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NONTRIVIAL SOLUTION FOR A THREE-POINT BOUNDARY-VALUE PROBLEM

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Abstract. In this paper, we study the existence of nontrivial solutions for the second-order three-point boundary-value problem

$$\begin{gathered} u'' + f(t,u) = 0, \quad 0 < t < 1, \\ u'(0) = 0, \quad u(1) = \alpha u'(\eta). \end{gathered}$$

where $\eta \in (0,1)$, $\alpha \in \mathbb{R}$, $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$. Under certain growth conditions on the nonlinearity f and by using Leray-Schauder nonlinear alternative, sufficient conditions for the existence of nontrivial solution are obtained. We illustrate the results obtained with some examples.

1. Introduction

This paper, we prove existence results for the following second-order three-point boundary value problem (BVP):

$$\begin{gathered} u'' + f(t,u) = 0, \quad 0 < t < 1, \\ u'(0) = 0, \quad u(1) = \alpha u'(\eta). \end{gathered} \tag{1.1}$$

where $\eta \in (0,1)$, $\alpha \in \mathbb{R}$, $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$.

The study of three-point BVP for certain nonlinear ordinary differential equations was initiated by Gupta [4]. Over the ten yeas, three-point boundary value problems have been extensively studied by many authors, for example, see [2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16], and references therein. But in the existing literature on the BVP (1.1) is few. Most of them studied the following three-point BVP

$$\begin{gathered} u'' + f(t,u) = 0, \quad 0 < t < 1, \\ u(0) = 0, \quad u(1) = \alpha u(\eta). \end{gathered}$$

(for example, Gupta [5], Ma [13], Liu [10], Webb [16], He and Ge [7]) or

$$\begin{gathered} u'' + f(t,u) = 0, \quad 0 < t < 1, \\ u'(0) = 0, \quad u(1) = \alpha u(\eta). \end{gathered}$$

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(for example, Liu [11] and Webb [16]). In a recent paper, Infante [9] investigated the BVP (1.1) for the first time. The aim of the present paper is to establish some simple criteria of the existence of nontrivial solution for the BVP (1.1). Note that we do not require any monotonicity and nonnegativity on f. The results we obtained are new.

The paper is organized as follows. In Section 2, we present two lemmas that will be used to prove the main results. In Section 3, we obtain some existence results for nontrivial solution of the BVP (1.1). Finally, in Section 4, as an application, we give some examples to illustrate the results we obtained.

2. Preliminaries

Let $E = C[0,1]$, with supremum norm $\|y\| = \sup_{t\in[0,1]} |y(t)|$ for any $y \in E$. A solution $u(t)$ of the BVP (1.1) is called nontrivial solution if $u(t) \not\equiv 0$. In arriving at our results, we need to state two preliminary results.

Lemma 2.1. *Let $y \in C[0,1]$, then the three-point BVP*

$$\begin{gathered} u'' + y(t) = 0, \quad 0 < t < 1, \\ u'(0) = 0, \quad u(1) = \alpha u'(\eta). \end{gathered}$$

has a unique solution

$$u(t) = \int_0^1 (1-s)y(s)ds - \int_0^t (t-s)y(s)ds - \alpha \int_0^\eta y(s)ds.$$

The proof of this lemma is easy, and we omit it.

Define the integral operator $T : E \to E$ by

$$\begin{aligned} Tu(t) = &\int_0^1 (1-s)f(s,u(s))ds - \int_0^t (t-s)f(s,u(s))ds \\ &- \alpha \int_0^\eta f(s,u(s))ds, \quad t \in [0,1]. \end{aligned} \tag{2.1}$$

By Lemma 2.1, the BVP (1.1) has a solution if and only if the operator T has a fixed point in E. So we only need to seek a fixed point of T in E. By Ascoli-Arzela Theorem, we can prove that T is a completely continuous operator. The key tool in our approach is the following Leray-Schauder nonlinear alternative (See [1]).

