.4}$$

for any locally bounded tested function $\phi \in W_{\rm loc}(\Omega_T) \cap W^{1,2}_{\rm loc}(0,T; W^{1,2}_0(K))$.

We can write (ii) in a way that is technically more convenient and involves the discrete time derivative. This can be accomplished by using the Friedrichs mollifier of a function (see [2] for more details). Then, we obtain the following result.

**Lemma 2.3.** *If $u$ is a solution of* (1.1) *in the sense of Definition 2.2, then for every subset $K$ of $\Omega$, and for any $h < t_1 \leq t_2 < T-h$, the*

$$\int_{t_1}^{t_2}\int_K \Big[u_{h,t}\varphi + \sum_{i=1}^N \Big(|\frac{\partial u}{\partial x_i}|^{p_i(x,t)-2}\frac{\partial u}{\partial x_i}\Big)_h \cdot \frac{\partial \varphi}{\partial x_i}\Big]\, dx\, dt = 0 \tag{2.5}$$

*holds for any locally bounded test function $\varphi \in W_{\rm loc}(\Omega_T) \cap W^{1,2}_{\rm loc}(0,T; W^{1,2}_0(K))$.*

*Proof.* As in [2], we introduce the regularization operator

$$I^{-h} f = f_{-h}(x,t) = h^{-1}\int_{t-h}^t \int_{|x-y|\leq h} f(y,\tau)\rho_h(x-y)\, dy\, d\tau. \tag{2.6}$$

Consider equation (2.4) with

$$\phi = I^{-h}(\varphi\chi), \quad \varphi \in W_{\rm loc}(\Omega_T) \cap W^{1,2}_{\rm loc}(0,T; W^{1,2}_0(K)).$$

Since

$$-\int_{t_1}^{t_2}\int_K u\frac{\partial I^{-h}(\varphi\chi)}{\partial t}\,dx\,dt = \int_{t_1}^{t_2}\int_K u_{h,t}\varphi\chi\,dx\,dt,$$

it follows that

$$\int_{t_1}^{t_2}\int_K \Big[u_{h,t}\varphi\chi + \sum_{i=1}^N \big(|\frac{\partial u}{\partial x_i}|^{p_i(x,t)-2}\frac{\partial u}{\partial x_i}\big)_h \cdot \frac{\partial(\varphi\chi)}{\partial x_i}\Big]\,dx\,dt = 0.$$

Passing here from $\chi \in C_0^\infty(t_1,t_2)$ to characteristic function of the segment $[t_1,t_2]$, we obtain the desired relation (2.5). □

2.4. **Regularity result.** To obtain the interior continuity of the solutions by means of intrinsic scaling, we need to consider a geometry that accommodates the degeneracy of the anisotropic parabolic equation (1.1). For this purpose let $(x_0,t_0)$ be an interior point of the space time domain $\Omega_T$, by translation and to simplify, assume $(x_0,t_0) = (0,0)$. Also, let $0 < R < 1$, be sufficiently small such that the cylinder

$$Q(R^2,R) = K_R \times (-R^2,0) := \{x : \max_{1\le i\le N}|x_i| < R\} \times (-R^2,0)$$

is a subset of $\Omega_T$, and define

$$\mu^+ = \operatorname{ess\,sup}_{Q(R^2,R)} u, \quad \mu^- = \operatorname{ess\,inf}_{Q(R^2,R)} u, \quad \omega = \operatorname{ess\,osc}_{Q(R^2,R)} u = \mu^+ - \mu^-.$$

Let $a_0 = (\omega/2^\lambda)^{2-p^-}$ be a positive real number, for some $\lambda > 1$ to be chosen later. We construct the cylinder

$$Q(a_0R^{p^+},R) = K_R \times (-a_0R^{p^+},0).$$

Under the assumption

$$R^{\frac{2-p^+}{2-p^-}} < \frac{\omega}{2^\lambda}, \tag{2.7}$$

the inclusion $Q(a_0R^{p^+},R) \subset Q(R^2,R)$ holds, and consequently we have

$$\operatorname{ess\,osc}_{Q(a_0R^{p^+},R)} u \le \omega.$$

**Remark 2.4.** *If* (2.7) *does not hold, then the essential oscillation $\omega$ approaches zero when the radius $R$ goes to zero, and then there is nothing to prove.*

To begin our approach, inside $Q(a_0R^{p^+},R)$ consider subcylinders of small size constructed as follows

$$(0,t^*) + Q(\theta R^{p^+},R), \quad \theta = \big(\frac{\omega}{2}\big)^{2-p^-}.$$

These are contained in $Q(a_0R^{p^+},R)$ if

$$(2^{p^--2} - 2^{\lambda(p^--2)})\frac{R^{p^+}}{\omega^{p^--2}} < t^* < 0.$$

For a given $\nu_0 \in (0,1)$, to be determined in terms of the data and $\omega$, either

$$|\{(x,t) \in (0,t^*) + Q(\theta R^{p^+},R) :\ u(x,t) < \mu^- + \frac{\omega}{2}\}| \le \nu_0|Q(\theta R^{p^+},R)| \tag{2.8}$$

or, noting that $\mu^+ - \frac{\omega}{2} = \mu^- + \frac{\omega}{2}$, and

$$\begin{aligned} &\big|\{(x,t) \in (0,t^*) + Q(\theta R^{p^+}, R) :\ u(x,t) > \mu^+ - \frac{\omega}{2}\}\big| \\ &\le (1-\nu_0)|Q(\theta R^{p^+}, R)|. \end{aligned} \tag{2.9}$$

The analysis of this alternative leads to the following result.

**Proposition 2.5.** *Assume that $\overline{p} < N$, then there exist positive numbers $\nu_0$, $\sigma \in (0,1)$, depending on the data and $\omega$, such that*

$$\operatorname{ess\,osc}_{Q(\theta(\frac{R}{8})^{p^+}, \frac{R}{8})} u \le \sigma\omega. \tag{2.10}$$

An immediate consequence is the following result.

**Theorem 2.6.** *Under the assumption that $\overline{p} < N$, any locally bounded weak solution of (1.1) is locally continuous in $\Omega_T$.*

**Remark 2.7.** Local continuity of the weak solutions in the elliptic case of our equation is also established for $\overline{p} < N$ (see [14, 21]). This assumption is necessary to apply (2.2). Moreover, the Hölder-continuity of the weak solutions for the isotropic case of our equation is established for $p \in (\frac{2N}{N+2}, \infty)$.

**Remark 2.8.** The proof of Theorem 2.6 follows from a slight modification of the arguments in Proposition 9 in [13]. From (2.10) one defines recursively a sequence $Q_n$ of nested and shrinking cylinders and a sequence $\omega_n$ converging to zero, such that

$$\operatorname{ess\,osc}_{Q_n} u \le \omega_n.$$

This is enough to obtain the continuity of $u$ but we are unable to derive a modulus since the constant $\sigma$ appearing in Proposition 2.5 depends on the oscillation of $\omega$.

