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AN ESTIMATE FOR SOLUTIONS TO THE SCHRÖDINGER EQUATION

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Abstract. In this note, we find a priori estimates in the L_2-norm for solutions to the Schrödinger equation with a parameter. It is shown that a constant occuring in the inequality does not depend on the value of the parameter. In particular, the estimate is valid for eigenfunctions associated with the Schrödinger operator with arbitrary boundary conditions.

1. Introduction

Asymptotic properties of solutions to the Schrödinger equation of second order elliptic equation with a parameter have been investigated in many papers; see for instance [3, 7, 9, 10, 14]. In particular, eigenfunctions and functions associated with the Schrödinger operator have been considered with various boundary conditions. The authors of these papers have studied the general elliptic operator of second order

$$\mathbb{L}u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}[a_{ij}(x)\frac{\partial u}{\partial x_j}] + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u$$

(or the Schrödinger operator) in an arbitrary domain G of $\mathbb{R}^n$.

Let $u^0(x)$ be a regular solution of the equation $\mathbb{L}u^0 + \lambda u^0 = 0$, and let $u^m(x)$ be a regular solution of the equation $\mathbb{L}u^m + \lambda u^m = u^{m-1}$ (m=1,2,...). Under some smoothness conditions on the coefficients and restrictions on the range of the spectral parameter λ, the following estimated has been established:

$$\|u^m\|_{L_p(K)} \le C|\lambda|^{\delta(m,n,p,q)}\|u^m\|_{L_q(K')} \tag{1.1}$$

where $1 \le q \le p \le \infty$, the constant C depends on the coefficients of the operator and on compact sets K and K' (with $K, K' \subset G$). When $n > 1$, K must lie strictly inside K'. However, when $n = 1$ this condition is omitted [7, 14]; i.e. K' can coincide with K and even K' can lie inside K. As done in [2], estimates of the type (1.1) can be applied to study the convergence of the spectral expansions corresponding to nonselfadjoint differential operators.

The main objective of the present paper is obtaining an estimate of the type (1.1) in the multidimensional case when the condition $K \subset K'$ is not satisfied. The methods used here are different from those used previously. To estimate a solution

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of an elliptic equation, we investigate a corresponding hyperbolic equation. This approach gives the possibility of using energy estimates for solutions of hyperbolic equations, the theorem on domain of dependence, and the generalized Kirchoff formula for the solution of the Cauchy problem. As is well known, the Kirchoff formula has a different form for n odd and for n even. When n is odd, the domain of dependence is a sphere, and not a ball. Since, we use this fact in our proof, the assumption that n is odd cannot be omitted. This leaves open the question of finding similar estimates for n even.

2. The main result

We consider the Schrödinger operator

$$Lu = \Delta u - q(x)u$$

defined on a bounded domain $\Omega \subset \mathbb{R}^n$ for odd $n > 1$. Here q is a complex-valued function continuous on the closure of Ω, $\bar{\Omega}$. Let d be the diameter of Ω, let λ be a complex number, and let

$$q_0 = \max_{x\in\bar{\Omega}} |q(x)|.$$

For $m = 0, 1, \dots$, let u^m be twice continuously differentiable functions on Ω satisfying

$$Lu^0 + \lambda u^0 = 0, \tag{2.1}$$

$$Lu^m + \lambda u^m = u^{m-1}. \tag{2.2}$$

Let $B(x, R)$ be the ball of radius R and centered at x in $\mathbb{R}^n$, and let $\partial\Omega$ denote the boundary of Ω. For x in Ω with $3R < \operatorname{dist}(x, \partial\Omega)$, we have the following an a priori estimate.

Theorem 2.1. *For any complex number λ, and any positive real numbers R, ε such that $3R + \varepsilon < \operatorname{dist}(x, \partial\Omega)$, we have*

$$\|u^m\|_{L_2(B(x,R))} \le C\|u^m\|_{L_2(B(x,3R+\varepsilon)\setminus B(x,R))} \tag{2.3}$$

for $m = 0, 1, \dots$ where $C = C(n, q_0, d, \varepsilon, m)$.