Lemma 2.2. *Let E be Banach space and Ω be a bounded open subset of E, $0 \in \Omega$ $T : \overline{\Omega} \to E$ be a completely continuous operator. Then, either there exists $x \in \partial\Omega$, $\lambda > 1$ such that $T(x) = \lambda x$, or there exists a fixed point $x^* \in \overline{\Omega}$.*

3. Main Results

In this section, we present and prove our main results.

Theorem 3.1. *Suppose $f(t,0) \not\equiv 0$, and there exist nonnegative functions $k, h \in L^1[0,1]$ such that*

$$\begin{gathered} |f(t,x)| \le k(t)|x| + h(t), \ \text{a.e. } (t,x) \in [0,1]\times\mathbb{R}, \\ 2\int_0^1 (1-s)k(s)ds + |\alpha| \int_0^\eta k(s)ds < 1. \end{gathered}$$

Then the BVP (1.1) has at least one nontrivial solution $u^* \in C[0,1]$.

Proof. Let

$$\begin{aligned} A &= 2\int_0^1 (1-s)k(s)ds + |\alpha| \int_0^\eta k(s)ds, \\ B &= 2\int_0^1 (1-s)h(s)ds + |\alpha| \int_0^\eta h(s)ds. \end{aligned}$$

Then $A < 1$. Since $f(t,0) \not\equiv 0$, there exists an interval $[\sigma,\tau] \subset [0,1]$ such that $\min_{\sigma\le t\le\tau} |f(t,0)| > 0$. On the other hand, from $h(t) \ge |f(t,0)|$, a.e. $t \in [0,1]$, we know that $B > 0$. Let $m = B(1-A)^{-1}$, $\Omega = \{u \in C[0,1] : \|u\| < m\}$. Suppose $u \in \partial\Omega$, $\lambda > 1$ such that $Tu = \lambda u$, then

$$\begin{aligned} \lambda m = \lambda\|u\| = \|Tu\| &= \max_{0\le t\le 1} |(Tu)(t)| \\ &\le \int_0^1 (1-s)|f(s,u(s))|ds + \max_{0\le t\le 1}\int_0^t (t-s)|f(s,u(s))|ds \\ &\quad + |\alpha| \int_0^\eta |f(s,u(s))|ds \\ &\le 2\int_0^1 (1-s)|f(s,u(s))|ds + |\alpha|\int_0^\eta |f(s,u(s))|ds \\ &\le \Big[2\int_0^1 (1-s)k(s)|u(s)|ds + |\alpha|\int_0^\eta k(s)|u(s)|ds\Big] \\ &\quad + \Big[2\int_0^1 (1-s)h(s)ds + |\alpha|\int_0^\eta h(s)ds\Big] \\ &\le A\|u\| + B = Am + B. \end{aligned}$$

Therefore,

$$\lambda \le A + \frac{B}{m} = A + \frac{B}{B(1-A)^{-1}} = A + (1-A) = 1,$$

this contradicts $\lambda > 1$. By Lemma 2.2, T has a fixed point $u^* \in \overline{\Omega}$. In view of $f(t,0) \not\equiv 0$, the BVP (1.1) has a nontrivial solution $u^* \in C[0,1]$. This completes the proof. □

Theorem 3.2. *Suppose $f(t,0) \not\equiv 0$, and there exist nonnegative functions $k, h \in L^1[0,1]$ such that*

$$|f(t,x)| \le k(t)|x| + h(t), \ \text{a.e. } (t,x) \in [0,1]\times\mathbb{R}.$$

If one of the following conditions is fulfilled:

(1) *There exists constant $p > 1$ such that*

$$\int_0^1 k^p(s)ds < \Big[\frac{(1+q)^{1/q}}{2+|\alpha|\ [\eta(1+q)]^{1/q}}\Big]^p, \quad \Big(\frac{1}{p}+\frac{1}{q}=1\Big).$$

