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EXISTENCE AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THE DIRICHLET PROBLEM FOR A NONLINEAR PSEUDOPARABOLIC EQUATION

LE THI PHUONG NGOC, DAO THI HAI YEN, NGUYEN THANH LONG

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Abstract. This article concerns the initial-boundary value problem for non-linear pseudo-parabolic equation

$$
\begin{gathered}
u_t - u_{xxt} - (1+\mu(u_x))u_{xx} + (1+\sigma(u_x))u = f(x,t), \quad 0<x<1,\ 0<t<T, \\
u(0,t) = u(1,t) = 0, \\
u(x,0) = \tilde{u}_0(x),
\end{gathered}
$$

where f, $\tilde{u}_0$, μ, σ are given functions. Using the Faedo-Galerkin method and the compactness method, we prove that there exists a unique weak solution u such that $u \in L^\infty(0,T;H_0^1 \cap H^2)$, $u' \in L^2(0,T;H_0^1)$ and $\|u\|_{L^\infty(Q_T)} \leq \max\{\|\tilde{u}_0\|_{L^\infty(\Omega)}, \|f\|_{L^\infty(Q_T)}\}$. Also we prove that the problem has a unique global solution with H^1-norm decaying exponentially as $t \to +\infty$. Finally, we establish the existence and uniqueness of a weak solution of the problem associated with a periodic condition.

1. Introduction

Consider the following initial-boundary value problem for the pseudo-parabolic equation arising in third-grade fluid flows

$$
u_t-(1+\mu(u_x))u_{xx}-\alpha u_{xxt}+\big(\gamma+\beta\sigma(u_x)\big)u = f(x,t), \quad 0<x<1,\ 0<t<T, \tag{1.1}
$$

with the boundary conditions

$$
u(0,t) = u(1,t) = 0, \tag{1.2}
$$

and with the initial condition

$$
u(x,0) = \tilde{u}_0(x), \tag{1.3}
$$

or the T-periodic condition

$$
u(x,0) = u(x,T), \tag{1.4}
$$

where $\alpha > 0$, $\beta > 0$, $\gamma > 0$ are given constants and f, $\tilde{u}_0$, μ, σ are given functions satisfying conditions specified later.

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The pseudo-parabolic equation

$$u_t - u_{xxt} = F(x, t, u_x, u_{xx}), \quad 0 < x < 1,\ t > 0 \tag{1.5}$$

with the initial condition $u(x,0) = \tilde{u}_0(x)$ and with the difrerent boundary conditions, has been extensively studied by many authors, see for example [2], [3], [6], [10], [14] among others and the references given therein. In these works, many results about existence, regularity, asymptotic behavior, and decay of solutions were obtained.

Equations of type (1.5) with a one time derivative appearing in the highest order term are called pseudo-parabolic or Sobolev equations, and arise in many areas of mathematics and physics. We refer to the monographs of Al'shin [1], and of Carroll [7] for references and results on pseudoparabolic or Sobolev type equations. Mathematical study of pseudo-parabolic equations goes back to works of Showalter in the seventies, since then, numerous of interesting results about linear and nonlinear pseudo-parabolic equations have been obtained. We also refer to [12] for asymptotic behavior and to [13] for nonlinear problems.

An important special case of the model is the Benjamin-Bona-Mahony-Burgers (BBMB) equation

$$u_t + u_x + uu_x - \nu u_{xx} - \alpha^2 u_{xxt} = 0, \tag{1.6}$$

it was studied by Amick et al in [2], where $\nu > 0$, $\alpha = 1$, $x \in \mathbb{R}$, $t \geq 0$. The authors proved that solution of (1.6) with initial data in $L^1 \cap H^2$ decays to zero in L^2 norm as $t \to +\infty$. With $\nu > 0$, $\alpha > 0$, $x \in [0,1]$, $t \geq 0$, the model has the form (1.6) was also investigated earlier by Bona and Dougalis in [6], where uniqueness, global existence and continuous dependence of solutions on initial and boundary data were established and the solutions were shown to depend continuously on $\nu \geq 0$ and on $\alpha > 0$.

The Benjamin-Bona-Mahony (BBM) equation is introduced in [5], in 1972, as a model for describing long - wave behavior. Since then, the periodic boundary value problems, the initial value problems and the initial boundary value problems, for various generalized BBM equations have been studied. On the other hand, many people have studied the large time behaviors of solutions to the initial value problems for various generalized BBM equations, such as BBMB equations with a Burgers-type dissipative term appended, see [14]. Medeiros and Miranda [10] studied another special case, namely

$$u_t + f(u)_x - u_{xxt} = g(x,t), \tag{1.7}$$

where $u = u(x,t)$, $0 < x < 1$, and $t \geq 0$ is the time. They proved existence, uniqueness of solutions for f in C^1 and regularity in the case $f(s) = s^2/2$. Arnold et al.[3] considered the following equation from the point of view of periodic solutions

$$-(au_{xt})_x + cu_t = -(\alpha u_x)_x + \beta u_x + \gamma, \quad x \in \mathbb{R},\ t \in [0,T]. \tag{1.8}$$

Here, the authors proved the existence, uniqueness and regularity of solutions under the hypothesis that α, β and γ are C^1-functions of x, t and u, and that they are bounded together with their first derivatives.

It is well known that equation (1.1) arises within frameworks of mathematical models in engineering and physical sciences on third-grade fluid flows, see [4, 8, 11] and references therein. For example, the following equation of motion for the unsteady flow of third-grade fluid over the rigid plate with porous medium is investigated

$$\rho\frac{\partial u}{\partial t}=\mu\frac{\partial^2 u}{\partial y^2}+\alpha_1\frac{\partial^3 u}{\partial y^2\partial t}+6\beta_3\Big(\frac{\partial u}{\partial y}\Big)^2\frac{\partial^2 u}{\partial y^2}-\frac{\phi}{k}\Big[\mu+\alpha_1\frac{\partial}{\partial t}+2\beta_3\Big(\frac{\partial u}{\partial y}\Big)^2\Big]u, \quad (1.9)$$

for $y>0$, $t>0$, where u is the velocity component, ρ is the density, μ the coefficient of viscosity, α_1 and β_3 are the material constants, see [4].

Motivated by the above mentioned works, because of mathematical context, we study of the existence, uniqueness and exponential decay of solutions for Dirichlet problem (1.1)-(1.3) or (1.4). This article is organized as follows. In section 2, under appropriate conditions of α, β, γ, f, $\tilde{u}_0$, μ, σ we prove the existence of a unique solution on $(0,T)$, for every $T>0$ and the boundedness of the solution. In section 3, we study exponential decay of solutions. In section 4, we prove the existence and uniqueness of a T-periodic weak solution.

2. Preliminaries

Without loss of generality, we consider Problem (1.1) – (1.3) with $\alpha=\beta=\gamma=1$.

We put $\Omega=(0,1)$ and denote the usual function spaces used in this paper by the notations $L^p=L^p(\Omega)$, $H^m=H^m(\Omega)$. Let $\langle\cdot,\cdot\rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2 and we denote by $\|\cdot\|_X$ the norm in the Banach space X. We call X' the dual space of X.

We denote by $L^p(0,T;X)$, $1\le p\le\infty$ for the Banach space of real functions $u:(0,T)\to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)}=\Big(\int_0^T\|u(t)\|_X^p dt\Big)^{1/p}<\infty \quad \text{for } 1\le p<\infty,$$

and

$$\|u\|_{L^\infty(0,T;X)}=\operatorname{ess\,sup}_{0<t<T}\|u(t)\|_X \quad \text{for } p=\infty.$$

On H^1, we shall use the norm

$$\|v\|_{H^1}=(\|v\|^2+\|v_x\|^2)^{1/2}.$$

The following lemma is well known.

Lemma 2.1. *The imbedding $H^1\hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$\|v\|_{C^0(\overline{\Omega})}\le\sqrt{2}\|v\|_{H^1} \quad \textit{for all } v\in H^1.$$

Remark 2.2. On H_0^1, $\|v\|_{H^1}$ and $\|v_x\|$ are equivalent norms. Furthermore,

$$\|v\|_{C^0(\overline{\Omega})}\le\|v_x\| \quad \text{for all } v\in H_0^1.$$

3. Existence and uniqueness theorem

Without losing of generality, we consider problem (1.1)-(1.3) with $\alpha=\beta=\gamma=1$.

$$\begin{gathered}u_t-u_{xxt}-\frac{\partial}{\partial x}(u_x+\bar{\mu}(u_x))+(1+\sigma(u_x))u=f(x,t), \quad 0<x<1,\ 0<t<T,\\ u(0,t)=u(1,t)=0,\\ u(x,0)=\tilde{u}_0(x),\end{gathered} \tag{3.1}$$

where $\bar{\mu}(y)=\int_0^y \mu(z)dz$, $y\in\mathbb{R}$.

