Electronic Journal of Differential Equations, Vol. 2020 (2020), No. 105, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

NONLINEAR DEGENERATE ELLIPTIC EQUATIONS IN WEIGHTED SOBOLEV SPACES

AHARROUCH BENALI, BENNOUNA JAOUAD

Abstract. We study the existence of solutions for the nonlinear degenerated elliptic problem

$$-\operatorname{div} a(x,u,\nabla u) = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded open set in $\mathbb{R}^N$, $N \geq 2$, a is a Carathéodory function having degenerate coercivity $a(x,u,\nabla u)\nabla u \geq \nu(x)b(|u|)|\nabla u|^p$, $1 < p < N$, $\nu(\cdot)$ is the weight function, b is continuous and $f \in L^r(\Omega)$.

1. Introduction

In this article we prove the existence of solutions for some nonlinear elliptic equations with principal part having degenerate coercivity. The model case is

$$\begin{gathered} -\operatorname{div}\Big(\frac{\nu(\cdot)|\nabla u|^{p-2}\nabla u}{(1-|u|)^\alpha}\Big) = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

with Ω a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, $p > 1$, $\alpha \geq 0$, $\nu(\cdot)$ is weight function defined on Ω and f a measurable function on whose summability we will make different assumptions. It is clear from the above example that the differential operator is defined on $W_0^{1,p}(\Omega,\nu)$, but that it may not be coercive on the same space as u near to 1. Because of this lack of coercivity, standard existence theorems for solutions of nonlinear elliptic equations cannot be applied. We consider the nonlinear degenerate elliptic problem

$$A(u) = -\operatorname{div}(a(x,u,\nabla u)) = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where, Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, $1 < p < N$, and $a : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function, such that the following assumption holds

$$a(x,s,\xi).\xi \geq \nu(x)b(|s|)|\xi|^p,$$

for almost every x in Ω, for every $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$, with

$$b(|s|) = 1/(1-|s|)^\alpha, \tag{1.2}$$

2010 *Mathematics Subject Classification.* 35J70, 46E30, 35J85.
Key words and phrases. Nonlinear degenerated elliptic operators; weighted Sobolev space; monotony and rearrangement methods.

Submitted December 4, 2019. Published October 12, 2020.

under various assumptions on f. As stated before, due to assumption (1.2), the operator A may not be coercive on $W_0^{1,p}(\Omega,\nu)$, when the solutions approach the critical values ± 1. To overcome this difficulties, we will reason by approximation, cutting by means of truncatures the nonlinearity $a(x,s,\xi)$ in order to get coercive differential operator on $W_0^{1,p}(\Omega,\nu)$, and give a sense to the equation when the solutions near to ± 1 and to manage the set $\{x \in \Omega: \ |u(x)| = 1\}$. For the case $\nu(\cdot)$ being a constant, the existence of solutions to problem (1.1) is proved in [11], when f a measurable function on whose summability have make different assumptions, the analogous problems was treated by many other authors. See, for example, [3, 4, 9, 10, 8] where problems such as

$$-\operatorname{div}\Big(\frac{1}{(1 \pm |u|)^{\alpha}} |\nabla u|^{p-2}\nabla u\Big) = f,$$

are considered.

This article is organized as follows: In section 2, we recall some preliminaries on Weighted Sobolev spaces and properties of rearrangement. In section 3, we first prove the propositions that we will use to prove some a priori estimates of the solutions, then we prove the existence of weak and entropy solution with respect to the summability of f.

2. Preliminaries

Assumptions. Let $b : [0,l[\to (0,\infty)$, with $l > 0$, be a continuous function such that

$$\lim_{s \to l^-} b(s) = +\infty\,. \tag{2.1}$$

We define

$$\begin{gathered}
A(s) = \int_0^s b(t)^{\frac{1}{p-1}}\,dt, \quad \text{for } s \in [0,l), \\
A(l^-) = \lim_{s \to l^-} \int_0^s b(t)^{\frac{1}{p-1}}\,dt = +\infty.
\end{gathered}$$

We study Dirichlet problems of the form

$$\begin{gathered}
-\operatorname{div} a(x,u,\nabla u) = f \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{2.2}$$

where Ω is a bounded open set in $\mathbb{R}^N$, $N \geq 2$, $1 < p < N$, and $a : \Omega \times (-l,l) \times \mathbb{R}^N \to \mathbb{R}^N$, is a Carathéodory function and $\nu : \Omega \to \mathbb{R}_+$ satisfies the following assumptions:

$$\begin{gathered}
a(x,s,\xi)\cdot\xi \geq b(|s|)\nu(x)|\xi|^p, \\
\nu \in L^r(\Omega), \quad r \geq 1, \quad \nu^{-1} \in L^t(\Omega), \quad t \geq N, \quad 1 + \frac{1}{t} < p < N(1 + \frac{1}{t}).
\end{gathered} \tag{2.3}$$

for a.e. $x \in \Omega$, for all $s \in (-l,l)$ and all $\xi \in \mathbb{R}^N$;

$$|a(x,s,\xi)| \leq \nu(x)[h(x) + b(|s|)|\xi|^{p-1}], \tag{2.4}$$

for a.e. $x \in \Omega$, for all $s \in (-l,l)$, for all $\xi \in \mathbb{R}^N$, and $h \in L^{p'}(\Omega,\nu)$;

$$(a(x,s,\xi) - a(x,s,\xi')) \cdot (\xi - \xi') > 0, \tag{2.5}$$

for a.e. $x \in \Omega$, for all $s \in (-l,l)$ and all $\xi \in \mathbb{R}^N$, $\xi \neq \xi'$. Moreover, f is a measurable function on whose summability we will make several assumptions.

For stating existence results in the next section, we need some classes of solutions.

Definition 2.1. We say that $u \in W_0^{1,p}(\Omega,\nu)$ is a weak solution to problem (2.2) if

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla\varphi\,dx = \int_\Omega f\varphi\,dx, \quad \forall\varphi\in W_0^{1,p}(\Omega,\nu). \tag{2.6}$$

Definition 2.2. A measurable function $u \in W_0^{1,p}(\Omega,\nu)$ is an entropy solution to problem (2.2) if

$$|u|\le l \quad \text{a.e. in } \Omega \tag{2.7}$$

and for all $0<k<l$,

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla T_k(u-\varphi)\,dx \le \int_\Omega f T_k(u-\varphi)\,dx, \tag{2.8}$$

for any $\varphi\in W_0^{1,p}(\Omega,\nu)\cap L^\infty(\Omega)$ such that $\|\varphi\|_{L^\infty(\Omega)}<l-k$.