(2) *There exists a constant $\mu > -1$ such that*

$$k(s) \le \frac{(1+\mu)(2+\mu)}{2+|\alpha|(2+\mu)\eta^{1+\mu}} s^\mu, \quad \text{a.e. } s \in [0,1],$$

$$\operatorname{meas}\Big\{s \in [0,1] : k(s) < \frac{(1+\mu)(2+\mu)}{2+|\alpha|(2+\mu)\eta^{1+\mu}} s^\mu\Big\} > 0.$$

(3) *There exists a constant* $\mu > -1$ *such that*

$$k(s) \le \frac{(1+\mu)(2+\mu)}{2(1+\mu)+|\alpha|(2+\mu)}(1-s)^{\mu}, \textit{a.e. } s \in [0,1],$$

$$\operatorname{meas}\big\{s \in [0,1] : k(s) < \frac{(1+\mu)(2+\mu)}{2(1+\mu)+|\alpha|(2+\mu)}(1-s)^{\mu}\big\} > 0.$$

(4) k *satisfies*

$$k(s) \le \frac{1}{1+|\alpha|\eta}, \quad \textit{a.e. } s \in [0,1],$$

$$\operatorname{meas}\big\{s \in [0,1] : k(s) < \frac{1}{1+|\alpha|\eta}\big\} > 0.$$

(5) f *satisfies*

$$\Lambda := \limsup_{|x|\to\infty} \max_{t\in[0,1]} \big|\frac{f(t,x)}{x}\big| < \frac{1}{1+|\alpha|\eta}.$$

Then the BVP (1.1) *has at least one nontrivial solution* $u^* \in C[0,1]$.

Proof. Let A be given in Theorem 3.1. In view of Theorem 3.1, we only need to prove $A < 1$.
(1) By using the Hölder inequality, we have

$$\begin{aligned}
A &\le \Big[\int_0^1 k^p(s)ds\Big]^{1/p}\Big\{2\Big[\int_0^1 (1-s)^q ds\Big]^{1/q} + |\alpha|\Big[\int_0^\eta 1^q ds\Big]^{1/q}\Big\}\\
&\le \Big[\int_0^1 k^p(s)ds\Big]^{1/p}\Big[2\big(\frac{1}{1+q}\big)^{1/q} + |\alpha|\ \eta^{1/q}\Big]\\
&< \frac{(1+q)^{1/q}}{2+|\alpha|\,[\eta(1+q)]^{1/q}} \cdot \frac{2+|\alpha|\,[\eta(1+q)]^{1/q}}{(1+q)^{1/q}} = 1.
\end{aligned}$$

(2) In this case, we have

$$\begin{aligned}
A &< \frac{(1+\mu)(2+\mu)}{2+|\alpha|(2+\mu)\eta^{1+\mu}}\Big[2\int_0^1 (1-s)s^{\mu}ds + |\alpha|\int_0^\eta s^{\mu}ds\Big]\\
&\le \frac{(1+\mu)(2+\mu)}{2+|\alpha|(2+\mu)\eta^{1+\mu}}\Big[\frac{2}{(1+\mu)(2+\mu)} + |\alpha|\cdot\frac{\eta^{1+\mu}}{1+\mu}\Big]\\
&= \frac{(1+\mu)(2+\mu)}{2+|\alpha|(2+\mu)\eta^{1+\mu}} \cdot \frac{2+|\alpha|(2+\mu)\eta^{1+\mu}}{(1+\mu)(2+\mu)} = 1.
\end{aligned}$$

(3) In this case, we have

$$\begin{aligned}
A &< \frac{(1+\mu)(2+\mu)}{2(1+\mu)+|\alpha|(2+\mu)}\Big[2\int_0^1 (1-s)^{1+\mu}ds + |\alpha|\int_0^\eta (1-s)^{\mu}ds\Big]\\
&= \frac{(1+\mu)(2+\mu)}{2(1+\mu)+|\alpha|(2+\mu)}\Big[\frac{2}{2+\mu} + |\alpha|\cdot\frac{1-(1-\eta)^{1+\mu}}{1+\mu}\Big]\\
&\le \frac{(1+\mu)(2+\mu)}{2(1+\mu)+|\alpha|(2+\mu)}\Big[\frac{2}{2+\mu} + |\alpha|\cdot\frac{1}{1+\mu}\Big]\\
&= \frac{(1+\mu)(2+\mu)}{2(1+\mu)+|\alpha|(2+\mu)} \cdot \frac{2(1+\mu)+|\alpha|(2+\mu)}{(1+\mu)(2+\mu)} = 1.
\end{aligned}$$

