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Existence of Positive Solutions for some Dirichlet Problems with an Asymptotically Homogeneous Operator *

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Abstract

Existence of positive radially symmetric solutions to a Dirichlet problem of the form

$$\begin{aligned} -\mathrm{div}(A(|Du|)Du) = f(u) \quad & \text{in } \Omega \\ u = 0 \quad & \text{on } \partial\Omega \end{aligned}$$

is studied by using blow-up techniques. It is proven here that by choosing the functions $sA(s)$ and $f(s)$ among a certain class called *asymptotically homogeneous*, the blow-up method still provides the a-priori bounds for positive solutions. Existence is proved then by using degree theory.

1 Introduction

In this paper we consider the existence of positive radially symmetric solutions for the problem

$$(D) \quad \begin{cases} -\mathrm{div}(A(|Du|)Du) = f(u) & \text{in } \Omega \\ \qquad\qquad u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega = B(0,R)$, $R > 0$, is the ball of radius R in $\mathbb{R}^N$ and the function $f : \mathbb{R} \to \mathbb{R}$ is continuous. For some functions $A : \mathbb{R} \to \mathbb{R}$, the radial solutions of (D) satisfy the nonlinear boundary value problem

$$(D_r) \quad \begin{cases} -(r^{N-1}\phi(u'))' = r^{N-1}f(u) & \text{in } (0,R) \\ \qquad u'(0) = 0 = u(R) \end{cases}$$

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where $r = |x|,\ x \in \mathbb{R}^N$ and $\phi : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism of $\mathbb{R}$, that is, an odd increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, given by $\phi(s) = sA(s)$. In (D_r), $'$ denotes derivative with respect to r.

In the rest of the paper we will deal with problem (D_r) in the "superlinear" case, that is, when

$$\lim_{s\to+\infty} \frac{f(s)}{\phi(s)} = +\infty. \tag{1}$$

and ϕ, f belong to a class of functions to be described later.

By a solution to this problem we will understand a function $u \in C^1[0,R]$ with $\phi(u') \in C^1[0,R]$ and such that (D_r) is satisfied.

It is well known that for the homogeneous case, that is when $\phi(s) = |s|^{p-2}s$, $p > 1$, the use of blow up techniques allows to transform the question of a-priori bounds for positive solutions to some superlinear problems into a problem of non-existence of positive solutions in $\mathbb{R}^N$ for a certain limiting equation. This limiting equation having the same left hand side nonlinear operator as the original equation, due to the homogeneity. See [GS] for the case of a scalar equation and $p = 2$, and [CMM] for the case of a system of p, q-Laplacians. See also [PvV] for related results.

The natural question of whether this method can be extended to cover the radial situation posed by problem (D_r), when the function ϕ is no longer homogeneous arises.

We give here a positive answer to this question by restricting the functions ϕ, f to a special class. This, being strongly motivated by some previous works done for the one dimensional case, see [GMZ1], [GMZ2], [GMZ3], and [U].

We now describe the class of functions ϕ, f we will consider in order to formulate our model problem. Throughout this paper we will assume that ϕ is an odd homeomorphism of $\mathbb{R}$ satisfying

$$\lim_{|s|\to\infty} \frac{\phi(\sigma s)}{\phi(s)} = \sigma^{p-1} \quad \text{for all} \quad \sigma \in \mathbb{R}_+ \tag{2}$$

for some $p > 1$ and f satisfies

$$\lim_{s\to+\infty} \frac{f(\sigma s)}{f(s)} = \sigma^{\delta} \qquad \text{for all} \qquad \sigma \in \mathbb{R}_+ \tag{3}$$

for some $\delta > 0$, where $\mathbb{R}_+ := [0, +\infty)$.

Conditions of this type, even without the monotonicity assumption on ϕ, have been very much used in Applied Probability in a different context than the one we will do here, see for instance [R], [S] and the references therein. Indeed from [R] or [S] we have the following general definition. Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function that satisfies

$$\lim_{s\to+\infty} \frac{h(\sigma s)}{h(s)} = \sigma^{q} \qquad \text{for all} \qquad \sigma \in \mathbb{R}_+. \tag{4}$$

We will say then that h is *asymptotically homogeneous with index* q, for short AH or q-AH. We point out that in [R], [S], functions h satisfying (4) are called *regularly varying of index* q, nevertheless from our point of view it will be more illustrative to call them asymptotically homogeneous.

In this sense if the function $\phi(s) = sA(s)$ in (D_r) is AH of index $p-1$, we will say that the corresponding operator in (D_r) or (D) is an *asymptotically homogeneous operator.*

Thus, with this notation, we will require that ϕ be a $(p-1)$-AH odd homeomorphism of $\mathbb{R}$ for some $p > 1$ and that the continuous function f be δ-AH for some $\delta > 0$.

Condition (1) implies that $\delta \geq p-1$, see Section 2. We observe that the case $\delta = p-1$ is indeed allowed as the following example shows:

$$\phi(s) = |s|^{p-2}s\frac{|s|}{\sqrt{1+s^2}}\ , \quad f(s) = |s|^{p-2}s\log(1+|s|).$$

For later use we define the following functions

$$\Phi(s) = \int_0^s \phi(\tau)d\tau\ , \qquad \Phi_*(s) = \int_0^s \phi^{-1}(\tau)d\tau. \tag{5}$$

and following [FP] or [PS] we define the Legendre transform of Φ by

$$H(s) = s\phi(s) - \Phi(s). \tag{6}$$

Also we will set

$$\phi_p(s) = |s|^{p-2}s \quad \text{for all} \quad s \in \mathbb{R} \quad \text{and} \quad p > 1,$$

and $p^* = \dfrac{p}{p-1}$.

We end this section by establishing the organization of this paper. In Section 2 we establish and prove our main result for existence of positive solutions. In Section 3 we show some examples that illustrate our results, and finally in the Appendix, we prove some properties of the class of asymptotically homogeneous functions that we use throughout this paper.