## 3. Local energy and logarithmic estimates

Let $\tau$ and $\rho$ be small such that $Q(\tau,\rho) \subset \Omega_T$, in addition let $\xi$ be a piecewise smooth cutoff function in $Q(\tau,\rho)$ such that

$$\xi \in [0,1], \quad |\frac{\partial \xi}{\partial x_i}| < \infty \quad \forall i = 1,\dots,N, \quad \text{and} \quad \xi(x,t) = 0 \text{ for } x \text{ outside } K_\rho.$$

**Proposition 3.1.** *Let $u$ be a local weak solution of (1.1) in $\Omega_T$, then there exists a positive constant $C$ such that, for every cylinder $Q(\tau,\rho) \subset \Omega_T$ and for every $k \in \mathbb{R}$, we have*

$$\begin{aligned} &\sup_{-\tau<t<0} \int_{K_\rho} (u-k)_\pm^2 \xi^{p^+}(x,t)\,dx + \sum_{i=1}^N \int_{-\tau}^0 \int_{K_\rho} |\frac{\partial}{\partial x_i}(u-k)_\pm|^{p_i(x,t)} \xi^{p^+}\,dx\,dt \\ &\le \int_{K_\rho} (u-k)_\pm^2 \xi^{p^+}(x,-\tau)\,dx + p^+ \int_{-\tau}^0 \int_{K_\rho} (u-k)_\pm^2 \xi^{p^+-1}\xi_t\,dx\,dt \\ &\quad + C\sum_{i=1}^N \int_{-\tau}^0 \int_{K_\rho} (u-k)_\pm^{p_i(x,t)} |\frac{\partial \xi}{\partial x_i}|^{p_i(x,t)}\,dx\,dt. \end{aligned} \tag{3.1}$$

*Proof.* In the weak formulation (2.5) we take the test function $\varphi = \pm(u_h-k)_\pm \xi^{p^+}$, where

$$(u_h-k)_- = (k-u_h)_+ = \max\{k-u, 0\},$$

and $u_h$ are regularizations of the form (2.3). Then integrate over $(-\tau, t)$, $t \in (-\tau, 0)$, and use Lemma 2.1. Now we estimating the various terms separately. The first term gives

$$
\begin{aligned}
&\int_{-\tau}^{t} \int_{K_\rho} u_{h,t} \varphi \, dx \, dt \\
&= \int_{-\tau}^{t} \int_{K_\rho} u_{h,t} (\pm (u_h - k)_\pm \xi^{p^+}) \, dx \, dt \\
&\to -\frac{p^+}{2} \int_{-\tau}^{t} \int_{K_\rho} (u-k)_\pm^2 \xi^{p^+ - 1} \xi_t \, dx \, dt + \frac{1}{2} \int_{K_\rho} (u-k)_\pm^2 \xi^{p^+}(x,t) \, dx \\
&\quad - \frac{1}{2} \int_{K_\rho} (u-k)_\pm^2 \xi^{p^+}(x,-\tau) \, dx, \qquad h \to 0 \,.
\end{aligned}
$$

For the remaining term, when $h \to 0$, we obtain

$$
\begin{aligned}
&\sum_{i=1}^{N} \int_{-\tau}^{t} \int_{K_\rho} |\frac{\partial u}{\partial x_i}|^{p_i(x,t)-2} \frac{\partial u}{\partial x_i} \big[ \frac{\partial}{\partial x_i} (\pm (u-k)_\pm) xi^{p^+} \pm p^+ (u-k)_\pm \xi^{p^+ - 1} \frac{\partial \xi}{\partial x_i} \big] \, dx \, dt \\
&\geq \sum_{i=1}^{N} \int_{-\tau}^{t} \int_{K_\rho} |\frac{\partial}{\partial x_i} (u-k)_\pm|^{p_i(x,t)} \xi^{p^+} \, dx \, dt \\
&\quad - p^+ \sum_{i=1}^{N} \int_{-\tau}^{t} \int_{K_\rho} |\frac{\partial}{\partial x_i} (u-k)_\pm|^{p_i(x,t)-1} (u-k)_\pm \xi^{p^+ - 1} |\frac{\partial \xi}{\partial x_i}| \, dx \, dt \\
&\geq \frac{1}{2} \sum_{i=1}^{N} \int_{-\tau}^{t} \int_{K_\rho} |\frac{\partial}{\partial x_i} (u-k)_\pm|^{p_i(x,t)} \xi^{p^+} \, dx \, dt \\
&\quad - C \sum_{i=1}^{N} \int_{-\tau}^{t} \int_{K_\rho} (u-k)_\pm^{p_i(x,t)} |\frac{\partial \xi}{\partial x_i}|^{p_i(x,t)} \, dx \, dt.
\end{aligned}
$$

Here we used Young's inequality, and the fact that $0 \leq \xi \leq 1$ and $\frac{p_i(x,t)}{p_i(x,t)-1} \geq \frac{p^+}{p^+ - 1}$ imply that $\xi^{\frac{p_i(x,t)(p^+ - 1)}{p_i(x,t)-1}} \leq \xi^{p^+}$, $\forall i = 1, \dots, N$. Hence, since $t \in (-\tau, 0)$ is arbitrary, we can combine both estimates to obtain (3.1). $\square$

Now, introduce the logarithmic function

$$
\psi^\pm(u) = \psi(H_k^\pm, (u-k)_\pm, c) = \Big( \ln \big( \frac{H_k^\pm}{H_k^\pm - (u-k)_\pm + c} \big) \Big)_+,
$$

where $H_k^\pm = \operatorname{ess\,sup}_{Q(\tau,\rho)} |(u-k)_\pm|$ and $0 < c < H_k^\pm$. In the cylinder $Q(\tau, \rho)$, we take a cutoff function satisfying $\xi \in [0,1]$, $|\frac{\partial \xi}{\partial x_i}| < \infty$ for $i = 1, \dots, N$ and $\xi$ is independent of $t \in (-\tau, 0)$.

**Proposition 3.2.** *Let $u$ be a local weak solution of (1.1) in $\Omega_T$, then there exists a positive constant $C$ such that for every cylinder $Q(\tau, \rho) \subset \Omega_T$ and for every level*

$k \in \mathbb{R}$,

$$\begin{aligned} &\operatorname{ess\,sup}_{-\tau<t<0} \int_{K_\rho\times\{t\}} [\psi^\pm(u)]^2 \xi^{p^+}\,dx \\ &\le \int_{K_\rho\times\{-\tau\}} [\psi^\pm(u)]^2 \xi^{p^+}\,dx \\ &\quad + C\sum_{i=1}^N \int_{-\tau}^0 \int_{K_\rho} \psi^\pm(u)[(\psi^\pm)'(u)]^{2-p_i(x,t)} |\frac{\partial \xi}{\partial x_i}|^{p_i(x,t)}\,dx\,dt. \end{aligned} \tag{3.2}$$

*Proof.* In (2.5) we take the test function $\varphi = 2\psi^\pm(u_h)[(\psi^\pm)'(u)]\xi^{p^+}$, and by direct computation we obtain

$$(\psi^\pm(u))'' = \{(\psi^\pm(u))'\}^2.$$

Therefore, we estimate the various terms separately, integrate in time over $(-\tau,t)$ for $t\in(-\tau,0)$, and use Lemma 2.1. The first term gives

$$\begin{aligned} &\int_{-\tau}^t \int_{K_\rho} u_{h,t}\big\{2\psi^\pm(u_h)[(\psi^\pm)'(u_h)]\xi^{p^+}\big\}\,dx\,dt \\ &= \int_{-\tau}^t \int_{K_\rho} (\psi^\pm(u_h)^2)_t \xi^{p^+}\,dx\,dt \\ &\to \int_{K_\rho\times\{t\}} [\psi^\pm(u)]^2\xi^{p^+}\,dx - \int_{K_\rho\times\{-\tau\}} [\psi^\pm(u)]^2\xi^{p^+}\,dx. \end{aligned} \tag{3.3}$$