Proof. We proceed by induction. Consider $m = 0$ and set $K = B(x, R)$, $K_0 = B(x, R+\varepsilon/8)$, $K_0' = B(x, R+3\varepsilon/8)$, $K' = B(x, 3R+5\varepsilon/8)$, $K'' = B(x, 3R+7\varepsilon/8)$ and $\hat{K} = B(x, 3R+\varepsilon)$. Let η and η_0 be cut off functions satisfying

$$\eta(x) = \begin{cases} 1, & \text{if } x \in K' \\ 0, & \text{if } x \notin K'', \end{cases} \quad \eta_0(x) = \begin{cases} 1, & \text{if } x \in K' \setminus K_0' \\ 0, & \text{if } x \notin K'' \text{ or } x \in K_0\,, \end{cases}$$

$\eta(x) = \eta_0(x)$ if $x \in K'' \setminus K'$, $\eta, \eta_0 \in C^\infty(\mathbb{R}^n)$, $0 \le \eta, \eta_0 \le 1$.

Let $\mu = \sqrt{\lambda}$ where $-\pi/2 < \arg\mu \le \pi/2$ and consider the function

$$\omega(t) = \begin{cases} e^{-i\mu t}, & \text{if } \operatorname{Im}\mu \ge 0 \\ e^{i\mu t}, & \text{if } \operatorname{Im}\mu < 0\,. \end{cases}$$

Clearly $\omega(0) = 1$, $|\omega'(0)| = |\mu|$ and $|\omega(t)| = e^{|Im\mu|t}$. Define the operator $\hat{L}$ by

$$\hat{L}\phi = \frac{\partial^2\phi}{\partial t^2} - \Delta\phi \tag{2.4}$$

for $(x,t) \in \mathbb{R}^{n+1}$. From (2.4) it follows that for all $(x,t) \in \Omega \times \mathbb{R}$,
$$\hat{L}(\omega(t)u^0(x)) = -q(x)u^0(x)\omega(t)\,. \tag{2.5}$$
Consider the following three Cauchy problems:
$$\begin{gathered} \hat{L}\phi = \hat{L}(\omega\eta u^0) \\ \phi(x,0) = \eta(x)u^0(x), \quad \phi_t(x,0) = \omega'(0)\eta(x)u^0(x)\,, \end{gathered} \tag{2.6}$$
$$\begin{gathered} \hat{L}\phi = -q\eta u^0\omega \\ \phi(x,0) = \eta_0(x)u^0(x), \quad \phi_t(x,0) = \omega'(0)\eta_0(x)u^0(x)\,, \end{gathered} \tag{2.7}$$
$$\begin{gathered} \hat{L}\phi = -q\eta u^0\omega \\ \phi(x,0) = \eta(x)u^0(x), \quad \phi_t(x,0) = \omega'(0)\eta(x)u^0(x)\,. \end{gathered} \tag{2.8}$$
Since $\hat{L}$ is the wave operator, from a results in [5], solutions to problems (2.6), (2.7), and (2.8) exist and are unique. Moreover the solution ϕ_2 of problem (2.7) satisfies the estimate
$$\begin{aligned} &\max_{0\le t\le T}(\|\phi_2(x,t)\|_{W_2^1(\mathbb{R}^n)} + \|\frac{\partial}{\partial t}\phi_2(x,t)\|_{L_2(\mathbb{R}^n)}) \\ &\le e^{c_1T}(\|\phi_2(x,0)\|_{W_2^1(\mathbb{R}^n)} + \|\frac{\partial}{\partial t}\phi_2(x,0)\|_{L_2(\mathbb{R}^n)} + \int_0^T |\omega(t)|dt\|q\eta u^0\|_{L_2(\mathbb{R}^n)}). \end{aligned}$$
This inequality and the definition of the cut off functions imply that
$$\begin{aligned} &\max_{0\le t\le T}(\|\phi_2(x,t)\|_{W_2^1(\mathbb{R}^n)} + \|\frac{\partial}{\partial t}\phi_2(x,t)\|_{L_2(\mathbb{R}^n)}) \\ &\qquad\le e^{c_1T}(\|\eta_0u^0\|_{W_2^1(K''\setminus K_0)} + |\mu|\|u^0\|_{L_2(K''\setminus K_0)} \\ &\qquad\quad + Tq_0e^{|Im\mu|T}\|u^0\|_{L_2(K'')}) \end{aligned} \tag{2.9}$$
where $T > 0$ is arbitrary. Let ϕ_1 and ϕ_3 be the solutions of (2.6) and (2.8), respectively. From (2.5) and the theorem on domain of dependance (see [5]) we have
$$\phi_1(x,t) = \phi_3(x,t) \tag{2.10}$$
for all $(x,t) \in K_0\times[0,\mathrm{dist}(K_0,\partial K')]$. Since $\omega(t)\eta(x)u^0(x)$ is the solution of problem (2.6) for $(x,t) \in \mathbb{R}^n \times [0,\infty)$, it follows that
$$\phi_3(x,t) = \omega(t)\eta(x)u^0(x) \tag{2.11}$$
for all $(x,t) \in K_0 \times [0,\mathrm{dist}(K_0,\partial K')]$.

