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MULTIPLE SOLUTIONS FOR SEMILINEAR ROBIN PROBLEMS WITH SUPERLINEAR REACTION AND NO SYMMETRIES

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Abstract. We study a semilinear Robin problem driven by the Laplacian with a parametric superlinear reaction. Using variational tools from the critical point theory with truncation and comparison techniques, critical groups and flow invariance arguments, we show the existence of seven nontrivial smooth solutions, all with sign information and ordered.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this article we study the parametric semilinear Robin problem

$$\begin{gathered} -\Delta u(z) + \xi(z)u(z) = f_\lambda(z, u(z)) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} + \beta(z)u = 0 \quad \text{on } \partial\Omega,\ \lambda \in (0, \widetilde{\lambda}). \end{gathered} \tag{1.1}$$

In this problem the potential function $\xi \in L^\infty(\Omega)$ and $\xi(z) \geq 0$ for a.a. $z \in \Omega$. The reaction $f_\lambda(z, x)$ is parametric with $\lambda > 0$ being the parameter, it is measurable in $(z, x) \in \Omega \times \mathbb{R}$ and it is C^1 in the x-variable. We assume that $f_\lambda(z, \cdot)$ exhibits superlinear growth as $x \to \pm\infty$, without satisfying the common in such cases Ambrosetti-Rabinowitz condition (the AR-condition for short). In the boundary condition $\frac{\partial u}{\partial n}$ denotes the normal derivative of u and it is defined via the Green's identity (see Papageorgiou-Rădulescu-Repovš [17, p. 35]. We know that if $u \in C^1(\overline{\Omega})$, then $\frac{\partial u}{\partial n} = (\nabla u, n)_{\mathbb{R}^N}$ with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. The boundary coefficient $\beta \in W^{1,\infty}(\partial\Omega)$ and $\beta(z) \geq 0$ for all $z \in \partial\Omega$.

Our aim is to prove a multiplicity theorem for problem (1.1) providing sign information for all the solutions, without imposing any symmetry conditions on $f_\lambda(z, \cdot)$. Using variational tools coming from the critical point theory, with truncation and comparison techniques, the use of critical groups and flow invariance arguments, we show that for all small values of the parameter $\lambda > 0$, problem (1.1) has at least seven nontrivial smooth solutions all with sign information (two positive, two negative and three nodal (sign changing) solutions). Moreover, the constant sign and nodal solutions are ordered.

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The study of superlinear elliptic problems was initiated with the seminal paper by Ambrosetti-Rabinowitz [2] and continued with the works of Wang [20], Miyagaki-Souto [12] (semilinear Dirichlet problems) and by Fang-Liu [5], Li-Yang [11], Sun [18], Aizicovici-Papageorgiou-Staicu [1] (nonlinear Dirichlet problems driven by the p-Laplacian). More recently, Mugnai-Papageorgiou [14] and Papageorgiou-Rădulescu [16], studied superlinear problems driven by nonlinear nonhomogeneous differential operators. All the aforementioned works produce at most three nontrivial solutions without providing sign information for all of them. Also, only Miyagaki-Souto [12] deal with a parametric problem with $f_\lambda(z,x) = \lambda f(z,x)$ and show that for all $\lambda > 0$ the problem has a nontrivial solution. Actually their asymptotic hypotheses on $f(z,\cdot)$ as $x \to \pm\infty$ and as $x \to 0$ (see hypotheses (f_1) an (f_4)) make the presence of the parameter $\lambda > 0$ irrelevant and for this reason their existence result is global in $\lambda > 0$. We also mention the works of Castro-Cossio-Vélez [4] and of Papageorgiou-Papalini [15], which study Dirichlet problems with an asymptotically linear reaction and prove the existence of seven nontrivial solutions without any symmetry conditions on $f(z,\cdot)$. The methods of proof differ in the two papers. Our approach here is closer to that of [15].

2. Mathematical background - hypotheses

The main spaces in the analysis of problem (1.1) are the Sobolev space $H^1(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the boundary Lebesgue spaces $L^r(\partial\Omega)$, $1 \le r < \infty$. By $\|\cdot\|$ we denote the norm of the Sobolev space $H^1(\Omega)$. We have

$$\|u\| = [\|u\|_2^2 + \|\nabla u\|_2^2]^{1/2} \quad \text{for all } u \in H^1(\Omega).$$

The space $C^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone $C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \ge 0 \text{ for all } z \in \overline{\Omega}\}$. This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. From the theory of Sobolev spaces, we know that there exists a unique continuous linear map $\widehat{\gamma}_0 : H^1(\Omega) \to L^2(\partial\Omega)$ such that

$$\widehat{\gamma}_0(u) = u\big|_{\partial\Omega} \quad \text{for all } u \in H^1(\Omega) \cap C(\overline{\Omega}).$$

This map is known as the "trace map" and it extends the notion of boundary values to all Sobolev functions. The trace map is actually compact into $L^r(\partial\Omega)$ for all $1 \le r < \frac{2(N-1)}{N-2}$ if $N \ge 3$ and into $L^r(\partial\Omega)$ for all $1 \le r < \infty$ if $N = 1, 2$. Moreover, the trace map is not surjective. In fact we have

$$\operatorname{im} \widehat{\gamma}_0 = H^{1/2,2}(\partial\Omega) \quad \text{and} \quad \ker \widehat{\gamma}_0 = H_0^1(\Omega).$$

In what follows, for the sake of notational simplicity we drop the use of the trace map $\widehat{\gamma}_0$. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

Let $u, v : \Omega \to \mathbb{R}$ be two measurable functions such that $u(z) \le v(z)$ for a.a. $z \in \Omega$. We introduce the following order intervals in the Sobolev space $H^1(\Omega)$:

$$\begin{gathered} [u,v] = \{h \in H^1(\Omega) : u(z) \le h(z) \le v(z) \text{ for a.a. } z \in \Omega\}, \\ [u) = \{h \in H^1(\Omega) : u(z) \le h(z) \text{ for a.a. } z \in \Omega\}, \\ \operatorname{int}_{C^1(\overline{\Omega})}[u,v] = \text{ the interior in } C^1(\overline{\Omega}) \text{ of } [u,v] \cap C^1(\overline{\Omega}). \end{gathered}$$

Also, for $u \in H^1(\Omega)$ we define $u^{\pm} = \max\{\pm u, 0\}$. We know that

$$u^{\pm} \in H^1(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

Given $f : \Omega\times\mathbb{R} \to \mathbb{R}$ a measurable function, by N_f we denote the Nemytski operator corresponding to $f(\cdot,\cdot)$; that is, for all $u : \Omega \to \mathbb{R}$ measurable $N_f(u)(\cdot) = f(\cdot, u(\cdot))$. Note that $z \to f(z, u(z))$ is measurable.

Our hypotheses on the potential term $\xi(\cdot)$ and the boundary coefficient $\beta(\cdot)$ are the following:

(H1) $\xi \in L^\infty(\Omega)$, $\beta \in W^{1,\infty}(\partial\Omega)$, $\xi(z) \geq 0$ for a.a. $z \in \Omega$, $\beta(z) \geq 0$ for all $z \in \partial\Omega$ and $\xi \not\equiv 0$ or $\beta \not\equiv 0$.

We introduce the C^1-functional $\gamma : H^1(\Omega) \to \mathbb{R}$ defined by

$$\gamma(u) = \|\nabla u\|_2^2 + \int_\Omega \xi(z)u^2\,dz + \int_{\partial\Omega} \beta(z)u^2 d\sigma \quad \text{for all } u \in H^1(\Omega).$$

On account of hypotheses (H1) and using Mugnai-Papageorgiou [13, Lemma 4.11] and Gasiński-Papageorgiou [8, Proposition 2.4], we have

$$c_0\|u\|^2 \leq \gamma(u) \quad \text{for some } c_0 > 0, \text{ all } u \in H^1(\Omega). \tag{2.1}$$

In our arguments we will employ the spectrum of the operator $u \to -\Delta u + \xi(z)u$ with the Robin boundary condition. So, we consider the linear eigenvalue problem

$$\begin{gathered} -\Delta u(z) + \xi(z)u(z) = \widehat{\lambda}u(z) \quad \text{in } \Omega,\\ \frac{\partial u}{\partial n} + \beta(z)u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.2}$$

Using the spectral theorem for linear, compact self-adjoint operators, we show that the spectrum of (2.2) consists of a sequence $\{\widehat{\lambda}_n\}_{n\in\mathbb{N}}$ of distinct eigenvalues such that $\widehat{\lambda}_n \to +\infty$. Also there is a corresponding sequence $\{\widehat{u}_n\}_{n\in\mathbb{N}} \subseteq H^1(\Omega)$ of eigenfunctions which form an orthonormal basis of $H^1(\Omega)$. By $E(\widehat{\lambda}_n)$ we denote the eigenspace corresponding to the eigenvalue $\widehat{\lambda}_n$, $n \in \mathbb{N}$. These items have the following properties:

(a) $E(\widehat{\lambda}_n)$ is finite dimensional and $E(\widehat{\lambda}_n) \subseteq C^1(\overline{\Omega})$ for all $n \in \mathbb{N}$ (see Wang [19]).
(b) Each eigenspace $E(\widehat{\lambda}_n)$ has the so-called "Unique Continuation Property" (UCP for short), namely if $u \in E(\widehat{\lambda}_n)$ and vanishes on a set of positive Lebesgue measure, then $u \equiv 0$.
(c) $H^1(\Omega) = \overline{\oplus_{n\in\mathbb{N}} E(\widehat{\lambda}_n)}$.
(d) Let

$$\widehat{\lambda}_1 = \inf\Big[\frac{\gamma(u)}{\|u\|_2^2} : u \in H^1(\Omega),\, u \neq 0\Big] > 0, \tag{2.3}$$

$$\widehat{\lambda}_n = \inf\Big[\frac{\gamma(u)}{\|u\|_2^2} : u \in \widehat{H}_n,\, u \neq 0\Big] = \sup\Big[\frac{\gamma(u)}{\|u\|_2^2} : u \in \overline{H}_n,\, u \neq 0\Big], \tag{2.4}$$

where $\overline{H}_n = \oplus_{k=1}^n E(\widehat{\lambda}_k)$, $\widehat{H}_n = \overline{\oplus_{k\geq n} E(\widehat{\lambda}_k)}$, $n \in \mathbb{N}$.

The infimum in (2.3) is realized on $E(\widehat{\lambda}_1)$, while both the infimum and the supremum in (2.4) are realized on $E(\widehat{\lambda}_n)$. The elements of $E(\widehat{\lambda}_1)$ do not change sign and $\widehat{\lambda}_1 > 0$ is simple. By $\widehat{u}_1$ we denote the positive, L^2-normalized (that is, $\|\widehat{u}_1\|_2 = 1$) eigenfunction corresponding to $\widehat{\lambda}_1$. We have that $\widehat{u}_1 \in \operatorname{int} C_+$. On the other hand the eigenfunctions corresponding to an eigenvalue $\widehat{\lambda}_n > 0$, $n \geq 2$, are all nodal (sign changing). For details, we refer to Gasiński-Papageorgiou [7].

Using the orthogonality of the eigenspaces, (2.3) and (2.4) and the UCP, we obtain the following result.

Lemma 2.1. (a) *If $n \in \mathbb{N}$, $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \geq \widehat{\lambda}_n$ for a.a. $z \in \Omega$ and the inequality is strict on a set of positive Lebesgue measure, then we can find $c_1 > 0$ such that*
$$c_1\|u\|^2 \leq \int_\Omega \vartheta(z)u^2\,dz - \gamma(u) \quad \text{for all } u \in \overline{H}_n.$$

(b) *If $n \in \mathbb{N}$, $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \leq \widehat{\lambda}_n$ for a.a. $z \in \Omega$ and the inequality is strict on a set of positive Lebesgue measure, then we can find $c_2 > 0$ such that*
$$c_2\|u\|^2 \leq \gamma(u) - \int_\Omega \vartheta(z)u^2\,dz \quad \text{for all } u \in \widehat{H}_n.$$

Let X be a Banach space and $\varphi \in C^1(X)$. We say that $\varphi(\cdot)$ satisfies the "C-condition", if it has the property

> Every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence.