as $h\to 0$. For the remaining term, we first let $h\to 0$, to obtain

$$\begin{aligned} &\sum_{i=1}^N \int_{-\tau}^t \int_{K_\rho} |\frac{\partial u}{\partial x_i}|^{p_i(x,t)-2}\frac{\partial u}{\partial x_i}\cdot\frac{\partial \varphi}{\partial x_i}\,dx\,dt \\ &= \sum_{i=1}^N \int_{-\tau}^t \int_{K_\rho} |\frac{\partial u}{\partial x_i}|^{p_i(x,t)-2}\frac{\partial u}{\partial x_i}\big[2\frac{\partial u}{\partial x_i}[(\psi^\pm)'(u)]^2\xi^{p^+} \\ &\quad + 2\frac{\partial u}{\partial x_i}\psi^\pm(u)[(\psi^\pm)'(u)]^2\xi^{p^+}\big]\,dx\,dt \\ &\quad + \sum_{i=1}^N \int_{-\tau}^t \int_{K_\rho} |\frac{\partial u}{\partial x_i}|^{p_i(x,t)-2}\frac{\partial u}{\partial x_i} 2p^+\psi^\pm(u)[(\psi^\pm)'(u_h)]\xi^{p^+-1}\frac{\partial \xi}{\partial x_i}\,dx\,dt \\ &\ge \sum_{i=1}^N \int_{-\tau}^t \int_{K_\rho} |\frac{\partial u}{\partial x_i}|^{p_i(x,t)} 2[(\psi^\pm)'(u)]^2(1+\psi^\pm(u)-\psi^\pm(u))\xi^{p^+}\,dx\,dt \\ &\quad - C\sum_{i=1}^N \int_{-\tau}^t \int_{K_\rho} \psi^\pm(u)[(\psi^\pm)'(u)]^{2-p_i(x,t)}|\frac{\partial \xi}{\partial x_i}|^{p_i(x,t)}\,dx\,dt \\ &\ge -C\sum_{i=1}^N \int_{-\tau}^t \int_{K_\rho} \psi^\pm(u)[(\psi^\pm)'(u)]^{2-p_i(x,t)}|\frac{\partial \xi}{\partial x_i}|^{p_i(x,t)}\,dx\,dt. \end{aligned}$$

Hence, since $t\in(-\tau,0)$ is arbitrary, we can combine both estimates to obtain (3.2). ◻

## 4. Continuity of the weak solutions

In this section we analyze the alternative and prove Proposition 2.5. Assume that (2.8) is satisfied. The following lemma determines the number $\nu_0$ and guarantees that the solution $u$ is above a smaller level within a smaller cylinder.

**Lemma 4.1.** *There exists $\nu_0 \in (0,1)$, depending on the data and $\omega$, such that if* (2.8) *holds then*

$$u(x,t) > \mu^- + \frac{\omega}{4} \quad a.e. \ in \ (0,t^*) + Q(\theta(\frac{R}{2})^{p^+}, \frac{R}{2}). \tag{4.1}$$

*Proof.* Up to translation we can assume that $(0,t^*) = (0,0)$. We define two decreasing sequences of positive numbers

$$R_n = \frac{R}{2} + \frac{R}{2^{n+1}}, \quad k_n = \mu^- + \frac{\omega}{4} + \frac{\omega}{2^{n+2}}, \quad n = 0,1,\dots$$

We construct the family of nested and shrinking cylinders $Q(\theta R_n^{p^+}, R_n)$, and let $0 \le \xi_n(x,t) \le 1$ be piecewise smooth functions in $Q(\theta R_n^{p^+}, R_n)$ such that

$$\xi_n = 1 \text{ in } Q(\theta R_{n+1}^{p^+}, R_{n+1}), \quad \xi_n = 0 \text{ on } \partial_p Q(\theta R_n^{p^+}, R_n),$$

$$|\frac{\partial \xi_n}{\partial x_i}| \le \frac{2^{\frac{(n+1)p^+}{p_i^+}}}{R^{\frac{p^+}{p_i^+}}}, \quad 0 < (\xi_n)_t \le \frac{2^{p^+(n+1)}}{\theta R^{p^+}}, \quad \forall i = 1,\dots,N.$$

Now, by using the energy inequality (3.1) for the functions $(u-k_n)_-$ we obtain

$$\begin{aligned}
&\sup_{-\theta R_n^{p^+} < t < 0} \int_{K_{R_n}} (u-k_n)_-^2 \xi_n^{p^+}(x,t)\,dx \\
&+ \sum_{i=1}^N \int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} |\frac{\partial}{\partial x_i}(u-k_n)_-|^{p_i^-} \xi_n^{p^+}\,dx\,dt \\
&\le C\Big( \int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} (u-k_n)_-^2 \xi_n^{p^+-1} (\xi_n)_t\,dx\,dt \\
&\quad + \sum_{i=1}^N \int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} (u-k_n)_-^{p_i(x,t)} |\frac{\partial \xi_n}{\partial x_i}|^{p_i(x,t)}\,dx\,dt \\
&\quad + \int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} \chi((u-k_n)_- > 0)\,dx\,dt \Big) \\
&\le C \frac{2^{p^+(n+1)}}{R^{p^+}} \Big( \frac{1}{\theta} \int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} (u-k_n)_-^2\,dx\,dt \\
&\quad + \sum_{i=1}^N \int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} (u-k_n)_-^{p_i(x,t)}\,dx\,dt \\
&\quad + \int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} \chi((u-k_n)_- > 0)\,dx\,dt \Big) \\
&\le C \frac{2^{p^+(n+1)}}{R^{p^+}} \Big( (\frac{\omega}{2})^{p^-} + (\frac{\omega}{2})^{p^+} + 1 \Big) \int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} \chi((u-k_n)_- > 0)\,dx\,dt
\end{aligned}$$

$$\le C\frac{2^{p^+(n+2)}}{R^{p^+}}\Big((\frac{\omega}{2})^{p^+}+1\Big)\int_{-\theta R_n^{p^+}}^0\int_{K_{R_n}}\chi((u-k_n)_->0)\,dx\,dt$$

By means of (2.7), this implies

$$\begin{aligned}&\sup_{-\theta R_n^{p^+}<t<0}\int_{K_{R_n}}(u-k_n)_-^2\xi_n^{p^+}(x,t)\,dx\\&+\sum_{i=1}^N\int_{-\theta R_n^{p^+}}^0\int_{K_{R_n}}|\frac{\partial}{\partial x_i}(u-k_n)_-|^{p_i^-}\xi_n^{p^+}\,dx\,dt\\&\le C\frac{2^{p^+(n+2)}}{R^{p^+}}(\frac{\omega}{2})^{p^+}\int_{-\theta R_n^{p^+}}^0\int_{K_{R_n}}\chi((u-k_n)_->0)\,dx\,dt,\end{aligned}$$

where $\chi_E$ denotes the characteristic function of the set $E$. Using the fact that $(u-k_n)_-=0$ or

$$(u-k_n)_-=(\mu^--u)+\frac{\omega}{4}+\frac{\omega}{2^{n+2}}\le\frac{\omega}{2},\tag{4.2}$$

we obtain

$$(u-k_n)_-^2\ge\theta(u-k_n)_-^{p^-}.\tag{4.3}$$

Then the above estimates reads

$$\begin{aligned}&\sup_{-\theta R_n^{p^+}<t<0}\int_{K_{R_n}}(u-k_n)_-^{p^-}\xi_n^{p^+}(x,t)\,dx\\&+\frac{1}{\theta}\sum_{i=1}^N\int_{-\theta R_n^{p^+}}^0\int_{K_{R_n}}\big|\frac{\partial}{\partial x_i}(u-k_n)_-\big|^{p_i^-}\xi_n^{p^+}\,dx\,dt\\&\le C\frac{2^{p^+(n+2)}}{R^{p^+}}(\frac{\omega}{2})^{p^+}\frac{1}{\theta}\int_{-\theta R_n^{p^+}}^0\int_{K_{R_n}}\chi((u-k_n)_->0)\,dx\,dt,\end{aligned}\tag{4.4}$$

Let us now consider the change of variable $\tilde{t}=\frac{t}{\theta}$ and define the functions

$$\tilde{u}(\cdot,\tilde{t})=u(\cdot,t),\quad\tilde{\xi_n}(\cdot,\tilde{t})=\xi_n(\cdot,t).$$