(4) In this case, we have

$$A < \frac{1}{1+|\alpha|\eta}\Big[2\int_0^1 (1-s)ds + |\alpha|\int_0^\eta ds\Big] = \frac{1}{1+|\alpha|\eta}\,(1+|\alpha|\eta) = 1.$$

(5) Let $\varepsilon = \frac{1}{2}(\frac{1}{1+|\alpha|\eta} - \Lambda)$, then there exists $c > 0$ such that

$$|f(t,x)| \le \big(\frac{1}{1+|\alpha|\eta} - \varepsilon\big)|x|, \quad (t,x) \in [0,1] \times \mathbb{R} \setminus (-c,c).$$

Set $M = \max\{|f(t,x)| : (t,x) \in [0,1] \times [-c,c]\}$, then

$$|f(t,x)| \le \big(\frac{1}{1+|\alpha|\eta} - \varepsilon\big)|x| + M, \quad (t,x) \in [0,1] \times \mathbb{R}.$$

Set $k(s) = \frac{1}{1+|\alpha|\eta} - \varepsilon$, $h(s) = M$, then (4) holds. This completes the proof. $\square$

Corollary 3.3. *Suppose $f(t,0) \not\equiv 0$, and there exist two nonnegative functions $k, h \in L^1[0,1]$ such that*

$$|f(t,x)| \le k(t)|x| + h(t), \ \text{a.e. } (t,x) \in [0,1] \times \mathbb{R}.$$

If one of the following conditions holds

(1) *There exists a constant $p > 1$ such that*

$$\int_0^1 k^p(s)ds < \Big[\frac{(1+q)^{1/q}}{2+|\alpha|\ (1+q)^{1/q}}\Big]^p, \quad \big(\frac{1}{p} + \frac{1}{q} = 1\big).$$

(2) *There exists a constant $\mu > -1$ such that*

$$k(s) \le \frac{(1+\mu)(2+\mu)}{2+|\alpha|(2+\mu)}s^\mu, \quad \text{a.e. } s \in [0,1],$$

$$\text{meas}\big\{s \in [0,1] : k(s) < \frac{(1+\mu)(2+\mu)}{2+|\alpha|(2+\mu)}s^\mu\big\} > 0.$$

(3) *k satisfies*

$$k(s) \le \frac{1}{1+|\alpha|}, \quad \text{a.e. } s \in [0,1],$$

$$\text{meas}\big\{s \in [0,1] : k(s) < \frac{1}{1+|\alpha|}\big\} > 0.$$

(4) *f satisfies*

$$\Lambda =: \limsup_{|x|\to\infty} \max_{t\in[0,1]} \big|\frac{f(t,x)}{x}\big| < \frac{1}{1+|\alpha|}.$$

Then the BVP (1.1) *has at least one nontrivial solution $u^* \in C[0,1]$.*

Proof. In this case, we have

$$A = 2\int_0^1 (1-s)k(s)ds + |\alpha|\int_0^\eta k(s)ds \le 2\int_0^1 (1-s)k(s)ds + |\alpha|\int_0^1 k(s)ds.$$

The rest of the proof is the same as in Theorem 3.2. $\square$

Corollary 3.4. *Suppose $f(t,0) \not\equiv 0$, and there exist two nonnegative functions $k, h \in L^1[0,1]$ such that*

$$|f(t,x)| \le k(t)|x| + h(t), \quad a.e.\ (t,x) \in [0,1] \times \mathbb{R}.$$

If one of the following conditions is holds.