This is a compactness-type condition on the functional $\varphi(\cdot)$ which compensates for the fact that the ambient space X need not be locally compact (in most cases X is infinite dimensional). Using this compactness-type condition on $\varphi(\cdot)$, we can prove a deformation theorem from which follow the minimax theorems of critical point theory (see [17, Sections 5.3, 5.4]).

Consider the following two sets:
$$K_\varphi = \{u \in X : \varphi'(u) = 0\} \quad \text{(the critical set of } \varphi),$$
$$\varphi^c = \{u \in X : \varphi(u) \leq c\} \quad \text{for any } c \in \mathbb{R}.$$

If $B \subseteq A \subseteq X$, then by $H_k(A, B)$, $k \in \mathbb{N}_0$, we denote the k^{th}-relative singular homology group for the pair (A, B) with integer coefficients. If $u \in K_\varphi$ is isolated, then the critical groups of φ at u are defined by
$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \in \mathbb{N}_0,$$
where U is a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property implies that this definition is independent of the choice of the isolating neighborhood.

If $\varphi \in C^1(X)$ satisfies the C-condition and $\varphi\big|_{K_\varphi}$ is bounded below, then for $c < \inf_{u\in K_\varphi} \varphi(u)$, the critical groups of φ at infinity are defined by
$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \in \mathbb{N}_0.$$
[17, Corollary 5.3.13, p. 392] implies that this definition is independent of the choice of $c < \inf_{K_\varphi} \varphi$.

Suppose that K_φ is finite and introduce the following quantities:
$$M(t, u) = \sum_{k\geq 0} \text{rank } C_k(\varphi, u)t^k,$$
$$P(t, \infty) = \sum_{k\geq 0} \text{rank } C_k(\varphi, \infty)t^k \quad \text{for all } t \in [0, 1], \text{ all } u \in K_\varphi.$$

The Morse relation says that

$$\sum_{u\in K_\varphi} M(t,u) = P(t,\infty) + (1+t)Q(t), \tag{2.5}$$

with $Q(t)$ being a formal series in t with nonnegative, integer coefficients (see [17]).

Now we introduce the hypotheses on the reaction $f_\lambda(z,x)$.

(H2) $f_\lambda : \Omega\times\mathbb{R}\to\mathbb{R}$ ($\lambda\in(0,\widetilde{\lambda})$) is a measurable function such that for all $\lambda\in(0,\widetilde{\lambda})$, a.a. $z\in\Omega$ $f_\lambda(z,\cdot)\in C^1(\mathbb{R})$, $f_\lambda(z,0)=0$ and

(i) $|(f_\lambda)'_x(z,x)|\le a_\lambda(z)+c|x|^{r-2}x$ for a.a. $z\in\Omega$, all $x\ge 0$, with $a_\lambda\in L^\infty(\Omega)$, $c>0$,

$$2<r<2^*=\begin{cases}\frac{2N}{N-2} & \text{if } N\ge 3\\ +\infty & \text{if } N=1,2\end{cases}$$

and $\|a_\lambda\|_\infty\to 0$ as $\lambda\to 0^+$;

(ii) if $F_\lambda(z,x)=\int_0^x f_\lambda(z,s)ds$, then $\lim_{x\to\pm\infty}\frac{F_\lambda(z,x)}{x^2}=+\infty$ uniformly for a.a. $z\in\Omega$;

(iii) if $e_\lambda(z,x)=f_\lambda(z,x)x-2F_\lambda(z,x)$, then there exists $\eta_\lambda\in L^1(\Omega)$ such that $e_\lambda(z,x)\le e_\lambda(z,y)+\eta_\lambda(z)$ for a.a. $z\in\Omega$, all $0\le x\le y$ or $y\le x\le 0$;

(iv) there exists $m\in\mathbb{N}$ such that

$$\begin{gathered}(f_\lambda)'_x(z,0)=\lim_{x\to 0}\frac{f_\lambda(z,x)}{x}\ \text{uniformly for a.a. } z\in\Omega,\text{ all }\lambda\in(0,\widetilde{\lambda}),\\ (f_\lambda)'_x(z,0)\in[\widehat{\lambda}_m,\widehat{\lambda}_{m+1}]\ \text{for a.a. } z\in\Omega,\\ (f_\lambda)'_x(\cdot,0)\not\equiv\widehat{\lambda}_m,\quad (f_\lambda)'_x(\cdot,0)\not\equiv\widehat{\lambda}_{m+1}.\end{gathered}$$

Remark 2.2. Hypotheses (H2)(ii),(iii) imply that for all $\lambda\in(0,\widetilde{\lambda})$

$$\lim_{x\to 0}\frac{f_\lambda(z,x)}{x}=+\infty\quad\text{uniformly for a.a. } z\in\Omega.$$

So, the reaction of problem (1.1) is superlinear. However, $f_\lambda(z,\cdot)$ need not satisfy the AR-condition as it is often the case in the literature (see Wang [20]). Recall that the AR-condition says that there exist $M>0$ and $p>2$ such that

$$\begin{gathered}0<pF_\lambda(z,x)\le f_\lambda(z,x)x\quad\text{for a.a. } z\in\Omega,\text{ all } |x|\ge M,\\ 0<\operatorname*{ess\,inf}_\Omega F_\lambda(\cdot,\pm M).\end{gathered}$$

Integrating the first inequality and using the second, we show that

$$c_3|x|^p\le F_\lambda(z,x)\quad\text{for a.a. } z\in\Omega,\text{ all } |x|\ge M,\text{ some } c_3>0.$$

Therefore the AR-condition dictates that $f_\lambda(z,\cdot)$ eventually has at least $(p-1)$-polynomial growth. This excludes from consideration nonlinearities with "slower" growth near $\pm\infty$ (see the example below). Here instead of the AR-condition, we employ the less restrictive quasimonotonicity hypothesis (H2)(iii). This condition is a slight generalization of a hypothesis employed by Li-Yang [11] (see also [14]). There are easy ways to verify this condition. So, if there exists $M>0$ such that for a.a. $z\in\Omega$

$$x\mapsto\frac{f_\lambda(z,x)}{x}\ \text{is nondecreasing on } [M,+\infty),$$

$$x \mapsto \frac{f_\lambda(z,x)}{x} \text{ is nonincreasing on } (-\infty, -M],$$

then the quasimonotonicity condition (H2)(iii) is satisfied. Hypothesis (H2)(iv) implies nonuniform nonresonance with respect to any spectral interval.

Example 2.3. The following function satisfies hypotheses (H2) but not the AR-condition. For the sake of simplicity, we drop the z-dependence:

$$f_\lambda(x) = \begin{cases} x \ln|x| - \lambda \sin\left(\frac{\eta}{\lambda}x\right) & \text{if } x < -1, \\ \lambda \sin\left(\frac{\eta}{\lambda}x\right) & \text{if } |x| \le 1 \\ x \ln|x| + \lambda \sin\left(\frac{\eta}{\lambda}x\right) & \text{if } 1 < x, \end{cases}$$

with $\eta \in (\widehat{\lambda}_m, \widehat{\lambda}_{m+1})$, $m \in \mathbb{N}$.

Finally note that on account of (H2)(i), we have

$$|f_\lambda(z,x)| \le \widehat{a}_\lambda(z) + \widehat{c}|x|^{r-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R} \tag{2.6}$$

with $\widehat{a}_\lambda \in L^\infty(\Omega)$, $\widehat{c} > 0$ and $\|\widehat{a}_\lambda\|_\infty \to 0$ as $\lambda \to 0^+$.

3. Solutions of constant sign

On account of hypothesis (H2)(iv), we can find $\delta_0 > 0$ such that

$$f_\lambda(z,x)x \ge 0 \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \le \delta_0, \text{ all } \lambda \in (0, \widetilde{\lambda}). \tag{3.1}$$

Proposition 3.1. *If hypotheses* (H1), (H2)(i), (H2)(iv) *hold, then there exists* $\lambda^* \in (0, \widetilde{\lambda})$ *such that for all* $\lambda \in (0, \lambda^*)$ *problem* (1.1) *has a positive solution* $u_0 \in \operatorname{int} C_+$ *and a negative solution* $v_0 \in -\operatorname{int} C_+$ *such that* $\|v_0\|_\infty, \|u_0\|_\infty \le \delta_0$.

Proof. We consider the auxiliary Robin problem

$$\begin{gathered} -\Delta u(z) + \xi(z)u(z) = 1 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} + \beta(z)u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.2}$$

We consider the operator $V \in \mathcal{L}(H^1(\Omega), H^1(\Omega)^*)$ defined by

$$\langle V(u), h\rangle = \int_\Omega (\nabla u, \nabla h)_{\mathbb{R}^N} \, dz + \int_\Omega \xi(z)uh \, dz + \int_{\partial\Omega} \beta(z)uh \, d\sigma$$

for all $u, h \in H^1(\Omega)$. From (2.1) it is clear that $V(\cdot)$ is strongly monotone (hence maximal monotone too), and $V(\cdot)$ is coercive.

It follows that $V(\cdot)$ is surjective [17, Corollary 2.8.7, p. 135]. So, there exists $\widetilde{u} \in H^1(\Omega) \setminus \{0\}$ such that $V(\widetilde{u}) = 1$.

The strong monotonicity of $V(\cdot)$ implies that this solution is unique. We have

$$\int_\Omega (\nabla\widetilde{u}, \nabla h)_{\mathbb{R}^N} \, dz + \int_\Omega \xi(z)\widetilde{u}h \, dz + \int_{\partial\Omega} \beta(z)\widetilde{u}h d\sigma = \int_\Omega h \, dz \tag{3.3}$$

for all $h \in H^1(\Omega)$ which implies that $\widetilde{u}$ is the unique solution of (3.2).

In (3.3) we use the test function $h = -\widetilde{u}^- \in H^1(\Omega)$ and obtain that

$$\begin{aligned} &\|\nabla\widetilde{u}^-\|_2^2 + \int_\Omega \xi(z)(\widetilde{u}^-)^2 \, dz + \int_{\partial\Omega} \beta(z)(\widetilde{u}^-)^2 d\sigma \le 0, \\ &\Rightarrow \ c_0\|\widetilde{u}^-\|^2 \le 0 \quad (\text{see } (2.1)), \\ &\Rightarrow \ \widetilde{u} \ge 0, \quad \widetilde{u} \ne 0. \end{aligned}$$

From the regularity theory of Wang [19], we have $\widetilde{u} \in C_+ \setminus \{0\}$. Moreover, from (3.2) and hypotheses (H1), we have

$$\begin{aligned}&\Delta \widetilde{u}(z) \le \|\xi\|_\infty \widetilde{u}(z) \quad \text{for a.a. } z \in \Omega,\\ &\Rightarrow \widetilde{u} \in \operatorname{int} C_+ \quad \text{(by Hopf's lemma)}.\end{aligned}$$

We claim that there exists $\lambda^* \in (0, \widetilde{\lambda})$ such that, if $\lambda \in (0, \lambda^*)$, then we can find $\widetilde{\xi} = \widetilde{\xi}(\lambda) \in \big(0, \frac{\delta_0}{\|\widetilde{u}\|_\infty}\big)$ such that

$$\|\widehat{a}_\lambda\|_\infty + \widehat{c}\,(\widetilde{\xi}\,\|\widetilde{u}\|_\infty)^{r-1} < \widetilde{\xi} \tag{3.4}$$

with $\widehat{a}_\lambda \in L^\infty(\Omega)$ and $\widehat{c} > 0$ as in (2.6) and $\delta_0 > 0$ as in (3.1).