Weighted Sobolev spaces. Let $1\le p<N$, and $\nu:\Omega\to\mathbb{R}$ be a weight function, i.e. a function which is measurable and positive almost everywhere in Ω. The weighted Lebesgue spaces $L^p(\Omega,\nu)$ is defined as

$$L^p(\Omega,\nu)=\big\{u : \text{measurable, real-valued function}, \int_\Omega \nu(x)|u(x)|^p\,dx<\infty\big\}.$$

which is a Banach space (uniformly convex and hence reflexive if $p>1$) equipped with the norm

$$\|u\|_{L^p(\Omega,\nu)}=\Big(\int_\Omega \nu(x)|u(x)|^p\,dx\Big)^{1/p}.$$

By $W^{1,p}(\Omega,\nu)$ we denote the completion of the space $C^1(\overline{\Omega})$ with respect to the norm

$$\|u\|_{W^{1,p}(\Omega,\nu)}=\|u\|_{L^p(\Omega,\nu)}+\||\nabla u|\|_{L^p(\Omega,\nu)}.$$

Moreover we denote by $W_0^{1,p}(\Omega,\nu)$ the closure of $C^1(\overline{\Omega})$ in $W^{1,p}(\Omega,\nu)$ which is normed by

$$\|u\|_{W_0^{1,p}(\Omega,\nu)}=\||\nabla u|\|_{L^p(\Omega,\nu)}.$$

We denote by $W^{-1,p'}(\Omega,1/\nu)$ the dual space of $W_0^{1,p}(\Omega,\nu)$; for more details see [16].

Rearrangement properties. We recall some definitions about decreasing rearrangement of functions. Let Ω be a bounded open set of $\mathbb{R}^N$ and $u:\Omega\to\mathbb{R}$ a measurable function.

Definition 2.3. The distribution function of u is defined as

$$\mu_u(t)=|\{x\in\Omega : |u(x)|>t\}|, \quad t\ge 0.$$

The function μ_u is decreasing and right continuous.

Definition 2.4. The decreasing rearrangement of u is defined as

$$u_*(s):=\sup\{t\ge 0:\mu_u(t)>s\}, s\ge 0.$$

The function u_* is the generalized inverse of μ_u. We recall that

$$\int_\Omega |u|^p\,dx = p\int_0^{+\infty} t^{p-1}\mu_u(t)dt, \quad \text{for } p\ge 1\,. \tag{2.9}$$

Then the L^p-norm, for $1\le p<+\infty$, is invariant with respect to rearrangement, that is,

$$\|u\|_{L^p(\Omega)}=\|u_*\|_{L^p[0,|\Omega|]}.$$

Moreover, if $u \in L^\infty(\Omega)$, by definition $u_*(0) = \operatorname{ess\,sup}_\Omega |u|$. For more details about rearrangements we refer the reader to [6, 13, 18]. We recall that a measurable function $u : \Omega \to \mathbb{R}$ belongs to the Marcinkiewicz space $M^p(\Omega)$ (or weak-L^p) if the distribution function μ_u satisfies

$$\mu_u(t) \le \frac{c}{t^r}, \quad \forall t > 0,$$

for some constant c. We observe that the above condition is equivalent to

$$u_*(s) \le \frac{c}{s^{1/r}}, \quad \forall s > 0,$$

and we define

$$\|u\|_{M^p(\Omega)} = \sup_{s>0} u_*(s) s^{1/r}.$$

We observe that the Marcinkiewicz spaces are "intermediate" between Lebesgue spaces. Indeed, it is not difficult to show that

$$L^p(\Omega) \subset M^p(\Omega) \subset L^q(\Omega),$$

for $1 \le q < p$. Now, we give a sense to the gradient of a function $u \in L^1(\Omega)$ such that the truncates of u are Sobolev functions.

Lemma 2.5 ([7]). *For each measurable function $u : \Omega \to \mathbb{R}$ such that for every $k > 0$ the truncated function $T_k(u)$ belong to $W^{1,1}_{\rm loc}(\Omega)$, there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that*

$$\nabla T_k(u) = v\chi_{|u|<k} \quad a.e.\ in\ \Omega. \tag{2.10}$$

Furthermore, $u \in W^{1,1}_0(\Omega)$ if and only if $v \in L^1_{\rm loc}(\Omega)$, and then $v = \nabla u$ in the usual weak sense.

Now we recall some Sobolev-type inequalities which will be used later.

Lemma 2.6 ([16]). *Let ν be a nonnegative function on Ω such that $\nu \in L^r(\Omega)$, $r \ge 1$, $\nu^{-1} \in L^t(\Omega)$, $t \ge N$. And let $p, p^\sharp$ be two real number that satisfy $t \ge N/p$, $1 + \frac{1}{t} < p < N(1 + \frac{1}{t})$, $1/p^\sharp = 1/p(1 + \frac{1}{t}) - \frac{1}{N}$. Then*

$$\|u\|_{p^\sharp} \le c_0 \|\nabla u\|_{L^p(\nu)}, \quad \forall u \in W^{1,p}_0(\Omega, \nu).$$

Lemma 2.7. *Suppose that $\lambda > 0$ and $1 \le \gamma < +\infty$. Let ψ a non-negative measurable function on $(0, +\infty)$. Then the*

$$\int_0^{+\infty} \Big(t^{-\lambda} \int_0^t \psi(s) ds\Big)^\gamma \frac{dt}{t} \le c \int_0^{+\infty} (t^{1-\lambda}\psi(t))^\gamma \frac{dt}{t},, \tag{2.11}$$

$$\int_0^{+\infty} \Big(t^{\lambda} \int_t^{+\infty} \psi(s) ds\Big)^\gamma \frac{dt}{t} \le c \int_0^{+\infty} (t^{1+\lambda}\psi(t))^\gamma \frac{dt}{t}. \tag{2.12}$$

Also we shall need the following proposition of weak approximation (see [5]). Let $u \in W^{1,p}_0(\Omega)$, and for $s \in [0, |\Omega|]$, let $G(s)$ be a measurable subset of Ω such that

$$|G(s)| = s$$
$$s_1 < s_2 \Rightarrow G(s_1) \subset G(s_2)$$
$$G(s) = \{x \in \Omega : |u(x)| > t\} \quad \text{if } s = \mu(t).$$

For a given a function $\varphi \in L^1(\Omega)$, we set

$$\phi(s) = \frac{d}{ds} \int_{G(s)} \varphi(x)\, dx.$$

Lemma 2.8 ([5]). *If $\varphi \in L^p(\Omega)$ with $p > 1$, then there exists a sequence $(\varphi(s))_n$, such that $\varphi_n^*(s) = \varphi^*(s)$ and $\varphi_n \rightharpoonup \phi$ weakly in $L^p(0, |\Omega|)$.*

3. Main result

The following Proposition gives a sufficient condition for the gradient of a function to belong to some Marcinkiewicz space, These are the generalized results of [7] in the Weighted Sobolev spaces $W_0^{1,p}(\Omega, \nu)$.

Proposition 3.1. *Let $1 < p < N$, and $u \in \mathcal{T}_0^{1,p}(\Omega, \nu)$ be such that*

$$\int_{\{|u|<k\}} |\nabla u|^p \nu(x)\, dx \leq M k^\lambda$$

for every $k > 0$. Then $u \in \mathcal{M}^{p_1}(\Omega)$ where $p_1 = p^\sharp(1 - \lambda/p)$. More precisely, there exists a c such that $\operatorname{meas}\{|u| > k\} = \operatorname{meas}\{x \in \Omega : |u(x)| > k\} \leq ck^{-p_1}$.