The weak formulation of (3.1) can be given in the following manner: Find $u(t)$ defined in the open set $(0,T)$ such that $u(t)$ satisfies the variational problem

$$\begin{aligned}&\langle u_t(t),w\rangle+\langle u_{xt}(t),w_x\rangle+\langle u_x(t)+\bar{\mu}(u_x(t)),w_x\rangle\\&+\langle(1+\sigma(u_x(t)))u(t),w\rangle=\langle f(t),w\rangle,\end{aligned}\tag{3.2}$$

for all $w\in H_0^1$ and the initial condition

$$u(0)=\tilde{u}_0.\tag{3.3}$$

We make the following assumptions:

(H1) $\tilde{u}_0\in H_0^1\cap H^2$;

(H2) $f\in L^2(0,T;H_0^1)$;

(H3) $\mu\in C^0(\mathbb{R};\mathbb{R})$ such that $\mu(0)=0$, $\mu(z)>0$, for all $z\in\mathbb{R}$, $z\neq0$;

(H4) $\sigma\in C^1(\mathbb{R};\mathbb{R})$ such that
 (i) $\sigma(0)=0$, $\sigma(z)>0$, $z\sigma'(z)>0$, for all $z\in\mathbb{R}$, $z\neq0$,
 (ii) $y(\int_0^y z\sigma'(z)dz)\leq y^2\sigma(y)$ for all $y\in\mathbb{R}$.

An example of the function σ satisfying (H4) is

$$\sigma(z)=|z|^q,$$

where $q>1$ is a constant. It is obvious that (H4) holds, because

$$\begin{gathered}\sigma(z)=|z|^q,\quad \sigma'(z)=q|z|^{q-2}z,\\ \sigma(0)=0,\quad \sigma(z)>0, z\sigma'(z)=q|z|^q>0,\quad \forall z\in\mathbb{R},\quad z\neq0,\end{gathered}$$

$$\begin{aligned}y\Big(\int_0^y z\sigma'(z)dz\Big)&=qy\Big(\int_0^y |z|^q dz\Big)=qy\frac{|y|^q y}{q+1}\\&=\frac{q}{q+1}|y|^{q+2}=\frac{q}{q+1}y^2\sigma(y)\leq y^2\sigma(y).\end{aligned}$$

Theorem 3.1. *Let $T>0$ and (H1)–(H4) hold. Then, problem (3.1) has a unique weak solution u satisfying*

$$u\in L^\infty(0,T;H_0^1\cap H^2),\quad u'\in L^2(0,T;H_0^1).\tag{3.4}$$

Furthermore, we have the estimate

$$\|u\|_{L^\infty(Q_T)}\leq\max\{\|\tilde{u}_0\|_{L^\infty(\Omega)},\|f\|_{L^\infty(Q_T)}\}.\tag{3.5}$$

Estimate (3.5) appears naturally, both physical and mathematical context, from the maximum principle in the study of partial differential equation of the kind of (3.1).

Proof. The proof consists of several steps.

Step 1: The Faedo-Galerkin approximation (introduced by Lions [9]). Consider a special orthonormal basis $\{w_j\}$ on $H_0^1: w_j(x)=\sqrt{2}\sin(j\pi x)$, $j\in\mathbb{N}$, formed by the eigenfunctions of the Laplacian $-\Delta=-\frac{\partial^2}{\partial x^2}$:

$$-\triangle w_j=\lambda_j w_j, w_j\in C^\infty([0,1]),\quad \lambda_j=(j\pi)^2,\quad j=1,2,\dots$$

Put

$$u_m(t)=\sum_{j=1}^m c_{mj}(t)w_j,\tag{3.6}$$

where the coefficients $c_{mj}(t)$ satisfy a system of nonlinear differential equations

$$\begin{gathered}\langle u_m'(t), w_j\rangle + \langle u_{mx}'(t), w_{jx}\rangle + \langle u_{mx}(t) + \bar{\mu}(u_{mx}(t)), w_{jx}\rangle \\ + \langle (1+\sigma(u_{mx}(t)))u_m(t), w_j\rangle = \langle f(t), w_j\rangle, \quad 1 \le j \le m, \\ u_m(0) = u_{0m},\end{gathered} \tag{3.7}$$

in which

$$u_{0m} = \sum_{j=1}^{m} \beta_{mj} w_j \to \tilde{u}_0 \quad \text{strongly in } H_0^1 \cap H^2. \tag{3.8}$$

System (3.7) can be rewritten in the form

$$\begin{gathered}c_{mi}'(t) + c_{mi}(t) + \frac{1}{1+\lambda_i}\big[\langle \bar{\mu}(u_{mx}(t)), w_{ix}\rangle + \langle \sigma(u_{mx}(t))u_m(t), w_i\rangle\big] \\ = \frac{1}{1+\lambda_i}\langle f(t), w_i\rangle, \\ c_{mi}(0) = \beta_{mi}, \quad 1 \le i \le m.\end{gathered} \tag{3.9}$$

It is clear that for each m there exists a solution $u_m(t)$ in form (3.6) which satisfies (3.7) almost everywhere on $0 \le t \le T_m$ for some T_m, $0 < T_m \le T$. The following estimates allow us to take $T_m = T$ for all m.

Step 2: A priori estimates.

(a) First estimate. Multiplying the j^{th} equation of $(3.7)_1$ by $c_{mj}(t)$ and summing up with respect to j, afterwards, integrating with respect to the time variable from 0 to t, we obtain after some rearrangements

$$S_m(t) = S_m(0) + 2\int_0^t \langle f(s), u_m(s)\rangle ds, \tag{3.10}$$

where

$$\begin{aligned}S_m(t) &= \|u_m(t)\|_{H^1}^2 + 2\int_0^t \|u_m(s)\|_{H^1}^2 ds \\ &\quad + 2\int_0^t \langle \bar{\mu}(u_{mx}(s)), u_{mx}(s)\rangle ds + 2\int_0^t \langle \sigma(u_{mx}(s)), u_m^2(s)\rangle ds.\end{aligned} \tag{3.11}$$

By $u_{0m} \to \tilde{u}_0$ strongly in $H_0^1 \cap H^2$, we deduce

$$S_m(0) = \|u_{0m}\|_{H^1}^2 \le \bar{S}_0 \quad \forall m \in \mathbb{N}, \tag{3.12}$$

where $\bar{S}_0$ always indicates a constant depending on $\tilde{u}_0$.

Note that

$$y\bar{\mu}(y) = y\int_0^y \mu(z)dz \ge 0, \quad \forall y \in \mathbb{R}.$$

On the other hand, we have

$$\begin{aligned}2\int_0^t \langle f(s), u_m(s)\rangle ds &\le \int_0^t \|f(s)\|^2 ds + \int_0^t \|u_m(s)\|^2 ds \\ &\le \int_0^T \|f(s)\|^2 ds + \frac{1}{2}S_m(t).\end{aligned} \tag{3.13}$$

Therefore,

$$S_m(t) \le 2\bar{S}_0 + 2\int_0^T \|f(s)\|^2 ds \le C_T^{(1)}. \tag{3.14}$$

(b) Second estimate. Next, by replacing w_j in $(3.7)_1$ by $-w_{jxx}$, we obtain that

$$\begin{aligned} &\langle u'_{mx}(t), w_{jx}\rangle + \langle \Delta u'_m(t), \Delta w_j\rangle + \langle \Delta u_m(t), \Delta w_j\rangle \\ &+ \langle u_{mx}(t), w_{jx}\rangle + \langle \mu(u_{mx}(t))\Delta u_m(t), \Delta w_j\rangle \\ &+ \langle \sigma'(u_{mx}(t))u_m(t)\Delta u_m(t) + \sigma(u_{mx}(t))u_{mx}(t), w_{jx}\rangle \\ &= \langle f_x(t), w_{jx}\rangle,\ 1 \le j \le m. \end{aligned} \tag{3.15}$$

Similar to $(3.7)_1$, we have

$$\begin{aligned} P_m(t) = P_m(0) &- 2\int_0^t \Big[\langle \sigma'(u_{mx}(s))u_m(s)\Delta u_m(s), u_{mx}(s)\rangle \\ &+ \langle \sigma(u_{mx}(s)), |u_{mx}(s)|^2\rangle\Big] ds + 2\int_0^t \langle f_x(s), u_{mx}(s)\rangle ds \\ &= P_m(0) + I_1 + I_2, \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} P_m(t) &= \|u_{mx}(t)\|^2 + \|\Delta u_m(t)\|^2 + 2\int_0^t (\|u_{mx}(s)\|^2 + \|\Delta u_m(s)\|^2) ds \\ &+ 2\int_0^t \langle \mu(u_{mx}(s)), |\Delta u_m(s)|^2\rangle ds. \end{aligned} \tag{3.17}$$

From $u_{0m} \to \tilde{u}_0$ strongly in $H_0^1 \cap H^2$, we deduce

$$P_m(0) = \|u_{mx}(0)\|^2 + \|\triangle u_m(0)\|^2 = \|u_{0mx}\|^2 + \|\triangle u_{0m}\|^2 \le \bar{P}_0 \quad \forall m \in \mathbb{N}, \tag{3.18}$$

where $\bar{P}_0$ always indicates a constant depending on $\tilde{u}_0$.