Arguing by contradiction, suppose that we can find $\lambda_n \to 0^+$ such that

$$\xi \le \|\widehat{a}_{\lambda_n}\|_\infty + \widehat{c}(\xi\|\widetilde{u}\|_\infty)^{r-1} \quad \text{for all } n \in \mathbb{N}, \text{ all } \xi \in \big(0, \frac{\delta_0}{\|\widetilde{u}\|_\infty}\big).$$

Passing to the limit as $n \to \infty$, we obtain

$$1 \le \widehat{c}\,\xi^{r-2}\|\widetilde{u}\|_\infty^{r-1} \quad \text{for all } \xi \in \big(0, \frac{\delta_0}{\|\widetilde{u}\|_\infty}\big).$$

Letting $\xi \to 0^+$, we have a contradiction. This proves (3.4).

Fix $\lambda \in (0, \lambda^*)$ and let $\overline{u} = \widetilde{\xi}\,\widetilde{u} \in \operatorname{int} C_+$. We consider the following truncation of the reaction

$$\widehat{f}_\lambda^+(z, x) = \begin{cases} f_\lambda(z, x^+) & \text{if } x \le \overline{u}(z), \\ f_\lambda(z, \overline{u}(z)) & \text{if } \overline{u}(z) < x. \end{cases} \tag{3.5}$$

This is a Carathéodory function. We set $\widehat{F}_\lambda^+(z, x) = \int_0^x \widehat{f}_\lambda^+(z, s)ds$ and consider the C^1-functional $\widehat{\varphi}_\lambda^+ : H^1(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}_\lambda^+(u) = \frac{1}{2}\gamma(u) - \int_\Omega \widehat{F}_\lambda^+(z, u)dz \quad \text{for all } u \in H^1(\Omega).$$

From (2.1) and (3.5) we see that $\widehat{\varphi}_\lambda^+(\cdot)$ is coercive. Also, the Sobolev embedding theorem and the compactness of the trace operator, imply that $\widehat{\varphi}_\lambda^+(\cdot)$ is sequentially weakly lower semicontinuous. Then by the Weierstrass-Tonelli theorem, we know that there exists $u_0 \in H^1(\Omega)$ such that

$$\widehat{\varphi}_\lambda^+(u_0) = \inf[\widehat{\varphi}_\lambda^+(u) : u \in H^1(\Omega)]. \tag{3.6}$$

On account of hypothesis (H2)(iv), we see that given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$F_\lambda^+(z, x) \ge \frac{1}{2}[(f_\lambda)'_x(z, 0) - \varepsilon]x^2 \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \le \delta. \tag{3.7}$$

We can always take $\delta \le \min_{\overline{\Omega}} \overline{u}$ (recall $\overline{u} \in \operatorname{int} C_+$). Choose $t \in (0, 1)$ small so that

$$t\widehat{u}_1(z) \in (0, \delta] \quad \text{for all } z \in \overline{\Omega} \text{ (recall } \widehat{u}_1 \in \operatorname{int} C_+). \tag{3.8}$$

We have

$$\begin{aligned}\widehat{\varphi}_\lambda^+(t\widehat{u}_1) &= \frac{t^2}{2}\Big[\gamma(\widehat{u}_1) - \int_\Omega \widehat{F}_\lambda^+(z, t\widehat{u}_1)dz\Big] \\ &\le \frac{t^2}{2}\Big[\int_\Omega \Big(\widehat{\lambda}_1 - (f_\lambda)'_x(z, 0)\Big)\,\widehat{u}_1^2 dz + \varepsilon\Big]\end{aligned}$$

(see (3.5), (3.7), (3.8) and recall $\|\widehat{u}_1\|_2 = 1$). Note that

$$\int_\Omega \Big((f_\lambda)'_x(z,0) - \widehat{\lambda}_1\Big)\widehat{u}_1^2 dz > 0 \quad (\text{since } \widehat{u}_1 \in \operatorname{int} C_+).$$

So, choosing $\varepsilon > 0$ small, we obtain

$$\begin{aligned}
&\widehat{\varphi}_\lambda^+(t\widehat{u}_1) < 0,\\
&\Rightarrow\ \widehat{\varphi}_\lambda^+(u_0) < 0 = \widehat{\varphi}_\lambda^+(0) \quad (\text{see } (3.6)),\\
&\Rightarrow\ u_0 \neq 0.
\end{aligned}$$

From (3.6) we have

$$(\widehat{\varphi}_\lambda^+)'_+(u_0) = 0 \ \Rightarrow\ \langle \gamma'(u_0), h\rangle = \int_\Omega \widehat{f}_\lambda^+(z,u_0) h\, dz \quad \text{for all } h \in H^1(\Omega). \tag{3.9}$$

In (3.9) first we choose $h = -u_0^- \in H^1(\Omega)$. Then

$$\begin{aligned}
&\gamma(u_0^-) = 0,\\
&\Rightarrow\ c_0\|u_0^-\|^2 \le 0 \quad (\text{see } (2.1)),\\
&\Rightarrow\ u_0 \ge 0,\, u_0 \neq 0.
\end{aligned}$$

Next in (3.9) we use the test function $h = (u_0 - \overline{u})^+ \in H^1(\Omega)$. Then

$$\begin{aligned}
\langle \gamma'(u_0), (u_0 - \overline{u})^+\rangle &= \int_\Omega \widehat{f}_\lambda^+(z,u_0)(u_0 - \overline{u})^+\, dz\\
&= \int_\Omega f_\lambda(z,\overline{u})(u_0 - \overline{u})^+\, dz \quad (\text{see } (3.5))\\
&\le \int_\Omega [\widehat{a}_\lambda(z) + \widehat{c}\,\overline{u}^{r-1}](u_0 - \overline{u})^+\, dz \quad (\text{see } (2.6))\\
&\le \int_\Omega \widetilde{\xi}(\overline{u} - u_0)^+\, dz \quad (\text{see } (3.4))\\
&= \langle \gamma'(\overline{u}), (u_0 - \overline{u})^+\rangle \quad (\text{since } \overline{u} = \widetilde{\xi}\,\widetilde{u}),\\
&\Rightarrow\ u_0 \le \overline{u}.
\end{aligned}$$

So, we have proved that

$$u_0 \in [0, \overline{u}],\ u_0 \neq 0. \tag{3.10}$$

From (3.10), (3.5) and (3.9), we infer that

$$\begin{gathered}
-\Delta u_0 + \xi(z) u_0 = f_\lambda(z,u_0) \quad \text{a.e. in } \Omega,\\
\frac{\partial u_0}{\partial n} + \beta(z) u_0 = 0 \quad \text{on } \partial\Omega,
\end{gathered}$$

implies that $u_0 \in \operatorname{int} C_+ \setminus \{0\}$ is a solution of (1.1), $\lambda \in (0, \lambda^*)$ (see Wang [19]). We have

$$\|\overline{u}\|_\infty = \widetilde{\xi}\,\|\widetilde{u}\|_\infty \le \delta_0 \ \Rightarrow\ \overline{u}(z) \le \delta_0 \quad \text{for all } z \in \overline{\Omega}.$$

So, from (3.1) we have

$$\begin{aligned}
&-\Delta u_0(z) + \xi(z) u_0(z) \ge 0 \quad \text{for a.a. } z \in \Omega,\\
&\Rightarrow\ \Delta u_0(z) \le \|\xi\|_\infty u_0(z) \quad \text{for a.a. } z \in \Omega,\\
&\Rightarrow\ u_0 \in \operatorname{int} C_+ \quad (\text{by Hopf's lemma}).
\end{aligned}$$

Also we have

$$\begin{aligned}&\Delta(\overline{u}-u_0)(z)\le\|\xi\|_\infty(\overline{u}-u_0)(z)\quad\text{for a.a. } z\in\Omega \text{ (see (3.4))},\\ &\Rightarrow\ \overline{u}-u_0\in\operatorname{int}C_+\quad\text{(again Hopf's lemma).}\end{aligned}$$

So, finally we can say that

$$u_0\in\operatorname{int}_{C^1(\overline{\Omega})}[0,\overline{u}]. \tag{3.11}$$

In a similar fashion working on the negative semiaxis with $\overline{v}=-\widetilde{\xi}\,\widetilde{u}\in-\operatorname{int}C_+$, we produce a negative solution v_0 such that

$$v_0\in-\operatorname{int}C_+,\quad\|v_0\|_\infty\le\delta_0,\quad v_0\in\operatorname{int}_{C^1(\overline{\Omega})}[\overline{v},0].$$

□

Now using these two solutions and the full set of hypotheses (H2), we will generate two more constant sign smooth solutions.

Proposition 3.2. *If* (H1), (H2) *hold, then for all* $\lambda\in(0,\lambda^*)$, *problem* (1.1) *has two more constant sign solutions* $\widehat{u}\in\operatorname{int}C_+$ *and* $\widehat{v}\in-\operatorname{int}C_+$ *such that* $\widehat{u}-u_0\in\operatorname{int}C_+$ *and* $v_0-\widehat{v}\in\operatorname{int}C_+$.

Proof. Let $u_0\in\operatorname{int}C_+$ be the positive solution of problem (1.1) ($\lambda\in(0,\lambda^*)$) produced in Proposition 3.1. We introduce the following truncation of $f_\lambda(z,\cdot)$:

$$g_\lambda^+(z,x)=\begin{cases}f_\lambda(z,u_0(z)) & \text{if } x\le u_0(z),\\ f_\lambda(z,x) & \text{if } u_0(z)<x.\end{cases} \tag{3.12}$$

This is a Carathéodory function. We set $G_\lambda^+(z,x)=\int_0^x g_\lambda^+(z,s)ds$ and consider the C^1-functional $\psi_\lambda^+:H^1(\Omega)\to\mathbb{R}$ defined by

$$\psi_\lambda^+(u)=\frac{1}{2}\gamma(u)-\int_\Omega G_\lambda^+(z,u)dz\quad\text{for all } u\in H^1(\Omega).$$

Using (3.8) and the regularity theory of Wang [19], we obtain easily that

$$K_{\psi_\lambda^+}\subseteq[u_0)\cap\operatorname{int}C_+. \tag{3.13}$$

From (3.12) and (3.13) we see that we may assume that

$$K_{\psi_\lambda^+}\cap[0,\overline{u}]=\{u_0\}. \tag{3.14}$$

Otherwise we already have a second positive smooth solution for problem (1.1).