It is easy to see that $\phi_3(x,t) - \phi_2(x,t)$ is a generalized solution of the Cauchy problem
$$\begin{gathered} \hat{L}\phi = 0 \\ \phi(x,0) = (\eta(x)-\eta_0(x))u^0(x), \quad \phi_t(x,0) = \omega'(0)(\eta(x)-\eta_0(x))u^0(x) \end{gathered} \tag{2.12}$$
in $\mathbb{R}^n \times [0,\infty)$. Note that the function $\eta - \eta_0$ is non zero only on K_0'. Let $u_h^0 = h^{-n}\int_{\mathbb{R}^n} u^0(y)\gamma(\frac{x-y}{h})dy$ where $\gamma \in C^\infty(\mathbb{R}^n)$, $0 \le \gamma \le 1$, $\gamma(y) = 0$ for $|y| \ge 1$, $\int_{\mathbb{R}^n}\gamma(y)dy = 1$, and we set $u^0(y) = 0$ for all $y \notin \hat{K}$. Let $\tilde{\phi}_h$ be the solution of the Cauchy problem
$$\begin{gathered} \hat{L}\phi = 0 \\ \phi(x,0) = (\eta(x)-\eta_0)u_h^0(x), \quad \phi_t(x,0) = \omega'(0)(\eta(x)-\eta_0(x))u_h^0(x) \end{gathered} \tag{2.13}$$

Since $(\eta-\eta_0)u_h^0 \in C^\infty(\mathbb{R}^n)$ it follows (see [11]) that $\tilde\phi_h$ is the classical solution of problem (13) and satisfies the generalized Kirchoff formula (see [11])

$$\tilde\phi_h(x,t)=\frac{\partial}{\partial t}Q_{\psi_0,h}(x,t)+Q_{\psi_1,h}(x,t)$$

where

$$\psi_{0,h}(x)=(\eta(x)-\eta_0(x))u_h^0(x),\quad \psi_{1,h}(x)=\omega'(0)(\eta(x)-\eta_0(x))u_h^0(x),$$
$$Q_\psi(x,t)=\frac{1}{2(2\pi)^{l+1}}\sum_{j=0}^{l}\frac{\partial^j}{\partial t^j}\frac{P_l^{(l-j)}(1)}{t^{2l+1-j}}\int_{S_t}\psi(y)dS_y$$

where S_t is the surface of the n–dimensional hypersphere of radius t and centre x, $l=(n-3)/2$, and P_l is the Legendre polynomial of degree l. Hence $\tilde\phi_h(x,t)=0$ for all $(x,t)\in K_0\times[2R+\varepsilon/2,\infty)$.

It follows from [13, Ch. 3, §18] that the solution $\phi_3-\phi_2$ of problem (2.12) is the limit of $\tilde\phi_h$ as $h\to 0$ and thus $\phi_3(x,t)=\phi_2(x,t)$ for all $(x,t)\in K_0\times[2R+\varepsilon/2,\infty)$. From this, (2.10), and since $\operatorname{dist}(K_0,\partial K')=2R+\varepsilon/2$ it follows that $\phi_2(x,t_0)=\phi_1(x,t_0)$ for all $x\in K_0$ and $t_0=2R+\varepsilon/2$. From this, (2.9) and (2.11) it follows that

$$\begin{aligned}&|\mu|e^{|Im\mu|t_0}\|u^0\|_{L_2(K_0)}\\ &\le e^{c_1t_0}(\|\eta_0u^0\|_{W_2^1(K''\setminus K_0)}+|\mu|\|u^0\|_{L_2(K''\setminus K_0)}\\ &\quad+q_0t_0e^{|Im\mu|t_0}\|u^0\|_{L_2(K'')}).\end{aligned}\tag{2.14}$$