Estimating I_1. Note that

$$\begin{aligned} &-2\langle \sigma'(u_{mx}(s))u_{mx}(s)\Delta u_m(s), u_m(s)\rangle \\ &= -2\int_0^1 \sigma'(u_{mx}(x,s))u_{mx}(x,s)\Delta u_m(x,s)u_m(x,s)\, dx \\ &= -2\int_0^1 u_m(x,s)\frac{\partial}{\partial x}\Big(\int_0^{u_{mx}(x,s)} z\sigma'(z)dz\Big)\, dx \\ &= -2\Big[u_m(x,s)\Big(\int_0^{u_{mx}(x,s)} z\sigma'(z)dz\Big)\Big|_0^1 \\ &\quad - \int_0^1 u_{mx}(x,s)\Big(\int_0^{u_{mx}(x,s)} z\sigma'(z)dz\Big)\, dx\Big] \\ &= 2\int_0^1 u_{mx}(x,s)\Big(\int_0^{u_{mx}(x,s)} z\sigma'(z)dz\Big)\, dx \\ &\le 2\int_0^1 u_{mx}^2(x,s)\sigma(u_{mx}(x,s))\, dx \\ &= 2\langle \sigma(u_{mx}(s)), |u_{mx}(s)|^2\rangle, \end{aligned} \tag{3.19}$$

since $y\big(\int_0^y z\sigma'(z)dz\big) \le y^2\sigma(y)$ for all $y \in \mathbb{R}$. Hence

$$\begin{aligned} I_1 = -2\int_0^t &\Big[\langle \sigma'(u_{mx}(s))u_m(s)\Delta u_m(s), u_{mx}(s)\rangle \\ &+ \langle \sigma(u_{mx}(s)), |u_{mx}(s)|^2\rangle\Big] ds \le 0. \end{aligned} \tag{3.20}$$

Estimating I_2.

$$
\begin{aligned}
I_2 = 2\int_0^t \langle f_x(s), u_{mx}(s)\rangle ds &\le \int_0^T \|f_x(s)\|\|u_{mx}(s)\|ds \\
&\le \int_0^T \|f_x(s)\|\sqrt{S_m(s)}ds \le \sqrt{C_T^{(1)}}\int_0^T \|f_x(s)\|ds.
\end{aligned} \tag{3.21}
$$

It follows from (3.16), (3.18), (3.20), (3.21) that

$$
P_m(t) \le \bar{P}_0 + \sqrt{C_T^{(1)}}\int_0^T \|f_x(s)\|ds \le C_T^{(2)}. \tag{3.22}
$$

(c) Third estimate. Multiplying the j^{th} equation of $(3.7)_1$ by $c'_{mj}(t)$ and summing up with respect to j, afterwards, integrating with respect to the time variable from 0 to t, we obtain after some rearrangements

$$
\begin{aligned}
Q_m(t) &= Q_m(0) - 2\int_0^t \langle \sigma(u_{mx}(s))u_m(s), u'_m(s)\rangle ds + 2\int_0^t \langle f(s), u'_m(s)\rangle ds \\
&= Q_m(0) + J_1 + J_2,
\end{aligned} \tag{3.23}
$$

where

$$
\begin{gathered}
Q_m(t) = \|u_m(t)\|_{H^1}^2 + 2\int_0^t \|u'_m(s)\|_{H^1}^2 ds + 2\int_0^1 \tilde{\mu}(u_{mx}(x,t))\,dx, \\
\tilde{\mu}(z) = \int_0^z \bar{\mu}(y)dy \ge 0 \quad \forall z \in \mathbb{R}.
\end{gathered} \tag{3.24}
$$

Estimating $Q_m(0)$. From $u_{0m} \to \tilde{u}_0$ strongly in $H_0^1 \cap H^2$, we can deduce the existence of a constant $\bar{Q}_0 > 0$ independent of m such that

$$
Q_m(0) = \|u_{0m}\|_{H^1}^2 + 2\int_0^1 \tilde{\mu}(u_{0mx}(x))\,dx \le \bar{Q}_0 \quad \forall m \in \mathbb{N}. \tag{3.25}
$$

Estimating J_1. By (3.22), we have

$$
\begin{aligned}
|u_{mx}(x,s)| &\le \|u_{mx}(s)\|_{C^0([0,1])} \le \sqrt{2}\|u_{mx}(s)\|_{H^1} \\
&\le \sqrt{2}\sqrt{\|u_{mx}(s)\|^2 + \|\Delta u_m(s)\|^2} \le \sqrt{2}\sqrt{2\|\Delta u_m(s)\|^2} \\
&\le 2\|\Delta u_m(s)\| \le 2\sqrt{P_m(s)} \le 2\sqrt{C_T^{(2)}}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
J_1 &= -2\int_0^t \langle \sigma(u_{mx}(s))u_m(s), u_m'(s)\rangle ds \\
&\le 2 \sup_{|z|\le 2\sqrt{C_T^{(2)}}} \sigma(z) \int_0^t \|u_m(s)\|\|u_m'(s)\| ds \\
&\le 2 \sup_{|z|\le 2\sqrt{C_T^{(2)}}} \sigma(z) \int_0^t \sqrt{S_m(s)}\|u_m'(s)\| ds \\
&\le 2\sqrt{C_T^{(1)}} \sup_{|z|\le 2\sqrt{C_T^{(2)}}} \sigma(z) \int_0^t \|u_m'(s)\| ds \\
&\le 2TC_T^{(1)} \sup_{|z|\le 2\sqrt{C_T^{(2)}}} \sigma^2(z) + \frac{1}{2}\int_0^t \|u_m'(s)\|^2 ds \\
&\le 2TC_T^{(1)} \sup_{|z|\le 2\sqrt{C_T^{(2)}}} \sigma^2(z) + \frac{1}{4}Q_m(t).
\end{aligned}
\tag{3.26}
$$

Estimating J_2.

$$
\begin{aligned}
J_2 &= 2\int_0^t \langle f(s), u_m'(s)\rangle ds \\
&\le 2\int_0^T \|f(s)\|^2 ds + \frac{1}{2}\int_0^t \|u_m'(s)\|^2 ds \\
&\le 2\int_0^T \|f(s)\|^2 ds + \frac{1}{4}Q_m(t).
\end{aligned}
\tag{3.27}
$$

Then, it follows from (3.23), (3.25)–(3.27) that

$$
Q_m(t) \le 2\Big(\bar{Q}_0 + 2TC_T^{(1)} \sup_{|z|\le 2\sqrt{C_T^{(2)}}} \sigma^2(z) + 2\int_0^T \|f(s)\|^2 ds\Big) \le C_T^{(3)}. \tag{3.28}
$$

Step 3: Limiting process. Thanks to (3.14), (3.22), (3.28) there exists a subsequence of $\{u_m\}$, still denoted by $\{u_m\}$ such that

$$
\begin{gathered}
u_m \to u \quad \text{in } L^\infty(0,T;H_0^1\cap H^2) \text{ weakly*}, \\
u_m' \to u' \quad \text{in } L^2(0,T;H_0^1) \text{ weakly}.
\end{gathered}
\tag{3.29}
$$

Using the compactness lemma of Lions [9, p.57], and applying Fischer-Riesz theorem, from (3.29), there exists a subsequence of $\{u_m\}$, denoted by the same symbol satisfying

$$
\begin{gathered}
u_m \to u \quad \text{strongly in } L^2(0,T;H_0^1) \text{ and a.e. in } Q_T, \\
u_{mx} \to u_x \quad \text{strongly in } L^2(Q_T) \text{ and a.e. in } Q_T.
\end{gathered}
\tag{3.30}
$$

Then, it follows from (3.30), that

$$
\begin{gathered}
\bar{\mu}(u_{mx}(x,t)) \to \bar{\mu}(u_x(x,t)) \quad \text{a.e., } (x,t) \text{ in } Q_T, \\
\sigma(u_{mx}(x,t))u_m(x,t) \to \sigma(u_x(x,t))u(x,t) \quad \text{a.e., } (x,t) \text{ in } Q_T.
\end{gathered}
\tag{3.31}
$$

On the other hand, by (3.22), we have

$$
\begin{aligned}
|u_{mx}(x,t)| &\leq \|u_{mx}(t)\|_{C^0([0,1])} \leq \sqrt{2}\|u_{mx}(t)\|_{H^1} \\
&\leq 2\|\triangle u_m(t)\| \leq 2\sqrt{P_m(t)} \leq 2\sqrt{C_T^{(2)}}; \\
|\bar{\mu}(u_{mx}(x,t))| &\leq \sup_{|z|\leq 2\sqrt{C_T^{(2)}}} |\bar{\mu}(z)| \leq C_T; \\
|\sigma(u_{mx}(x,t))u_m(x,t)| &\leq \|u_{mx}(t)\||\sigma(u_{mx}(x,t))| \\
&\leq \sqrt{C_T^{(2)}} \sup_{|z|\leq 2\sqrt{C_T^{(2)}}} |\sigma(z)| \leq C_T.
\end{aligned}
\tag{3.32}
$$