Proof. For $k > 0$, from (2.3), we have

$$\|T_k(u)\|_{p^\sharp} \leq c_1 \|\nabla T_k(u)\|_{L^p(\nu)} \leq c_1 k^{\lambda/p}.$$

For $0 < \varepsilon \leq k$, we have $\{x \in \Omega : |u| > \varepsilon\} = \{x \in \Omega : |T_k(u)| > \varepsilon\}$. Hence

$$\operatorname{meas}\{|u| > \varepsilon\} \leq \Big(\frac{\|T_k(u)\|_{p^\sharp}}{\varepsilon}\Big)^{p^\sharp} \leq c_1 k^{\lambda p^\sharp/p} \varepsilon^{-p^\sharp}.$$

Setting $\varepsilon = k$, we obtain $\operatorname{meas}\{|u| > \varepsilon\} \leq c_1 k^{-p_1}$, where $p_1 = p^\sharp(1 - \lambda/p)$. □

Proposition 3.2. *Let $1 < p < N$, and $u \in \mathcal{T}_0^{1,p}(\Omega, \nu)$ be such that*

$$\int_{\{|u|<k\}} |\nabla u|^p \nu(x)\, dx \leq M k^\lambda$$

for every $k > 0$. Then $\nu^{1/p}\nabla u \in \mathcal{M}^{p_2}(\Omega)$ where $p_2 = pp_1/(\lambda + p_1)$. More precisely, there exists a c such that $\operatorname{meas}\{\nu^{1/p}|\nabla u| > h\} \leq ch^{-p_2}$.

Proof. For $k, h > 0$. Set $\phi(k, \alpha) = \operatorname{meas}\{\nu(x)|\nabla u|^p > \alpha,\ |u| > k\}$. From Proposition 3.1 we have

$$\phi(k, 0) \leq c_1 k^{-p_1}.$$

Using that the function $\alpha \mapsto \phi(k, \alpha)$ is non-increasing, for $k, \lambda > 0$ we obtain

$$\begin{aligned}
\phi(0, \alpha) \leq& \frac{1}{\alpha} \int_0^\alpha \phi(0, s) ds \\
=& \frac{1}{\alpha} \int_0^\alpha \phi(0, s) + \phi(k, 0) - \phi(k, 0) ds \\
\leq& \phi(k, 0) + \frac{1}{\alpha} \int_0^\alpha \phi(0, s) - \phi(k, 0) ds \\
\leq& \phi(k, 0) + \frac{1}{\alpha} \int_0^\alpha \phi(0, s) - \phi(k, s) ds.
\end{aligned} \tag{3.1}$$

Since $\phi(0, s) - \phi(k, s) = \operatorname{meas}\{\nu(x)|\nabla u|^p > s,\ |u| < k\}$ we have

$$\frac{1}{\alpha} \int_0^\alpha \phi(0, s) - \phi(k, s) ds = \frac{1}{\alpha} \int_{|u|<k} \nu(x) |\nabla u|^p\, dx \leq c \frac{k^\lambda}{\alpha},$$

which by (3.1) gives

$$\phi(0, \alpha) \leq c_1 k^{-p_1} + c_2 \frac{k^\lambda}{\alpha}\,. \tag{3.2}$$

By minimizing (3.2) in k and setting $\alpha = h^p$ we obtain

$$\operatorname{meas}\{\nu^{1/p}|\nabla u| > k\} \le c h^{-pp_1/(\lambda+p_1)}$$

□

3.1. **A priori estimate.** Let ε be positive and sufficiently small. We consider the problem

$$\begin{gathered} -\operatorname{div} a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) = f_\varepsilon \quad \text{in } \Omega, \\ u_\varepsilon = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.3}$$

where $a_\varepsilon(x, s, \xi) = a(x, T_{l-\varepsilon}(s), \xi)$, with $x \in \Omega, s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$ and $f_\varepsilon \in L^\infty(\Omega)$. We use some classical results (see, for example [1, 2]) to assure that problem (3.3) has at least one solution $u_\varepsilon \in W_0^{1,p}(\Omega, \nu) \cap L^\infty(\Omega)$. Then, we define $b_\varepsilon(t) = b(T_{l-\varepsilon}(t))$ for all $t \in [0, +\infty)$, and

$$A_\varepsilon(s) = \int_0^s b_\varepsilon(r)^{1/(p-1)} dr.$$

First, we prove an integral inequality for weak solutions of problem (3.3).

Proposition 3.3. *Let u_ε be a weak solution of (3.3). Then*

$$A_\varepsilon(u_\varepsilon^*(s)) \le C_N \int_s^{|\Omega|} r^{-p'/N'} [D(r)]^{p'/p} \Big(\int_0^r f_\varepsilon^*(\sigma) d\sigma \Big)^{p'/p} dr, \quad s \in [0, |\Omega|], \tag{3.4}$$

where $D : [0, |\Omega|] \to \mathbb{R}$ is a measurable function such that

$$\int_{|u_\varepsilon| > y} \nu^{-t}(x)\, dx = \int_0^{\mu(y)} (D(r))^t\, dr.$$

Proof. Let $\phi = T_h(u_\varepsilon - T_\theta(u_\varepsilon))$ be a test function in (3.3). Then we have

$$\frac{1}{h} \int_{\theta < |u_\varepsilon| \le \theta + h} b(|u_\varepsilon|) \nu(x) |\nabla u_\varepsilon|^p\, dx \le \int_{|u_\varepsilon| > \theta} |f|\, dx$$

Applying Hardy-Littlewood inequality and passing to the limit on h to 0, we obtain

$$b(\theta) \Big(-\frac{d}{d\theta} \int_{|u_\varepsilon| > \theta} \nu(x) |\nabla u_\varepsilon|^p\, dx \Big) \le \int_0^{\mu_{u_\varepsilon}(\theta)} f_\varepsilon^*(s) ds. \tag{3.5}$$

On the other hand by Hölder inequality, we obtain

$$\begin{aligned} -\frac{d}{d\theta} \int_{|u_\varepsilon| > \theta} |\nabla u_\varepsilon|\, dx &\le \Big(-\frac{d}{d\theta} \int_{|u_\varepsilon| > \theta} \nu(x) |\nabla u_\varepsilon|^p\, dx \Big)^{1/p} \\ &\quad \times \Big(-\frac{d}{d\theta} \int_{|u_\varepsilon| > \theta} \nu^{-p'/p}(x)\, dx \Big)^{1/p'} \\ &\le \Big(-\frac{d}{d\theta} \int_{|u_\varepsilon| > \theta} \nu(x) |\nabla u_\varepsilon|^p\, dx \Big)^{1/p} \\ &\quad \times \Big(-\frac{d}{d\theta} \int_{|u_\varepsilon| > \theta} \nu^{-t}(x)\, dx \Big)^{1/r_1 p'} (-\mu'_{u_\varepsilon}(\theta))^{1/r_2 p'}. \end{aligned} \tag{3.6}$$

where $1/r_1 + 1/r_2 = 1$ and $p' r_1 / p = t$. By Lemma 2.8, since $\nu^{-1} \in L^t(\Omega), t > 1$ there exists $D \in L^t([0, |\Omega|])$ such that

$$-\frac{d}{d\theta} \int_{|u_\varepsilon| > \theta} \nu^{-t}(x)\, dx = -\mu'_{u_\varepsilon}(\theta) [D(\mu_{u_\varepsilon}(\theta))]^t.$$