We introduce the following truncation of $g_\lambda^+(z,\cdot)$:

$$\widehat{g}_\lambda^+(z,x)=\begin{cases}g_\lambda^+(z,x) & \text{if } x\le\overline{u}(z),\\ g_\lambda^+(z,\overline{u}(z)) & \text{if } \overline{u}(z)<x.\end{cases} \tag{3.15}$$

This is a Carathéodory function. We set $\widehat{G}_\lambda^+(z,x)=\int_0^x\widehat{g}_\lambda^+(z,s)ds$ and consider the C^1-functional $\widehat{\psi}_\lambda^+:H^1(\Omega)\to\mathbb{R}$ defined by

$$\widehat{\psi}_\lambda^+(u)=\frac{1}{2}\gamma(u)-\int_\Omega\widehat{G}_\lambda^+(z,u)dz\quad\text{for all } u\in H^1(\Omega).$$

From (2.1) and (3.15) it is clear that $\widehat{\psi}_\lambda^+(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_0\in H^1(\Omega)$ such that

$$\widehat{\psi}_\lambda^+(\widetilde{u}_0)=\inf[\widehat{\psi}_\lambda^+(u):u\in H^1(\Omega)]. \tag{3.16}$$

Using (3.12) and (3.15), we can easily show that

$$K_{\widehat{\psi}_\lambda^+} \subseteq [u_0, \overline{u}] \cap \operatorname{int} C_+. \tag{3.17}$$

Also we have

$$\psi_\lambda^+\big|_{[0,\overline{u}]} = \widehat{\psi}_\lambda^+\big|_{[0,\overline{u}]} \quad \text{and} \quad (\psi_\lambda^+)'\big|_{[0,\overline{u}]} = (\widehat{\psi}_\lambda^+)'\big|_{[0,\overline{u}]}. \tag{3.18}$$

From (3.16) we have $\widetilde{u}_0 \in K_{\widehat{\psi}_\lambda^+}$. From (3.17), (3.18) and (3.14) it follows that $\widetilde{u}_0 = u_0 \in \operatorname{int} C_+$. Since $u_0 \in \operatorname{int}_{C^1(\overline{\Omega})}[0, \overline{u}]$ (see (3.11)), from (3.16) and (3.18) it follows that

$$\begin{aligned} &u_0 \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } \psi_\lambda^+, \\ &\Rightarrow u_0 \text{ is a local } H^1(\Omega)\text{-minimizer of } \psi_\lambda^+ \text{ (see Brezis-Nirenberg [3])}. \end{aligned} \tag{3.19}$$

From (3.13) and (3.12), we see that we can assume that

$$K_{\psi_\lambda^+} \text{ is finite.} \tag{3.20}$$

Otherwise we already have a whole sequence of distinct positive smooth solutions of (1.1), all strictly bigger than u_0 and so we are done. Then (3.19), (3.20) and Papageorgiou-Rădulescu-Repovš [17, Theorem 5.7.6, p. 449] imply that we can find $\rho \in (0, 1)$ small such that

$$\psi_\lambda^+(u_0) < \inf[\psi_\lambda^+(u) : \|u - u_0\| = \rho] = m_\lambda^+. \tag{3.21}$$

Hypothesis (H2)(ii) implies that if $u \in \operatorname{int} C_+$, then

$$\psi_\lambda^+(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{3.22}$$

Claim: ψ_λ^+ satisfies the C-condition. We consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq H^1(\Omega)$ such that

$$|\psi_\lambda^+(u_n)| \le c_3 \quad \text{for some } c_3 > 0, \text{ all } n \in \mathbb{N}, \tag{3.23}$$

$$(1 + \|u_n\|)(\psi_\lambda^+)'(u_n) \to 0 \text{ in } H^1(\Omega)^* \text{ as } n \to \infty. \tag{3.24}$$

From (3.24) we have

$$\big|\langle \gamma'(u_n), h\rangle - \int_\Omega g_\lambda^+(z, u_n) h \, dz\big| \le \varepsilon_n \frac{\|h\|}{1 + \|u_n\|} \tag{3.25}$$

for all $h \in H^1(\Omega)$, with $\varepsilon_n \to 0^+$. In (3.25) we use the test function $h = -u_n^- \in H^1(\Omega)$. Then

$$\begin{aligned} &\gamma(u_n^-) \le c_4 \|u_n^-\| \quad \text{for some } c_4 > 0, \text{ all } n \in \mathbb{N} \text{ (see (3.12))}, \\ &\Rightarrow \{u_n^-\}_{n\in\mathbb{N}} \subseteq H^1(\Omega) \text{ is bounded (see (2.1))}. \end{aligned} \tag{3.26}$$

Using (3.26) in (3.23), we obtain

$$\gamma(u_n^+) - \int_\Omega 2G_\lambda^+(z, u_n^+) dz \le c_5 \quad \text{for some } c_5 > 0, \text{ all } n \in \mathbb{N}. \tag{3.27}$$

On the other hand, if in (3.25) we use the test function $h = u_n^+ \in H^1(\Omega)$ then

$$-\gamma(u_n^+) + \int_\Omega g_\lambda^+(z, u_n^+) u_n^+ dz \le \varepsilon_n \quad \text{for all } n \in \mathbb{N}. \tag{3.28}$$

We add (3.27) and (3.28) and obtain

$$\begin{aligned}
&\int_\Omega [g_\lambda^+(z,u_n^+)u_n^+ - 2G_\lambda^+(z,u_n^+)]dz \le c_6 \quad \text{for some } c_6>0, \text{ all } n\in\mathbb{N},\\
&\Rightarrow \int_\Omega [f_\lambda(z,u_n^+)u_n^+ - 2F_\lambda(z,u_n^+)]dz \le c_7 \quad \text{for some } c_7>0, \text{ all } n\in\mathbb{N},\\
&\Rightarrow \int_\Omega e_\lambda(z,u_n^+)dz \le c_7 \quad \text{for all } n\in\mathbb{N}. \qquad (3.29)
\end{aligned}$$

Using (3.29) we will show that $\{u_n^+\}_{n\in\mathbb{N}} \subseteq H^1(\Omega)$ is bounded.

We argue by contradiction. So, suppose that at least for a subsequence, we have

$$\|u_n^+\| \to \infty \quad \text{as } n\to\infty. \qquad (3.30)$$

Let $y_n = \frac{u_n^+}{\|u_n^+\|}$, $n\in\mathbb{N}$. Then $\|y_n\|=1$, $y_n\ge 0$ for all $n\in\mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } H^1(\Omega) \text{ and } y_n\to y \text{ in } L^r(\Omega),\ y_n\to y \text{ in } L^2(\partial\Omega),\ y\ge 0. \qquad (3.31)$$

First we assume that $y\ne 0$. Let $\Omega_+=\{z\in\Omega : y(z)>0\}$. Then from (3.31) we see that $|\Omega_+|_N>0$ with $|\cdot|_N$ denoting the Lebesgue measure on $\mathbb{R}^N$. We have

$$\begin{aligned}
&u_n^+(z)\to+\infty \quad \text{for a.a. } z\in\Omega_+,\\
&\Rightarrow \frac{F_\lambda(z,u_n^+(z))}{u_n^+(z)^2}\to+\infty \quad \text{for a.a. } z\in\Omega_+ \text{ (see hypothesis (H2)(ii))},\\
&\Rightarrow \frac{F_\lambda(z,u_n^+(z))}{\|u_n^+\|^2}\to+\infty \quad \text{for a.a. } z\in\Omega_+.
\end{aligned}$$

Then using Fatou's lemma, we have

$$\begin{aligned}
&\int_{\Omega_+} \frac{F_\lambda(z,u_n^+)}{\|u_n^+\|^2}dz\to+\infty,\\
&\Rightarrow \int_{\Omega_+} \frac{G_\lambda^+(z,u_n^+)}{\|u_n^+\|^2}dz\to+\infty \quad \text{(see (3.12))}. \qquad (3.32)
\end{aligned}$$

On account of hypotheses (H2)(i),(ii), we have

$$\begin{aligned}
&F_\lambda(z,x)\ge -c_8 \quad \text{for a.a. } z\in\Omega, \text{ all } x\in\mathbb{R}, \text{ some } c_8>0,\\
&\Rightarrow G_\lambda^+(z,x)\ge -c_9 \quad \text{for a.a. } z\in\Omega, \text{ all } x\in\mathbb{R}, \text{ some } c_9>0 \text{ (see (3.12))}. \qquad (3.33)
\end{aligned}$$

So, we have

$$\begin{aligned}
\int_\Omega \frac{G_\lambda^+(z,u_n^+)}{\|u_n^+\|^2}dz &= \int_{\Omega_+} \frac{G_\lambda^+(z,u_n^+)}{\|u_n^+\|^2}dz + \int_{\Omega\setminus\Omega_+} \frac{G_\lambda^+(z,u_n^+)}{\|u_n^+\|^2}dz\\
&\ge \int_{\Omega_+} \frac{G_\lambda^+(z,u_n^+)}{\|u_n^+\|^2}dz - c_9|\Omega|_N \quad \text{(see (3.33))},\\
&\Rightarrow \int_\Omega \frac{G_\lambda^+(z,u_n^+)}{\|u_n^+\|^2}dz\to+\infty \quad \text{(see (3.32))}. \qquad (3.34)
\end{aligned}$$

From (3.23) and (3.26), we have

$$\int_\Omega \frac{G_\lambda^+(z,u_n^+)}{\|u_n^+\|^2}dz \le c_{10}+\frac{1}{2}\gamma(u_n^+) \quad \text{for some } c_{10}>0, \text{ all } n\in\mathbb{N}. \qquad (3.35)$$

Comparing (3.34) and (3.35) we have a contradiction.

Now assume that $y = 0$. With $k > 0$, let $v_n = (2k)^{1/2} y_n$, $n \in \mathbb{N}$. We have

$$v_n \xrightarrow{w} 0 \text{ in } H^1(\Omega) \text{ and } v_n \to 0 \text{ in } L^2(\Omega) \text{ and in } L^2(\partial\Omega) \text{ (see (3.31))}. \tag{3.36}$$

Let $t_n \in [0, 1]$ be such that

$$\psi_\lambda^+(t_n u_n^+) = \max[\psi_\lambda^+(t u_n^+) : 0 \le t \le 1]. \tag{3.37}$$

On account of (3.30), we can find $n_0 \in \mathbb{N}$ such that

$$0 < (2k)^{1/2} \frac{1}{\|u_n^+\|} \le 1 \quad \text{for all } n \in \mathbb{N},\ n \ge n_0. \tag{3.38}$$

We have

$$\psi_\lambda^+(t_n u_n^+) \ge \psi_\lambda^+(v_n) = k - \int_\Omega G_\lambda^+(z, v_n) dz \quad \text{for all } n \ge n_0 \text{ (see (3.38))}. \tag{3.39}$$

From (3.36) we see that

$$\int_\Omega G_\lambda^+(z, v_n) dz \to 0 \quad \text{as } n \to \infty.$$

So, from (3.39) we see that we can find $n_1 \in \mathbb{N}$, $n_1 \ge n_0$ such that

$$\psi_\lambda^+(t_n u_n^+) \ge \frac{k}{2} \quad \text{for all } n \ge n_1.$$

Since $k > 0$ is arbitrary, it follows that

$$\psi_\lambda^+(t_n u_n^+) \to +\infty \quad \text{as } n \to \infty. \tag{3.40}$$

We know that

$$\psi_\lambda^+(0) = 0 \quad \text{and} \quad \psi_\lambda^+(u_n^+) \le c_{11} \quad \text{for some } c_{11} > 0, \text{ all } n \in \mathbb{N} \tag{3.41}$$

(see (3.23), (3.26)). From (3.40) and (3.41) it follows that we can find $n_2 \in \mathbb{N}$ such that

$$t_n \in (0, 1) \quad \text{for all } n \ge n_2. \tag{3.42}$$

From (3.37) and (3.42), we infer that

$$\begin{aligned}
&\frac{d}{dt} \psi_\lambda^+(t u_n^+)\Big|_{t = t_n} = 0 \quad \text{for } n \ge n_2, \\
&\Rightarrow \langle (\psi_\lambda^+)'(t_n u_n^+), t_n u_n^+ \rangle = 0 \quad \text{for } n \ge n_2 \\
&\text{(by the chain rule and see (3.42))}, \\
&\Rightarrow \gamma(t_n u_n^+) = \int_\Omega g_\lambda^+(z, t_n u_n^+)(t_n u_n^+) dz \quad \text{for all } n \ge n_2.
\end{aligned} \tag{3.43}$$

From (3.12) we see that

$$\int_\Omega g_\lambda^+(z, t_n u_n^+)(t_n u_n^+) dz \le \int_\Omega f_\lambda(z, t_n u_n^+)(t_n u_n^+) dz + c_{12} \tag{3.44}$$

for some $c_{12} > 0$, all $n \in \mathbb{N}$. Also from (3.42) and hypothesis (H2)(iii), we have

$$\begin{aligned}
&\int_\Omega e_\lambda(z, t_n u_n^+)\, dz \le \int_\Omega e_\lambda(z, u_n^+)\, dz + \|\eta_\lambda\|_1 \le c_{13} \\
&\text{for some } c_{13} > 0, \text{ all } n \ge n_2 \text{ (see (3.29))}, \\
&\Rightarrow \int_\Omega f_\lambda(z, t_n u_n^+)(t_n u_n^+) dz \le c_{13} + \int_\Omega 2F_\lambda(z, t_n u_n^+) dz \quad \text{for all } n \ge n_2.
\end{aligned} \tag{3.45}$$