Applying the dominated convergence theorem, from (3.31), (3.32) we obtain

$$
\begin{gathered}
\bar{\mu}(u_{mx}) \to \bar{\mu}(u_x) \quad \text{strongly in } L^2(Q_T), \\
\sigma(u_{mx})u_m \to \sigma(u_x)u \quad \text{strongly in } L^2(Q_T).
\end{gathered}
\tag{3.33}
$$

Passing to the limit in (3.7) by (3.8), (3.29), (3.30) and (3.33), we have u satisfying

$$
\begin{gathered}
\langle u_t(t), w\rangle + \langle u_{xt}(t), w_x\rangle + \langle u_x(t) + \bar{\mu}(u_x(t)), w_x\rangle + \langle (1+\sigma(u_x(t)))u(t), w\rangle \\
= \langle f(t), w\rangle, \quad \forall w \in H_0^1, \\
u(0) = \tilde{u}_0.
\end{gathered}
\tag{3.34}
$$

Furthermore,

$$
u \in L^\infty(0,T; H_0^1 \cap H^2), \quad u' \in L^2(0,T; H_0^1).
$$

Step 4: Uniqueness of the solution. Let u and v be two weak solutions of (3.1) such that

$$
u, v \in L^\infty(0,T; H_0^1 \cap H^2), \quad u', v' \in L^2(0,T; H_0^1). \tag{3.35}
$$

Then $w = u - v$ satisfies

$$
\begin{gathered}
\langle w_t(t), y\rangle + \langle w_{xt}(t), y_x\rangle + \langle w_x(t), y_x\rangle + \langle w(t), y\rangle \\
+ \langle \bar{\mu}(u_x(t)) - \bar{\mu}(v_x(t)), y_x\rangle + \langle \sigma(u_x(t))u - \sigma(v_x(t))v, y\rangle = 0, \quad \forall y \in H_0^1, \\
w(0) = 0, \\
u, v, w \in L^\infty(0,T; H_0^1 \cap H^2), \quad u_t, v_t, w_t \in L^2(0,T; H_0^1).
\end{gathered}
\tag{3.36}
$$

Take $y = w = u - v$, in $(3.36)_1$ and integrating with respect to t, we obtain

$$
\begin{aligned}
\rho(t) &= -2\int_0^t \langle \bar{\mu}(u_x(s)) - \bar{\mu}(v_x(s)), w_x(s)\rangle ds \\
&\quad - 2\int_0^t \langle \sigma(u_x(s))u(s) - \sigma(v_x(s))v(s), w(s)\rangle ds \\
&= \rho_1(t) + \rho_2(t),
\end{aligned}
\tag{3.37}
$$

where

$$
\rho(t) = \|w(t)\|_{H^1}^2 + 2\int_0^t \|w(s)\|_{H^1}^2 ds. \tag{3.38}
$$

Estimating $\rho_1(t)$. Using the monotonicity of the function $z \mapsto \bar{\mu}(z)$, we obtain

$$\rho_1(t) = -2\int_0^t \langle \bar{\mu}(u_x(s)) - \bar{\mu}(v_x(s)), w_x(s)\rangle ds \le 0. \tag{3.39}$$

Estimating $\rho_2(t)$. We have

$$\begin{aligned} w &= [\sigma(u_x)w + (\sigma(u_x) - \sigma(v_x))v]w \\ &= \sigma(u_x)w^2 + (\sigma(u_x) - \sigma(v_x))vw \\ &\ge (\sigma(u_x) - \sigma(v_x))vw. \end{aligned} \tag{3.40}$$

This implies

$$\begin{aligned} \rho_2(t) &= -2\int_0^t \langle \sigma(u_x(s))u(s) - \sigma(v_x(s))v(s), w(s)\rangle ds \\ &\le -2\int_0^t \langle [\sigma(u_x(s)) - \sigma(v_x(s))]\, v(s), w(s)\rangle ds \\ &\le 2\int_0^t \|\, [\sigma(u_x(s)) - \sigma(v_x(s))]\, v(s)\| \|w(s)\| ds \\ &\le 2\int_0^t \|\sigma(u_x(s)) - \sigma(v_x(s))\| \|v_x(s)\| \|w(s)\| ds. \end{aligned} \tag{3.41}$$

Put $M = \|u\|_{L^\infty(0,T;H_0^1\cap H^2)} + \|v\|_{L^\infty(0,T;H_0^1\cap H^2)}$ and $L_M = \sup_{|z|\le M} |\sigma'(z)|$, we have

$$|\sigma(u_x) - \sigma(v_x)| \le L_M |w_x|. \tag{3.42}$$

Hence

$$\begin{aligned} \rho_2(t) &\le 2L_M \int_0^t \|w_x(s)\| \|v_x(s)\| \|w(s)\| ds \\ &\le 2ML_M \int_0^t \|w_x(s)\| \|w(s)\| ds \\ &\le ML_M \int_0^t \rho(s) ds. \end{aligned} \tag{3.43}$$

Then, from (3.37), (3.39), (3.43) it follows that

$$\rho(t) \le ML_M \int_0^t \rho(s) ds. \tag{3.44}$$

By Gronwall's lemma, (3.44) leads to $\rho(t) = 0$, i.e., $w = u - v = 0$.

Step 5: Proof of the estimate(3.5). First, let us assume that

$$u_0(x) \le M, \quad \text{a. e., } x \in \Omega, \text{ and } \max\{\|\tilde{u}_0\|_{L^\infty}, \|f\|_{L^\infty(Q_T)}\} \le M. \tag{3.45}$$

Then $z = u - M$ satisfies the initial and boundary value

$$\begin{gathered} z_t - z_{xxt} - \frac{\partial}{\partial x}(z_x + \bar{\mu}(z_x)) + z + (z + M)\sigma(z_x) \\ = f(x,t) - M, \quad 0 < x < 1,\ 0 < t < T, \\ z(0,t) = z(1,t) = -M, \\ z(x,0) = \tilde{u}_0(x) - M. \end{gathered} \tag{3.46}$$

Multiplying equation (3.46) by $v \in H_0^1$, then integrating by parts with respect to variable x, after some rearrangements, one has

$$\begin{aligned}
&\langle z_t(t), v\rangle + \langle z_{xt}(t), v_x\rangle + \langle z_x(t) + \bar{\mu}(z_x(t)), v_x\rangle \\
&+ \langle z(t) + (z(t)+M)\sigma(z_x(t)), v\rangle \\
&= \langle f(t) - M, v\rangle, \quad \text{for all } v \in H_0^1.
\end{aligned} \tag{3.47}$$

From assumption (H1)–(H4) we deduce that the solution of the initial and boundary value problem (3.1) satisfies $u \in L^\infty(0,T; H_0^1 \cap H^2)$, $u' \in L^2(0,T; H_0^1)$, so that we are allowed to take $v = z^+ = \frac{1}{2}(|z| + z)$ in (3.47)). Thus, it follows that

$$\begin{aligned}
&\langle z_t(t), z^+(t)\rangle + \langle z_{xt}(t), z_x^+(t)\rangle + \langle z_x(t) + \bar{\mu}(z_x(t)), z_x^+(t)\rangle \\
&+ \langle z(t) + (z(t)+M)\sigma(z_x(t)), z^+(t)\rangle \\
&= \langle f(t) - M, z^+(t)\rangle.
\end{aligned} \tag{3.48}$$

Hence

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|z^+(t)\|^2 + \|z_x^+(t)\|^2) + \|z_x^+(t)\|^2 + \|z^+(t)\|^2 \\
&= -\langle \bar{\mu}(z_x^+(t)), z_x^+(t)\rangle - \langle (z^+(t)+M)\sigma(z_x^+(t)), z^+(t)\rangle \\
&\quad + \langle f(t) - M, z^+(t)\rangle \le 0,
\end{aligned} \tag{3.49}$$

since $M \ge \max\{\|\tilde{u}_0\|_{L^\infty}, \|f\|_{L^\infty(Q_T)}\}$ and

$$\begin{aligned}
\langle z_t(t), z^+(t)\rangle &= \int_0^1 z_t(x,t) z^+(x,t)\, dx = \int_{0,z>0}^1 (z^+(x,t))_t\, z^+(x,t)\, dx \\
&= \frac{1}{2}\frac{d}{dt}\int_{0,z>0}^1 |z^+(x,t)|^2\, dx = \frac{1}{2}\frac{d}{dt}\int_0^1 |z^+(x,t)|^2\, dx \\
&= \frac{1}{2}\frac{d}{dt}\|z^+(t)\|^2,
\end{aligned} \tag{3.50}$$

and on the domain $z > 0$ we have $z^+ = z$, $z_x = (z^+)_x$ and $z_t = (z^+)_t$.