Then, for

$$A_n=\int_{-R_n^{p^+}}^0\int_{K_{R_n}}\chi((\tilde{u}-k_n)_->0)\,dx\,d\tilde{t},$$

inequality (4.4) becomes

$$\begin{aligned}&\sup_{-R_n^{p^+}<\tilde{t}<0}\int_{K_{R_n}}(\tilde{u}-k_n)_-^{p^-}\tilde{\xi_n}^{p^+}(x,\tilde{t})\,dx\\&+\sum_{i=1}^N\int_{-R_n^{p^+}}^0\int_{K_{R_n}}|\frac{\partial}{\partial x_i}(\tilde{u}-k_n)_-\big|^{p_i^-}\tilde{\xi_n}^{p^+}\,dx\,d\tilde{t}\\&\le C\frac{2^{p^+(n+2)}}{R^{p^+}}(\frac{\omega}{2})^{p^+}A_n.\end{aligned}\tag{4.5}$$

From the definition of $k_n$, we have

$$(\frac{\omega}{2^{n+3}})^{\overline{p}}A_{n+1}=|k_n-k_{n+1}|^{\overline{p}}A_{n+1}\le\int_{-R_n^{p^+}}^0\int_{K_{R_n}}(\tilde{u}-k_n)_-^{\overline{p}}\tilde{\xi_n}^{\beta}\,dx\,d\tilde{t}.$$

Now we use Hölder's inequality with exponents $\frac{N}{N-\overline{p}}$ and $\frac{N}{\overline{p}}$ to obtain

$$(\frac{\omega}{2^{n+3}})^{\overline{p}}A_{n+1}\le C\int_{-R_n^{p^+}}^0\Big(\int_{K_{R_n}}\big((\tilde u-k_n)_-\tilde{\xi_n}^{\beta/\overline{p}}\big)^{p^*}dx\Big)^{\overline{p}/p^*}d\tilde tA_n^{\overline{p}/N},$$

where $p^*$ is defined in (2.1). So, by the anisotropic Sobolev inequality (2.2), we have

$$\begin{aligned}
(\frac{\omega}{2^{n+3}})^{\overline{p}}A_{n+1}&\le C\int_{-R_n^{p^+}}^0\prod_{i=1}^N\Big\{\int_{K_{R_n}}|\frac{\partial}{\partial x_i}[(\tilde u-k_n)_-\tilde{\xi_n}^{\beta/\overline{p}}]|^{p_i^-}dx\Big\}^{\frac{\overline{p}}{Np_i^-}}d\tilde tA_n^{\overline{p}/N}\\
&\le C\prod_{i=1}^N\Big\{\int_{-R_n^{p^+}}^0\int_{K_{R_n}}|\frac{\partial}{\partial x_i}(\tilde u-k_n)_-|^{p_i^-}\tilde{\xi_n}^{\frac{\beta p_i^-}{\overline{p}}}\,dx\,d\tilde t\\
&\quad+\int_{-R_n^{p^+}}^0\int_{K_{R_n}}(\tilde u-k_n)_-^{p_i^-}\big|\frac{\partial\tilde{\xi_n}}{\partial x_i}\big|^{p_i^-}\,dx\,d\tilde t\Big\}^{\frac{\overline{p}}{Np_i^-}}A_n^{\overline{p}/N}.
\end{aligned}$$

Since $0\le\xi_n(x,t)\le 1$, we choose $\beta$ such that $p^+\le\frac{\beta p_i^-}{\overline{p}}$, for $i=1,2,\dots,N$. Therefore, using (4.5) we obtain

$$(\frac{\omega}{2^{n+3}})^{\overline{p}}A_{n+1}\le C\frac{2^{p^+(n+2)}}{R^{p^+}}(\frac{\omega}{2})^{p^+}A_n^{1+\frac{\overline{p}}{N}}. \tag{4.6}$$

A direct calculation leads to

$$\frac{|Q(R_n^{p^+},R_n)|^{1+\frac{\overline{p}}{N}}}{|Q(R_{n+1}^{p^+},R_{n+1})|}\le 2^{P^++N}R^{\overline{p}(1+\frac{\overline{p}}{N})}. \tag{4.7}$$

Next, if we define

$$X_n=\frac{A_n}{Q(R_n^{p^+},R_n)},$$

we obtain the recursive relation

$$X_{n+1}\le C4^{np^+}(\frac{\omega}{2})^{p^+-\overline{p}}X_n^{1+\frac{\overline{p}}{N}}.$$

Thus, [12, Lemma 4.1 in Chapter I] implies that if

$$X_0\le\big[C(\frac{\omega}{2})^{p^+-\overline{p}}\big]^{-N/\overline{p}}4^{-p^+(\frac{N}{\overline{p}})^2}=\nu_0, \tag{4.8}$$

then

$$X_n\to 0. \tag{4.9}$$

But (4.8) is nothing but the assumption (2.8). Hence, the result easily follows from (4.9). □

Now consider the time level $-\hat t=t^*-\theta(\frac{R}{2})^{p^+}$, then from the conclusion of Lemma 4.1, we have

$$u(x,-\hat t)>\mu^-+\frac{\omega}{4}\quad\text{a.e. for } x\in K_{\frac{R}{2}}.$$

We will use this time level as an initial condition to bring the information up to $t=0$, and therefore to obtain an analogous inequality in a in a smaller cylinder. The first step in this direction is given by the following lemma.

**Lemma 4.2.** *For every $\nu_1 \in (0,1)$, there exists a positive integer $s_1$ depending on the data and $\omega$, such that*

$$|\{x \in K_{R/4} : u(x,t) < \mu^- + \frac{\omega}{2^{s_1}}\}| \leq \nu_1 |K_{R/4}|, \quad \forall t \in (-\hat{t}, 0). \tag{4.10}$$

*Proof.* Consider the cylinder $Q(\hat{t}, R/2)$ and write the logarithmic estimate (3.2) over this cylinder, for the function $(u-k)_-$, with

$$k = \mu^- + \frac{\omega}{4} \quad \text{and} \quad c = \frac{\omega}{2^{n+2}},$$

where $n$ is to be chosen later. We define $H_k^-$ such that

$$k - u \leq H_k^- = \operatorname{ess\,sup}_{Q(\hat{t}, \frac{R}{2})} |(u - \mu^- - \frac{\omega}{4})_-| \leq \frac{\omega}{4}. \tag{4.11}$$

Assuming $H_k^- \leq \frac{\omega}{8}$ (else the result is trivial). Then the logarithmic function $\psi^-$ is well defined and satisfies the inequality

$$\psi^- \leq n \ln(2) \quad \text{since } \frac{H_k^-}{H_k^- + u - k + c} \leq \frac{\frac{\omega}{4}}{c} = 2^n, \tag{4.12}$$

and, for $u \neq -k + c$,

$$0 \leq (\psi^-)' \leq \frac{1}{H_k^- + u - k + c} \leq \frac{1}{c}, \tag{4.13}$$

and

$$|(\psi^-)'(u)|^{2-p^-} = (H_k^- + u - k + c)^{p^- - 2} \leq (\frac{\omega}{2})^{p^- - 2}. \tag{4.14}$$

For $t = -\hat{t}$, by Lemma 4.1, we have $u(x, -\hat{t}) > k$, and therefore

$$[\psi^-(u)](x, -\hat{t}) = 0 \quad \text{for } x \in K_{\frac{R}{2}}.$$

To obtain the estimate, we choose a cutoff function $0 < \xi(x) \leq 1$, defined on $K_{\frac{R}{2}}$, such that

$$\xi = 1 \text{ in } \; K_{\frac{R}{2}} \quad \text{and} \quad |\frac{\partial \xi}{\partial x_i}| \leq (\frac{8}{R})^{\frac{p^+}{p_i^+}}, \text{ for } i = 1, 2, \dots, N.$$