We return to (3.44) and using (3.45) and (3.12), we obtain

$$\int_\Omega g_\lambda^+(z,t_nu_n^+)(t_nu_n^+)dz \le c_{14} + \int_\Omega 2G_\lambda^+(z,t_nu_n^+)\,dz \tag{3.46}$$

for some $c_{14}>0$, all $n\ge n_2$. If in (3.43) we use (3.46), then

$$2\psi_\lambda^+(t_nu_n^+) \le c_{14} \quad \text{for all } n\ge n_2. \tag{3.47}$$

Comparing (3.40) and (3.47), we have a contradiction. Therefore $\{u_n^+\}_{n\in\mathbb{N}}\subseteq H^1(\Omega)$ is bounded, hence $\{u_n\}_{n\in\mathbb{N}}\subseteq H^1(\Omega)$ is bounded (see (3.26)). So, we can assume that

$$u_n \xrightarrow{w} u \text{ in } H^1(\Omega) \text{ and } u_n\to u \text{ in } L^r(\Omega) \text{ and in } L^2(\partial\Omega). \tag{3.48}$$

In (3.25), we choose $h=u_n-u\in H^1(\Omega)$, pass to the limit as $n\to\infty$ and use (3.48). Then

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega(\nabla u_n,\nabla u_n-\nabla u)_{\mathbb{R}^N}\,dz=0,\\
&\Rightarrow\ \|\nabla u_n\|_2\to\|\nabla u\|_2,\\
&\Rightarrow\ u_n\to u \quad \text{in } H^1(\Omega)
\end{aligned}$$

(see (3.48) and use the Kadec-Klee property, see [7], p. 911).

This proves the Claim.

On account of (3.21), (3.22) and the Claim, we can apply the mountain pass theorem. So, we can find $\widehat{u}\in H^1(\Omega)$ such that

$$\widehat{u}\in K_{\psi_\lambda^+}\subseteq[u_0)\cap\operatorname{int}C_+ \text{ (see (3.13))},\quad m_\lambda^+\le\psi_\lambda^+(\widehat{u}) \text{ (see (3.21))}.$$

So, $\widehat{u}\ne u_0$ (see (3.21)), $u_0\le\widehat{u}$ and $\widehat{u}\in\operatorname{int}C_+$ is the second positive solution of (1.1) ($\lambda\in(0,\lambda^*)$), distinct from u_0.

If $\rho=\|\widehat{u}\|_\infty$, then on account of hypothesis (H2)(i), we can find $\widehat{\xi}_\rho>0$ such that

$$\begin{aligned}
&f_x'(z,x)x^2\ge-\widehat{\xi}_\rho x^2 \quad \text{for a.a. } z\in\Omega,\text{ all } |x|\le\rho,\\
&\Rightarrow\ x\to f(z,x)+\widehat{\xi}_\rho x \text{ is nondecreasing on } [-\rho,\rho].
\end{aligned}$$

We have

$$\begin{aligned}
-\Delta\widehat{u}+[\xi(z)+\widehat{\xi}_\rho]\widehat{u}&=f_\lambda(z,\widehat{u})+\widehat{\xi}_\rho\widehat{u}\\
&\ge f_\lambda(z,u_0)+\widehat{\xi}_\rho u_0 \quad \text{(recall } u_0\le\widehat{u})\\
&=-\Delta u_0+[\xi(z)+\widehat{\xi}_\rho]u_0,\\
&\Rightarrow\ \widehat{u}-u_0\in\operatorname{int}C_+ \text{ (by Hopf's lemma).}
\end{aligned}$$

Similarly using this time $v_0\in-\operatorname{int}C_+$ and reasoning as above (working this time on the negative semiaxis), we produce a second negative solution $\widehat{v}$ of problem (1.1) ($\lambda\in(0,\lambda^*)$), such that

$$\widehat{v}\in-\operatorname{int}C_+,\quad v_0-\widehat{v}\in\operatorname{int}C_+.$$

□

We will show that there exist extremal constant sign solutions, that is, a smallest positive solution $u_*^\lambda\in\operatorname{int}C_+$ and a biggest negative solution $v_*^\lambda\in-\operatorname{int}C_+$. We will use these extremal (barrier) solutions in Section 4 in order to produce nodal solutions.

We introduce the following two sets:

$$\begin{aligned} S_+(\lambda) &= \text{ set of positive solutions of (1.1)},\\ S_-(\lambda) &= \text{ set of negative solutions of (1.1)}. \end{aligned}$$

We already know from Proposition 3.1, that assuming (H2)(i), (H2)(iv),

$$\emptyset \neq S_+(\lambda) \subseteq \operatorname{int} C_+,\ \emptyset \neq S_-(\lambda) \subseteq -\operatorname{int} C_+, \quad \text{for all } \lambda > 0 \text{ small}.$$

From hypotheses (H2)(i), (H2)(iv), we see that given $\varepsilon > 0$, we can find $c_{15} = c_{15}(\varepsilon) > 0$ such that

$$f_\lambda(z,x)x \geq [(f_\lambda)'_x(z,0) - \varepsilon]x^2 - c_{15}|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{3.49}$$

Motivated by the unilateral growth restriction on $f_\lambda(z,\cdot)$, we consider the auxiliary Robin problem

$$\begin{gathered} -\Delta u + \xi(z)u = [(f_\lambda)'_x(z,0) - \varepsilon]u - c_{15}|u|^{r-2}u \quad \text{in } \Omega,\\ \frac{\partial u}{\partial n} + \beta(z)u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.50}$$

Proposition 3.3. *If* (H1) *hold and* $\lambda > 0$*, then for all* $\varepsilon > 0$ *small problem* (3.50) *has a unique positive solution* $w_+^\lambda \in \operatorname{int} C_+$ *and since the equation is odd* $w_-^\lambda = -w_+^\lambda \in -\operatorname{int} C_+$ *is the unique negative solution of* (3.50)*.*

Proof. First we show the existence of a positive solution. To this end, we introduce the C^1-functional $\tau_\lambda^+ : H^1(\Omega) \to \mathbb{R}$ defined by

$$\tau_\lambda^+(u) = \frac{1}{2}\gamma(u) + \frac{c_{15}}{r}\|u^+\|_r^r - \frac{1}{2}\int_\Omega [(f_\lambda)'_x(z,0) - \varepsilon](u^+)^2 dz \quad \text{for all } u \in H^1(\Omega).$$

Since $r > 2$, this functional is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $w_+^\lambda \in H^1(\Omega)$ such that

$$\tau_\lambda^+(w_+^\lambda) = \inf[\tau_\lambda^+(u) : u \in H^1(\Omega)]. \tag{3.51}$$

Let $t \in (0,1)$. We have

$$\begin{aligned} \tau_\lambda^+(t\widehat{u}_1) &= \frac{t^2}{2}\gamma(\widehat{u}_1) + \frac{c_{15}t^r}{r}\|\widehat{u}_1\|_r^r - \frac{t^2}{2}\int_\Omega [(f_\lambda)'_x(z,0) - \varepsilon]\widehat{u}_1^2 dz\\ &= \frac{t^2}{2}\Big[\int_\Omega [\widehat{\lambda}_1 - (f_\lambda)'_x(z,0)]\widehat{u}_1^2 dz + \varepsilon\Big] + \frac{c_{15}t^r}{r}\|\widehat{u}_1\|_r^r. \end{aligned} \tag{3.52}$$

Hypothesis (H2)(iv) and the fact that $\widehat{u}_1 \in \operatorname{int} C_+$ imply that

$$\vartheta_\lambda = \int_\Omega [(f_\lambda)'_x(z,0) - \widehat{\lambda}_1]\widehat{u}_1^2 dz > 0.$$

So, from (3.52) and choosing $\varepsilon > 0$ small, we have

$$\tau_\lambda^+(t\widehat{u}_1) \leq c_{16}t^r - c_{17}t^2 \quad \text{for some } c_{16}, c_{17} > 0.$$

Since $r > 2$, choosing $t \in (0,1)$ small we have

$$\begin{aligned} &\tau_\lambda^+(t\widehat{u}_1) < 0,\\ &\Rightarrow\ \tau_\lambda^+(w_+^\lambda) < 0 = \tau_\lambda^+(0) \quad \text{(see (3.51))},\\ &\Rightarrow\ w_+^\lambda \neq 0. \end{aligned}$$

From (3.51), we have

$$(\tau_\lambda^+)'(w_+^\lambda) = 0,$$

$$\Rightarrow \langle \gamma'(w_+^\lambda), h\rangle = \int_\Omega \big([(f_\lambda)'_x(z,0)-\varepsilon](w_+^\lambda)^+ - c_{15}(w_+^\lambda)^{r-1}\big)\, h\, dz \tag{3.53}$$

for all $h \in H^1(\Omega)$.

In (3.53) we use the test function $h = -(w_+^\lambda)^- \in H^1(\Omega)$. Then

$$\begin{aligned} &\gamma((w_+^\lambda)^-) = 0, \\ &\Rightarrow\ w_+^\lambda \ge 0,\ w_+^\lambda \ne 0 \quad \text{(see (2.1))}. \end{aligned} \tag{3.54}$$

Therefore w_+^λ is a positive solution of (3.50) ($\lambda > 0$ and $\varepsilon > 0$ small) and from the regularity theory of Wang [19], we have

$$w_+^\lambda \in C_+ \setminus \{0\} \quad \text{(see (3.54))}.$$

From the equation we have

$$\begin{aligned} &\Delta w_+^\lambda \le [\|\xi\|_\infty + c_{15}\|w_+^\lambda\|_\infty^{r-2}] w_+^\lambda \quad \text{in } \Omega, \\ &\Rightarrow\ w_+^\lambda \in \operatorname{int} C_+. \end{aligned} \tag{3.55}$$

Next we show the uniqueness of the positive solution of (3.50). Suppose that $\widehat{w}_+^\lambda \in H^1(\Omega)$ is another positive solution. Again we have $\widehat{w}_+^\lambda \in \operatorname{int} C_+$ (see (3.55)). Let $t > 0$ be the biggest positive real such that

$$t\widehat{w}_+^\lambda \le w_+^\lambda \quad \text{(see [17], p. 274)}. \tag{3.56}$$

Suppose that $0 < t < 1$. Let $\rho_\lambda = \|w_+^\lambda\|_\infty$. On account of hypothesis (H2)(i), we see that we can find $\widehat{\xi}_{\rho_\lambda} > 0$ such that for a.a. $z \in \Omega$ the function $x \to [(f_\lambda)'_x(z,0) - \varepsilon]x - c_{15}x^{r-1} + \widehat{\xi}_{\rho_\lambda} x$ is nondecreasing on $[0, \rho_\lambda]$. We have

$$\begin{aligned} &-\Delta(t\widehat{w}_+^\lambda) + [\xi(z) + \widehat{\xi}_{\rho_\lambda}](t\widehat{w}_+^\lambda) \\ &= t\Big[((f_\lambda)'_x(z,0)-\varepsilon)\widehat{w}_+^\lambda - c_{15}(\widehat{w}_+^\lambda)^{r-1} + \widehat{\xi}_{\rho_\lambda}\widehat{w}_+^\lambda\Big] \\ &\le ((f_\lambda)'_x(z,0)-\varepsilon)(t\widehat{w}_+^\lambda) - c_{15}(t\widehat{w}_+^\lambda)^{r-1} + \widehat{\xi}_{\rho_\lambda}(t\widehat{w}_+^\lambda) \\ &\quad \text{(since } 0 < t < 1 \text{ and } r > 2) \\ &\le ((f_\lambda)'_x(z,0)-\varepsilon)w_+^\lambda - c_{15}(w_+^\lambda)^{r-1} + \widehat{\xi}_{\rho_\lambda}\widehat{w}_+^\lambda \quad \text{(see (3.56))} \\ &= -\Delta w_+^\lambda + [\xi(z) + \widehat{\xi}_{\rho_\lambda}] w_+^\lambda, \\ &\Rightarrow\ \Delta(w_+^\lambda - t\widehat{w}_+^\lambda) \le [\|\xi\|_\infty + \widehat{\xi}_{\rho_\lambda}](w_+^\lambda - t\widehat{w}_+^\lambda), \\ &\Rightarrow\ w_+^\lambda - t\widehat{w}_+^\lambda \in \operatorname{int} C_+ \quad \text{(by Hopf's lemma)}. \end{aligned}$$