2 Existence of positive solutions.

In this section we will show that problem (D_r), that we recall next

$$(D_r) \quad \begin{cases} -(r^{N-1}\phi(u'))' = r^{N-1}f(u) \quad \text{in } (0,R) \\ \\ u'(0) = 0 = u(R), \end{cases}$$

has positive solutions assuming that the homeomorphism ϕ satisfies (2) and the nonlinearity f satisfies (3), $sf(s) \geq 0$ for $s \geq 0$ and it is superlinear with respect to ϕ.

Let Φ, Φ_* and H as in (5) and (6) respectively. We have that

$$H(s) = \Phi_*(\phi(s)), \tag{7}$$

and that H is an even function of s. Since Φ_* is p^*-AH and ϕ is $(p-1)$-AH, it is easy to see, by using Proposition 4.1 in the Appendix that

$$\lim_{s\to+\infty} \frac{H(\sigma s)}{H(s)} = \sigma^p \quad \text{for all } \sigma \in \mathbb{R}_+, \tag{8}$$

and hence H is p-AH.

Let $F(s) = \int_0^s f(\tau)d\tau$. We note that from Karamata's theorem, see [R, Theorem 0.6], by using (2) and (3) it follows that

$$\lim_{s\to+\infty} \frac{\Phi(s)}{s\phi(s)} = \frac{1}{p} \quad \text{and} \quad \lim_{s\to+\infty} \frac{F(s)}{sf(s)} = \frac{1}{\delta+1}. \tag{9}$$

Now, from (9), given $\varepsilon > 0$ there exists $s_0 > 0$ such that for all $s \ge s_0$

$$\delta + 1 - \varepsilon < \frac{sF'(s)}{F(s)} < \delta + 1 + \varepsilon.$$

Hence, solving this differential inequality we obtain

$$A_1 s^{\delta-\varepsilon} \le f(s) \le A_2 s^{\delta+\varepsilon} \quad \text{for all } s \ge s_0. \tag{10}$$

Similarly, it can be shown that

$$A_3 s^{p-1-\varepsilon} \le \phi(s) \le A_4 s^{p-1+\varepsilon} \quad \text{for all } s \ge s_0 \tag{11}$$

where $A_i := A_i(\varepsilon) > 0 \;\; i = 1, \ldots, 4$. Hence (10) and (11) yield that

$$\frac{f(s)}{\phi(s)} \le \frac{A_2}{A_3} s^{\delta-(p-1)+2\varepsilon} \tag{12}$$

and thus we see that (1) implies that $\delta \ge p - 1$.

Next, and for later purposes we consider the equation

$$H(z) = F(s). \tag{13}$$

Since F is strictly increasing for s greater than some $s_0 > 0$, and $F(s) \to +\infty$ as $s \to +\infty$ it is clear that for each $s > s_0$, equation (13) will have a unique solution which we denote by $z(s)$. Define now $g : (s_0, +\infty) \to \mathbb{R}^+$, where $\mathbb{R}^+ := (0, +\infty)$ by

$$g(s) = \frac{z(s)}{s}. \tag{14}$$

We have the following proposition.

Proposition 2.1 *If (1) holds, then $g(s) \to +\infty$ as $s \to +\infty$.*

Proof. We have that
$$F(s) = H(sg(s))$$
thus, if $g(s_n) \leq M$ for some sequence $\{s_n\}$, $s_n \to +\infty$, then
$$\frac{F(s_n)}{H(s_n)} \leq \frac{H(s_n M)}{H(s_n)} \to M^p \quad \text{as} \quad n \to \infty. \tag{15}$$
On the other hand,
$$\frac{F(s)}{H(s)} = \frac{F(s)}{\Phi(s)} \cdot \frac{\Phi(s)}{\Phi_*(\phi(s))}$$
and
$$\frac{\Phi(s)}{\Phi_*(\phi(s))} = \frac{\Phi(s)}{s\phi(s) - \Phi(s)} = \frac{1}{\frac{s\phi(s)}{\Phi(s)} - 1},$$
hence, from (9),
$$\lim_{s\to+\infty} \frac{\Phi(s)}{\Phi_*(\phi(s))} = \frac{1}{p-1}.$$
Thus, (15) contradicts (1) by L'Hôpital's rule.

Now we establish our main existence result.

Theorem 2.1 *Suppose that ϕ is an increasing odd homeomorphism of $\mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ is continuous , satisfies $sf(s) \geq 0$, and is ultimately increasing. Assume also that ϕ and f satisfy the superlinear condition (1) and that there exist p, with $1 < p < N$, and $\delta > 0$ such that*

(i) $\displaystyle\lim_{s\to+\infty} \frac{\phi(\sigma s)}{\phi(s)} = \sigma^{p-1}$ for all $\sigma \in \mathbb{R}_+$.

(ii) $\displaystyle\lim_{s\to+\infty} \frac{f(s\sigma)}{f(s)} = \sigma^{\delta}$, for all $\sigma \in \mathbb{R}_+$.

(iii) $\displaystyle\lim_{t\to 0} \frac{\phi(t)}{f(t)} = +\infty$ and $\displaystyle\liminf_{t\to 0} \frac{\phi(t\sigma)}{\phi(t)} > 0$, for every $\sigma \in \mathbb{R}^+$,

(iv) $\displaystyle\delta < \frac{N(p-1)+p}{N-p}$.

Then problem (D_r) has a positive solution.

The proof of this theorem will be done in three steps. We note that finding positive solutions of problem (D_r) is equivalent to finding nontrivial solutions to the problem
$$(A) \qquad \begin{cases} -(r^{N-1}\phi(u'))' = r^{N-1} f(|u|) & \text{in } (0,R) \\ u'(0) = 0 = u(R) \end{cases}$$
Indeed, if $u(r)$ is a nontrivial solution of (A), then $u'(R) < 0$ for all $r \in (0,R)$ and since $u(R) = 0$ we find that $u(r) > 0$, for all $r \in (0,R)$. This shows that $u(r)$ is a positive solution of problem (D_r).

Step 1. Abstract formulation of problem (A)**.** Let $C_\#$ denote the closed subspace of $C[0,R]$ defined by

$$C_\# = \{u \in C[0,R] \ : \ u(R)=0\},$$

then $C_\#$ is a Banach space for the norm $\| \ \| := \| \ \|_\infty$.