Gathering these estimates in (3.2), and using that

$$\hat{t} \leq \big(\frac{\omega}{2^\lambda}\big)^{2-p^-} R^{p^+}, \tag{4.15}$$

we arrive at

$$\begin{aligned}
&\operatorname{ess\,sup}_{-\hat{t}<t<0} \int_{K_{\frac{R}{2}} \times \{t\}} [\psi^-(u)]^2 \xi^{p^+} \, dx \\
&\leq C \sum_{i=1}^N \int_{-\hat{t}}^0 \int_{K_{\frac{R}{2}}} \psi^-(u) [(\psi^-)'(u)]^{2-p_i(x,t)} |\frac{\partial \xi}{\partial x_i}|^{p_i(x,t)} \, dx \, dt \\
&\leq C \; n \ln(2) (\frac{\omega}{2})^{p^- - 2} (\frac{8}{R})^{p^+} \hat{t} |K_{R/4}| \\
&\leq C \; n (\frac{\omega}{2})^{p^- - 2} (\frac{8}{R})^{p^+} \big(\frac{\omega}{2^\lambda}\big)^{2-p^-} R^{p^+} |K_{R/4}| \\
&\leq C n 2^{\lambda(p^- - 2)} |K_{R/4}|.
\end{aligned} \tag{4.16}$$

The left hand side of (4.16) is estimated from below integrating over the smaller set
$$S=\{x\in K_{R/4},\ u(x,t)<\mu^-+\frac{\omega}{2^{n+2}}\}\subset K_{\frac{R}{2}},\ t\in(-\hat t,0).$$
On such set, $\xi=1$ and $\psi^-\geq((n-1)\ln(2))$, because
$$\frac{H_k^-}{H_k^-+u-k+\frac{\omega}{2^{n+2}}}\geq\frac{\frac{\omega}{4}}{\frac{\omega}{4}+u-k+\frac{\omega}{2^{n+2}}}=\frac{\frac{\omega}{4}}{u-\mu^-+\frac{\omega}{2^{n+2}}}\geq\frac{\frac{\omega}{4}}{2^{n+2}}=2^{n-1}.$$
Putting this in (4.16), we obtain that for all $t\in(-\hat t,0)$,
$$|S|\leq C\frac{n}{(n-1)^2}2^{\lambda(p^--2)}|K_{R/4}|.$$
The proof is complete once we choose $s_1=n+2$ with $n>1+\frac{2C}{\nu_1}2^{\lambda(p^--2)}$. □

The conclusion of Lemma 4.2 will be employed to deduce that, within the cylinder $Q(\hat t,\frac{R}{8})$, the set where $u$ is away from its infimum is arbitrarily small.

**Lemma 4.3.** *There exists $1<s_2\in\mathbb{N}$, depending on the data and $\omega$, such that*
$$u(x,t)>\mu^-+\frac{\omega}{2^{s_2+1}}\quad a.e.\ (x,t)\in Q(\hat t,\frac{R}{8}).\tag{4.17}$$

*Proof.* Define two decreasing sequences of positive numbers
$$R_n=\frac{R}{8}+\frac{R}{2^{n+3}},\ k_n=\mu^-+\frac{\omega}{2^{s_2+1}}+\frac{\omega}{2^{s_2+1+n}},\quad n=0,1,\dots.$$
We construct the family of nested and shrinking cylinders $Q(\hat t,R_n)$, and letting $0\leq\xi_n(x)\leq1$ be piecewise smooth functions in $K_{R_n}$ that equal one on $K_{R_{n+1}}$ and
$$|\frac{\partial\xi_n}{\partial x_i}|\leq\frac{2^{(n+4)p^+/p_i^+}}{R^{p^+/p_i^+}}\quad\text{for } i=1,\dots,N.$$
Lemma 4.1 implies that $(u-k_n)_-(x,-\hat t)=0$ in $K_{R_n}$. Now, since $(u-k_n)_-\leq\omega/2^{s_2}$, by using (4.15) and letting $s_2>\lambda+\frac{p^+}{p^--2}$ we obtain
$$(u-k_n)_-^2\geq\frac{\hat t}{(\frac{R}{2})^{p^+}}(u-k_n)_-^{p^-}.$$

Therefore, with these choices, and by applying the local energy inequalities (3.1) on the functions $(u-k_n)_-$, we obtain
$$\begin{aligned}&\frac{\hat t}{(\frac{R}{2})^{p^+}}\sup_{-\hat t<t<0}\int_{K_{R_n}\times\{t\}}(u-k_n)_-^{p^-}\xi_n^{p^+}\,dx\\&+\sum_{i=1}^N\int_{-\hat t}^0\int_{K_{R_n}}|\frac{\partial}{\partial x_i}(u-k_n)_-|^{p_i^-}\xi_n^{p^+}\,dx\,dt\\&\leq C\Big(\sum_{i=1}^N\int_{-\hat t}^0\int_{K_{R_n}}(u-k_n)_-^{p_i^+}|\frac{\partial\xi_n}{\partial x_i}|^{p_i^+}\,dx\,dt\\&\quad+\int_{-\hat t}^0\int_{K_{R_n}}\chi((u-k_n)_->0)\,dx\,dt\Big)\\&\leq C\frac{2^{np^+}}{R^{p^+}}(\frac{\omega}{2^{s_2}})^{p^+}\int_{-\hat t}^0\int_{K_{R_n}}\chi((u-k_n)_->0)\,dx\,dt.\end{aligned}\tag{4.18}$$

We divide by $\frac{\hat{t}}{(\frac{R}{2})^{p^+}}$, and introduce the change of variable $\widetilde{t} = t(\frac{R}{2})^{p^+}/\hat{t}$. As in the proof of Lemma 4.1, we arrive at

$$\Big(\frac{\omega}{2^{s_2+2+n}}\Big)^{\overline{p}} A_{n+1} \le C\frac{2^{np^+}}{R^{p^+}}\Big(\frac{\omega}{2^{s_2}}\Big)^{p^+} A_n^{1+\frac{\overline{p}}{N}}, \tag{4.19}$$

where

$$A_n = \int_{-(\frac{R}{2})^{p^+}}^{0} \int_{K_{R_n}} \chi((\widetilde{u} - k_n)_- > 0)\,dx\,d\tilde{t}.$$

Here we have considered $\widetilde{u}(x,\bar{t}) = u(x,t)$. Next, we define the numbers

$$X_n = \frac{A_n}{Q((\frac{R}{2})^{p^+}, R_n)}.$$

Dividing (4.19) by $Q((\frac{R}{2})^{p^+}, R_{n+1})$, we obtain the recursive relation

$$X_{n+1} \le C4^{np^+}\Big(\frac{\omega}{2^{s_2}}\Big)^{p^+-\overline{p}} X_n^{1+\frac{\overline{p}}{N}}.$$

Therefore, [12, Lemma 4.1 in Chapter I] implies that if

$$X_0 \le \Big[C\Big(\frac{\omega}{2^{s_2}}\Big)^{p^+-\overline{p}}\Big]^{-N/\overline{p}} 4^{-p^+(\frac{N}{\overline{p}})^2} = \nu_1, \tag{4.20}$$

then

$$X_n \to 0. \tag{4.21}$$

By applying Lemma 4.2 with $s_1 := s_2$ we obtain easily (4.20). Hence, the result easily follows from (4.21). □

As an immediate consequence we obtain the reduction of the oscillation of $u$.

**Corollary 4.4.** *There exists a constant $\sigma_0 \in (0,1)$, depending only on the data and $\omega$, such that if (2.8) holds then*

$$\operatorname{ess\,osc}_{Q(\theta(R/8)^{p^+}, R/8)} u \le \sigma_0\omega. \tag{4.22}$$

*Proof.* The proof follows since $Q(\theta(\frac{R}{8})^{p^+}, \frac{R}{8}) \subset Q(\hat{t}, \frac{R}{8})$, where we have $\sigma_0 = 1 - \frac{1}{2^{s_2+1}}$. □

Assume that (2.8) does not hold, then (2.9) holds. Even in this case, we are able to deduce a result analogous to Corollary 4.4.