Integrating (3.49), we obtain

$$\|z^+(t)\|^2 + \|z_x^+(t)\|^2 \le \|z^+(0)\|^2 + \|z_x^+(0)\|^2. \tag{3.51}$$

Since $z^+(x,0) = (u(x,0) - M)^+ = (\tilde{u}_0(x) - M)^+ = 0$, $z_x^+(x,0) = 0$, we obtain $\|z^+(t)\|^2 + \|z_x^+(t)\|^2 = 0$. Thus $z^+ = 0$ and $u(x,t) \le M$, for a.e. $(x,t) \in Q_T$.

The case $-M \le u_0(x)$, a.e., $x \in \Omega$, and $M \ge \max\{\|\tilde{u}_0\|_{L^\infty}, \|f\|_{L^\infty(Q_T)}\}$ can be dealt with, in the same manner as above, by considering $z = u + M$ and $z^- = \frac{1}{2}(|z| - z)$, we also obtain $z^- = 0$ and hence $u(x,t) \ge -M$, for a.e. $(x,t) \in Q_T$.

From the above, one obtains $|u(x,t)| \le M$, a.e. $(x,t) \in Q_T$, i.e.,

$$\|u\|_{L^\infty(Q_T)} \le M, \tag{3.52}$$

for all $M \ge \max\{\|\tilde{u}_0\|_{L^\infty}, \|f\|_{L^\infty(Q_T)}\}$. This implies (3.5). The proof is complete. $\square$

4. Exponential decay of solutions

This section investigates the decay of the solution of (3.1). For this purpose, we make the following assumption.

- (H5) $f \in L^2(\mathbb{R}_+; H_0^1)$ and there exist two constants $C_0 > 0$, $\gamma_0 > 0$ such that $\|f(t)\| \le C_0 e^{-\gamma_0 t}$, for all $t \ge 0$.

Theorem 4.1. *Assume that* (H1), (H3)–(H5) *hold. Then, problem* (3.1) *has a unique weak solution u satisfying*

$$u \in L^\infty(0,T;H_0^1 \cap H^2), \quad u' \in L^2(0,T;H_0^1) \quad \textit{for all } T > 0, \tag{4.1}$$

and there exist positive constants C, γ such that

$$\|u(t)\|_{H^1} \le C\exp(-\gamma t) \quad \textit{for all } t \ge 0. \tag{4.2}$$

Proof. Multiplying the j^{th} equation of $(3.7)_1$ by $c_{mj}(t)$ and summing with respect to j, after some rearrangements, we obtain

$$\begin{aligned}&\frac{d}{dt}\|u_m(t)\|_{H^1}^2 + 2\|u_m(t)\|_{H^1}^2 + 2\langle\bar{\mu}(u_{mx}(t)), u_{mx}(t)\rangle + 2\langle\sigma(u_{mx}(t)), u_m^2(t)\rangle \\ &= 2\langle f(t), u_m(t)\rangle.\end{aligned} \tag{4.3}$$

Note that

$$\begin{aligned}2\langle f(t), u_m(t)\rangle &\le 2\|f(t)\|\|u_m(t)\| \le 2\|f(t)\|\|u_m(t)\|_{H^1} \\ &\le \frac{1}{2\delta}\|f(t)\|^2 + 2\delta\|u_m(t)\|_{H^1}^2,\end{aligned} \tag{4.4}$$

for all $\delta > 0$.

It follows from (4.3), (4.4) that

$$\begin{aligned}&\frac{d}{dt}\|u_m(t)\|_{H^1}^2 + 2(1-\delta)\|u_m(t)\|_{H^1}^2 \\ &\le \frac{1}{2\delta}\|f(t)\|^2 \le \frac{1}{2\delta}C_0^2 e^{-2\gamma_0 t}, \quad \text{for all } \delta > 0.\end{aligned} \tag{4.5}$$

Choose δ and γ such that

$$0 < \delta < 1, \quad 0 < \gamma < \min\{1-\delta, \gamma_0\}. \tag{4.6}$$

Then from (4.5), (4.6) we have

$$\frac{d}{dt}\|u_m(t)\|_{H^1}^2 + 2\gamma\|u_m(t)\|_{H^1}^2 \le \frac{1}{2\delta}C_0^2 e^{-2\gamma_0 t}. \tag{4.7}$$

Integrating (4.7), we obtain

$$\|u_m(t)\|_{H^1}^2 \le \Big(\|\tilde{u}_0\|_{H^1}^2 + \frac{C_0^2}{4\delta(\gamma_0 - \gamma)}\Big)e^{-2\gamma t}. \tag{4.8}$$

Letting $m \to +\infty$ in (4.8), we obtain

$$\begin{aligned}\|u(t)\|_{H^1}^2 &\le \liminf_{m\to+\infty}\|u_m(t)\|_{H^1}^2 \\ &\le \Big(\|\tilde{u}_0\|_{H^1}^2 + \frac{C_0^2}{4\delta(\gamma_0 - \gamma)}\Big)e^{-2\gamma t}, \quad \text{for all } t \ge 0.\end{aligned} \tag{4.9}$$

This implies (4.2), and completes the proof. □

5. Existence and uniqueness of a T-periodic weak solution

In this section, we shall consider problem (1.1), (1.2), (1.4) with the constants $\alpha = \beta = \gamma = 1$,

$$\begin{gathered}u_t - u_{xxt} - (1+\mu(u_x))u_{xx} + (1+\sigma(u_x))u = f(x,t), \quad 0 < x < 1,\ 0 < t < T, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = u(x,T).\end{gathered} \tag{5.1}$$

We make the following assumptions:

(H6) f is T-periodic in t, i.e., $f(x,0)=f(x,T)$.

Remark 5.1. The weak formulation of problem (5.1) can be given in the following manner: Find $u\in L^\infty(0,T;H_0^1\cap H^2)$ with $u'\in L^2(0,T;H_0^1)$, such that u satisfies the variational equation

$$\begin{gathered}
\int_0^T \langle u'(t)+u(t),w(t)\rangle\,dt+\int_0^T\langle u_x'(t)+u_x(t),w_x(t)\rangle\,dt\\
+\int_0^T\langle\bar\mu(u_x(t)),w_x(t)\rangle\,dt+\int_0^T\langle\sigma(u(t),u_x(t))u(t),w(t)\rangle\,dt\\
=\int_0^T\langle f(t),w(t)\rangle\,dt,\quad \text{for all } w\in L^2(0,T;H_0^1),\\
u(0)=u(T).
\end{gathered}\tag{5.2}$$

Theorem 5.2. *Let $T>0$ and* (H2), (H3), (H4), (H6) *hold. Then problem* (5.1) *has a weak solution u such that*

$$u\in L^\infty(0,T;H_0^1\cap H^2)\ \text{and}\ u'\in L^2(0,T;H_0^1).\tag{5.3}$$

Furthermore, if $\|u\|_{L^\infty(0,T;H_0^1\cap H^2)}\le R$, with $R\sup_{|z|\le\sqrt{2}R}|\sigma'(z)|<2$, then the solution is unique.

Proof. The proof consists of several steps.

Step 1: Consider the basis $\{w_j\}$ as above. Let W_m be the linear space generated by $w_1,w_2,\dots,w_m$. We consider the following problem.

Find a function $u_m(t)$ in the form (3.6) satisfying the nonlinear differential equation system $(3.7)_1$ and the T-periodic condition

$$u_m(0)=u_m(T).\tag{5.4}$$

We consider an initial value problem given by (3.7), where u_{0m} is given in W_m.

It is clear that for each m, there exists a solution $u_m(t)$ in the form (3.6) which satisfies (3.7) almost everywhere on $0\le t\le T_m$ for some T_m, $0<T_m\le T$. The following a priori estimates allow us to take $T_m=T$ for all m.

Step 2: A priori estimates. Multiplying the j^{th} equation of $(3.7)_1$ by $c_{mj}(t)$ and summing with respect to j, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|u_m(t)\|_{H^1}^2+2\|u_m(t)\|_{H^1}^2+2\langle\bar\mu(u_{mx}(t)),u_{mx}(t)\rangle\\
&+2\|\sqrt{\sigma(u_{mx}(t))}u_m(t)\|^2\\
&=2\langle f(t),u_m(t)\rangle.
\end{aligned}\tag{5.5}$$

We estimate without difficulty the term $2\langle f(t),u_m(t)\rangle$ as follows

$$2\langle f(t),u_m(t)\rangle\le\frac{1}{2\delta_1}\|f(t)\|^2+2\delta_1\|u_m(t)\|^2\le\frac{1}{2\delta_1}\|f(t)\|^2+2\delta_1\|u_m(t)\|_{H^1}^2,\tag{5.6}$$

for all δ_1, $0<\delta_1<1$.