(1) *There exists a constant $p > 1$ such that*

$$\int_0^1 k^p(s)ds < \Big[\frac{1}{2+|\alpha|} \cdot \big(\frac{1+q}{2^{1+q}-1}\big)^{1/q}\Big]^p, \quad \big(\frac{1}{p} + \frac{1}{q} = 1\big).$$

(2) *There exists a constant $\mu > -1$ such that*

$$k(s) \le \frac{(1+\mu)(2+\mu)}{(2+|\alpha|)(\mu+3)} s^\mu, \quad a.e.\ s \in [0,1],$$
$$\operatorname{meas}\big\{s \in [0,1] : k(s) < \frac{(1+\mu)(2+\mu)}{(2+|\alpha|)(\mu+3)} s^\mu\big\} > 0.$$

(3) *There exists a constant $\mu > -2$ such that*

$$k(s) \le \frac{(2+\mu)}{(2+|\alpha|)(2^{2+\mu}-1)}(2-s)^\mu, \quad a.e.\ s \in [0,1],$$
$$\operatorname{meas}\big\{s \in [0,1] : k(s) < \frac{(2+\mu)}{(2+|\alpha|)(2^{2+\mu}-1)}(2-s)^\mu\big\} > 0.$$

Then the BVP (1.1) has at least one nontrivial solution $u^* \in C[0,1]$.

Proof. In this case,

$$\begin{aligned} A &= 2\int_0^1 (1-s)k(s)ds + |\alpha| \int_0^\eta k(s)ds \\ &\le 2\int_0^1 (1-s)k(s)ds + |\alpha| \int_0^1 k(s)ds \\ &\le (2+|\alpha|) \int_0^1 (2-s)k(s)ds. \end{aligned}$$

(1) Using the Hölder inequality,

$$\begin{aligned} A &\le (2+|\alpha|) \int_0^1 (2-s)k(s)ds \\ &\le (2+|\alpha|)\Big[\int_0^1 k^p(s)ds\Big]^{1/p}\Big[\int_0^1 (2-s)^q ds\Big]^{1/q} \\ &< (2+|\alpha|) \cdot \frac{1}{2+|\alpha|}\big(\frac{1+q}{2^{1+q}-1}\big)^{1/q} \cdot \big(\frac{2^{1+q}-1}{1+q}\big)^{1/q} = 1. \end{aligned}$$

(2) In this case, we have

$$\begin{aligned} A &\le (2+|\alpha|) \int_0^1 (2-s)k(s)ds \\ &< (2+|\alpha|) \cdot \frac{(1+\mu)(2+\mu)}{(2+|\alpha|)(\mu+3)} \int_0^1 (2-s)s^\mu ds \\ &= \frac{(1+\mu)(2+\mu)}{\mu+3} \cdot \frac{\mu+3}{(1+\mu)(2+\mu)} = 1. \end{aligned}$$

(3) In this case,

$$\begin{aligned}
A &\le (2+|\alpha|)\int_0^1 (2-s)k(s)ds \\
&< (2+|\alpha|)\cdot\frac{(2+\mu)}{(2+|\alpha|)(2^{2+\mu}-1)}\int_0^1 (2-s)^{1+\mu}ds \\
&= \frac{2+\mu}{2^{2+\mu}-1}\cdot\frac{2^{2+\mu}-1}{2+\mu} = 1.
\end{aligned}$$

The proof is complete. □

4. Examples

In this section, in order to illustrate our results, we consider some examples.