But this contradicts the maximality of $t > 0$ in (3.56). Hence $1 \le t$ and so we have $\widehat{w}_+^\lambda \le w_+^\lambda$. Interchanging the roles of $\widehat{w}_+^\lambda$ and w_+^λ in the above argument, we obtain $w_+^\lambda \le \widehat{w}_+^\lambda$ and so we conclude that $w_+^\lambda = \widehat{w}_+^\lambda$. This proves the uniqueness of the positive solution of problem (3.50) ($\lambda > 0$, $\varepsilon > 0$ small). Problem (3.50) is odd. Therefore $w_-^\lambda = -w_+^\lambda \in -\operatorname{int} C_+$ is the unique negative solution of (3.50). □

These solutions provide bounds for the sets $S_+(\lambda)$ and $S_-(\lambda)$.

Proposition 3.4. *If* (H1), (H2)(i), (H2)(iv) *hold and* $\lambda \in (0, \lambda^*)$, *then* $w_+^\lambda \le u$ *for all* $u \in S_+(\lambda)$ *and* $v \le w_-^\lambda$ *for all* $v \in S_-(\lambda)$.

Proof. Let $u \in S_+(\lambda) \subseteq \operatorname{int} C_+$ and for $\varepsilon > 0$ small as in Proposition 3.3, we consider the Carathéodory function $\vartheta^+_\lambda(z,x)$ defined by

$$\vartheta^+_\lambda(z,x) = \begin{cases} [(f_\lambda)'_x(z,0) - \varepsilon](x^+) - c_{15}(x^+)^{r-1} & \text{if } x \le u(z), \\ [(f_\lambda)'_x(z,0) - \varepsilon]u(z) - c_{15}u(z)^{r-1} & \text{if } u(z) < x. \end{cases} \tag{3.57}$$

We set $\Theta^+_\lambda(z,x) = \int_0^x \vartheta^+_\lambda(z,s)ds$ and introduce the C^1-functional $b^+_\lambda : H^1(\Omega) \to \mathbb{R}$ defined by

$$b^+_\lambda(u) = \frac{1}{2}\gamma(u) - \int_\Omega \Theta^+_\lambda(z,u)dz \quad \text{for all } u \in H^1(\Omega).$$

From (2.1) and (3.57) we see that $b^+_\lambda(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widehat{w}^\lambda_+ \in H^1(\Omega)$ such that

$$b^+_\lambda(\widehat{w}^\lambda_+) = \inf[b^+_\lambda(u) : u \in H^1(\Omega)]. \tag{3.58}$$

As in the proof of Proposition 3.3, we have

$$b^+_\lambda(\widehat{w}^\lambda_+) < 0 = b^+_\lambda(0) \ \Rightarrow\ \widehat{w}^\lambda_+ \ne 0.$$

From (3.58), we have

$$\begin{aligned} &(b^+_\lambda)'(\widehat{w}^\lambda_+) = 0 \\ &\Rightarrow\ \langle \gamma'(\widehat{w}^\lambda_+), h\rangle = \int_\Omega \vartheta^+_\lambda(z,\widehat{w}^\lambda_+)h\,dz \quad \text{for all } h \in H^1(\Omega). \end{aligned} \tag{3.59}$$

Here first we choose $h = -(\widehat{w}^\lambda_+)^- \in H^1(\Omega)$ and obtain

$$\begin{gathered} c_0\|(\widehat{w}^\lambda_+)^-\|^2 \le 0 \quad (\text{see } (2.1)), \\ \Rightarrow\ \widehat{w}^\lambda_+ \ge 0, \quad \widehat{w}^\lambda_+ \ne 0. \end{gathered}$$

Then in (3.59) we use the test function $h = (\widehat{w}^\lambda_+ - u)^+ \in H^1(\Omega)$. We have

$$\begin{aligned} &\langle \gamma'(\widehat{w}^\lambda_+), (\widehat{w}^\lambda_+ - u)^+\rangle = \int_\Omega \big([(f_\lambda)'_x(z,0) - \varepsilon]u - c_{15}u^{r-1}\big)(\widehat{w}^\lambda_+ - u)^+dz \ (\text{see } (3.57)) \\ &\le \int_\Omega f_\lambda(z,u)(\widehat{w}^\lambda_+ - u)^+dz \quad (\text{see } (3.49)) \\ &= \langle \gamma'(u), (\widehat{w}^\lambda_+ - u)^+\rangle \quad (\text{since } u \in S_+(\lambda)), \\ &\Rightarrow\ \widehat{w}^\lambda_+ \le u. \end{aligned}$$

So, we can say that

$$\widehat{w}^\lambda_+ \in [0,u],\ \widehat{w}^\lambda_+ \ne 0. \tag{3.60}$$

Then from (3.60), (3.57) and (3.59) it follows that $\widehat{w}^\lambda_+$ is a positive solution $\widehat{w}^\lambda_+$ of (3.50). Proposition 3.3 implies that $\widehat{w}^\lambda_+ = w^\lambda_+$. Therefore

$$w^\lambda_+ \le u \quad \text{for all } u \in S_+(\lambda).$$

In a similar fashion we show that $v \le w^\lambda_-$ for all $v \in S_-(\lambda)$. □

These bounds lead to the existence of extremal constant sign solutions.

Proposition 3.5. *If* (H1), (H2)(i), (H2)(iv) *hold and* $\lambda \in (0,\lambda^*)$, *then problem* (1.1) *has a smallest positive solution* $u^\lambda_* \in \operatorname{int} C_+$ *(that is,* $u^\lambda_* \le u$ *for all* $u \in S_+(\lambda)$*) and a biggest negative solution* $v^\lambda_* \in -\operatorname{int} C_+$ *(that is,* $v \le v^\lambda_*$ *for all* $v \in S_-(\lambda)$*).*

Proof. From Filippakis-Papageorgiou [6], we know that $S_+(\lambda)$ is downward directed (that is, if $u_1, u_2 \in S_+(\lambda)$, then we can find $u \in S_+(\lambda)$ such that $u \le u_1$, $u \le u_2$). Then using Hu-Papageorgiou [10, Lemma 3.10, p. 178], we can find a decreasing sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq H^1(\Omega)$ such that

$$\inf_{n\in\mathbb{N}} u_n = \inf S_+(\lambda).$$

We have

$$\langle \gamma'(u_n), h\rangle = \int_\Omega f_\lambda(z, u_n) h \, dz \quad \text{for all } h \in H^1(\Omega), \tag{3.61}$$

$$\widehat{w}_+^\lambda \le u_n \le u_1 \quad \text{for all } n \in \mathbb{N} \text{ (see Proposition 3.4)}. \tag{3.62}$$

If in (3.61) we choose the test function $h = u_n \in H^1(\Omega)$ and use (3.62) and (2.6), we infer that $\{u_n\}_{n\in\mathbb{N}} \subseteq H^1(\Omega)$ is bounded. We assume that

$$u_n \xrightarrow{w} u_*^\lambda \text{ in } H^1(\Omega) \text{ and } u_n \to u_*^\lambda \text{ in } L^2(\Omega) \text{ and in } L^2(\partial\Omega). \tag{3.63}$$

In (3.61) we choose $h = u_n - u_*^\lambda \in H^1(\Omega)$, pass to the limit as $n \to \infty$ and use (3.63). Then

$$\begin{aligned}
&\lim_{n\to\infty} \int_\Omega (\nabla u_n, \nabla u_n - \nabla u_*^\lambda)_{\mathbb{R}^N} \, dz = 0,\\
&\Rightarrow \ \|\nabla u_n\|_2 \to \|\nabla u_*^\lambda\|_2,\\
&\Rightarrow \ u_n \to u_*^\lambda \quad \text{in } H^1(\Omega)
\end{aligned}$$

(by the Kadec-Klee property of Hilbert spaces). So, in the limit as $n \to \infty$, we have

$$\begin{gathered}
\langle \gamma'(u_*^\lambda), h\rangle = \int_\Omega f_\lambda(z, u_*^\lambda) h \, dz \quad \text{for all } h \in H^1(\Omega),\\
\widehat{w}_+^\lambda \le u_*^\lambda \quad \text{(see (3.62))}.
\end{gathered}$$

Therefore $u_*^\lambda \in S_+(\lambda)$ and $u_*^\lambda = \inf S_+(\lambda)$. Similarly we produce $v_*^\lambda \in S_-(\lambda) \subseteq \operatorname{int} C_+$ such that $v_*^\lambda = \sup S_-(\lambda)$. Note that $S_-(\lambda)$ is upward directed (that is, if $v_1, v_2 \in S_-(\lambda)$, then we can find $v \in S_-(\lambda)$ such that $v_1 \le v$, $v_2 \le v$). $\square$

4. Nodal solutions

In this section we show the existence of nodal (sign changing) solutions for problem (1.1) ($\lambda \in (0, \lambda^*)$). We produce three nodal solutions. Our strategy to obtain these nodal solutions, is the following. Using truncations, we focus on the order interval $[v_*^\lambda, u_*^\lambda]$. On account of the extremality of u_*^λ and v_*^λ, any solution of (1.1) in this order interval distinct from u_*^λ, v_*^λ and 0, will be nodal. The first nodal solution is obtained using the critical point theory (the mountain pass theorem). The second is established using the theory of critical groups. Finally for the third nodal solution, we employ a flow invariance argument using the gradient flow.

For this strategy to work, we need to slightly strengthen hypothesis (H2)(iv). So, the new conditions on $f_\lambda(z, x)$ are the following:

(H2') Same as hypotheses (H2) only now in (H2)(iv), $m \in \mathbb{N}$ and $m \ge 2$.

We start implementing this strategy. So, we introduce the following truncation of $f_\lambda(z,\cdot)$:

$$\widehat{g}_\lambda(z,x)=\begin{cases} f_\lambda(z,v_*^\lambda(z)) & \text{if } x<v_*^\lambda(z),\\ f_\lambda(z,x) & \text{if } v_*^\lambda(z)\le x\le u_*^\lambda(z),\\ f_\lambda(z,u_*^\lambda(z)) & \text{if } u_*^\lambda(z)<x. \end{cases} \tag{4.1}$$

This is a Carathéodory function. Also, we consider the positive and negative truncations of $\widehat{g}_\lambda(z,\cdot)$, namely the Carathéodory functions $\widehat{g}_\lambda^\pm(z,x)$ defined by

$$\widehat{g}_\lambda^\pm(z,x)=\widehat{g}_\lambda(z,\pm x^\pm). \tag{4.2}$$

We set $\widehat{G}_\lambda(z,x)=\int_0^x\widehat{g}_\lambda(z,s)ds$, $\widehat{G}_\lambda^\pm(z,x)=\int_0^x\widehat{g}_\lambda^\pm(z,s)ds$ and consider the C^1-functionals $j_\lambda,j_\lambda^\pm:H^1(\Omega)\to\mathbb{R}$ defined by

$$\begin{gathered} j_\lambda(u)=\frac{1}{2}\gamma(u)-\int_\Omega\widehat{G}_\lambda(z,u)dz,\\ j_\lambda^\pm(u)=\frac{1}{2}\gamma(u)-\int_\Omega\widehat{G}_\lambda^\pm(z,u)dz\quad\text{for all } u\in H^1(\Omega). \end{gathered}$$

Using (4.1), (4.2), the regularity theory of Wang [19] and the extremality of u_*^λ and v_*^λ, we have the following result.