Hence, from (5.5), (5.6) it follows that

$$
\begin{aligned}
&\frac{d}{dt}\|u_m(t)\|_{H^1}^2 + 2(1-\delta_1)\|u_m(t)\|_{H^1}^2 + 2\langle\bar{\mu}(u_{mx}(t)), u_{mx}(t)\rangle \\
&+ 2\|\sqrt{\sigma(u_{mx}(t))}u_m(t)\|^2 \\
&\leq \frac{1}{2\delta_1}\|f(t)\|^2.
\end{aligned}
\tag{5.7}
$$

Next, multiplying the j^{th} equation of (3.14) by $c_{mj}(t)$ and summing with respect to j, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\|u_{mx}(t)\|_{H^1}^2 + 2\|u_{mx}(t)\|_{H^1}^2 + 2\|\sqrt{\mu(u_{mx}(t))}\triangle u_m(t)\|^2 \\
&+ 2\langle\sigma'(u_{mx}(t))u_m(t)\Delta u_m(t) + \sigma(u_{mx}(t))u_{mx}(t), u_{mx}(t)\rangle \\
&= 2\langle f_x(t), u_{mx}(t)\rangle.
\end{aligned}
\tag{5.8}
$$

Similarly, we have

$$
\begin{aligned}
&2\langle\sigma'(u_{mx}(t))u_m(t)\Delta u_m(t) + \sigma(u_{mx}(t))u_{mx}(t), u_{mx}(t)\rangle \\
&= 2\int_0^1 u_m(x,t)u_{mx}(x,t)\sigma'(u_{mx}(x,t))\Delta u_m(x,t)\,dx \\
&\quad + 2\int_0^1 u_{mx}^2(x,t)\sigma(u_{mx}(x,t))\,dx \\
&= 2\int_0^1 u_m(x,t)\frac{\partial}{\partial x}\Big(\int_0^{u_{mx}(x,t)} y\sigma'(y)\Big)\,dx + 2\int_0^1 u_{mx}^2(x,t)\sigma(u_{mx}(x,t))\,dx \\
&= -2\int_0^1 u_{mx}(x,t)\Big(\int_0^{u_{mx}(x,t)} y\sigma'(y)\Big)\,dx + 2\int_0^1 u_{mx}^2(x,t)\sigma(u_{mx}(x,t))\,dx \\
&= 2\int_0^1 \Big[u_{mx}^2(x,t)\sigma(u_{mx}(x,t)) - u_{mx}(x,t)\Big(\int_0^{u_{mx}(x,t)} y\sigma'(y)\Big)\Big]\,dx \geq 0,
\end{aligned}
\tag{5.9}
$$

and this implies

$$
\begin{aligned}
&\frac{d}{dt}\|u_{mx}(t)\|_{H^1}^2 + 2(1-\delta_1)\|u_{mx}(t)\|_{H^1}^2 + 2\|\sqrt{\mu(u_{mx}(t))}\triangle u_m(t)\|^2 \\
&\leq \frac{1}{2\delta_1}\|f_x(t)\|^2,
\end{aligned}
\tag{5.10}
$$

for all δ_1, $0 < \delta_1 < 1$.

It follows from (5.7), (5.10) that

$$
\begin{aligned}
&\frac{d}{dt}\left[\|u_m(t)\|_{H^1}^2 + \|u_{mx}(t)\|_{H^1}^2\right] + 2(1-\delta_1)(\|u_m(t)\|_{H^1}^2 + \|u_{mx}(t)\|_{H^1}^2) \\
&\leq \frac{1}{2\delta_1}\|f(t)\|_{H^1}^2.
\end{aligned}
\tag{5.11}
$$

Integrating (5.11), we have

$$\begin{aligned}
&\|u_m(t)\|_{H^1}^2 + \|u_{mx}(t)\|_{H^1}^2 \\
&\le \Big(\|u_{0m}\|_{H^1}^2 + \|u_{0mx}\|_{H^1}^2 - R^2\Big)e^{-2(1-\delta_1)t} \\
&\quad + \Big(R^2 + \frac{1}{2\delta_1}\int_0^t e^{2(1-\delta_1)s}\|f(s)\|_{H^1}^2 ds\Big)e^{-2(1-\delta_1)t} \\
&\le \Big(\|u_{0m}\|_{H^1}^2 + \|u_{0mx}\|_{H^1}^2 - R^2\Big)e^{-2(1-\delta_1)t} + R^2,
\end{aligned} \tag{5.12}$$

where $R^2 = \sup_{0\le t\le T} R_1(t)$,

$$R_1(t) = \begin{cases} \frac{1}{2\delta_1}\frac{1}{e^{2(1-\delta_1)t}-1}\int_0^t e^{2(1-\delta_1)s}\|f(s)\|_{H^1}^2 ds, & 0 < t \le T, \\ \frac{1}{4\delta_1(1-\delta_1)}\|f(0)\|_{H^1}^2, & t = 0. \end{cases} \tag{5.13}$$

Therefore, if we choose u_{0m} such that $\|u_{0m}\|_{H^1}^2 + \|u_{0mx}\|_{H^1}^2 \le R^2$, we obtain from (5.12) that

$$\|u_m(t)\|_{H^1}^2 + \|u_{mx}(t)\|_{H^1}^2 \le R^2, \quad \text{i.e., } T_m = T \text{ for all } m. \tag{5.14}$$

Let $\bar{B}_m(0,R)$ be a closed ball in the space W_m of linear combinations of the functions $w_1, w_2, \dots, w_m$, with the norm

$$\|u_{0m}\|_* = \sqrt{\|u_{0m}\|_{H^1}^2 + \|u_{0mx}\|_{H^1}^2}.$$

Let us define

$$\begin{gathered}
\mathcal{F}_m : \bar{B}_m(0,R) \to \bar{B}_m(0,R) \\
u_{0m} \mapsto u_m(T).
\end{gathered} \tag{5.15}$$

We prove that $\mathcal{F}_m$ is continuous. Let $u_{0m}, \bar{u}_{0m} \in \bar{B}_m(0,R)$ and let $y_m(t) = u_m(t) - \bar{u}_m(t)$, where $u_m(t)$ and $\bar{u}_m(t)$ are solutions of the system $(3.7)_1$ on $[0,T]$ satisfying the initial conditions $u_m(0) = u_{0m}$ and $\bar{u}_m(0) = \bar{u}_{0m}$, respectively. Then, $y_m(t)$ satisfies the differential equation

$$\begin{aligned}
&\langle y_m'(t) + y_m(t), w_j\rangle + \langle y_{mx}'(t) + y_{mx}(t), w_{jx}\rangle \\
&+ \langle \bar{\mu}(u_{mx}(t)) - \bar{\mu}(\bar{u}_{mx}(t)), w_{jx}\rangle \\
&+ \langle \sigma(u_{mx}(t))u_m(t) - \sigma(\bar{u}_{mx}(t))\bar{u}_m(t), w_j\rangle = 0,
\end{aligned} \tag{5.16}$$

$1 \le j \le m$, with initial condition

$$y_m(0) = u_{0m} - \bar{u}_{0m}. \tag{5.17}$$

Using the same arguments as before, we can show that

$$\begin{aligned}
&\frac{d}{dt}\|y_m(t)\|_{H^1}^2 + 2\|y_m(t)\|_{H^1}^2 + 2\langle \bar{\mu}(u_{mx}(t)) - \bar{\mu}(\bar{u}_{mx}(t)), y_{mx}(t)\rangle \\
&+ 2\langle \sigma(u_{mx}(t))u_m(t) - \sigma(\bar{u}_{mx}(t))\bar{u}_m(t), y_m(t)\rangle = 0.
\end{aligned} \tag{5.18}$$

On the other hand, we have

$$\langle \bar{\mu}(u_{mx}(t)) - \bar{\mu}(\bar{u}_{mx}(t)), y_{mx}(t)\rangle \ge 0; \tag{5.19}$$

$$\begin{aligned}
&2\langle \sigma(u_{mx}(t))u_m(t) - \sigma(\bar{u}_{mx}(t))\bar{u}_m(t), y_m(t)\rangle \\
&= 2\|\sqrt{\sigma(u_{mx}(t))}y_m(t)\|^2 + 2\langle \sigma(u_{mx}(t)) - \sigma(\bar{u}_{mx}(t)), \bar{u}_m(t)y_m(t)\rangle.
\end{aligned} \tag{5.20}$$

Putting $\tilde{K}_R = \sup_{|z|\leq\sqrt{2}R} |\sigma'(z)|$, we have

$$\begin{aligned} &2\langle\sigma(u_{mx}(t)) - \sigma(\bar{u}_{mx}(t)), \bar{u}_m(t)y_m(t)\rangle \\ &\leq 2\|\bar{u}_{mx}(t)\|\|y_m(t)\|\|\sigma(u_{mx}(t)) - \sigma(\bar{u}_{mx}(t))\| \\ &\leq 2\tilde{K}_R\|\bar{u}_{mx}(t)\|\|y_m(t)\|\|y_{mx}(t)\| \\ &\leq \tilde{K}_R\|\bar{u}_{mx}(t)\|\|y_m(t)\|^2_{H^1} \leq R\tilde{K}_R\|y_m(t)\|^2_{H^1}. \end{aligned} \tag{5.21}$$

It follows from (5.18)-(5.21) that

$$\frac{d}{dt}\|y_m(t)\|^2_{H^1} + (2 - R\tilde{K}_R)\|y_m(t)\|^2_{H^1} \leq 0. \tag{5.22}$$