Example 4.1. Consider the three-point BVP

$$\begin{gathered}
u'' + (t-t^2)|u|\sin u - t^2 u + t^3 - 2\sin t = 0, \quad 0<t<1, \\
u'(0) = 0, \quad u(1) = 4u'(1/2).
\end{gathered} \tag{4.1}$$

Set $\alpha = 4$, $\eta = \frac{1}{2}$, and

$$\begin{gathered}
f(t,x) = (t-t^2)|x|\sin x - t^2 x + t^3 - 2\sin t, \\
k(t) = t, \quad h(t) = t^3 + 2\sin t.
\end{gathered}$$

It is easy to prove that $k, h \in L^1[0,1]$ are nonnegative functions and

$$f(t,x) \le k(t)|x| + h(t), \quad (t,x) \in [0,1]\times\mathbb{R}.$$

and

$$A = 2\int_0^1 (1-s)k(s)ds + |\alpha|\int_0^\eta k(s)ds = \frac{5}{6} < 1.$$

Hence, by Theorem 3.1, the BVP (4.1) has at least one nontrivial solution u^* in $C[0,1]$.

Example 4.2. Consider the three-point BVP

$$\begin{gathered}
u'' + \frac{2\sqrt{t}u^3}{3+u^4}e^{-|sin(u^2-t)|} + 3e^t - 2\sin t = 0, \quad 0<t<1, \\
u'(0) = 0, \quad u(1) = \sqrt{3}u'(1/4).
\end{gathered} \tag{4.2}$$

Set $\alpha = \sqrt{3}$, $\eta = \frac{1}{4}$, and

$$\begin{gathered}
f(t,x) = \frac{2\sqrt{t}x^3}{3+x^4}e^{-|\sin(x^2-t)|} + 3e^t - 2\sin t, \\
k(t) = \sqrt{\frac{t}{3}}, \quad h(t) = 3e^t + 2\sin t.
\end{gathered}$$

It is easy to prove that $k, h \in L^1[0,1]$ are nonnegative functions and

$$f(t,x) \le k(t)|x| + h(t), \quad (t,x) \in [0,1]\times\mathbb{R}.$$

Let $p = q = 2$, then

$$\int_0^1 k^p(s)ds = \int_0^1 \frac{1}{3}s\,ds = \frac{1}{6},$$
$$\Big[\frac{(1+q)^{1/q}}{2+|\alpha|[\eta(1+q)]^{1/q}}\Big]^p = \frac{12}{49}.$$

Therefore,

$$\int_0^1 k^p(s)ds < \Big[\frac{(1+q)^{1/q}}{2+|\alpha|[\eta(1+q)]^{1/q}}\Big]^p.$$

Thus, by Theorem 3.2 (1), the BVP (4.2) has at least one nontrivial solution u^* in $C[0,1]$.

Example 4.3. Consider the three-point BVP

$$\begin{gathered} u'' + \frac{u^2e^{-t}}{3(1+u^2)(1+2e^u)\sqrt{t}} + \frac{1}{7\sqrt{t}}u + e^t - \sqrt{t}\cos t = 0, \quad 0<t<1, \\ u'(0) = 0, \quad u(1) = \frac{1}{3}u'(1/4). \end{gathered} \tag{4.3}$$

Set $\alpha = \frac{1}{3}$, $\eta = \frac{1}{4}$, and

$$f(t,x) = \frac{x^2e^{-t}}{3(1+x^2)(1+2e^x)\sqrt{t}} + \frac{1}{7\sqrt{t}}x + e^t - \sqrt{t}\cos t,$$
$$k(t) = \frac{1}{6\sqrt{t}} + \frac{1}{7\sqrt{t}}, \quad h(t) = e^t + \sqrt{t}\cos t.$$

It is easy to prove that $k, h \in L^1[0,1]$ are nonnegative functions, and

$$f(t,x) \le k(t)|x| + h(t), \quad \text{a.e. } (t,x) \in [0,1]\times\mathbb{R},$$

Let $\mu = -\frac{1}{2}$, then

$$\frac{(1+\mu)(2+\mu)}{2+|\alpha|(2+\mu)\eta^{1+\mu}} = \frac{1}{3}.$$

Therefore,

$$\begin{aligned} k(s) &= \frac{1}{6\sqrt{s}} + \frac{1}{7\sqrt{s}} < \frac{1}{6\sqrt{s}} + \frac{1}{6\sqrt{s}} = \frac{1}{3}s^{-\frac{1}{2}} \\ &= \frac{(1+\mu)(2+\mu)}{2+|\alpha|(2+\mu)\eta^{1+\mu}}s^\mu, \quad \text{a.e. } s\in[0,1], \end{aligned}$$

$$\operatorname{meas}\big\{s\in[0,1] : k(s) < \frac{(1+\mu)(2+\mu)}{2+|\alpha|(2+\mu)\eta^{1+\mu}}s^\mu\big\} = 1 > 0.$$