By direct verification it can be seen that u is a solution to (A) if and only if u is a fixed point of the operator $T_0 : C_\# \to C_\#$ defined by

$$T_0(u)(r) = \int_r^R \phi^{-1}\Big[\frac{1}{s^{N-1}}\int_0^s \xi^{N-1} f(|u(\xi)|)d\xi\Big]ds. \tag{16}$$

Define now the operator $T : C_\# \times \mathbb{R}_+ \to C_\#$, by

$$T(u,\tau)(r) = \int_r^R \phi^{-1}\Big[\frac{1}{s^{N-1}}\int_0^s \xi^{N-1}(f(|u(\xi)|)+\tau)d\xi\Big]ds \tag{17}$$

We have that T sends bounded sets of $C_\# \times \mathbb{R}_+$ into bounded sets of $C_\#$ and that $T(u,0) = T_0(u)$. We prove now the following.

Proposition 2.2 *The operator T is completely continuous.*

Proof. Let $\{(u_n,\tau_n)\}$ be a bounded sequence in $C_\# \times \mathbb{R}_+$, say

$$\|u_n\|_\infty + \tau_n \le C, \quad \text{for all } n \in \mathbb{N}, \tag{18}$$

and set

$$v_n = T(u_n,\tau_n), \quad n \in \mathbb{N}.$$

We want to show that $\{v_n\}$ has a convergent subsequence. By (18) and (17), $v_n \in C^1[0,R]$, for all $n \in \mathbb{N}$ and satisfies

$$\begin{aligned} |\phi(v_n'(r))| &= \frac{1}{r^{N-1}}\int_0^r \xi^{N-1}(f(|u_n(\xi)|)+\tau_n)d\xi \\ &\le \frac{\tilde{C}R}{N}, \end{aligned}$$

where $\tilde{C}$ is a positive constant. Thus the sequence $\{v_n'\}$ is bounded and since the sequence $\{v_n\}$ is bounded also, the existence of a convergent subsequence follows from the Ascoli Arzèla's Theorem.

To show now that T is continuous, let $\{(u_n,\tau_n)\}$ be a sequence in $C_\# \times \mathbb{R}_+$ converging to $(u,\tau) \in C_\# \times \mathbb{R}_+$ and set

$$v_n(r) = \int_r^R h_n(s)ds \quad n \in \mathbb{N},$$

where

$$h_n(s) = \phi^{-1}\Big[\frac{1}{s^{N-1}}\int_0^s \xi^{N-1}(f(|u_n(\xi)|) + \tau_n)d\xi\Big].$$

Clearly we have that $h_n(s) \to h(s)$ for each $s \in [0, R]$, where

$$h(s) := \phi^{-1}\Big[\frac{1}{s^{N-1}}\int_0^s \xi^{N-1}(f(|u(\xi)|) + \tau)d\xi\Big].$$

Thus, since $\{h_n\}$ is bounded, it follows from Lebesgue's dominated convergence theorem that

$$\|h_n - h\|_{L^1(0,R)} \to 0 \text{ as } n \to \infty.$$

If

$$v(r) := \int_r^R h(s)ds,$$

then

$$\|v_n - v\| \le \|h_n - h\|_{L^1(0,R)}$$

and hence

$$T(u_n, \tau_n) = v_n \to v = T(u, \tau) \quad \text{ as } n \to \infty.$$

This concludes the proof of proposition 2.2. □

Step 2. A-priori bounds. We will show here that solutions $(u, \tau) \in C_\# \times \mathbb{R}_+$ of the equation

$$u = T(u, \tau) \tag{19}$$

are a priori bounded. This will be done by using *blow-up* techniques.

We first prove the following.

Proposition 2.3 *Suppose that there exists a sequence $\{(u_n, \tau_n)\}$ of solutions of (19) such that*

$$\|u_n\| + \tau_n \to \infty \text{ as } n \to \infty, \tag{20}$$

then

(i) $\|u_n\| \to \infty$ *as* $n \to \infty$*; and*

(ii) $\dfrac{\tau_n}{f(\|u_n\|)} \to 0.$

Proof. We have that for each $n \in \mathbb{N}$ the pair (u_n, τ_n) satisfies

$$u_n(r) = \int_r^R \phi^{-1}\Big[\frac{1}{s^{N-1}}\int_0^s \xi^{N-1}(f(|u_n|) + \tau_n)d\xi\Big]\, ds$$

and thus

$$||u_n|| = u_n(0) \geq \int_{R/2}^R \phi^{-1}(\frac{s\tau_n}{N})\, ds \geq \frac{R}{2}\phi^{-1}(\frac{R\tau_n}{2N}) \tag{21}$$

from which (i) follows. To show (ii) we have that by (21) and large n,

$$\frac{R}{2N}\frac{\tau_n}{f(\|u_n\|)} \leq \frac{\phi(\frac{2}{R}\|u_n\|)}{f(\|u_n\|)} = \frac{\phi(\frac{2}{R}\|u_n\|)}{\phi(\|u_n\|)}\frac{\phi(\|u_n\|)}{f(\|u_n\|)}. \tag{22}$$

Thus by (1), and the fact that ϕ is (p-1)-AH, we obtain that

$$\frac{\tau_n}{f(\|u_n\|)} \to 0 \text{ as } n \to \infty. \quad \square$$

.

We will now prove that solutions to (19) are *a-priori* bounded.