Since the compact set $K''\setminus(K_0\setminus\partial K_0)$ lies strictly inside the compact set $\hat K\setminus(K\setminus\partial K)$ it follows from [8, page 95] that

$$\|\eta_0u^0\|_{W_2^1(K''\setminus K_0)}\le\|\eta_0\|_{C^1(K''\setminus K_0)}\|u^0\|_{W_2^1(K''\setminus K_0)}\le c_2|\mu|\|u^0\|_{L_2(\hat K\setminus K)}.\tag{2.15}$$

Moreover, from (2.14) and (2.15) we obtain

$$\begin{aligned}\|u^0\|_{L_2(K)}&\le e^{c_1d}(c_3\|u^0\|_{L_2(\hat K\setminus K)}+\frac{dq_0}{|\mu|}\|u^0\|_{L_2(\hat K)})\\ &\le e^{c_1d}(c_3\|u^0\|_{L_2(\hat K\setminus K)}+\frac{dq_0}{|\mu|}\|u^0\|_{L_2(K)}+\frac{dq_0}{|\mu|}\|u^0\|_{L_2(\hat K\setminus K)}).\end{aligned}$$

If $|\mu|>2dq_0e^{c_1d}$ then (2.3) follows.

Now assume that the conclusion of Theorem 2.1 holds for $0\le k\le m-1$, $|\mu|>M$ where M is a constant. We show that it holds if $k=m$, $|\mu|>\tilde M$ where $\tilde M$ is sufficiently large. Consider the case $\operatorname{Im}\mu<0$. We define the function

$$\omega^m(x,t)=\sum_{j=0}^{m}e^{i\mu t}P_j(t,\mu)u^{m-j}(x),$$

where

$$P_j(t,\mu)=\frac{1}{2^{j+1}j!\mu^j}\sum_{k=0}^{j-1}\frac{i^{k-j}(j+k-1)!t^{j-k}}{2^kk!(j-k-1)!\mu^k},$$

for $1\le j\le m$, and $P_0(t,\mu)=1/2$. It is easy to show that $\omega^m(x,0)=u^m(x)/2$, $\omega_t^m(x,0)=\mu F(x)$, where

$$F(x)=\sum_{j=0}^{m}\frac{(2j-2)!u^{m-j}(x)}{i4^jj!(j-1)!\mu^{2j}};$$

here we formally set $(-1)! = -1$, and $(-2)! = 1/2$. An easy calculation gives

$$\hat{L}(\omega^m) = -q\omega^m,$$

for all $x \in \Omega$ and $t > 0$.

An argument similar to that used in the derivation of (2.14) gives the estimate

$$\begin{aligned} \|\omega_t^m(x,t_0)\|_{L_2(K_0)} \le e^{c_1 t_0}(\|\eta_0 u^m\|_{W_2^1(K''\setminus K_0)} + |\mu|\|F\|_{L_2(K''\setminus K_0)} \\ + q_0 \int_0^{t_0} \|\omega^m(x,t)\|_{L_2(K'')} dt) \end{aligned} \tag{2.16}$$

where $t_0 = 2R + \varepsilon/2$. It follows from [4, Th. 1] that

$$\begin{aligned} q_0 \int_0^{t_0} \|\omega^m(x,t)\|_{L_2(K'')} dt &\le q_0 t_0 e^{|Im\mu| t_0} \max_{0\le t\le t_0} \|\sum_{j=0}^m P_j(t,\mu) u^{m-j}(x)\|_{L_2(K'')} \\ &\le c_2 q_0 t_0 e^{|Im\mu|t_0} \|u^m\|_{L_2(\hat{K})}. \end{aligned} \tag{2.17}$$