Integrating inequality (5.22), we obtain

$$\|y_m(T)\|^2_{H^1} \leq e^{(R\tilde{K}_R-2)T}\|u_{0m} - \bar{u}_{0m}\|^2_{H^1},$$

or

$$\|\mathcal{F}_m(u_{0m}) - \mathcal{F}_m(\bar{u}_{0m})\|_{H^1} \leq \exp\Big(\big(\frac{1}{2}R\tilde{K}_R - 1\big)T\Big)\|u_{0m} - \bar{u}_{0m}\|_{H^1}. \tag{5.23}$$

Note that, on W_m, $\|v_{0m}\|_{H^1}$ and $\|v_{0m}\|_* = \sqrt{\|v_{0m}\|^2_{H^1} + \|v_{0mx}\|^2_{H^1}}$ are equivalent norms, hence, there exist two constants $D_{1m} > 0$, $D_{2m} > 0$ such that

$$D_{1m}\|v_{0m}\|_* \leq \|v_{0m}\|_{H^1} \leq D_{2m}\|v_{0m}\|_* \quad \text{for all } v_{0m} \in W_m. \tag{5.24}$$

It follows from (5.23), (5.24) that

$$\|\mathcal{F}_m(u_{0m}) - \mathcal{F}_m(\bar{u}_{0m})\|_* \leq \frac{D_{2m}}{D_{1m}}\exp((\frac{1}{2}R\tilde{K}_R - 1)T)\|u_{0m} - \bar{u}_{0m}\|_* \tag{5.25}$$

for all u_{0m}, $\bar{u}_{0m} \in W_m$.

Hence, $\mathcal{F}_m : \bar{B}_m(0,R) \to \bar{B}_m(0,R)$ is continuous. Applying the fixed point theorem of Brouwer, we have (for every m) a function $u_{0m} \in \bar{B}_m(0,R)$ such that the solution of the initial value problem (3.7) is a T-periodic solution of the system $(3.7)_1$. This solution satisfies the inequality (5.14) a.e., in $[0,T]$ and consequently, by (5.11) we have

$$\begin{aligned} &\|u_m(t)\|^2_{H^1} + \|u_{mx}(t)\|^2_{H^1} + 2(1-\delta_1)\int_0^t (\|u_m(s)\|^2_{H^1} + \|u_{mx}(s)\|^2_{H^1})ds \\ &\leq R^2 + \frac{1}{2\delta_1}\int_0^T \|f(s)\|^2_{H^1}ds \leq C_T. \end{aligned} \tag{5.26}$$

On the other hand, we multiplying the j^{th} equation of $(3.7)_1$ by $c'_{mj}(t)$ and summing up with respect to j, afterwards, integrating with respect to the time variable from 0 to T, we obtain after some rearrangements

$$\begin{aligned} &2\int_0^T \|u'_m(t)\|^2_{H^1}dt + \int_0^T \frac{d}{dt}\Big[\|u_m(t)\|^2_{H^1} + 2\int_0^1 \tilde{\mu}(u_{mx}(x,t))\,dx\Big]\,dt \\ &+ 2\int_0^T \langle\sigma(u_{mx}(t))u_m(t), u'_m(t)\rangle\,dt \\ &= 2\int_0^T \langle f(t), u'_m(t)\rangle\,dt, \end{aligned} \tag{5.27}$$

where $\tilde{\mu}(z) = \int_0^z \bar{\mu}(y)dy \geq 0$ for all $z \in \mathbb{R}$.

From (5.4), we obtain

$$\begin{aligned}
&\int_0^T \frac{d}{dt}\Big[\|u_m(t)\|_{H^1}^2 + 2\int_0^1 \tilde{\mu}(u_{mx}(x,t))\,dx\Big]\,dt \\
&= \|u_m(T)\|_{H^1}^2 - \|u_m(0)\|_{H^1}^2 + 2\int_0^1 [\tilde{\mu}(u_{mx}(x,T)) - \tilde{\mu}(u_{mx}(x,0))]\,dx = 0.
\end{aligned} \tag{5.28}$$

Moreover,

$$\begin{aligned}
2\int_0^T \langle f(t), u_m'(t)\rangle\,dt &\le 2\int_0^T \|f(t)\|\|u_m'(t)\|\,dt \\
&\le 2\int_0^T \|f(t)\|^2 dt + \frac{1}{2}\int_0^T \|u_m'(t)\|^2 dt.
\end{aligned} \tag{5.29}$$

Putting $\sigma_R = \sup_{|z|\le\sqrt{2}R} \sigma(z)$, we have

$$\begin{aligned}
&2\int_0^T \langle \sigma(u_{mx}(t))u_m(t), u_m'(t)\rangle\,dt \\
&\le 2\sigma_R \int_0^T \|u_m(t)\|\|u_m'(t)\|\,dt \\
&\le 2R\sigma_R \int_0^T \|u_m'(t)\|\,dt \le 2TR^2\sigma_R^2 + \frac{1}{2}\int_0^T \|u_m'(t)\|^2 dt.
\end{aligned} \tag{5.30}$$

It follows from (5.27), (5.28), (5.29) and (5.30), that

$$\int_0^T \|u_m'(t)\|_{H^1}^2 dt \le 2TR^2\sigma_R^2 + 2\int_0^T \|f(t)\|^2 dt \le C_T, \tag{5.31}$$

for all $m \in \mathbb{N}$, for all $t \in [0,T]$, where C_T always indicates a bound depending on T.

Step 3: The limiting process. By (5.14) and (5.31) we deduce that, there exists a subsequence of $\{u_m\}$, still denoted by $\{u_m\}$ such that

$$\begin{gathered}
u_m \to u \quad \text{in } L^\infty(0,T; H_0^1 \cap H^2) \text{ weakly*}, \\
u_m' \to u' \quad \text{in } L^2(0,T; H_0^1) \text{ weakly}.
\end{gathered} \tag{5.32}$$

From (5.4), we obtain

$$u(0) = u(T). \tag{5.33}$$

Using the compactness lemma of Lions [9, p.57] and applying Fischer-Riesz theorem, from (5.32), there exists a subsequence of $\{u_m\}$, denoted by the same symbol satisfying

$$\begin{gathered}
u_m \to u \quad \text{strongly in } L^2(0,T; H_0^1) \text{ and a.e. in } Q_T, \\
u_{mx} \to u_x \quad \text{strongly in } L^2(Q_T) \text{ and a.e. in } Q_T.
\end{gathered} \tag{5.34}$$

Applying an argument similar to the one used in the proof of Theorem 3.1, we have

$$\begin{gathered}
\bar{\mu}(u_{mx}) \to \bar{\mu}(u_x) \quad \text{strongly in } L^2(Q_T), \\
\sigma(u_{mx})u_m \to \sigma(u_x)u \quad \text{strongly in } L^2(Q_T).
\end{gathered} \tag{5.35}$$

Denote by $\{\zeta_i,\ i=1,2,\dots\}$ the orthonormal base in the real Hilbert space $L^2(0,T)$. The set $\{\zeta_i w_j,\ i,\ j=1,2,\dots\}$ forms an orthonormal base in $L^2(0,T;H_0^1)$. From $(3.7)_1$ we have

$$\begin{aligned}&\int_0^T \langle u_m'(t)+u_m(t), w_j\zeta_i(t)\rangle\,dt+\int_0^T \langle u_{mx}'(t)+u_{mx}(t), w_{jx}\zeta_i(t)\rangle\,dt\\&+\int_0^T \langle \bar{\mu}(u_{mx}(t)), w_{jx}\zeta_i(t)\rangle\,dt+\int_0^T \langle \sigma(u_{mx}(t))u_m(t), w_j\zeta_i(t)\rangle\,dt\\&=\int_0^T \langle f(t), w_j\zeta_i(t)\rangle\,dt,\end{aligned}\tag{5.36}$$

for all i, j, $1\le j\le m$, $i\in\mathbb{N}$.