Lemma 2.1 *Suppose $(u, \tau) \in C_\# \times \mathbb{R}_+$ is a solution of (19), then there is a constant C, independent of u and τ, such that*

$$\|u\| + \tau \leq C \tag{23}$$

Proof. We argue by contradiction and thus we assume there is a sequence $\{(u_n, \tau_n)\}$ in $C_\# \times \mathbb{R}_+$ such that (u_n, τ_n) satisfies (19) and such that

$$\|u_n\| + \tau_n \to \infty \text{ as } n \to \infty.$$

In order to simplify the writing, we set $t_n := ||u_n||$ and $z_n := z(t_n)$ for each $n \in \mathbb{N}$, where the function $z(\cdot)$ is defined in (13). Let us consider the change of variables

$$\begin{cases} y &= \dfrac{z_n}{t_n}r \\ w_n(y) &= \dfrac{u_n(r)}{t_n}. \end{cases}$$

Then from (19), we find that w_n satisfies

$$-\frac{d}{dy}(y^{N-1}\phi(z_n\dot{w}_n(y)))z_n = t_n y^{N-1}(f(t_n w_n(y)) + \tau_n) \tag{24}$$

$$w_n(0) = 1,\ \dot{w}_n(0) = 0,\ w_n(R\frac{z_n}{t_n}) = 0, \tag{25}$$

where here and henceforth $\dot{(\ )} := \dfrac{d}{dy}(\ )$. We note that $\dfrac{z_n}{t_n} = g(t_n)$, with g as defined in (14), and thus from propositions 2.1 and 2.3, $\dfrac{z_n}{t_n} \to +\infty$ as $n \to \infty$.

Let $M_0 > 0$ be a constant. In the next argument we will suppose that $Rg(t_n) > M_0$, for all n, by passing to a subsequence if necessary.

Dividing both sides of (24) by y^{N-1}, we may re-write it as

$$\frac{d}{dy}[\phi(z_n\dot{w}_n(y))]z_n + (f(t_nw_n(y)) + \tau_n)t_n = -\frac{N-1}{y}\phi(z_n\dot{w}_n(y))z_n,$$

and on multiplying both sides of this equation by $\dot{w}_n$ we obtain that

$$\frac{d}{dy}[H(z_n\dot{w}_n(y)) + F(t_nw_n(y)) + \tau_nt_nw_n(y)] \le 0. \tag{26}$$

Hence, by integrating (26) on $(0, y)$, we find that

$$H(z_n\dot{w}_n(y)) + F(t_nw_n(y)) + \tau_nt_nw_n(y) \le F(t_n) + \tau_nt_n,$$

and thus

$$\begin{aligned} H(z_n\dot{w}_n(y)) &\le F(t_n)\Big[1 + \frac{\tau_nt_n}{F(t_n)}\Big] \\ &= H(z_n)\Big[1 + \frac{\tau_n}{f(t_n)}\frac{t_nf(t_n)}{F(t_n)}\Big]. \end{aligned}$$

But $\dfrac{\tau_n}{f(t_n)} \to 0$ and $\dfrac{t_nf(t_n)}{F(t_n)} \to \delta + 1$ as $n \to \infty$, hence there exists a constant $C > 0$ such that

$$H(z_n\dot{w}_n(y)) \le CH(z_n).$$

This implies that

$$|\dot{w}_n(y)| \le \frac{H^{-1}(CH(z_n))}{H^{-1}(H(z_n))}, \tag{27}$$

and since from (8) and Proposition 4.1 in the Appendix

$$\frac{H^{-1}(Cs)}{H^{-1}(s)} \to C^{\frac{1}{p}} \quad \text{as} \quad s \to +\infty,$$

there exists a constant $C_1 > 0$ such that

$$|\dot{w}_n(y)| \le C_1 \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and all} \quad y \in [0, M_0].$$

Thus the sequence $\{w_n\}$ is equicontinuous. Since it is also uniformly bounded, an application of Ascoli Arzèla's theorem yields that $\{w_n\}$ contains a convergent subsequence, which we denote again by $\{w_n\}$, say $w_n \to w$ in $C[0, M_0]$ as $n \to \infty$. Integrating (24) on $[0, y] \subset [0, M_0]$, we find that

$$-\phi(z_n \dot{w}_n(y)) = \frac{t_n f(t_n)}{z_n} h_n(y) \tag{28}$$

where

$$h_n(y) = \frac{1}{y^{N-1}} \int_0^y s^{N-1} \Big(\frac{f(t_n w_n(s))}{f(t_n)} + \frac{\tau_n}{f(t_n)} \Big)\, ds. \tag{29}$$

We show now that $\{h_n(y)\}$ is a convergent sequence for each $y \in [0, M_0]$ by an application of Lebesgue's dominated convergence theorem.

Using that f is ultimately increasing, say for $x \geq x_1 > 0$, we have that there is a $x_0 > 0$ such that

$$\frac{f(\sigma x)}{f(x)} \leq 1 \tag{30}$$

for all $x \geq x_0$ and for all $\sigma \in [0, 1]$. Indeed there is a unique $x_0 \geq x_1$ such that

$$f(x_0) = \max_{s \in [0, x_1]} f(s) := M.$$

Let $\sigma \in (0, 1)$ and consider the term $\dfrac{f(\sigma x)}{f(x)}$ for $x \geq x_0$. If $\sigma x \geq x_0$, then $f(\sigma x) \leq f(x)$ and thus (30) holds. If now $\sigma x < x_0$, then

$$f(\sigma x) \leq M = f(x_0) \leq f(x)$$

and again (30) holds.

Thus from (30) we have that

$$\frac{f(t_n w_n(s))}{f(t_n)} \leq 1 \tag{31}$$

for all $s \in [0, M_0]$ and large n. In particular this implies that $\{h_n(y)\}$ is a bounded sequence. We will show next that for each $s_0 \in [0, M_0]$

$$\lim_{n \to \infty} \frac{f(t_n w_n(s_0))}{f(t_n)} = (w(s_0))^{\delta}. \tag{32}$$

We know that $w_n(s_0) \to w(s_0)$ as $n \to \infty$. If $w(s_0) > 0$, then for large n, $t_n w(s_0) > x_0$ and (32) follows from the fact that f is ultimately increasing and (ii). If $w(s_0) = 0$, we have to distinguish the two cases corresponding to the sequence $\{t_n w_n(s_0)\}$ being bounded or not. We only show that

$$\lim_{n \to \infty} \frac{f(t_n w_n(s_0))}{f(t_n)} = 0 \tag{33}$$

for the latter situation. We argue by contradiction and thus we suppose that

$$\limsup_{n\to\infty} \frac{f(t_n w_n(s_0))}{f(t_n)} = \mu_0 > 0.$$

Note that by (31), $\mu_0 \leq 1$. We have that there is subsequence $\{n_k\}$ of positive integers such that

$$\lim_{k\to\infty} \frac{f(t_{n_k} w_{n_k}(s_0))}{f(t_{n_k})} = \mu_0$$

with $\{t_{n_k} w_{n_k}(s_0)\}$ an unbounded sequence. Thus, passing to a subsequence if necessary, we can suppose that $t_{n_k} w_{n_k}(s_0) > x_0$.