Moreover, as in [4], we obtain

$$\|F\|_{L_2(K''\setminus K_0)} \le c_3 \|u^m\|_{L_2(\hat{K}\setminus K)}. \tag{2.18}$$

Since the compact set $K''\setminus(K_0\setminus\partial K_0)$ lies strictly inside the compact set $\hat{K}\setminus(K\setminus\partial K)$ it follows as in [8] that

$$\|\eta_0 u^m\|_{W_2^1(K''\setminus K_0)} \le \|\eta_0\|_{C^1(K''\setminus K_0)} \|u^m\|_{W_2^1(K''\setminus K_0)} \le c_4 |\mu| \|u^m\|_{L_2(\hat{K}\setminus K)}. \tag{2.19}$$

Using (2.16)–(2.19), we obtain

$$\|\omega_t^m(x,t_0)\|_{L_2(K_0)} \le e^{c_1 t_0}(c_5|\mu|\|u^m\|_{L_2(\hat{K}\setminus K)} + c_6 q_0 t_0 e^{|Im\mu|t_0} \|u^m\|_{L_2(\hat{K})}). \tag{2.20}$$

An easy calculation gives

$$\omega_t^m(x,t) = \sum_{j=0}^m e^{i\mu t} R_j(t) u^{m-j}(x) \tag{2.21}$$

where

$$R_j(t) = \frac{1}{2^{j+1} j! \mu^j} \sum_{k=0}^j \frac{i^{k-j-1}(j+k-2)!(k+j-(j-k)^2)t^{t-k}}{2^k k!(j-k)!\mu^{k-1}}$$

for $1 \le j \le m$, and $R_0(t) = i\mu/2$. From (2.20) and (2.21) it follows that

$$\begin{aligned} \|u^m\|_{L_2(K_0)} \le c_6 \sum_{j=1}^m |\mu|^{-j} \|u^{m-j}\|_{L_2(K_0)} \\ + e^{c_1 t_0}(c_8 \|u^m\|_{L_2(\hat{K}\setminus K)} + c_8 q_0 t_0 |\mu|^{-j} \|u^m\|_{L_2(\hat{K})}). \end{aligned} \tag{2.22}$$

From the induction hypothesis and the a posteriori estimates in [4] we obtain

$$\sum_{j=1}^m |\mu|^{-j} \|u^{m-j}\|_{L_2(K_0)} \le c_9 \sum_{j=1}^m |\mu|^{-j} \|u^{m-j}\|_{L_2(K''\setminus K_0)} \le c_{10} \|u^m\|_{L_2(\hat{K}\setminus K)}. \tag{2.23}$$

Clearly

$$\|u^m\|_{L_2(\hat{K})} \le \|u^m\|_{L_2(K)} + \|u^m\|_{L_2(\hat{K}\setminus K)}. \tag{2.24}$$

From (2.22)–(2.24), it follows that

$$\|u^m\|_{L_2(K)} \le e^{c_{11} d}(\|u^m\|_{L_2(\hat{K}\setminus K)} + |\mu|^{-1}\|u^m\|_{L_2(K)} + |\mu|^{-1}\|u^m\|_{L_2(\hat{K}\setminus K)}).$$

If $|\mu| > \max(2e^{c_{11}d}, M)$ then (2.3) follows.

The case $\operatorname{Im}\mu \geq 0$ follows by a similar argument, provided we replace the function $\omega^m(x,t)$ by

$$\psi^m(x,t) = \sum_{j=0}^{m} e^{-i\mu t} Q_j(t,\mu) u^{m-j}(x),$$

where

$$Q_j(t,\mu) = \frac{1}{2^{j+1} j! \mu^j} \sum_{k=0}^{j-1} \frac{(-i)^{k-j}(j+k-1)! t^{j-k}}{2^k k!(j-k-1)!\mu^k}$$

for $1 \leq j \leq m$; here $Q_0(t,\mu) = 1/2$.