For i and j fixed, we deduce from (5.32) that

$$\begin{gathered}\int_0^T \langle u_m'(t)+u_m(t), w_j\zeta_i(t)\rangle\,dt\to\int_0^T \langle u'(t)+u(t), w_j\zeta_i(t)\rangle\,dt,\\ \int_0^T \langle u_{mx}'(t)+u_{mx}(t), w_{jx}\zeta_i(t)\rangle\,dt\to\int_0^T \langle u_x'(t)+u_x(t), w_{jx}\zeta_i(t)\rangle\,dt.\end{gathered}\tag{5.37}$$

Furthermore, by (5.35), we have

$$\begin{gathered}\int_0^T \langle \bar{\mu}(u_{mx}(t)), w_{jx}\zeta_i(t)\rangle\,dt\to\int_0^T \langle \bar{\mu}(u_x(t)), w_{jx}\zeta_i(t)\rangle\,dt,\\ \int_0^T \langle \sigma(u_{mx}(t))u_m(t), w_j\zeta_i(t)\rangle\,dt\to\int_0^T \langle \sigma(u_x(t))u(t), w_j\zeta_i(t)\rangle\,dt.\end{gathered}\tag{5.38}$$

Passing to the limit in (5.36) by (5.37), (5.38), we have

$$\begin{aligned}&\int_0^T \langle u'(t)+u(t), w_j\zeta_i(t)\rangle\,dt+\int_0^T \langle u_x'(t)+u_x(t), w_{jx}\zeta_i(t)\rangle\,dt\\&+\int_0^T \langle \bar{\mu}(u_x(t)), w_{jx}\zeta_i(t)\rangle\,dt+\int_0^T \langle \sigma(u_x(t))u(t), w_j\zeta_i(t)\rangle\,dt\\&=\int_0^T \langle f(t), w_j\zeta_i(t)\rangle\,dt.\end{aligned}\tag{5.39}$$

This equation holds for every i, $j\in\mathbb{N}$, i.e., the equation

$$\begin{aligned}&\int_0^T \langle u'(t)+u(t), w(t)\rangle\,dt+\int_0^T \langle u_x'(t)+u_x(t), w_x(t)\rangle\,dt\\&+\int_0^T \langle \bar{\mu}(u_x(t)), w_x(t)\rangle\,dt+\int_0^T \langle \sigma(u_x(t))u(t), w(t)\rangle\,dt\\&=\int_0^T \langle f(t), w(t)\rangle\,dt,\quad \text{for all } w\in L^2(0,T;H_0^1),\end{aligned}\tag{5.40}$$

is satisfied.

Step 4: Uniqueness of the solutions. Let u and $\bar{u}$ be two solutions of (5.2) such that $\|u\|_{L^\infty(0,T;H_0^1\cap H^2)}\le R$, $\|\bar{u}\|_{L^\infty(0,T;H_0^1\cap H^2)}\le R$, with $R\sup_{|z|\le\sqrt{2}R}|\sigma'(z)|<2$.

Then $v = u - \bar{u}$ satisfies

$$
\begin{gathered}
\int_0^T \langle v'(t) + v(t), w(t)\rangle\,dt + \int_0^T \langle v'_x(t) + v_x(t), w_x(t)\rangle\,dt \\
+ \int_0^T \langle \bar{\mu}(u_x(t)) - \bar{\mu}(\bar{u}_x(t)), w_x(t)\rangle\,dt \\
+ \int_0^T \langle \sigma(u_x(t))u(t) - \sigma(\bar{u}_x(t))\bar{u}(t), w(t)\rangle\,dt = 0, \quad \forall w \in L^2(0,T;H_0^1), \\
v(0) = v(T), \\
v, u, \bar{u} \in L^\infty(0,T;H_0^1 \cap H^2), \quad v', u', \bar{u}' \in L^2(0,T;H_0^1).
\end{gathered} \tag{5.41}
$$

Taking $w = v$ in $(5.41)_1$ and using $(5.41)_2$, we obtain

$$
\int_0^T \langle v'(t), v(t)\rangle\,dt = \frac{1}{2}\|v(T)\|^2 - \frac{1}{2}\|v(0)\|^2 = 0; \tag{5.42}
$$

$$
\int_0^T \langle v'_x(t), v_x(t)\rangle\,dt = \frac{1}{2}\|v_x(T)\|^2 - \frac{1}{2}\|v_x(0)\|^2 = 0; \tag{5.43}
$$

$$
\int_0^T \langle \bar{\mu}(u_x(t)) - \bar{\mu}(\bar{u}_x(t)), v_x(t)\rangle\,dt \geq 0; \tag{5.44}
$$

$$
\begin{aligned}
&\int_0^T \langle \sigma(u_x(t))u(t) - \sigma(\bar{u}_x(t))\bar{u}(t), v(t)\rangle\,dt \\
&= \int_0^T \|\sqrt{\sigma(u_x(t))}v(t)\|^2 dt + \int_0^T \langle [\sigma(u_x(t)) - \sigma(\bar{u}_x(t))]\bar{u}(t), v(t)\rangle\,dt.
\end{aligned} \tag{5.45}
$$

As for (5.21), we have

$$
\int_0^T \langle [\sigma(u_x(t)) - \sigma(\bar{u}_x(t))]\bar{u}(t), v(t)\rangle\,dt \leq \frac{1}{2}R\tilde{K}_R \int_0^T \|v(t)\|_{H^1}^2 dt, \tag{5.46}
$$

with $\tilde{K}_R = \sup_{|z|\leq\sqrt{2}R} |\sigma'(z)|$. Hence

$$
\begin{aligned}
&\int_0^T \|v(t)\|_{H^1}^2 dt + \int_0^T \langle \bar{\mu}(u_x(t)) - \bar{\mu}(\bar{u}_x(t)), v_x(t)\rangle\,dt + \int_0^T \|\sqrt{\sigma(u_x(t))}v(t)\|^2 dt \\
&\leq \frac{1}{2}R\tilde{K}_R \int_0^T \|v(t)\|_{H^1}^2 dt.
\end{aligned} \tag{5.47}
$$

By $\frac{1}{2}R\tilde{K}_R = \frac{1}{2}R \sup\limits_{|z|\leq\sqrt{2}R} |\sigma'(z)| < 1$, we deduce from (5.47) that $\int_0^T \|v(t)\|_{H^1}^2 dt = 0$, i.e., $v = u - \bar{u} = 0$. This completes the proof. $\square$

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References

[1] Alexander B. Al'shin, , Maxim O. Korpusov; Alexey G. Sveshnikov; *Blow-up in nonlinear Sobolev type equations*, De Gruyter Series in Nonlinear Analysis and Applications, 15. Walter de Gruyter & Co., Berlin, 2011.

[2] C. J. Amick, J. L. Bona, M. E. Schonbek; *Decay of solutions of some nonlinear wave equations*, J. Differential Equa. **81** (1) (1989), 1-49.

[3] D. N. Arnold, J. Douglas Jr, V. Thomee; *Superconvergence of a finite element approximation to the solution of a Sobolev equation in a single space variable*, Math. Comp. **36** (1981), 53-63.
[4] T. Aziz, F. M. Mah; *A Note on the Solutions of Some Nonlinear Equations Arising in Third-Grade Fluid Flows: An Exact Approach*, The Scientific World Journal, **2014** (2014), Article ID 109128, 7 pages.
[5] T. B. Benjamin, J. L. Bona, J. J. Mahony; *Model equations for long waves in nonlinear dispersive systems*, Phil. Trans. Roy. Soc. Lond. Ser. A **272** (1972), 47-78.
[6] J. L. Bona, V. A. Dougalis; *An initial and boundary value problem for a model equation for propagation of long waves,* J. Math. Anal. Appl. **75** (1980), 503-522.
[7] Robert Wayne Carroll, Ralph E. Showalter; *Singular and Degenerate Cauchy problems*, Mathematics in Science and Engineering, Vol. **127**. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1976.
[8] T. Hayat, F. Shahzad, M. Ayub; *Analytical solution for the steady flow of the third grade fluid in a porous half space*, Applied Mathematical Modelling, **31** (11) (2007), 2424-2432.
[9] J. L. Lions; *Quelques méthodes de ré solution des problèmes aux limites nonlinéaires*, Dunod; Gauthier - Villars, Paris, 1969.
[10] L. A. Medeiros, M. M. Miranda; *Weak solutions for a nonlinear dispersive equation*, J. Math. Anal. Appl. **59** (1977), 432-441.
[11] M. Sajid, T. Hayat; *Series solution for steady flow of a third grade fluid through porous space*, Transport in Porous Media, **71** (2) (2008), 173-183.
[12] R. E. Showalter, T. W. Ting; *Asymptotic behavior of solutions of pseudo-parabolic partial differential equations*, Annali di Matematica Pura ed Applicata, **90** (4) (1971), 241-258.
[13] R. E. Showalter; *Existence and representation theorems for a semilinear Sobolev equation in Banach space*, SIAM J. Math. Anal., **3** (1972), 527-543.
[14] L. Zhang; *Decay of solution of generalized Benjamin-Bona-Mahony-Burgers equations in n-space dimensions*, Nonlinear Anal. TMA. **25** (12) (1995), 1343-1369.

Le Thi Phuong Ngoc
University of Khanh Hoa, 01 Nguyen Chanh Str., Nha Trang City, Vietnam
E-mail address: ngoc1966@gmail.com

Dao Thi Hai Yen
Department of Natural Science, Phu Yen University, 18 Tran Phu Str., Ward 7, Tuy Hoa City, Vietnam.
Department of Mathematics and Computer Science, VNUHCM - University of Science, 227 Nguyen Van Cu Str., Dist. 5, HoChiMinh City, Vietnam
E-mail address: haiyennbh@yahoo.com

Nguyen Thanh Long
Department of Mathematics and Computer Science, VNUHCM - University of Science, 227 Nguyen Van Cu Str., Dist. 5, HoChiMinh City, Vietnam
E-mail address: longnt2@gmail.com