Now, since $w_{n_k}(s_0) \to 0$, given $\varepsilon > 0$ there is a $k_0 := k_0(\varepsilon, \mu_0)$ such that

$$t_{n_k} w_{n_k}(s_0) \leq \varepsilon t_{n_k} \mu_0, \quad \text{for all} \quad k > k_0,$$

then

$$\frac{f(t_{n_k} w_{n_k}(s_0))}{f(t_{n_k})} \leq \frac{f(t_{n_k} \varepsilon \mu_0)}{f(t_{n_k})}. \tag{34}$$

But

$$\lim_{k\to\infty} \frac{f(t_{n_k} \varepsilon \mu_0)}{f(t_{n_k})} = (\varepsilon\mu_0)^\delta \leq \varepsilon^\delta,$$

since f is $\delta-$AH, thus letting $k \to \infty$ in (34) we find that

$$\mu_0 \leq \varepsilon^\delta,$$

which is a contradiction, and then (33) holds.

Applying Lebesgue's dominated convergence theorem to the right hand side of (29) we conclude that $\{h_n(y)\}$ converges to

$$h(y) = \frac{1}{y^{N-1}} \int_0^y s^{N-1} w^\delta(s) ds,$$

for each $y \in [0, M_0]$.

Solving for $\dot{w}_n(y)$ in (28) we find

$$-\dot{w}_n(y) = \frac{\phi^{-1}(\alpha_n(y)\phi(z_n))}{\phi^{-1}(\phi(z_n))} \qquad \text{for} \qquad y \in (0, M_0] \tag{35}$$

where

$$\alpha_n(y) = \frac{t_n f(t_n)}{z_n \phi(z_n)} h_n(y) = \frac{t_n f(t_n)}{F(t_n)} \cdot \frac{H(z_n)}{z_n \phi(z_n)} h_n(y).$$

From (6) and (9) it follows that

$$\frac{t_n f(t_n)}{F(t_n)} \cdot \frac{H(z_n)}{z_n \phi(z_n)} \to (\delta+1)\Big(1-\frac{1}{p}\Big) =: \beta \quad \text{as } n \to \infty,$$

Thus for each $y \in [0, M_0]$

$$\lim_{n\to\infty} \alpha_n(y) = \beta h(y). \tag{36}$$

Integrating (35) from 0 to $y \in (0, M_0]$, we obtain

$$1 - w_n(y) = \int_0^y \frac{\phi^{-1}(\alpha_n(s)\phi(z_n))}{\phi^{-1}(\phi(z_n))} ds, \tag{37}$$

and by using (36), Proposition 4.1, and Lebesgue's dominated convergence theorem, and by letting $n \to \infty$ in (37), we find that

$$1 - w(y) = \int_0^y \beta^{p^*-1}(h(s))^{p^*-1} ds. \tag{38}$$

Then, differentiating (38) we obtain

$$-w'(y) = \beta^{p^*-1}(h(y))^{p^*-1},$$

which yields

$$-\phi_p(w'(y)) = \frac{\beta}{y^{N-1}} \int_0^y s^{N-1} w^\delta(s) ds.$$

Thus w is a nonnegative nontrivial solution in $[0, M_0]$ to the initial value problem

$$-[y^{N-1}\phi_p(w'(y))]' = \beta y^{N-1} w^\delta(y) \tag{39}$$

$$w'(0) = 0, \quad w(0) = 1. \tag{40}$$

By using next a diagonal iterative scheme, see for example the last part of the proof of [CMM, Proposition 4.1], w can be extended to all $\mathbb{R}_+$, as a nonnegative solution of (39)-(40). Furthermore, and arguing like in [CMM], it can be shown that w is indeed a positive solution of class $C^2(0,+\infty)$ of (39)-(40). Since $\delta < \dfrac{N(p-1)+p}{N-p}$, this is a contradiction in the case that $\delta > p-1$, see [NS] or [CMM]. In case that $\delta = p-1$, it is well known (see for example [DM, Lemma 5.3]), that every solution of (39)-(40) with $\beta > 0$ is oscillatory in $(0,+\infty)$ and hence the contradiction. Thus, lemma 2.1 is proved. $\square$

Step 3. Proof of Theorem 2.1. From Lemma 2.1, if (u,τ) is a solution of (19), i.e.,

$$u = T(u,\tau)$$

then $\|u\| \le C$ and $0 \le \tau \le C$, where C is a positive constant. Thus if $B(0,R_1)$ denotes the ball centered at 0 in $C_\#$ with radius $R_1 > C$, we have that

$$u \ne T(u,\tau)$$

for any $(u,\tau) \in \partial B(0,R_1) \times [0,R_1]$. Hence if I denotes the identity in $C_\#$ we have that the Leray-Schauder degree of the operator

$$I - T(\cdot,\tau) : \overline{B(0,R)} \to C_\#$$

is well defined for every $\tau \in [0,R_1]$. Then, by the properties of the Leray-Schauder degree, we have that

$$\begin{aligned} \deg_{LS}(I - T(\cdot,\tau), B(0,R_1), 0) &= \deg_{LS}(I - T(\cdot,R_1), B(0,R_1), 0) \\ &= 0, \end{aligned} \tag{41}$$

since (19) does not have solutions on $\overline{B(0,R_1))} \times \{R_1\}$. Thus from (41) and the fact that $T(u,0) = T_0(u)$

$$\deg_{LS}(I - T_0, B(0,R_1), 0) = 0. \tag{42}$$

Next, let us define the operator $S : [0,1] \times C_\# \to C_\#$,

$$S(\lambda,u) = \int_r^R \phi^{-1}[\frac{\lambda}{s^{N-1}} \int_0^s \xi^{N-1} f(|u(\xi)|) d\xi] ds. \tag{43}$$

Then as in step 1, it can be proved that S is a completely continuous operator. We note that $S(1,\cdot) = T_0$.