Hence, by Theorem 3.2 (2), the BVP (4.3) has at least one nontrivial solution $u^* \in C[0,1]$.

Example 4.4. Consider the three-point BVP

$$\begin{gathered} u'' + \frac{u^2e^{-u^2}}{3(1+u^2)\sqrt[4]{1-t}} - 3e^{-t} + \sqrt{\sin t} = 0, \quad 0<t<1, \\ u'(0) = 0, \quad u(1) = -3u'(1/4). \end{gathered} \tag{4.4}$$

Set $\eta = \frac{1}{4}, \alpha = -3$, and

$$f(t,x) = \frac{x^2 e^{-x^2}}{3(1+x^2)\sqrt[4]{1-t}} - 3e^{-t} + \sqrt{\sin t},$$
$$k(t) = \frac{1}{6\sqrt[4]{1-t}}, \quad h(t) = 3e^{-t} + \sqrt{\sin t}.$$

It is easy to prove that $k, h \in L^1[0,1]$ are nonnegative functions, and

$$f(t,x) \le k(t)|x| + h(t), \quad \text{a.e.}(t,x) \in [0,1] \times \mathbb{R},$$

Let $\mu = -\frac{1}{4}$. Then

$$\frac{(1+\mu)(2+\mu)}{2(1+\mu) + |\alpha|(2+\mu)} = \frac{7}{36}.$$

Therefore,

$$k(s) = \frac{1}{6\sqrt[4]{1-s}} < \frac{7}{36\sqrt[4]{1-s}} = \frac{(1+\mu)(2+\mu)}{2(1+\mu) + |\alpha|(2+\mu)}(1-s)^{-\frac{1}{4}}, \quad \text{a.e.} s \in [0,1],$$
$$\operatorname{meas}\big\{s \in [0,1] : k(s) < \frac{(1+\mu)(2+\mu)}{2(1+\mu) + |\alpha|(2+\mu)}(1-s)^{-\frac{1}{4}}\big\} = 1 > 0.$$

Hence, by Theorem 3.2 (3), the BVP (4.4) has at least one nontrivial solution $u^* \in C[0,1]$.

Example 4.5. Consider the three-point BVP

$$\begin{gathered} u'' + \frac{t^2 u^2 e^{-u^2}}{t^2 + u^2} - \cos e^t + 3\sin^2 t = 0, \quad 0 < t < 1, \\ u'(0) = 0, \quad u(1) = 3u'(1/3). \end{gathered} \tag{4.5}$$

Set $\eta = \frac{1}{3}$, $\alpha = 3$, and

$$f(t,x) = \frac{t^2 x^2 e^{-x^2}}{t^2 + x^2} - \cos e^t + 3\sin^2 t,$$
$$k(t) = \frac{t}{2}, \; h(t) = \cos e^t + 3\sin^2 t.$$

Then it is easy to prove that $k, h \in L^1[0,1]$ are nonnegative functions, and

$$f(t,x) \le k(t)|x| + h(t), \quad \text{a.e. } (t,x) \in [0,1] \times \mathbb{R},$$
$$k(s) = \frac{s}{2} \le \frac{1}{1 + |\alpha|\eta} = \frac{1}{2}, \quad s \in [0,1],$$
$$\operatorname{meas}\big\{s \in [0,1] : k(s) < \frac{1}{1+|\alpha|\eta}\big\} = 1 > 0.$$

Hence, by Theorem 3.2 (4), the BVP (4.5) has at least one nontrivial solution $u^* \in C[0,1]$.

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