Then inequality (3.6), becomes

$$
\begin{aligned}
-\frac{d}{d\theta}\int_{|u_\varepsilon|>\theta}|\nabla u_\varepsilon|\,dx \le \Big(-\frac{d}{d\theta}\int_{|u_\varepsilon|>\theta}\nu(x)|\nabla u_\varepsilon|^p\,dx\Big)^{1/p} \\
\times\Big((-\mu'_{u_\varepsilon}(\theta))^{1/p'}[D(\mu_{u_\varepsilon}(\theta))]^{t/r_1p'}\Big).
\end{aligned} \tag{3.7}
$$

From isoperimetric inequality and Fleming-Rishel formula (see [15]), it follows that

$$
\begin{aligned}
C_N b(\theta)^{1/p}(\mu_{u_\varepsilon}(\theta))^{1/N'} \le \Big(-\frac{d}{d\theta}\int_{|u_\varepsilon|>\theta}\nu(x)|\nabla u_\varepsilon|^p\,dx\Big)^{1/p} \\
\times\Big((-\mu'_{u_\varepsilon}(\theta))^{1/p'}[D(\mu_{u_\varepsilon}(\theta))]^{t/r_1p'}b(\theta)^{1/p}\Big),
\end{aligned} \tag{3.8}
$$

which by (3.5) gives

$$
b(\theta)^{1/(p-1)} \le C_N(\mu_{u_\varepsilon}(\theta))^{-p'/N'}(-\mu'_{u_\varepsilon}(\theta))[D(\mu_{u_\varepsilon}(\theta))]^{t/r_1}\Big(\int_0^{\mu_{u\varepsilon}(\theta)} f^*_\varepsilon(s)\,ds\Big)^{p'/p}
$$

integrating between 0 and $u_*(s)$ we obtain

$$
\begin{aligned}
A(u_*(s)) \le C_N\int_0^{u_*(s)}\Big[(\mu_{u_\varepsilon}(\theta))^{-p'/N'}(-\mu'_{u_\varepsilon}(\theta))[D(\mu_{u_\varepsilon}(\theta))]^{t/r_1} \\
\times\Big(\int_0^{\mu_{u_\varepsilon}(\theta)} f^*_\varepsilon(s)ds\Big)^{p'/p}\Big]\,d\theta,
\end{aligned} \tag{3.9}
$$

which gives the results. □

Remark 3.4. Since $1+\frac{1}{t} < p < N(1+\frac{1}{t})$, and $t \ge N/p$, we have $qp'/p \ge 1$ and $q/r'_1 \ge 1$, where $r_1 = t(p-1)$, which allows us to apply the Proposition 2.11 and Proposition 2.12 to prove estimation (3.10) and (3.11), below.

Proposition 3.5. *Let u_ε be a solution of* (3.3).

(a) *If $1 < r < tN/(tp-N)$, then*

$$
\|(A_\varepsilon(|u_\varepsilon|))^q\|_{L^1(\Omega)} \le c\|f\|^{qp'/p}_{L^r(\Omega)}; \tag{3.10}
$$

where $q = rtN(p-1)/(t(N-rp)+rN)$.

(b) *If $r = 1$, then*

$$
\|A_\varepsilon(|u_\varepsilon|)\|_{M^{Nt(p-1)/(N+t(N-p))}} \le c\|f\|^{p'/p}_{L^1(\Omega)}\|D\|^{p'/p}_{L^t[0,|\Omega|]}\,. \tag{3.11}
$$

Proof. Case $1 < r < tN/(tp-N)$. Let us observe that A_ε being monotone, by Proposition 3.3, properties of rearrangements, (2.12) and (2.11), we obtain

$$
\begin{aligned}
\|(A_\varepsilon(|u_\varepsilon|))^q\|_{L^1(\Omega)} &\le C_N\int_0^{+\infty}\Big[\int_s^{|\Omega|} r^{-p'/N'}[D(r)]^{p'/p}\Big(\int_0^r f_*(\sigma)d\sigma\Big)^{p'/p}dr\Big]^q ds \\
&\le C_N\int_0^{+\infty}\Big[\int_s^{|\Omega|} r^{-\frac{p'r'_1}{N'}}\Big(\int_0^r f_*(\sigma)d\sigma\Big)^{\frac{p'r'_1}{p}}dr\Big]^{\frac{q}{r'_1}} ds \\
&\le C_N\int_0^{+\infty}\Big[s^{\frac{r'_1}{q}}\int_s^{|\Omega|} r^{-\frac{p'r'_1}{N'}}\Big(\int_0^r f_*(\sigma)d\sigma\Big)^{\frac{p'r'_1}{p}}dr\Big]^{\frac{q}{r'_1}} \frac{ds}{s} \\
&\le C_N\int_0^{+\infty}\Big[s^{(\frac{r'_1+q}{q}-\frac{p'r'_1}{N'})\frac{p}{p'r'_1}}\int_0^s f_*(\sigma)d\sigma\Big]^{\frac{qp'}{p}}\frac{ds}{s}
\end{aligned}
$$

$$
\begin{aligned}
&\le C_N \int_0^{+\infty} \Big[s^{(\frac{r_1'+q}{q}-\frac{p'r_1'}{'N})\frac{p}{p'r_1'}+1} f_*(s)\Big]^{\frac{qp'}{p}} \frac{ds}{s}\\
&\le C_N \int_0^{+\infty} \Big[s^{(\frac{r_1'+q}{q}-\frac{p'r_1'}{N'})\frac{p}{p'r_1'}+1-\frac{p}{qp'}} f_*(s)\Big]^{\frac{qp'}{p}} ds,
\end{aligned}
$$

where $\frac{qp'}{p} \ge 1$, $\frac{p'r_1}{p} = t$, and C_N a constant that vary from line to line. Since $f_\varepsilon \in M^r(\Omega)$ we conclude that

$$
\begin{aligned}
\|(A_\varepsilon(|u_\varepsilon|))^q\|_{L^1(\Omega)} &\le C_N \int_0^{+\infty} (f_*(s))^{-rq(\frac{1}{r_1'}-\frac{p'}{N'}+\frac{p'}{p})+\frac{qp'}{p}} ds \\
&\le C_N \|f_*\|^r_{L^r([0,|\Omega|])}.
\end{aligned}
\tag{3.12}
$$

where

$$
r = -rq(\frac{1}{r_1'} - \frac{p'}{N'} + \frac{p'}{p}) + \frac{qp'}{p}, \quad q = \frac{rtN(p-1)}{t(N-rp)+rN}.
$$