Claim. There exist an $\varepsilon > 0$ such that the equation

$$u = S(\lambda,u) \tag{44}$$

has no solution (u,λ) with $u \in \partial B(0,\varepsilon)$ and $\lambda \in [0,1]$.

Proof of the claim. We argue by contradiction and thus we assume that there are sequences $\{u_n\}$ and $\{\lambda_n\}$ with $\|u_n\| = \varepsilon_n \to 0$ as $n \to \infty$ and $\lambda_n \in [0,1]$ such that (u_n,λ_n) satisfies (44) for each $n \in \mathbb{N}$. We have that (u_n,λ_n) satisfies

$$u_n(r) = \int_r^R \phi^{-1}\Big[\frac{\lambda_n}{s^{N-1}} \int_0^s \xi^{N-1} f(|u_n(\xi)|) d\xi\Big] ds$$

which, by the first in (iii), implies that for n large

$$\varepsilon_n \leq \phi^{-1}\Big(\phi(\varepsilon_n)\frac{\mu R}{N}\Big)R,$$

where μ is a positive arbitrarily small number. Thus

$$\phi\Big(\frac{\varepsilon_n}{R}\Big) \leq \frac{\mu R}{N}\phi(\varepsilon_n). \tag{45}$$

If $R \leq 1$, we immediately reach a contradiction. If now $R > 1$, let us set $\sigma = 1/R$, then

$$\frac{\phi\Big(\sigma\varepsilon_n\Big)}{\phi\Big(\varepsilon_n\Big)} \leq \frac{\mu R}{N},$$

and we reach a contradiction by the second of (iii) and the fact that μ is arbitrary. Thus the claim holds.

It follows from this claim and the properties of the Leray Schauder degree that for $\varepsilon > 0$ small,

$$d_{LS}(I - S(\lambda,\cdot), B(0,\varepsilon), 0) = \text{ constant } \quad \text{for all} \quad \lambda \in [0,1].$$

Thus

$$d_{LS}(I - T_0, B(0,\varepsilon), 0) = d_{LS}(I, B(0,\varepsilon), 0) = 1,$$

and then by (42) and (44) and the excision property of the Leray-Schauder degree we obtain that there must be a solution of the equation

$$u = T_0(u)$$

in $B(0,R_1) \setminus \overline{B(0,\varepsilon)}$. This concludes the proof of the theorem. $\square$

3 Examples.

In this section we wish to show by mean of simple examples the applicability of our main theorem.

Example 3.1. Let ϕ be defined by

$$\phi(s) = \sum_{i=1}^{n} \alpha_i \phi_{p_i}(s)$$

where for simplicity we assume $\alpha_i > 0$ for $i = 1, n$. Also $p_{i+1} > p_i > 1$, for $i = 1, ..., n-1$.

In a similar form let f be defined by

$$f(s) = \sum_{j=1}^{m} \beta_j |s|^{\delta_j - 1} s,$$

where again for simplicity we assume $\beta_j > 0$ for $j = 1, \dots, m$, with $\delta_{j+1} > \delta_j > 1$, for $j = 1, ..., m-1$.

Then it is clear that

$$\lim_{s\to+\infty} \frac{\phi(\sigma s)}{\phi(s)} = \sigma^{p_n - 1} \quad \text{and} \quad \lim_{s\to 0} \frac{\phi(\sigma s)}{\phi(s)} = \sigma^{p_1 - 1} \quad \text{for all} \quad \sigma \in \mathbb{R}_+,$$

and that

$$\lim_{s\to+\infty} \frac{f(\sigma s)}{f(s)} = \sigma^{\delta_m}, \quad \text{for all} \quad \sigma \in \mathbb{R}_+.$$

It can also be verified that if $\delta_1 > p_1 - 1$, then

$$\lim_{s\to 0} \frac{\phi(s)}{f(s)} = +\infty.$$

Thus, if $N > p_n$, and

$$p_n - 1 < \delta_m < \frac{N(p_n - 1) + p_n}{N - p_n}$$

all the conditions of Theorem 2.1 are satisfied and hence for ϕ and f as defined above, problem (D_r) has a positive solution, and therefore the problem

$$\begin{cases} -\operatorname{div}(\sum\limits_{i=1}^{n} \alpha_i |Du|^{p_i - 2} Du) = \sum\limits_{j=1}^{m} \beta_j |u|^{\delta_j - 1} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a positive radial solution of class C^1.

In the first example the operator ϕ and the nonlinearity f, are asymptotic to powers at $+\infty$, in the sense that, $f(s)$ or $\phi(s)$ divided by a suitable power of s tends to a constant as $s \to +\infty$, as it can be directly checked. We will give next an example where both ϕ and f are not asymptotic to a power.

Example 3.2. Let us define the increasing homeomorphism ψ of $\mathbb{R}$ and the function g by

$$\psi(s) = \phi_q(s) \log(1 + |s|) \quad \text{and} \quad g(s) = |s|^{\mu - 1} \log(1 + |s|) s,$$

with $q > 1$ and $\mu > 0$. Then it can be checked that

$$\lim_{s\to+\infty} \frac{\psi(\sigma s)}{\psi(s)} = \sigma^{q-1} \quad \text{and} \quad \lim_{s\to 0} \frac{\psi(\sigma s)}{\psi(s)} = \sigma^{q} \quad \text{for all} \quad \sigma \in \mathbb{R}_+,$$

$$\lim_{s\to+\infty}\frac{g(\sigma s)}{g(s)}=\sigma^{\mu}\quad\text{for all}\quad\sigma\in\mathbb{R}_+,$$

where neither ψ nor g are asymptotic to a power at $+\infty$. Also, it can be directly checked that if $\mu > q-1$, then

$$\lim_{s\to+\infty}\frac{g(s)}{\phi(s)}=+\infty\quad\text{and}\quad\lim_{s\to 0}\frac{\phi(s)}{g(s)}=+\infty.$$

Thus, if $N > q$, and

$$q-1<\mu<\frac{N(q-1)+q}{N-q},$$

we have that all the conditions of Theorem 2.1 are satisfied with ψ and g in the place of ϕ and f and hence (D_r) has a positive solution, and therefore the problem

$$\begin{cases} -\operatorname{div}(|Du|^{q-2}\log(1+|Du|)Du)=|u|^{\mu-1}\log(1+|u|)u & \text{in }\Omega\\ u=0 & \text{on }\partial\Omega\end{cases}$$

has a positive radial solution of class C^1.