**Lemma 4.5.** *Assume that (2.9) holds, then there exists a time level*

$$t_0 \in [t^* - \theta R^{p^+}, t^* - \frac{\nu_0}{2}\theta R^{p^+}], \tag{4.23}$$

*such that*

$$|\{x \in K_R,\ u(x,t_0) > \mu^+ - \frac{\omega}{2}\}| \le \Big(\frac{1-\nu_0}{1-\frac{\nu_0}{2}}\Big)|K_R|. \tag{4.24}$$

*Proof.* In fact, if (4.24) does not hold, then also (2.9) does not hold. □

This lemma shows that at the time level $t_0$, the portion of the cube $K_R$ where $u(x)$ is close to its supremum is small. The next lemma claims that this indeed occurs for all time levels near the top of the cylinder $(0, t^*) + Q(\theta R^{p^+}, R)$.

**Lemma 4.6.** *There exists $1 < s_3 \in \mathbb{N}$, depending on the data and $\omega$, such that, for all $t \in [t^* - \frac{\nu_0}{2}\theta R^{p^+}, t^*]$,*

$$|\{x \in K_R : u(x,t) > \mu^+ - \frac{\omega}{2^{s_3}}\}| \le (1 - (\frac{\nu_0}{2})^2)|K_R|. \tag{4.25}$$

*Proof.* Consider the cylinder $K_R \times (t_0, t^*)$, and the level $k = \mu^+ - \frac{\omega}{2}$. Then we define

$$u - k \le H_k^+ = \operatorname{ess\,sup}_{K_R \times (t_0,t^*)} |(u - \mu^+ + \frac{\omega}{2})_+| \le \frac{\omega}{2}. \tag{4.26}$$

Assuming that $H_k^+ > \frac{\omega}{4}$ (otherwise there is nothing to prove). Select $n \in \mathbb{N}$ big enough so that

$$0 < c = \frac{\omega}{2^{n+1}} < H_k^+.$$

Then the logarithmic function $\psi^+$ is well defined and satisfies the inequalities

$$\psi^+ \le n \ln(2) \quad \text{since } \frac{H_k^+}{H_k^+ - u + k + c} \le \frac{\frac{\omega}{4}}{c} = 2^n, \tag{4.27}$$

and, for $u \neq k + c$,

$$0 \le (\psi^+)' \le \frac{1}{H_k^+ - u + k + c} \le \frac{1}{c}, \tag{4.28}$$

and

$$|(\psi^+)'(u)|^{2-p^-} = (H_k^+ - u + k + c)^{p^- - 2} \le (\frac{\omega}{2})^{p^- - 2}. \tag{4.29}$$

In the logarithmic inequality (3.2) applied to the function $(u - k)_+$, let $x \mapsto \xi(x)$ be a smooth cutoff function defined in $K_R$ such that for some $\pi \in (0, 1)$

$$0 \le \xi \le 1 \text{ in } K_R, \quad \xi = 1 \text{ on } K_{(1-\pi)R},$$
$$|\frac{\partial \xi}{\partial x_i}| \le (\pi R)^{-\frac{p^+}{p_i^+}}, \text{for } i = 1, \dots, N.$$

Gathering these estimates in (3.2), using Lemma 4.5 and that

$$t^* - t \le \theta R^{p^+}, \tag{4.30}$$

we arrive at

$$\begin{aligned}
&\operatorname{ess\,sup}_{t_0<t<t^*} \int_{K_R \times \{t\}} [\psi^+(u)]^2 \xi^{p^+} dx \le \int_{K_R \times \{t_0\}} [\psi^+(u)]^2 \xi^{p^+} dx \\
&+ C \sum_{i=1}^N \int_{t_0}^{t*} \int_{K_R} \psi^+(u)[(\psi^+)'(u)]^{2-p_i(x,t)} |\frac{\partial \xi}{\partial x_i}|^{p_i(x,t)} \, dx \, dt \\
&\le n^2 (\ln 2)^2 \big(\frac{1 - \nu_0}{1 - \frac{\nu_0}{2}}\big)|K_R| + Cn \ln(2)(\frac{\omega}{2})^{p^- - 2}(\pi R)^{-p^+}(t^* - t_0)|K_R| \\
&\le n^2 (\ln 2)^2 \big(\frac{1 - \nu_0}{1 - \frac{\nu_0}{2}}\big)|K_R| + Cn \ln(2)(\frac{\omega}{2})^{p^- - 2}(\pi R)^{-p^+}\theta R^{p^+}|K_R| \\
&\le n^2 (\ln 2)^2 \big(\frac{1 - \nu_0}{1 - \frac{\nu_0}{2}}\big)|K_R| + C\frac{n}{\pi^{p^+}}|K_R|.
\end{aligned} \tag{4.31}$$

The left hand side is estimated below by integrating over the smaller set

$$S = \{x \in K_{(1-\pi)R} : u(x,t) > \mu^+ - \frac{\omega}{2^{n+1}}\} \subset K_R.$$

On such set, $\xi = 1$ and $\psi^+ \geq (n-1)\ln 2$, because

$$\frac{H_k^+}{H_k^+ - u + k + c} \geq \frac{\frac{\omega}{2}}{\frac{\omega}{2} - u + k + \frac{\omega}{2^{n+1}}} \geq \frac{\frac{\omega}{2}}{\frac{\omega}{2^n}} \geq 2^{n-1},$$

since one has $-u + \mu^+ < \omega/2^n$. Therefore for all $t \in (t_0, t^*)$,

$$|S| \leq \big\{ \big(\frac{n}{n-1}\big)^2 \big(\frac{1-\nu_0}{1-\frac{\nu_0}{2}}\big) + \frac{C}{n\pi^{p^+}} \big\} |K_R|.$$

Consequently, for all $t \in (t_0, t^*)$,

$$\begin{aligned} &|\{x \in K_R : u(x,t) > \mu^+ - \frac{\omega}{2^{n+1}}\}| \\ &\leq |S| + N\pi |K_R| \\ &\leq \big\{ \big(\frac{n}{n-1}\big)^2 \big(\frac{1-\nu_0}{1-\frac{\nu_0}{2}}\big) + \frac{c}{n\pi^{p^+}} + N\pi \big\} |K_R|. \end{aligned} \tag{4.32}$$

The proof will be complete once we choose $\pi$ so small that $N\pi \leq \frac{3}{8}\nu_0^2$, then n so large that

$$\frac{C}{n\pi^{p^+}} \leq \frac{3}{8}\nu_0^2 \quad \text{and} \quad \big(\frac{n}{n-1}\big)^2 \leq (1 - \frac{\nu_0}{2})(1+\nu_0) < 1,$$

and finally take $s_3 = n + 1$. □

Recalling that $t_0 \in [t^* - \theta R^{p^+}, t^* - \frac{\nu_0}{2}\theta R^{p^+}]$ and choosing $\lambda$ so that $2^{(\lambda-1)(p^- - 2)} \geq 2$, the previous lemma immediately implies the following lemma.

**Lemma 4.7.** *There exists $1 < s_3 \in \mathbb{N}$, depending on the data and $\omega$, such that for all $t \in (-\frac{a_0}{2}R^{p^+}, 0)$,*

$$|\{x \in K_R,\ u(x,t) > \mu^+ - \frac{\omega}{2^{s_3}}\}| \leq (1 - (\frac{\nu_0}{2})^2)|K_R|. \tag{4.33}$$

From Lemma 4.7 we deduce that within the cylinder $Q(a_0 R^{p^+}, R)$, the set where $u$ is close to its supremum is arbitrarily small.