Case $r = 1$. By Proposition 3.3, and Hölder inequality, we have

$$
\begin{aligned}
A_\varepsilon(u_*(s)) &\le C_N \int_s^{|\Omega|} r^{-p'/N'} [D(r)]^{p'/p} \Big(\int_0^r f_*(\sigma) d\sigma\Big)^{p'/p} dr \\
&\le C_N \|D\|_{L^t[0,|\Omega|]} \Big(\int_s^{|\Omega|} r^{-\frac{p't(p-1)}{N'(tp-t-1)}}\Big)^{\frac{tp-t-1}{t(p-1)}} \\
&\le C_N \|D\|_{L^t[0,|\Omega|]} s^{1-\frac{p't(p-1)}{N'(tp-t-1)}}
\end{aligned}
$$

which implies the result. □

Remark 3.6. Since $p/N < 1 + \frac{1}{t}$, (see (2.3)), we have

$$
\frac{Ntp}{Nt(p-1) - N + tp} > 1.
$$

Proposition 3.7. *Let u_ε be a solution of* (3.3).

(a) *If* $\frac{Ntp}{Nt(p-1)-N+tp} < r < \frac{tN}{tp-N}$, *then*

$$
\|\nabla A_\varepsilon(|u_\varepsilon|)\|_{L^p(\Omega,\nu)} \le c_1\,. \tag{3.13}
$$

(b) *If*

$$
\max\big(1, \frac{tNp}{Nt(p-1)p + pt - N}\big) < r < \frac{tNp}{Nt(p-1) + pt - N},
$$

then

$$
\|\nabla A_\varepsilon(|u_\varepsilon|)\|_{L^\beta(\Omega,\nu^{\beta/p})} \le c_2, \tag{3.14}
$$

where $\beta = \frac{rNt(p-1)p}{rN+Ntp-ptr}$.

(c) *If*

$$
1 \le r \le \max\big(1, \frac{tNp}{Nt(p-1)p + pt - N}\big),
$$

then

$$
\|\nu^{1/p} \nabla A_\varepsilon(|u_\varepsilon|)\|_{M^\beta(\Omega)} \le c_3, \tag{3.15}
$$

where $\beta = \frac{rNt(p-1)p}{rN+Ntp-ptr}$.

Proof. Let u_ε is a solution of (3.3), by the definition of A_ε we can use as test function $v=[T_h(A_\varepsilon(|u_\varepsilon|)-T_\theta(A_\varepsilon(|u_\varepsilon|)]\operatorname{sign}(u_\varepsilon)$ and obtain

$$\int_{\theta<A_\varepsilon(|u_\varepsilon|)\le\theta+h}\nu(x)|\nabla A_\varepsilon(|u_\varepsilon|)|^p\,dx\le\int_{A_\varepsilon(|u_\varepsilon|)>\theta}|f_\varepsilon|\,dx,\tag{3.16}$$

Case 1: $\frac{Ntp}{Nt(p-1)-N+tp}<r<\frac{tN}{tp-N}$. Passing to the limit in (3.16), we obtain

$$\frac{d}{d\theta}\int_{A_\varepsilon(|u_\varepsilon|)\le\theta}\nu(x)|\nabla A_\varepsilon(|u_\varepsilon|)|^p\,dx\le\int_0^{\mu_\varepsilon(\theta)}f_\varepsilon^*(s)ds,\tag{3.17}$$

where we have denoted with $\mu_\varepsilon(\theta)$ the distribution functions of $A_\varepsilon(|u_\varepsilon|)$. Integrating (3.17) between 0 and $+\infty$ and using a Hölder inequality, we have

$$\begin{aligned}\int_\Omega\nu(x)|\nabla A_\varepsilon(|u_\varepsilon|)|^p\,dx&\le\int_0^{+\infty}d\theta\int_0^{\mu_\varepsilon(\theta)}f_\varepsilon^*(s)ds\\&=\int_0^{|\Omega|}A_\varepsilon(u_\varepsilon^*(s))f_\varepsilon^*(s)ds\\&\le\|f\|_{L^r(\Omega)}.\|A_\varepsilon(|u_\varepsilon|)\|_{L^{r'}(\Omega)}.\end{aligned}\tag{3.18}$$

We observe that if r is such that $\frac{Nt}{Nt(p-1)-N+pt}\le r<\frac{tN}{tp-N}$, by (3.10) the right-hand side of the above inequality is controlled by a constant depending on the norm of f_ε in $L^r(\Omega)$; so by (3.18) inequality (3.13) follows.

Case 2: $\max\big(1,\frac{tNp}{Nt(p-1)p+pt-N}\big)<r<\frac{tNp}{Nt(p-1)+pt-N}$. Applying the Hölder inequality in (3.16) and reasoning as before, we obtain

$$\begin{aligned}&\int_\Omega|\nabla A_\varepsilon(|u_\varepsilon|)|^\beta\nu^{\beta/p}(x)\,dx\\&\le\int_0^{+\infty}\Big(\int_0^{\mu_\varepsilon(\theta)}f_\varepsilon^*(s)ds\Big)^{\beta/p}(-\mu_\varepsilon'(\theta))^{1-\frac{\beta}{p}}d\theta\\&\le\Big(\int_0^{+\infty}(1+\theta)^q(-\mu_\varepsilon'(\theta))d\theta\Big)^{1-\frac{\beta}{p}}\\&\quad\times\Big(\int_0^{+\infty}(1+\theta)^{q(1-\frac{p}{\beta})}\Big(\int_0^{\mu_\varepsilon(\theta)}f_\varepsilon^*(s)ds\Big)d\theta\Big)^{\beta/p}.\end{aligned}\tag{3.19}$$

By the properties of rearrangements, we can write the first integral on the right-hand side of (3.19) as

$$\int_0^{+\infty}(1+\theta)^q(-\mu_\varepsilon'(\theta))d\theta=\int_0^{|\Omega|}(1+A_\varepsilon(u_\varepsilon^*))^q ds,\tag{3.20}$$

and by (3.10) this quantity is bounded by a constant depending on the norm of f_ε in $L^r(\Omega)$. On the other hand, integrating by parts the second integral on the right-hand side of (3.19) we have

$$\begin{aligned}&\int_0^{+\infty}(1+\theta)^{q(1-\frac{p}{\beta})}\Big(\int_0^{\mu_\varepsilon(\theta)}f_\varepsilon^*(s)ds\Big)d\theta\\&\le c\int_0^{|\Omega|}f_\varepsilon^*(s)[(1+A_\varepsilon(u_\varepsilon^*))^{(q(1-\frac{p}{\beta})+1)}]ds\\&\le c\|f_\varepsilon\|_{L^r(\Omega)}\Big[\int_0^{|\Omega|}[(1+A_\varepsilon(u_\varepsilon^*))^q]ds\Big]^{1-\frac{1}{r}}.\end{aligned}\tag{3.21}$$

Applying again (3.10), by (3.19) it follows the estimate (3.14).