Proposition 4.1. *If* (H1), (H2)(i), (H2)(iv) *hold and* $\lambda\in(0,\lambda^*)$, *then* $K_{j_\lambda}\subseteq[v_*^\lambda,u_*^\lambda]\cap C^1(\overline{\Omega})$, $K_{j_\lambda^+}=\{0,u_*^\lambda\}$, $K_{j_\lambda^-}=\{0,v_*^\lambda\}$.

It is clear from this proposition that we can assume that

$$K_{j_\lambda}\subseteq C^1(\overline{\Omega}) \text{ is finite}. \tag{4.3}$$

Otherwise we already have a sequence of distinct nodal smooth solutions and so we are done.

Proposition 4.2. *If* (H1), (H2)(i), (H2)(iv) *hold and* $\lambda\in(0,\lambda^*)$, *then* $u_*^\lambda\in\operatorname{int}C_+$ *and* $v_*^\lambda\in-\operatorname{int}C_+$ *are local minimizers of* $j_\lambda(\cdot)$.

Proof. From (4.1), (4.2) and (2.1), it is clear that $j_\lambda^+(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_*^\lambda\in H^1(\Omega)$ such that

$$j_\lambda^+(\widetilde{u}_*^\lambda)=\inf[j_\lambda^+(u):u\in H^1(\Omega)]. \tag{4.4}$$

As in the proof of Proposition 3.1, using (3.49), we show that

$$\begin{gathered} j_\lambda^+(\widetilde{u}_*^\lambda)<0=j_\lambda^+(0),\\ \Rightarrow\ \widetilde{u}_*^\lambda\neq 0. \end{gathered}$$

We have that $\widetilde{u}_*^\lambda\in K_{j_\lambda^+}\setminus\{0\}$ (see (4.4)). Hence from Proposition 4.1 it follows that

$$\widetilde{u}_*^\lambda=u_*^\lambda\in\operatorname{int}C_+. \tag{4.5}$$

It is clear from (4.1) and (4.2) that $j_\lambda\big|_{C_+}=j_\lambda^+\big|_{C_+}$. Then from (4.4) and (4.5) we infer that

$$\begin{aligned} &u_*^\lambda \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } j_\lambda,\\ &\Rightarrow\ u_*^\lambda \text{ is a local } H^1(\Omega)\text{-minimizer of } j_\lambda \text{ (see Brezis-Nirenberg [3])}. \end{aligned}$$

Similarly for $v_*^\lambda\in-\operatorname{int}C_+$, using this time the functional $j_\lambda^-(\cdot)$. ∎

We may assume that

$$j_\lambda(v_*^\lambda) \le j_\lambda(u_*^\lambda). \tag{4.6}$$

The analysis remains the same if the opposite inequality holds.

Proposition 4.3. *If* (H1), (H2) *hold and* $\lambda \in (0, \lambda^*)$, *then* $C_k(j_\lambda, 0) = \delta_{k,d_m}\mathbb{Z}$ *for all* $k \in \mathbb{N}_0$ *with* $d_m = \dim \overline{H}_m = \dim \oplus_{k=1}^m E(\widehat{\lambda}_k)$.

Proof. Let $\varphi_\lambda : H^1(\Omega) \to \mathbb{R}$ be the energy (Euler) functional for problem (1.1) defined by

$$\varphi_\lambda(u) = \frac{1}{2}\gamma(u) - \int_\Omega F_\lambda(z,u)dz \quad \text{for all } u \in H^1(\Omega).$$

Evidently $\varphi_\lambda \in C^2(H^1(\Omega))$. We consider the homotopy $h_\lambda(t,u)$ defined by

$$h_\lambda(t,u) = (1-t)\varphi_\lambda(u) + tj_\lambda(u) \quad \text{for all } (t,u) \in [0,1] \times H^1(\Omega).$$

Suppose we could find $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,1]$, $\{u_n\}_{n\in\mathbb{N}} \subseteq H^1(\Omega)$ such that

$$t_n \to t, u_n \to 0 \text{ in } H^1(\Omega) \text{ and } (h_\lambda)'_u(t_n,u_n) = 0 \quad \text{for all } n \in \mathbb{N}. \tag{4.7}$$

From the equality in (4.7), we have that

$$\langle \gamma'(u_n), h\rangle = (1-t_n)\int_\Omega f_\lambda(z,u_n)h\,dz + t_n\int_\Omega \widehat{g}_\lambda(z,u_n)h\,dz,$$

for all $h \in H^1(\Omega)$, all $n \in \mathbb{N}$, imply

$$\begin{gathered} -\Delta u_n + \xi(z)u_n = (1-t_n)f_\lambda(z,u_n) + t_n\widehat{g}_\lambda(z,u_n) \quad \text{in } \Omega, \\ \frac{\partial u_n}{\partial n} + \beta(z)u_n = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.8}$$

From (4.8) and the regularity theory of Wang [19], we know that there exist $\alpha \in (0,1)$ and $c_{16} > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}), \quad \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \le c_{16} \quad \text{for all } n \in \mathbb{N}. \tag{4.9}$$

Then from (4.9) and the compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$, we have

$$\begin{aligned} &u_n \to 0 \text{ in } C^1(\overline{\Omega}) \text{ (see (4.7))}, \\ &\Rightarrow\ u_n \in [v_*^\lambda, u_*^\lambda] \quad \text{for all } n \ge n_0, \\ &\Rightarrow\ \{u_n\}_{n\ge n_0} \subseteq K_{j_\lambda} \quad \text{(see Proposition 4.1)}, \end{aligned}$$

which contradicts (4.3). Therefore (4.7) can not happen and the homotopy invariance property of critical groups (see [17, p. 505]) implies that

$$C_k(\varphi_\lambda, 0) = C_k(j_\lambda, 0) \quad \text{for all } k \in \mathbb{N}_0. \tag{4.10}$$

Recall that $\varphi_\lambda \in C^2(H^1(\Omega))$ and

$$\langle \varphi''_\lambda(u)h, v\rangle = \langle \gamma(h), v\rangle - \int_\Omega (f_\lambda)'_x(z,u)hv\,dz \quad \text{for all } u,h,v \in H^1(\Omega).$$

Then on account of Proposition 4.3, the Morse index of $u = 0$ is $d_m = \dim \overline{H}_m = \dim \oplus_{k=1}^m E(\widehat{\lambda}_k)$. By [17, Proposition 6.2.6, p. 479], we have

$$\begin{aligned} &C_k(\varphi_\lambda, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0, \\ &\Rightarrow\ C_k(j_\lambda, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \text{ (see (4.10))}. \end{aligned}$$

∎

Now we are ready to produce the first two nodal solutions. As we already mentioned the first will be obtained using critical point theory, while for the second we will use the theory of critical groups. At this point we need to use the stronger hypotheses (H2').

Proposition 4.4. *If* (H1), (H2') *hold and* $\lambda \in (0, \lambda^*)$*, then problem* (1.1) *has at least two nodal solutions* $y_0, \widehat{y} \in C^1(\overline{\Omega})$ *such that* $y_0, \widehat{y} \in \operatorname{int}_{C^1(\overline{\Omega})}[v_*^\lambda, u_*^\lambda]$.

Proof. Using (4.3), (4.6), Proposition 4.2 and [17, Theorem 5.7.6, p. 449], we know that we can find $\rho \in (0,1)$ small such that

$$j_\lambda(v_*^\lambda) \le j_\lambda(u_*^\lambda) < \inf[j_\lambda(u) : \|u - u_*^\lambda\| = \rho] = m_\lambda, \quad \|v_*^\lambda - u_*^\lambda\| > \rho. \tag{4.11}$$

Also, since $j_\lambda(\cdot)$ is coercive (see (4.1)), we know that

$$j_\lambda(\cdot) \text{ satisfies the } C\text{-condition} \tag{4.12}$$

(see [17, Proposition 5.1.15, p. 369]). Then (4.11), (4.12) permit the use of the mountain pass theorem. So, we can find $y_0 \in H^1(\Omega)$ such that

$$y_0 \in K_{j_\lambda} \subseteq [v_*^\lambda, u_*^\lambda] \cap C^1(\overline{\Omega}) \quad \text{(see Proposition 4.1)}, \; m_\lambda \le j_\lambda(y_0). \tag{4.13}$$

From (4.13) and (4.11) we see that

$$y_0 \in [v_*^\lambda, u_*^\lambda] \cap C^1(\overline{\Omega}) \text{ is a solution of (1.1)}, \; y_0 \notin \{u_*^\lambda, v_*^\lambda\}. \tag{4.14}$$

The solution y_0 is a critical point of j_λ of mountain pass type. Hence [17, Theorem 6.5.8, p. 527] implies that

$$C_1(j_\lambda, y_0) \neq 0. \tag{4.15}$$

From Proposition 4.3, we know that

$$C_k(j_\lambda, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0, \text{ with } d_m \ge 2. \tag{4.16}$$

Comparing (4.15) and (4.16), we see that $y_0 \neq 0$ and so $y_0 \in C^1(\overline{\Omega})$ is a nodal solution of problem (1.1).

Let $\rho = \max\{\|v_*^\lambda\|_\infty, \|u_*^\lambda\|_\infty\}$. Using (H2') (i), (H2')(iv), we can find $\widetilde{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$

$$x \to f_\lambda(z,x) + \widetilde{\xi}_\rho x \text{ is nondecreasing on } [-\rho, \rho]. \tag{4.17}$$

We have

$$\begin{aligned}
-\Delta y_0 + [\xi(z) + \widetilde{\xi}_\rho] y_0 &= f_\lambda(z, y_0) + \widetilde{\xi}_\rho y_0 \\
&\le f_\lambda(z, u_*^\lambda) + \widetilde{\xi}_\rho u_*^\lambda \quad \text{(see (4.14), (4.17))} \\
&= -\Delta u_*^\lambda + [\xi(z) + \widetilde{\xi}_\rho] u_*^\lambda, \\
&\Rightarrow \; \Delta(u_*^\lambda - y_0) + [\|\xi\|_\infty + \widetilde{\xi}_\rho](u_*^\lambda - y_0) \quad \text{in } \Omega, \\
&\Rightarrow \; u_*^\lambda - y_0 \in \operatorname{int} C_+.
\end{aligned}$$

Similarly we show that $y_0 - v_*^\lambda \in \operatorname{int} C_+$. Therefore we conclude that

$$y_0 \in \operatorname{int}_{C^1(\overline{\Omega})}[v_*^\lambda, u_*^\lambda]. \tag{4.18}$$

As in the proof of Proposition 4.3, using the homotopy $h_\lambda(t,u)$ and (4.18), we show that

$$C_k(\varphi_\lambda, y_0) = C_k(j_\lambda, y_0) \quad \text{for all } k \in \mathbb{N}_0. \tag{4.19}$$

Since $\varphi_\lambda \in C^2(H^1(\Omega))$, from (4.19), (4.15) and [17, Proposition 6.5.9, p. 529], we have

$$\begin{aligned} &C_k(\varphi_\lambda, y_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0, \\ &\Rightarrow\ C_k(j_\lambda, y_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \text{ (see (4.19)).} \end{aligned} \tag{4.20}$$

From Proposition 4.2 we know that u_*^λ, v_*^λ are local minimizers of $j_\lambda(\cdot)$. Hence

$$C_k(j_\lambda, u_*^\lambda) = C_k(j_\lambda, v_*^\lambda) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{4.21}$$

Also, since $j_\lambda(\cdot)$ is coercive (see (4.1)), we have

$$C_k(j_\lambda, \infty) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \tag{4.22}$$

(see [17, Proposition 6.2.24, p. 491]). Suppose that $K_{j_\lambda} = \{0, y_0, u_*^\lambda, v_*^\lambda\}$. From (4.16), (4.20), (4.21), (4.22) and the Morse relation with $t = -1$ (see (2.5)), we have

$$\begin{aligned} &(-1)^{\mathrm{d_m}} + (-1)^1 + 2(-1)^0 = (-1)^0, \\ &\Rightarrow\ (-1)^{\mathrm{d_m}} = 0, \text{ a contradiction.} \end{aligned}$$

Therefore, there exists $\widehat{y} \in K_{j_\lambda} \subseteq [v_*^\lambda, u_*^\lambda] \cap C^1(\overline{\Omega})$ (see Proposition 4.1), with $\widehat{y} \notin \{0, y_0, u_*^\lambda, v_*^\lambda\}$. Hence $\widehat{y} \in C^1(\overline{\Omega})$ is a second nodal solution of (1.1) distinct from y_0. Moreover, as we did for y_0 earlier in this proof, we show that $\widehat{y} \in \mathrm{int}_{C^1(\overline{\Omega})}[v_*^\lambda, u_*^\lambda]$. □

Next using a flow invariance argument inspired from the works of Papageorgiou-Papalini [15] and He-Guo-Huang-Lei [9], we produce a third nodal solution.