**Lemma 4.8.** *For every $v_1 \in (0,1)$, there exists $s_3 \leq \lambda \in \mathbb{N}$ depending on the data and $\omega$, such that*

$$|\{(x,t) \in Q(\frac{a_0}{2}R^{p^+}, R) : u(x,t) > \mu^+ - \frac{\omega}{2^\lambda}\}| \leq \nu_1 |Q(\frac{a_0}{2}R^{p^+}, R)|. \tag{4.34}$$

*Proof.* Consider the cylinder $Q(a_0 R^{p^+}, 2R)$ and the level $k = \mu^+ - \frac{\omega}{2^s}$, for $s_3 \leq s \leq \lambda$, consider also the local energy estimates (3.1) for the functions $(u-k)_+$, where $0 \leq \xi(x,t) \leq 1$ is a smooth cutoff function defined in $Q(a_0 R^{p^+}, 2R)$ and satisfying

$$\begin{gathered} \xi = 1 \text{ in } Q(\frac{a_0}{2}R^{p^+}, R), \quad \xi = 0 \text{ on } \partial_p Q(a_0 R^{p^+}, 2R), \\ |\frac{\partial \xi}{\partial x_i}| \leq \frac{1}{R^{p^+/p_i^+}} \text{ for } i = 1, \dots, N, \quad 0 < \xi_t \leq \frac{2}{a_0 R^{p^+}}. \end{gathered}$$

Neglecting the first term on the left hand side of (3.1), and using the indicated choices, we obtain

$$
\begin{aligned}
&\sum_{i=1}^{N}\iint_{Q(\frac{a_0}{2}R^{p^+},R)}|\frac{\partial}{\partial x_i}(u-k)_+|^{p^-}\,dx\,dt\\
&\le C\Big(\sum_{i=1}^{N}\iint_{Q(a_0R^{p^+},2R)}(u-k)_+^{p_i^+}|\frac{\partial \xi}{\partial x_i}|^{p_i^+}\\
&\quad+\iint_{Q(a_0R^{p^+},2R)}(u-k)_+^2\xi_t\,dx\,dt\\
&\quad+\iint_{Q(a_0R^{p^+},2R)}\chi((u-k)_+>0)\,dx\,dt\Big)\\
&\le C\Big(\frac{1}{R^{p^+}}\sum_{i=1}^{N}\iint_{Q(\frac{a_0}{2}R^{p^+},R)}(u-k)_+^{p_i^+}\,dx\,dt\\
&\quad+\frac{1}{a_0R^{p^+}}\iint_{Q(\frac{a_0}{2}R^{p^+},R)}(u-k)_+^2\,dx\,dt\\
&\quad+\iint_{Q(\frac{a_0}{2}R^{p^+},R)}\chi((u-k)_+>0)\,dx\,dt\Big)\\
&\le C\Big(\frac{1}{R^{p^+}}\big(\frac{\omega}{2^s}\big)^{p^+}+\frac{1}{R^{p^+}}\big(\frac{\omega}{2^\lambda}\big)^{p^--2}\big(\frac{\omega}{2^s}\big)^2+1\Big)|Q(\frac{a_0}{2}R^{p^+},R)|\\
&\le \frac{C}{R^{p^+}}\big(\frac{\omega}{2^s}\big)^{p^+}|Q(\frac{a_0}{2}R^{p^+},R)|.
\end{aligned}
\tag{4.35}
$$

Here, we used (2.7) and the fact that $s\le\lambda$.

Next, for each $s\le\lambda$, introduce the two complimentary sets

$$
A_s(t)=\{x\in K_R: u(x,t)>\mu^+-\frac{\omega}{2^s}\},
$$
$$
K_R-A_s(t)=\{x\in K_R: u(x,t)\le\mu^+-\frac{\omega}{2^s}\},
$$

and let

$$
A_s=\int_{-a_0R^{p^+}/2}^{0}A_s(t)\,dt.
$$

Now, consider the doubly truncated function such that for all $t\in(-\frac{a_0}{2}R^{p^+},0)$

$$
v_s=\begin{cases}0 & \text{for } u(x,t)<\mu^+-\frac{\omega}{2^s},\\ u-(\mu^+-\frac{\omega}{2^s}) & \text{for } \mu^+-\frac{\omega}{2^s}\le u(x,t)\le\mu^+-\frac{\omega}{2^{s+1}},\\ \frac{\omega}{2^{s+1}} & \text{for } \mu^+-\frac{\omega}{2^{s+1}}\le u(x,t).\end{cases}
\tag{4.36}
$$

By construction $v_s$ vanishes on $K_R-A_s(t)$. Selecting two points $x=(x_1,\dots,x_N,t)$ in $A_s$ and $y=(y_1,\dots,y_N,t)$ in $K_R-A_s(t)$, we construct a polygonal joining $x$ and $y$ with sides parallel to the coordinate axes, for example $P_N=x$ and

$$
P_{N-1}=(x_1,\dots,x_{N-1},y_N),\quad P_{N-2}=(x_1,\dots,y_{N-1},y_N),\dots,
$$
$$
P_1=(x_1,y_2,\dots,y_N),\quad P_0=(y_1,\dots,y_N).
$$

By elementary computations, we have

$$
v_s(x,t)=[v_s(P_N,t)-v_s(P_{N-1},t)]+\dots+[v_s(P_1,t)-v_s(P_0,t)]
$$

$$\begin{aligned}
&= \int_{y_N}^{x_N} \frac{\partial}{\partial x_N} v_s(x_1, \dots, x_{N-1}, \zeta, t)\, d\zeta \\
&\quad + \int_{y_{N-1}}^{x_{N-1}} \frac{\partial}{\partial x_{N-1}} v_s(x_1, \dots, x_{N-2}, \zeta, y_N, t)\, d\zeta + \dots. \\
&\quad + \int_{y_1}^{x_1} \frac{\partial}{\partial x_1} v_s(\zeta, y_2, \dots, y_N, t)\, d\zeta \\
&\le \sum_{i=1}^{N} \int_{-R}^{R} |\frac{\partial}{\partial x_i} v_s(x_1, \dots, \zeta, \dots, y_N, t)|\, d\zeta.
\end{aligned}$$

Integrate in $dx$ over $A_s(t)$ and in $dy$ over $K_R - A_s(t)$, and take into account Lemma 4.7 to obtain

$$\big(\frac{\nu_0}{2}\big)^2 |K_R| \int_{K_R} v_s dx \le 2R|K_R| \sum_{i=1}^{N} \int_{K_R} |\frac{\partial v_s}{\partial x_i}| dx.$$

Therefore, by the definitions of $A_s(t)$ and $v_s$, we have

$$\frac{\omega}{2^{s+1}} |A_{s+1}(t)| \le \frac{C\ R}{\nu_0^2} \sum_{i=1}^{N} \int_{A_s(t)-A_{s+1}(t)} |\frac{\partial u}{\partial x_i}| dx.$$

Integrating for $t \in (-\frac{a_0}{2} R^{p^+}, 0)$, and using (4.35) we conclude that

$$\begin{aligned}
\frac{\omega}{2^{s+1}} |A_{s+1}| &\le \frac{C\ R}{\nu_0^2} \sum_{i=1}^{N} \iint_{A_s - A_{s+1}} |\frac{\partial u}{\partial x_i}|\, dx\, dt \\
&\le \frac{CR}{\nu_0^2} \sum_{i=1}^{N} \iint_{A_s} |\frac{\partial u}{\partial x_i}|^{p^-}\, dx\, dt)^{1/p^-} |A_s - A_{s+1}|^{\frac{p^- - 1}{p^-}} \qquad (4.37) \\
&\le \frac{C}{\nu_0^2} \big(\frac{\omega}{2^s}\big)^{p^+/p^-} |Q(\frac{a_0}{2} R^{p^+}, R)|^{1/p^-} |A_s - A_{s+1}|^{\frac{p^- - 1}{p^-}}.
\end{aligned}$$

If $s$ is large enough so that $\big(\frac{\omega}{2^s}\big)^{\frac{p^+}{p^-}} \frac{2^{s+1}}{\omega} < 1$, from (4.37) we obtain

$$|A_{s+1}| \le \frac{C}{\nu_0^2} |Q(\frac{a_0}{2} R^{p^+}, R)|^{1/p^-} |A_s - A_{s+1}|^{\frac{p^- - 1}{p^-}}, \qquad (4.38)$$

for all $s_3 \le s \le \lambda$. According to the previous energy estimates we obtain, for $s = s_3, s_3 + 1, \dots, \lambda - 1$,

$$|A_{s+1}|^{\frac{p^-}{p^- - 1}} \le C(\nu_0)^{\frac{-2p^-}{p^- - 1}} |Q(\frac{a_0}{2} R^{p^+}, R)|^{\frac{1}{p^- - 1}} |A_s - A_{s+1}|,$$

and we then add these inequalities for $s = s_3, s_3 + 1, \dots, \lambda - 1$.