It follows from Proposition 4.1 *(ii)* in the Appendix that the composition of two asymptotically homogeneous operators is also asymptotically homogeneous. We use this fact to obtain a third example as a combination of the previous two.

Example 3.3. Define the functions θ and h as follows

$$\theta(s)=(\phi\circ\psi)(s)\quad\text{and}\quad h=f\circ g$$

where $\phi,\ f$ are as in example 3.1 and $\psi,\ g$ as in example 3.2, then in particular θ is an odd increasing homeomorphism of $\mathbb{R}$.

By Proposition 4.1 *(ii)*, we have immediately that θ is $(r-1)$-AH and h is ρ-AH, with

$$r=(p_n-1)(q-1)+1\quad\text{and}\quad\rho=\delta_m\mu.$$

It can be directly verified that

$$\lim_{s\to+0}\frac{\theta(\sigma s)}{\theta(s)}=\sigma^{(q-1)(p_1-1)}\quad\text{for all}\quad\sigma\in\mathbb{R}_+,$$

and if

$$\mu\delta_1-(q-1)(p_1-1)\geq 0\quad\text{and}\quad\delta_1\geq(p_1-1)$$

with at least one of the inequalities strict, then

$$\lim_{s\to 0}\frac{\theta(s)}{h(s)}=+\infty.$$

Finally, if

$$(q-1)(p_n-1) < \delta_m \mu < \frac{N(q-1)(p_n-1)+(q-1)(p_n-1)+1}{N-(q-1)(p_n-1)-1},$$

with $\delta_m \geq (p_n - 1)$, we have that all the conditions of Theorem 2.1 are fulfilled with θ and h in the place of ϕ and f in that theorem. Hence (D_r) has a positive solution and therefore the problem

$$\begin{cases} -\mathrm{div}(\sum\limits_{i=1}^{n} \alpha_i |Du|^{(p_i-1)(q-1)-1} (\log(1+|Du|)^{p_i-1} Du) \\ \qquad = \sum\limits_{j=1}^{m} \beta_j |u|^{\mu\delta_j - 1} (\log(1+|u|)^{\delta_j} u \quad \text{in} \quad \Omega \\ \qquad u = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

has a positive radial solution of class C^1.

Remark. For simplicity the right hand side functions in the above examples have been chosen as increasing odd homeomorphisms of $\mathbb{R}$. It is nevertheless clear how to modify these functions so that they satisfy the asymptotic conditions required by Theorem 2.1.

4 Appendix

Here, and for the sake of completeness, we briefly state and prove some of the properties of AH functions we have used. For other properties of AH functions we have used, we refer to [R] or [S].

Our first proposition shows that if ϕ is AH then so is ϕ^{-1}, and that the composition of AH functions is also AH.

Proposition 4.1 *(i) Suppose that ϕ is an increasing odd homeomorphism of $\mathbb{R}$ that is (p-1)-AH, then*

$$\lim_{x\to+\infty} \frac{\phi^{-1}(\sigma x)}{\phi^{-1}(x)} = \phi_{p^*}(\sigma) \quad \text{for all } \sigma \in \mathbb{R}. \tag{46}$$

(ii) Suppose $\chi,\ \psi : \mathbb{R}_+ \to \mathbb{R}_+$ are (p-1)-AH and (q-1)-AH respectively, with χ ultimately increasing, and $\chi(s),\ \psi(s) \to +\infty$ as $s \to +\infty$. Then $\chi \circ \psi$ is (r-1)-AH, with $r = (p-1)(q-1)+1$.

Proof . (i) It suffices to prove the result for $\sigma \in (0,1)$. Hence, let $\sigma \in (0,1)$ be a fixed number and $\{x_n\}$ a sequence such that $x_n \to +\infty$ as $n \to \infty$. The sequence $\{\dfrac{\phi^{-1}(\sigma x_n)}{\phi^{-1}(x_n)}\}$ is a bounded sequence and thus it contains a convergent subsequence, that we called the same, say

$$\lim_{n\to\infty} \frac{\phi^{-1}(\sigma x_n)}{\phi^{-1}(x_n)} = L \in [0,1].$$

Then, given $\varepsilon > 0$, we can find an $n_0 \in \mathbb{N}$ such that

$$L - \varepsilon < \frac{\phi^{-1}(\sigma x_n)}{\phi^{-1}(x_n)} < L + \varepsilon \qquad \text{for all} \quad n \geq n_0.$$

Setting $t_n = \phi^{-1}(x_n), n \in \mathbb{N}$, and using that ϕ is increasing, we find that

$$\phi_p(L-\varepsilon) \leq \sigma \leq \phi_p(L+\varepsilon)$$

from where (i) follows.

To show (ii), let $\sigma \in (0,1)$ and $\varepsilon > 0$ be given. Then, there is $s_0 = s_0(\sigma)$ such that

$$-\varepsilon + \phi_q(\sigma) < \frac{\psi(\sigma s)}{\psi(s)} < \varepsilon + \phi_q(\sigma), \tag{47}$$

for all $s > s_0$ where we take $\varepsilon < \phi_q(\sigma)$. Since χ is ultimately increasing, we have from (47) that

$$\frac{\chi((\phi_q(\sigma)-\varepsilon)\psi(s))}{\chi(\psi(s))} \leq \frac{\chi(\psi(\sigma s))}{\chi(\psi(s))} \leq \frac{\chi((\phi_q(\sigma)+\varepsilon)\psi(s))}{\chi(\psi(s))} \tag{48}$$

for large s. Thus, letting s go to $+\infty$ in (48) we find that

$$\begin{aligned} \phi_p(\phi_q(\sigma)-\varepsilon) &\leq \liminf_{s\to+\infty} \frac{\chi(\psi(\sigma s))}{\chi(\psi(s))} \\ &\leq \limsup_{s\to+\infty} \frac{\chi(\psi(\sigma s))}{\chi(\psi(s))} \leq \phi_p(\phi_q(\sigma)+\varepsilon). \end{aligned}$$