We now prove Theorem 2.1 for $|\lambda| \leq \lambda_0$ for arbitrary positive λ_0. Assume the conclusion of Theorem 2.1 fails for some m. Then there exists $\varepsilon > 0$ such that there exist sequences $C_k > 0$, $x_k \in \Omega$ and $R_k > 0$ with the following properties: $\lim_{k\to\infty} C_k = \infty$; $3R_k + \varepsilon < \operatorname{dist}(x_k, \partial\Omega)$; for each k there exist functions $u_k^l(x)$, $0 \leq l \leq m$, satisfying (2.1) and (2.2) and for which there exists λ_k with $|\lambda_k| \leq \lambda_0$ and

$$\|u_k^m\|_{L_2(B(x,R_k)} > C_k \|u_k^m\|_{L_2(B(x_k,3R_k+\varepsilon)\setminus B(x_k,R_k))}. \tag{2.25}$$

Since the domain Ω is bounded and $0 < R_k \leq d/3$, there exists a subsequence C_{k_p}, $p = 1, 2, \cdots$, such that $\lim_{p\to\infty} x_{k_p} = \tilde{x}$ and $\lim_{p\to\infty} R_{k_p} = \tilde{R}$, where $0 \leq \tilde{R} \leq d/3$ and $3\tilde{R} + \varepsilon \leq \operatorname{dist}(\tilde{x}, \partial\Omega)$. From this it follows that $\operatorname{dist}(\tilde{x}, \partial\Omega) \geq \varepsilon$.

Consider the balls $B(\tilde{x}, \tilde{R}+\sigma)$ and $B(\tilde{x}, 3(\tilde{R}+\sigma)+\varepsilon/2)$ where $\sigma = \varepsilon/100$. It is easy to see that for sufficiently large p_0 and any $p \geq p_0$ $B(x_{k_p}, R_{k_p}) \subset B(\tilde{x}, \tilde{R}+\sigma)$ and $B(\tilde{x}, 3(\tilde{R}+\sigma)+\varepsilon/2) \subset B(x_{k_p}, 3R_{k_p}+\varepsilon)$. From this it follows that

$$B(\tilde{x}, 3(\tilde{R}+\sigma)+\varepsilon/2) \setminus B(\tilde{x}, \tilde{R}+\sigma) \subset B(x_{k_p}, 3R_{k_p}+\varepsilon) \setminus B(x_{k_p}, R_{k_p}).$$

Thus we obtain

$$\|u_{k_p}^m\|_{L_2(B(\tilde{x},\tilde{R}+\sigma))} \geq \|u_{k_p}^m\|_{L_2(B(x_{k_p},R_{k_p}))}$$

and

$$\|u_{k_p}^m\|_{L_2(B(\tilde{x},3(\tilde{R}+\sigma)+\varepsilon/2)\setminus B(\tilde{x},\tilde{R}+\sigma))} \leq \|u_{k_p}^m\|_{L_2(B(x_{k_p},3R_{k_p}+\varepsilon)\setminus B(x_{k_p},R_{k_p}))}.$$

From the last two inequalities and from (2.25), it follows that

$$\|u_{k_p}^m\|_{L_2(B(\tilde{x},\tilde{R}+\sigma))} > C_{k_p} \|u_{k_p}^m\|_{L_2(B(\tilde{x},3(\tilde{R}+\sigma)+\varepsilon/2)\setminus B(\tilde{x},\tilde{R}+\sigma))}$$

for $p \geq p_0$.

Set $\tilde{K} = B(\tilde{x}, \tilde{R}+\sigma)$ and $\hat{K} = B(\tilde{x}, 3(\tilde{R}+\sigma)+\varepsilon/2)$. Thus there is a sequence of functions u_i^m, $i = 1, 2, \dots$ such that

$$\|u_i^m\|^2_{L_2(\tilde{K})} > \hat{C}_i^2 \|u_i^m\|^2_{L_2(\hat{K}\setminus\tilde{K})}, \tag{2.26}$$

where $\lim_{i\to\infty} \hat{C}_i = \infty$. Without loss of generality we may assume that

$$\|u_i^m\|_{L_2(\hat{K})} = 1 \tag{2.27}$$

for $i = 1, 2, \dots$. Set $K_j = B(\tilde{x}, \tilde{R}+\sigma+j\varepsilon/20)$ for $j = 1, \dots, 5$. Since $|\lambda_i| \leq \lambda_0$, it follows from (2.27) and [8, page 95] that

$$\|u_i^l\|_{W_2^1(K_5)} \leq c_1$$

for all i and $0 \leq l \leq m$. In view of this and [6, Ch. 3, §7] it follows that