Case 3: $1 \le r \le \max\big(1, \frac{tNp}{Nt(p-1)p+pt-N}\big)$. Integrating inequality (3.17) between 0 and k, we obtain

$$\int_{A_\varepsilon(|u_\varepsilon|)\le k} \nu(x)|\nabla A_\varepsilon(|u_\varepsilon|)|^p\,dx \le \int_0^k d\theta \int_0^{\mu_\varepsilon(\theta)} f_\varepsilon^*(s)ds. \tag{3.22}$$

If $r=1$, from (3.22) we obtain

$$\int_{A_\varepsilon(|u_\varepsilon|)\le k} \nu(x)|\nabla A_\varepsilon(|u_\varepsilon|)|^p\,dx \le k\|f_\varepsilon\|_{L^1(\Omega)}.$$

by (3.11) and (2.3) we obtain the assertion.

If $1 \le r \le \max(1, \frac{tNp}{Nt(p-1)p+pt-N})$, then by (3.10) it follows that $A_\varepsilon(|u_\varepsilon|) \in M^q(\Omega)$, with $q = \frac{rNt(p-1)}{tN+rN-ptr}$; so we obtain

$$\int_{A_\varepsilon(|u_\varepsilon|)\le k} \nu(x)|\nabla A_\varepsilon(|u_\varepsilon|)|^p\,dx \le ck^{1-\frac{q}{r'}}$$

by Proposition 3.2, we conclude the result. □

Replacing $\nabla A_\epsilon(|u_\epsilon|)$ by ∇u_ϵ the above estimates also hold; furthermore it follows that

$$\int_\Omega \nu(x)|\nabla u_\epsilon|^\gamma\,dx \le c,$$

with $\gamma < \frac{Nt(p-1)}{tN+N-t}$, where c is a constant depending on the $L^1(\Omega)$ norm of f_ε. Using (3.5), the $T_k(u_\varepsilon)$ are uniformly bounded in $W_0^{1,p}(\Omega,\nu)$ for any $k>0$. Hence, there exists a function $u \in W_0^{1,\gamma}(\Omega,\nu)$ such that

$$u_\varepsilon \to u \quad \text{a.e. in } \Omega, \tag{3.23}$$

and, for any $k>0$,

$$T_k(u_\varepsilon) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega,\nu). \tag{3.24}$$

Remark 3.8. Choosing $k>l$, we have

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega,\nu). \tag{3.25}$$

Indeed, let us suppose $f \in L^1(\Omega)$. Using $T_{2l}(|u_\varepsilon|) - T_l(|u_\varepsilon|)$ as test function in (3.3), by (2.3) we obtain

$$b(l-\varepsilon)\int_\Omega (T_{2l}(|u_\varepsilon|) - T_l(|u_\varepsilon|))^{p^\sharp}\,dx \le l\|f_\varepsilon\|_{L^1(\Omega)}.$$

Letting $\varepsilon \to 0$, from condition (2.1), we conclude that, for almost all x in Ω, $|u| \le l$, which give the result by (3.24).

Next we prove a lemma needed for proving the existence result.

Lemma 3.9. *Let u_ε be a weak solution to problem* (3.3). *Suppose $f \in L^1(\Omega)$, and let $f_\varepsilon \in L^\infty(\Omega)$ be such that $f_\varepsilon \to f$ in $L^1(\Omega)$. Then*

$$\nabla u_\varepsilon \to \nabla u \quad \text{a.e. in } \{|u|<l\}.$$

Proof. We adapt the proof[presented in [11]. By Remark 3.8, we have $u_\varepsilon \to u$ in measure. We will prove that $u_\varepsilon \to u$ in measure on $\{|u| < m\}$. Let $\lambda > 0$ and $\eta > 0$ for $0 < k < l$, and $M > 0$, we set

$$\begin{aligned}
E_1 =& \{|u| < l\} \cap (\{|\nabla u_\varepsilon| > M\} \cup \{|\nabla u| > M\} \cup \{|u_\varepsilon| > k\} \cup \{|u| > k\}),\\
E_2 =& \{|u| < l\} \cap \{|u_\varepsilon - u| > \eta\},\\
E_3 =& \{|u_\varepsilon - u| \le \eta, |\nabla u_\varepsilon| \le M, |\nabla u| \le M, |u_\varepsilon| \le k, |u| \le k, |\nabla(u_\varepsilon - u)| \ge \lambda\}\\
&\cap \{|u| < l\}.
\end{aligned}$$

Observe that $\{|u| < l\} \cap \{|\nabla u_\varepsilon| \ge \lambda\} \subset E_1 \cup E_2 \cup E_3$.

Since u_ε and ∇u_ε are bounded in $L^1(\Omega)$, for any $\sigma > 0$ we can fix M and $k < l$ such that $|E_1| < \sigma/3$ independently of ε. By the monotonicity Assumption (2.5), there exists a real valued function γ such that

$$\begin{gathered}
\operatorname{meas}(\{x \in \Omega : \gamma(x) = 0\}) = 0,\\
(a(x, s, \xi) - a(x, s, \xi'))(\xi - \xi') \ge \gamma(x),
\end{gathered}$$

for any $s \in (-l, l), \xi, \xi' \in \mathbb{R}^N, |s| \le k, |\xi|, |\xi'| \le M$, and $|\xi - \xi'| \ge \lambda$. Denoting by χ_η the characteristic function of $[0, \eta]$, we obtain

$$\begin{aligned}
\int_{E_3} \gamma(x)\,dx &\le \int_{E_3} [a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) - a_\varepsilon(x, u_\varepsilon, \nabla u)](\nabla u_\varepsilon - u)\,dx\\
&\le \int_{\{|u_\varepsilon| \le k, |u| \le k\}} \Big[\Big(a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) - a_\varepsilon(x, u_\varepsilon, \nabla T_k(u))\Big)\\
&\quad \times \Big(\nabla u_\varepsilon - T_k(u))\chi_\eta(|u_\varepsilon - T_k(u)|\Big)\Big]\,dx\\
&\le \int_\Omega \Big[\Big(a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) - a_\varepsilon(x, u_\varepsilon, \nabla T_k(u))\Big)\\
&\quad \times \Big(\nabla u_\varepsilon - T_k(u))\chi_\eta(|u_\varepsilon - T_k(u)|\Big)\Big]\,dx\\
&\le \int_\Omega a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)(\nabla u_\varepsilon - T_k(u))\chi_\eta(|u_\varepsilon - T_k(u)|)\,dx\\
&\quad - \int_\Omega a_\varepsilon(x, u_\varepsilon, \nabla T_k(u)) \cdot (\nabla u_\varepsilon - T_k(u))\chi_\eta(|u_\varepsilon - T_k(u)|)\,dx\\
&:= J_1 - J_2.
\end{aligned}$$

For the term J_1, using $T_\eta(u_\varepsilon - T_k(u))$, we have

$$|J_1| = \Big|\int_\Omega f_\varepsilon T_\eta(|u_\varepsilon - T_k(u)|)\,dx\Big| \le \eta\|f\|_{L^1(\Omega)}.$$