Then (ii) follows, since $0 < \varepsilon < \phi_q(\sigma)$ is arbitrary. □

The final part of this section is dedicated to the following natural question, which is interesting in its own. What happens if instead of (2) we assume that

$$\text{(He)} \qquad \lim_{s\to+\infty} \frac{\phi(\sigma s)}{\phi(s)} = e(\sigma) \quad \text{for each} \quad \sigma \in \mathbb{R}_+,$$

where e is not supposed a-priori to be a positive power. If (He) is assumed, and since $\dfrac{\phi(\sigma s)}{\phi(s)}$ is an increasing function of σ, then e is nondecreasing and hence locally integrable. We will show in our next proposition that a simple condition on the average of e in $[0,1]$ implies that e is a positive power. Let us set

$$\overline{e} = \int_0^1 e(\sigma)\ d\sigma,$$

then we have

Proposition 4.2 *Suppose that $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that $\psi(s) > 0$ for $s > 0$ with $\psi(s) \to +\infty$ as $s \to +\infty$, and satisfying (He) with ψ in the place of ϕ. Then*

$$\overline{e} \in (0,1) \qquad \textit{if and only if} \qquad e(\sigma) = \sigma^{p-1} \quad \textit{for all} \quad \sigma \in \mathbb{R}_+, \tag{49}$$

where $p > 1$. In this case $\overline{e} = \dfrac{1}{p}$.

Proof. We only have to show that $\overline{e} \in (0,1)$ implies that $e(\sigma) = \sigma^{p-1}$. We will first prove the proposition in the case that ψ is increasing. Let $\sigma \in (0,1]$. Then for $x > 0$,

$$\frac{\Psi(\sigma x)}{x\psi(x)} = \int_0^{\sigma x} \frac{\psi(\tau)d\tau}{x\psi(x)}, \tag{50}$$

where $\Psi(x) := \int_0^x \psi(\tau)d\tau$. Making the change of variables $\tau = xs$ in the integral of (50), we find that

$$\frac{\Psi(\sigma x)}{x\psi(x)} = \int_0^{\sigma} \frac{\psi(xs)}{\psi(x)} ds. \tag{51}$$

Since $\dfrac{\psi(xs)}{\psi(x)} \leq 1$, for each $s \in [0,1]$, from Lebesgue's dominated convergence theorem it follows that

$$\lim_{x\to+\infty} \frac{\Psi(\sigma x)}{x\psi(x)} = \int_0^{\sigma} e(s)ds := E(\sigma). \tag{52}$$

In particular, for $\sigma = 1$,

$$\lim_{x\to+\infty} \frac{\Psi(x)}{x\psi(x)} = E(1) = \bar{e}. \tag{53}$$

On the other hand, since Ψ is of class C^1, by L'Hôpital's rule it follows that

$$\lim_{x\to+\infty} \frac{\Psi(\sigma x)}{\Psi(x)} = \sigma\ e(\sigma). \tag{54}$$

Letting $x \to +\infty$ in

$$\frac{\Psi(\sigma x)}{x\psi(x)} = \frac{\Psi(\sigma x)}{\Psi(x)} \frac{\Psi(x)}{x\psi(x)}\ , \tag{55}$$

and using (52), (53), (54), it follows that

$$E(\sigma) = \bar{e}\sigma e(\sigma),$$

thus

$$e(\sigma) = \frac{E(\sigma)}{\bar{e}\sigma}\ ,$$

and hence e is continuous in $(0, 1]$. Then E satisfies the differential equation

$$\frac{E'(\sigma)}{E(\sigma)} = \frac{1}{\bar{e}\sigma}. \tag{56}$$

Now, $\bar{e} = E(1) \in (0, 1)$, and the fact that E is nondecreasing in $[0, 1]$ imply the existence of a $\sigma_* \in [0, 1]$ such that

$$\sigma_* = \inf\{\sigma \in (0, 1] \mid E(\sigma) > 0\}.$$

Integrating (56) on $[\sigma, 1]$, with $\sigma \in [\sigma_*, 1]$, we find that

$$E(\sigma) = \frac{1}{p}\sigma^p, \tag{57}$$

with $p = \frac{1}{\bar{e}}$. Clearly then, σ_* must be zero and (57) holds for all $\sigma \in [0, 1]$. Thus

$$e(\sigma) = E'(\sigma) = \sigma^{p-1},$$

for all $\sigma \in [0, 1]$, and $p = \frac{1}{\bar{e}} > 1$.

That $e(\sigma) = \sigma^{p-1}$ holds for all $\sigma \in (1, +\infty)$ follows by setting $\mu = 1/\sigma$ in (He).

We prove now the result for the general case. To this end we consider $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $\Psi(s) = \int_0^s \psi(\tau)d\tau$. By (He) and an application of L'Hôpital's rule we have that

$$\lim_{s\to+\infty} \frac{\Psi(\sigma s)}{\Psi(s)} = \sigma e(\sigma) := E(\sigma) \quad \text{for all} \quad \sigma \in \mathbb{R}_+.$$

Let now $\overline{E} := \int_0^1 \sigma e(\sigma)\ d\sigma$, then $\overline{e} \in (0, 1)$ implies that $\overline{E} \in (0, 1)$. Thus by the previous argument, $E(\sigma) = \sigma^{q-1}$ with $q = \frac{1}{\overline{E}} > 1$. It follows that

$$e(\sigma) = \frac{E(\sigma)}{\sigma} = \sigma^{q-2}, \quad \text{for all} \quad \sigma \in \mathbb{R}^+.$$

Using then that

$$\overline{E} = \overline{e} - \int_0^1 (\int_0^t e(\sigma)\ d\sigma\ )dt$$

we find that $q = p+1$ for some $p > 1$. This concludes the proof of the proposition. $\square$

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