Since $\mu^+ - \frac{\omega}{2^{s+1}} \le \mu^+ - \frac{\omega}{2^\lambda}$, and $A_{s+1} \ge A_\lambda$, we have

$$\sum_{s=s_3}^{\lambda-1} A_{s+1}^{\frac{p^-}{p^- - 1}} \ge (\lambda - s_3) A_\lambda^{\frac{p^-}{p^- - 1}}.$$

Also note that $\sum_{s=s_3}^{\lambda-1} |A_s - A_{s+1}| \le |Q(\frac{a_0}{2} R^{p^+}, R)|$. Collecting these results, we arrive at

$$A_\lambda \le \frac{C}{(\lambda - s_3)^{\frac{p^- - 1}{p^-}}} (\nu_0)^{-2} |Q(\frac{a_0}{2} R^{p^+}, R)|.$$

The proof is complete once we choose $s_3 < \lambda \in \mathbb{N}$ sufficiently large so that

$$\frac{C}{(\lambda - s_3)^{\frac{p^- -1}{p^-}}} (\nu_0)^{-2} \le \nu_1.$$

□

**Lemma 4.9.** *The number $v_1 \in (0,1)$ can be chosen (and consequently $\lambda$), such that*

$$u(x,t) \le \mu^+ - \frac{\omega}{2^{\lambda+1}} \quad a.e.\ (x,t) \in Q\big(\frac{a_0}{2}(\frac{R}{2})^{p^+}, R\big). \tag{4.39}$$

*Proof.* Define two decreasing sequences of positive numbers

$$R_n = \frac{R}{2} + \frac{R}{2^{n+1}}, \quad k_n = \mu^+ - \frac{\omega}{2^{\lambda+1}} - \frac{\omega}{2^{\lambda+1+n}}, \quad n = 0, 1, \dots.$$

Now, consider the local energy estimates (3.1) for the functions $(u - k_n)_+$ over the constructed family of nested and shrinking cylinders $Q(\frac{a_0}{2} R_n^{p^+}, R_n)$, where $0 \le \xi_n(x,t) \le 1$ are smooth functions defined in $Q(\frac{a_0}{2} R_n^{p^+}, R_n)$ such that

$$\begin{gathered}
\xi_n = 1 \text{ in } Q(\frac{a_0}{2} R_{n+1}^{p^+}, R_{n+1}), \quad \xi_n = 0 \text{ on } \partial_p Q(\frac{a_0}{2} R_n^{p^+}, R_n), \\
|\frac{\partial \xi_n}{\partial x_i}| \le (\frac{2^{n+1}}{R})^{\frac{p^+}{p_i^+}} \text{ for } i = 1, \dots, N, \quad 0 < (\xi_n)_t \le \frac{2^{p^+(n+1)}}{\frac{a_0}{2} R^{p^+}}.
\end{gathered}$$

Since $(u - k_n)_+^2 \ge a_0 (u - k_n)_+^{p^-}$, we obtain

$$\begin{aligned}
& a_0 \operatorname{ess\,sup}_{-\frac{a_0}{2} R_n^{p^+} < t < 0} \int_{K_{R_n} \times \{t\}} (u - k_n)_+^{p^-} \xi_n^{p^+} \, dx \\
& + \sum_{i=1}^N \int_{-\frac{a_0}{2} R_n^{p^+}}^0 \int_{K_{R_n}} |\frac{\partial}{\partial x_i} (u - k_n)_+|^{p_i^-} \xi_n^{p^+} \, dx \, dt \\
& \le C \frac{2^{p^+(n+1)}}{R^{p^+}} \Big( \frac{1}{a_0} \int_{-\frac{a_0}{2} R_n^{p^+}}^0 \int_{K_{R_n}} (u - k_n)_+^2 \, dx \, dt \\
& \quad + \sum_{i=1}^N \int_{-\frac{a_0}{2} R_n^{p^+}}^0 \int_{K_{R_n}} (u - k_n)_+^{p_i(x,t)} \, dx \, dt \\
& \quad + \int_{-\frac{a_0}{2} R_n^{p^+}}^0 \int_{K_{R_n}} \chi((u - k_n)_+ > 0) \, dx \, dt \Big) \\
& \le C \frac{2^{p^+(n+1)}}{R^{p^+}} \big( \frac{\omega}{2^\lambda} \big)^{p^+} \int_{-\frac{a_0}{2} R_n^{p^+}}^0 \int_{K_{R_n}} \chi((u - k_n)_+ > 0) \, dx \, dt.
\end{aligned}$$

We divide by $a_0$ throughout the above inequality, and introduce the change of variable $\widetilde{t} = \frac{t}{\frac{a_0}{2}}$. Using the same tools as in Lemma 4.1, we arrive at the inequality

$$\big( \frac{\omega}{2^{\lambda+2+n}} \big)^{\overline{p}} A_{n+1} \le C \frac{2^{np^+}}{R^{p^+}} \big( \frac{\omega}{2^\lambda} \big)^{p^+} A_n^{1+\frac{\overline{p}}{N}} \tag{4.40}$$

where

$$A_n = \iint\limits_{Q(R_n^{p^+}, R_n)} \chi((\widetilde{u} - k_n)_+ > 0) \, dx \, d\widetilde{t},$$

here we considered $\widetilde{u}(x,\bar{t}) = u(x,t)$ and $\widetilde{\xi_n}(x,\widetilde{t}) = \xi_n(x,t)$. Next, if we denote $X_n = \frac{A_n}{|Q(R_n^{p^+}, R_n)|}$, then we obtain

$$X_{n+1} \le C4^{np^+}\big(\frac{\omega}{2^\lambda}\big)^{p^+-\overline{p}} X_n^{1+\frac{\overline{p}}{N}}.$$

Therefore, by using [12, Lemma 4.1 in Chapter I], the result is proved if we assume that

$$X_0 \le \big[C\big(\frac{\omega}{2^\lambda}\big)^{p^+-\overline{p}}\big]^{-N/\overline{p}} 4^{-p^+(\frac{N}{\overline{p}})^2} = \nu_1. \tag{4.41}$$

For this value of $\nu_1$, Lemma 4.8 implies that $X_0 \le \nu_1$. Hence, we can conclude that $X_n \to 0$ when $n \to +\infty$ and the result follows. □

As an immediate consequence we obtain the reduction of the oscillation of $u$ in the second case.

**Corollary 4.10.** *There exists a constant $\sigma_1 \in (0,1)$, depending only on the data and $\omega$, such that if (2.9) holds then*

$$\operatorname{ess\,osc}_{Q(\frac{a_0}{2}(\frac{R}{2})^{p^+}, \frac{R}{2})} u \le \sigma_1\omega. \tag{4.42}$$

The proof of the above corollary follows by choosing $\sigma_1 = 1 - \frac{1}{2^{\lambda+1}}$. Now, we are able to prove Proposition 2.5, recalling the conclusions of Corollaries 4.4 and 4.10 and since $\theta(\frac{R}{8})^{p^+} \le \frac{a_0}{2}(\frac{R}{2})^{p^+}$, we obtain that

$$\operatorname{ess\,osc}_{Q(\theta(\frac{R}{8})^{p^+}, \frac{R}{8})} u \le \sigma\omega,$$

where $\sigma = \max\{\sigma_0, \sigma_1\}$.

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