$$\|u_i^l\|_{W_2^2(K_4)} \leq c_2. \tag{2.28}$$

It follows from this, [11, Th. 2.5.1], and Rellich's Lemma [12], that there exists a subsequence of u_i^0, $u_{i_j}^0$, $j = 1, 2, \dots$, such that $\lim_{j\to\infty} \lambda_{i_j} = \tilde{\lambda}$ and

$$\lim_{j\to\infty} \|u_{i_j}^0 - \tilde{u}^0\|_{W_2^1(K_3)} = 0$$

for some $\tilde{u}^0 \in W_2^1(K_3)$. It follows from this that $\tilde{u}^0$ is a generalized solution of

$$L\tilde{u}^0 + \tilde{\lambda}\tilde{u}^0 = 0$$

in $K_3 \setminus \partial K_3$. From this and [6, Ch. 3, §10] it follows that $\tilde{u}^0 \in W_2^2(K_2)$. Similarly we find a further subsequence of i_j denoted p_j, such that $\lim_{j\to\infty} \lambda_{p_j} = \tilde{\lambda}$ and

$$\lim_{j\to\infty} \|u_{p_j}^1 - \tilde{u}^1\|_{W_2^1(K_3)} = 0$$

for some $\tilde{u}^1 \in W_2^1(K_3)$. By a similar argument to that above it follows that $\tilde{u}^1$ is a generalized solution of

$$L\tilde{u}^1 + \tilde{\lambda}\tilde{u}^1 = \tilde{u}^0$$

in $K_3 \setminus \partial K_3$ and that $\tilde{u}^1 \in W_2^2(K_2)$. Repeating this argument we obtain $\tilde{u}^l \in W_2^2(K_2)$, $0 \le l \le m$ which are generalized solutions of

$$\begin{gathered} L\tilde{u}^{m-l} + \tilde{\lambda}\tilde{u}^{m-l} = \tilde{u}^{m-l-1}, \quad 0 \le l \le m-1, \\ L\tilde{u}^0 + \tilde{\lambda}\tilde{u}^0 = 0 \end{gathered} \tag{2.29}$$

in $K_3 \setminus \partial K_3$. From (2.26) and (2.27), it follows that

$$\lim_{i\to\infty} \|u_i^m\|_{L_2(\hat{K}\setminus\tilde{K})} = 0.$$

It follows that

$$\lim_{i\to\infty} \|u_i^m\|_{L_2(K_1\setminus\tilde{K})} = 0$$

and therefore $\|\tilde{u}^m\|_{L_2(K_1\setminus\tilde{K})} = 0$. From this and (2.29) it follows that

$$\|\tilde{u}^{m-l}\|_{L_2(K_1\setminus\tilde{K})} = 0, \tag{2.30}$$

for $1 \le l \le m$. Setting $l = m$ in (2.30) and using results from [1, pages 235-236] we obtain $\|\tilde{u}^0\|_{L_2(K_1)} = 0$. Setting $l = m-1$ in (2.30), from a results of [1, pages 235-236], it follows that $\|\tilde{u}^1\|_{L_2(K_1)} = 0$. Repeating this argument we obtain

$$\|\tilde{u}^m\|_{L_2(K_1)} = 0. \tag{2.31}$$

From (2.26) and (2.27), it follows that $\lim_{i\to\infty} \|u_i^m\|_{L_2(\tilde{K})} = 1$ and thus

$$\|\tilde{u}^m\|_{L_2(\tilde{K})} = 1.$$

This contradicts (2.31). The proof is complete. □

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Remark. From (2.3) it follows that

$$\|u^m\|_{L_2(B(x,3R+\varepsilon))} \le \tilde{C}\|u^m\|_{L_2(B(x,3R+\varepsilon)\setminus B(x,R))} \tag{2.32}$$

where $\tilde{C} = \sqrt{C^2+1}$; i.e. the L_2-norm of any solution to the Schrödinger equation with a parameter on a ball can be estimated by the L_2-norm of the same solution on some compact subset of the ball. Furthermore, the constant $\tilde{C}$ in (2.32) does not depend on the value of the parameter λ.

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