Choosing $\eta > 0$ such that $k + \eta < l$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$,

$$a_\varepsilon(x, u_\varepsilon, \nabla T_k(u)) = a(x, u_\varepsilon, \nabla T_k(u)) \quad \text{in } \{x \in \Omega : |u_\varepsilon - T_k(u)| \le \eta\};$$

and since $\{x \in \Omega : |u_\varepsilon - T_k(u)| \le \eta\} \subset \{x \in \Omega : |u_\varepsilon| \le k + \eta\}$ we obtain

$$\begin{aligned}
J_2 &= \int_\Omega a(x, u_\varepsilon, \nabla T_k(u)) \cdot \nabla T_\eta(u_\varepsilon - T_k(u))\,dx\\
&= \int_\Omega a(x, T_{k+\eta}(u_\varepsilon), \nabla T_k(u)) \cdot (\nabla T_{k+\eta}(u_\varepsilon - T_k(u)))\chi_\eta(|u_\varepsilon - T_k(u)|)\,dx.
\end{aligned}$$

By (3.24), it follows that

$$T_{k+\eta}(u_\varepsilon) \rightharpoonup T_{k+\eta}(u) \quad \text{weakly in } W_0^{1,p}(\Omega,\nu),$$

on the other hand

$$|a(x, T_{k+\eta}(u_\varepsilon), \nabla T_k(u))| \le b(|T_{k+\eta}(u_\varepsilon|))\nu(x)|\nabla T_{k+\eta}(u)|^{p-1}$$

using Vitali's theorem we have

$$a(x, T_{k+\eta}(u_\varepsilon), \nabla T_k(u)) \to a(x, T_{k+\eta}(u), \nabla T_k(u)) \quad \text{strongly in } L^{p'}(\Omega, \nu^{-1/(p-1)}).$$

Letting ε and η tend to 0 respectively in J_2, we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_\Omega a(x, u_\varepsilon, \nabla T_k(u)) \cdot \nabla T_\eta(u_\varepsilon - T_k(u))\,dx \\
&= \int_\Omega a(x, T_{k+\eta}(u), \nabla T_k(u)) \cdot (\nabla T_{k+\eta}(u - T_k(u)))\chi_\eta(|u_\varepsilon - T_k(u)|)\,dx,
\end{aligned}$$

and

$$\lim_{\eta\to 0} \int_\Omega a(x, T_{k+\eta}(u), \nabla T_k(u)) \cdot (\nabla T_{k+\eta}(u - T_k(u)))\chi_\eta(|u_\varepsilon - T_k(u)|)\,dx = 0.$$

For η small enough $\eta\|f\|_{L^1(\Omega)} < \delta/2$, by Kolmogorov theorem, we have $|E_3| < \sigma$ independently of ε. Fix η, by the fact that $u_\varepsilon \to u$ in measure, we choose ε_1 such that $|E_2| < \eta$ for $\varepsilon \le \varepsilon_1$. This implies that $\nabla u_\varepsilon \to \nabla u$ in measure in $\{|u| < l\}$, consequently

$$\nabla u_\varepsilon \to \nabla u \quad \text{a.e. in } \{|u| < l\}.$$

□

We observe that since $u_\varepsilon \to u$ a.e. in Ω (see (3.23)), we have

$$\{x \in \Omega : |u(x)| = l\} = \Big\{x \in \Omega : \lim_{\varepsilon\to 0} \int_0^{|u_\varepsilon(x)|} b_\varepsilon(t) \ge \int_0^l b(t)\,dt\Big\}. \tag{3.26}$$

Theorem 3.10. *Let f be a function in $L^r(\Omega)$, with $r > tN/(tp-N)$. Assume that* (2.1)–(2.5) *hold. Then there exists a weak solution $u \in W_0^{1,p}(\Omega,\nu)$ of problem* (2.2) *such that $\|u\|_{L^\infty(\Omega)} < l$.*

Proof. For $f_\varepsilon = f$ with $\varepsilon > 0$. By classical results see for example [2, 1]) there exists a solution $u_\varepsilon \in W_0^{1,p}(\Omega,\nu)$ of the approximated problem (2.2). Estimate (3.4) implies

$$A_\varepsilon(\|u_\varepsilon\|_{L^\infty}) \le C(f) = C_N \int_0^{|\Omega|} r^{-p'/N'}[D(r)]^{p'/p}\Big(\int_0^r f_\varepsilon^*(\sigma)d\sigma\Big)^{p'/p} dr. \tag{3.27}$$

Since A is bijective in $[0, l)$, we can take $B = A^{-1}(C(f))$ and then we choose $\varepsilon_0 > 0$ such that $b(s) \le b(l-\varepsilon)$ for any $s \in [0, B]$. By definition of b_ε and A_ε we have, for any $\varepsilon < \varepsilon_0$,

$$A_\varepsilon(s) = A(s), \quad s \in [0, B].$$

Moreover, being A_ε increasing, it follows that, for any $\varepsilon < \varepsilon_0$,

$$A_\varepsilon(s) \le C(f) \Leftrightarrow s \in [0, B],$$

so by (3.27) we obtain

$$\|u_\varepsilon\|_{L^\infty} \le B < l.$$

By (2.3) and Lemma 3.9, we have

$$a_\varepsilon(x, u_{\varepsilon_1}(x), \nabla u_{\varepsilon_1}(x)) \to a(x, u, \nabla u) \quad \text{strongly in } L^{p'}(\Omega, \nu^{-1/(p-1)}),$$
$$f_\varepsilon \to f \quad \text{strongly in } L^\infty(\Omega).$$

Passing to the limit in the weak formulation of problem (3.3), we conclude that u is a weak solution of (2.2), which satisfies $\|u\|_{L^\infty(\Omega)} < l$. ◻

Theorem 3.11. *Let $f \in L^r(\Omega)$, with $\frac{Ntp}{Nt(p-1)-N+tp} < r < \frac{tN}{tp-N}$. Under hypothesis* (2.1)-(2.5)*, there exists a weak solution $u \in W_0^{1,p}(\Omega, \nu)$ of problem* (2.2)*, such that* $\operatorname{meas}(\{x \in \Omega : |u(x)| = l\}) = 0$.

Proof. Let $u_\varepsilon \in W_0^{1,p}(\Omega, \nu)$ be a weak solution to the approximated problem (3.3). By Remark (3.8), we have $u_\varepsilon \to u$ a.e. in Ω, since $A(l^-) = +\infty$, (3.26) implies that

$$A_\varepsilon(|u_\varepsilon|) \to A(|u|) \quad \text{a.e. in } \Omega. \tag{3.28}$$

By (3.13) and (3.28), we obtain

$$A_\varepsilon(|u_\varepsilon|) \to A(|u|) \quad \text{weakly in } W_0^{1,p}(\Omega, \nu), \tag{3.29}$$

Since $A(|u|)$ is bounded in $L^1(\Omega)$ and $\operatorname{meas}(\{x \in \Omega :\ |u(x)| = l\}) = 0$, by (2.3) we have

$$a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \to a(x, u, \nabla u) \quad \text{a.e. } \Omega.$$

On the other hand by (2.3) and (3.13)

$$|a_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)| \quad \text{is bounded in } L^{p'}(\Omega, \nu^{-1/(p-1)});$$

passing to the limit in the weak formulation (3.3), we obtain

$$\int_\Omega a(x, u, \nabla u) \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx, \quad \text{for all } \varphi \in W_0^{1,p}(\Omega, \nu).$$

◻

Theorem 3.12. *Let $f \in L^r(\Omega)$, with $1 \le r < \frac{Ntp}{Nt(p-1)-N+tp}$. Under hypothesis* (2.1) − (2.5)*, there exists a solution $u \in W_0^{1,p}(\Omega, \nu)$ of problem* (2.2)*, in the sense of Definition* (2.2) *such that* $\operatorname{meas}(\{x \in \Omega : |u(x)| = l\}) = 0$.