Proposition 4.5. *If* (H1), (H2') *hold and* $\lambda \in (0, \lambda^*)$, *then problem* (1.1) *has a nodal solution* $\widetilde{y} \in C^1(\overline{\Omega})$ *such that* $\widetilde{y} \notin \mathrm{int}_{C^1(\overline{\Omega})}[v_*^\lambda, u_*^\lambda]$.

Proof. On $H^1(\Omega)$ we consider the inner product

$$(u,h)_0 = \int_\Omega (\nabla u, \nabla h)_{\mathbb{R}^N} dz + \int_\Omega \xi(z) uh\, dz + \int_{\partial\Omega} \beta(z) uh d\sigma \quad \text{for all } u, h \in H^1(\Omega).$$

Let $\|\cdot\|_0$ denote the corresponding norm (that is, $\|u\|_0 = (u,u)_0^{1/2}$). It is clear from hypotheses (H1) and (2.1), that $\|\cdot\|_0$ is equivalent to $\|\cdot\|$ (the usual norm on $H^1(\Omega)$). Consider the operator $K : H^1(\Omega) \to H^1(\Omega)$ defined by

$$K(u) = A(u) + \xi(\cdot)u + \beta(\cdot)\widehat{\gamma}_0(u),$$

where $A \in \mathcal{L}(H^1(\Omega), H^1(\Omega)^*)$ is defined by

$$\langle A(u), h\rangle = \int_\Omega (\nabla u, \nabla h)_{\mathbb{R}^N} dz \quad \text{for all } u, h \in H^1(\Omega)$$

and $\widehat{\gamma}_0(\cdot)$ is the trace operator (see Section 2). We have that $K \in \mathcal{L}(H^1(\Omega), H^1(\Omega)^*)$ and

$$\langle K(u), h\rangle \geq c_{17}\|u\|_0^2 \quad \text{for some } c_{17} > 0, \text{ all } u \in H^1(\Omega).$$

Hence by Banach's theorem, we have

$$L = K^{-1} \in \mathcal{L}(H^1(\Omega), H^1(\Omega)^*).$$

We set $E = L \circ N_f$. Then $E : H^1(\Omega) \to H^1(\Omega)$ is continuous and the regularity theory (see Wang [19]), implies that

$$E(H^1(\Omega)) \subseteq C^1(\overline{\Omega}).$$

Moreover, the Sobolev embedding theorem and the strong maximum principle imply that

$$E(\cdot) \text{ is compact and is strictly increasing,} \tag{4.23}$$

that is, $v \le u$, $v \ne u \Rightarrow E(u) - E(v) \in \operatorname{int} C_+$.

We know that $\varphi_\lambda \in C^2(H^1(\Omega))$. The gradient $\nabla\varphi_\lambda$ is defined by

$$\langle \varphi_\lambda'(u), h\rangle = (\nabla\varphi_\lambda(u), h)_0 \quad \text{for all } u, h \in H^1(\Omega).$$

Then we have

$$\nabla\varphi_\lambda = \mathrm{id} - E. \tag{4.24}$$

We consider the negative gradient flow $\sigma_\lambda(\cdot, u)$ defined by the abstract Cauchy problem

$$\frac{d}{dt}\sigma_\lambda(t,u) = -\nabla\varphi_\lambda(\sigma_\lambda(t,u)), \quad t \ge 0, \quad \sigma_\lambda(0,u) = u. \tag{4.25}$$

Using (4.24) we can rewrite (4.25) as follows

$$\frac{d}{dt}\sigma_\lambda(t,u) + \sigma_\lambda(t,u) = E(\sigma_\lambda(t,u)), \quad t \ge 0, \quad \sigma_\lambda(0,u) = u.$$

So, the flow is global (see [7, p. 618]) and it is given by

$$\sigma_\lambda(t,u) = e^{-t}u + \int_0^t e^{-(t-s)}E(\sigma_\lambda(s,u))ds.$$

Now we introduce the set

$$B_1^\lambda = \{u \in C^1(\overline{\Omega}) : \exists\, t_0 > 0 \text{ such that for all } t \ge t_0\ \sigma_\lambda(t,u) \in \operatorname{int}_{C^1(\overline{\Omega})}[v_*^\lambda, u_*^\lambda]\}.$$

We know that $0 \in B_1^\lambda$ (see (4.23)) and the continuous dependence of the flow on the initial condition, implies that $B_1^\lambda \subseteq C^1(\overline{\Omega})$ is open. Moreover, the semigroup property of the flow implies that B_1^λ is flow invariant.

Claim 1: ∂B_1^λ is flow invariant. We argue indirectly. So, suppose that Claim 1 is not true. This means that we can find $\widehat{u} \in \partial B_1^\lambda$ and $\widehat{t} > 0$ such that $\sigma_\lambda(\widehat{t}, \widehat{u}) \notin \partial B_1^\lambda$. There are two possibilities. In the first $\sigma_\lambda(\widehat{t}, \widehat{u}) \in B_1^\lambda$ (recall B_1^λ is open). But then the semigroup property of the flow implies that $\widehat{u} \in B_1^\lambda$, a contradiction since $\widehat{u} \in \partial B_1^\lambda$. In the second possibility we have $\sigma_\lambda(\widehat{t}, \widehat{u}) \notin \overline{B_1^\lambda} = B_1^\lambda \cup \partial B_1^\lambda$. Since $\widehat{u} \in \partial B_1^\lambda$, we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq B_1^\lambda$ such that $u_n \to \widehat{u}$. Then

$$\begin{aligned} &\sigma_\lambda(\widehat{t}, u_n) \to \sigma_\lambda(\widehat{t}, \widehat{u}) \in B_1^\lambda, \\ &\Rightarrow\ \sigma_\lambda(\widehat{t}, u_n) \notin \overline{B_1^\lambda} \quad \text{for all } n \ge n_0. \end{aligned} \tag{4.26}$$

On the other hand, the flow invariance of B_1^λ implies that $\sigma_\lambda(\widehat{t}, u_n) \in \overline{B_1^\lambda}$ for all $n \in \mathbb{N}$ and this contradicts (4.26). This proves Claim 1.

Next we introduce a second set in $C^1(\overline{\Omega})$, namely the set

$$B_2^\lambda = \{u \in C^1(\overline{\Omega}) : \exists\, \widetilde{t} > 0 \text{ such that } \sigma_\lambda(t,u) \in \operatorname{int} C_+ \text{ for all } t \ge \widetilde{t}\}.$$

This set too is open and from (4.23) we have

$$C_+ \setminus \{0\} \subseteq B_2^\lambda \quad \text{and} \quad 0 \in \partial B_2^\lambda.$$

Moreover, as we did for ∂B_1^λ in Claim 1, we can show that ∂B_2^λ is flow invariant. Note that $\partial B_1^\lambda \cap \partial B_2^\lambda \ne \emptyset$ and so we can define

$$\widehat{c}_0 = \inf[\varphi_\lambda(u) : u \in \partial B_1^\lambda \cap \partial B_2^\lambda]. \tag{4.27}$$

Claim 2: $\widehat{c}_0$ is a critical value of φ_λ. Since σ_λ is the negative (descent) flow for φ_λ, we have

$$-\infty < \inf\{\varphi_\lambda(u) : u \in [v_*^\lambda, u_*^\lambda\} \le \widehat{c}_0.$$

Recall that φ_λ satisfies the C-condition. So, $\varphi_\lambda(K_{\varphi_\lambda})$ is closed. Then if $\widehat{c}_0$ is not a critical value of φ_λ, we can find $\varepsilon > 0$ small such that

$$(\widehat{c}_0 - \varepsilon, \widehat{c}_0 + \varepsilon) \cap \varphi_\lambda(K_{\varphi_\lambda}) = \emptyset.$$

Let $\widehat{u} \in \partial B_1^\lambda \cap \partial B_2^\lambda$ such that $\varphi_\lambda(\widehat{u}) \le \widehat{c}_0 + \frac{\varepsilon}{2}$ (see (4.27)). From the deformation theorem (see [7], p. 636), by taking $\varepsilon > 0$ even smaller if necessary, we have

$$\varphi_\lambda(\sigma_\lambda(1, \widehat{u})) \le \widehat{c}_0 - \frac{\varepsilon}{2}. \tag{4.28}$$

The flow invariance of $\partial B_1^\lambda \cap \partial B_2^\lambda$ (see Claim 1), implies that

$$\sigma_\lambda(1, \widehat{u}) \in \partial B_1^\lambda \cap \partial B_2^\lambda. \tag{4.29}$$

But then (4.28), (4.29) and (4.27), lead to a contradiction. This proves Claim 2.

Using Claim 2, we can find $\widetilde{y} \in \partial B_1^\lambda \cap \partial B_2^\lambda$ such that $\widetilde{y} \in K_{\varphi_\lambda}$. Then $\widetilde{y} \notin \{0, y_0, \widehat{y}\}$ (see Proposition 4.4). Also we have $\widetilde{y} \notin (\operatorname{int} C_+ \cup (-\operatorname{int} C_+))$, which implies that it is nodal. Moreover, we have $\widetilde{y} \notin \operatorname{int}_{C^1(\overline{\Omega})}[v_*^\lambda, u_*^\lambda]$. □

Therefore we can state the following multiplicity theorem for problem (1.1).

Theorem 4.6. *If* (H1), (H2') *hold, then for all* $\lambda > 0$ *small, problem* (1.1) *has at least seven nontrivial smooth solutions:*

- $u_0, \widehat{u} \in \operatorname{int} C_+$ *with* $\widehat{u} - u_0 \in \operatorname{int} C_+$,
- $v_0, \widehat{v} \in -\operatorname{int} C_+$ *with* $v_0 - \widehat{v} \in \operatorname{int} C_+$,
- $y_0, \widehat{y}, \widetilde{y} \in C^1(\overline{\Omega})$ *nodal with* $y_0, \widehat{y} \in \operatorname{int}_{C^1(\overline{\Omega})}[v_0, u_0]$, $\widetilde{y} \notin \operatorname{int}_{C^1(\overline{\Omega})}[v_0, u_0]$.

It seems that seven is the maximum number of solutions with sign information that we can have without imposing any symmetry hypotheses on $f_\lambda(z, \cdot)$.

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