Proof. Let u_ε be a weak solution of the approximate problem (3.3), by passing to the limit we can show that $|u| < l$ a.e. in Ω. Take $T_k(u_\varepsilon - \varphi)$, with $\varphi \in W_0^{1,p}(\Omega, \nu) \cap L^\infty(\Omega)$ as test function in (3.3) we obtain

$$\begin{aligned} &\int_{|u_\varepsilon - \varphi| \le k} a(x, T_{l-\varepsilon}(u_\varepsilon), \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \, dx \\ &- \int_{|u_\varepsilon - \varphi| \le k} a(x, T_{l-\varepsilon}(u_\varepsilon), \nabla u_\varepsilon) \cdot \nabla \varphi \, dx \\ &= \int_\Omega f_\varepsilon T_k(u_\varepsilon - \varphi) \, dx. \end{aligned} \tag{3.30}$$

Since $\{|u_\varepsilon - \varphi|\} \subseteq \{|u_\varepsilon| \le k + \|\varphi\|_{L^\infty(\Omega)} = M\}$, for $1 < k < l$ and $\|\varphi\|_{L^\infty(\Omega)} < l - k$, we obtain $M < l$ and consequently $|a(x, T_M(u_\varepsilon), \nabla T_M(u_\varepsilon))|$ is bounded in $L^{p'}(\Omega, \nu^{-1/(p-1)})$, and

$$\lim_{\varepsilon \to 0} \int_{|u_\varepsilon - \varphi| \le k} a(x, T_{l-\varepsilon}(u_\varepsilon), \nabla u_\varepsilon) \cdot \nabla \varphi \, dx = \int_{|u - \varphi| \le k} a(x, u, \nabla u) \cdot \nabla \varphi \, dx. \tag{3.31}$$

Moreover since f_ε strongly convergent to f in $L^1(\Omega)$, and $T_k(u_\varepsilon - \varphi)$ weakly* convergent to $T_k(u-\varphi)$ in $L^\infty(\Omega)$, we have

$$\lim_{\varepsilon\to 0}\int_\Omega f_\varepsilon T_k(u_\varepsilon-\varphi)\,dx = \int_\Omega fT_k(u-\varphi)\,dx. \tag{3.32}$$

On the other hand $a(x, T_{l-\varepsilon}(u_\varepsilon), \nabla u_\varepsilon)\cdot\nabla u_\varepsilon$ being non-negative, and almost everywhere convergent to $a(x,u,\nabla u)\cdot\nabla u$, by Fatou's lemma we conclude that

$$\liminf_{\varepsilon\to 0}\int_{|u_\varepsilon-\varphi|\le k} a(x,T_{l-\varepsilon}(u_\varepsilon),\nabla u_\varepsilon)\cdot\nabla u_\varepsilon\,dx \le \int_{|u-\varphi|\le k} a(x,u,\nabla u)\cdot\nabla u\,dx. \tag{3.33}$$

Combining (3.31), (3.32) and (3.33) we obtain

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla T_k(u-\varphi)\,dx \le \int_\Omega fT_k(u-\varphi)\,dx, \quad \text{for all } \varphi\in W_0^{1,p}(\Omega,\nu).$$

□

References

[1] L. Aharouch, Y. Akdim, E. Azroul; *Quasilinear degenerate elliptic unilateral problems*, Abstract and Applied Analysis, 1 (2005) 11–31.
[2] Y. Akdim, E. Azroul, A. Benkirane; *Existence of solutions for quasilinear degenerate elliptic equations*, Electronic J. Diff. Equ., 2001 (2001) No. 71, 1–19
[3] A. Alvino, L. Boccardo, V. Ferone, L. Orsina, G. Trombetti; *Existence results for nonlinear elliptic equations with degenerate coercivity*, Ann. Mat. Pura Appl., (4) 182 (1) (2003) 53–79.
[4] A. Alvino, V. Ferone, G. Trombetti; *A priori estimates for a class of non-uniformly elliptic equations*, Atti Semin. Mat. Fis. Univ. Modena, 46 (Suppl.) (1998) 381–391.
[5] A. Alvino and Trombetti; *Sulle migliori costanti di maggiorazione per una classe di equazioni ellittiche degeneri.* Ricerche Mat., 1978, 27, 413–428.
[6] C. Bandle; *Isoperimetric inequalities and applications*, in: Monographs and Studies in Math., Pitman, London; 1980.
[7] Ph. Benilan, L. Boccardo, T. Gallouet, R. Gariepy, M. Pierre, J. L. Vazquez; *An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci.; 4 (1995), 241-273.
[8] A. Benkirane, J. Bennouna; *Existence of solutions for nonlinear elliptic degenerate equations*, Nonlinear Analysis, 54 (2003) 9–37.
[9] D. Blanchard, O. Guibé; *Infinite valued solutions of non-uniformly elliptic problems*, Anal. Appl., 2(3) (2004) 227–246.
[10] L. Boccardo, A. Dall'Aglio, L. Orsina; *Existence and regularity results for some degenerate elliptic equations*, Atti. Semin. Mat. Fis. Univ. Modena, 46(suppl.) (1998) 51–81.
[11] F. Della Pietra, G. di Blasio; *Existence results for nonlinear elliptic problems with unbounded coefficients*, Nonlinear Analysis, 71 (2009) 72–87.
[12] R. Di Nardo, F. Feo; *Existence and uniqueness for nonlinear anisotropic elliptic equations*, Arch. Math., 102 (2014), 141–153.
[13] G. H. Hardy, J. L. Littlewood, G. Polya; *Inequalities*, Cambridge Univ. Press; 1964.
[14] J.-L. Lions; *Quelques méthodes de réesolution des problèmes aux limites non linéaires*, Dunod et Gauthier-Villars, Paris, 1984.
[15] J. Mossino; *Inégalities isopérimètriques et applications en physique, Collection travaux en cours*, Herman, Paris; 1984.
[16] M. K. W Murthy, G. Stampacchia; *Boundary value problems for some degenerate elliptic operators*, Ann. Mat. Pura Appl.; 80 (1968), 1–122.
[17] A. Porretta, S. Segura de Leon; *Nonlinear elliptic equations having a gradient term with natural growth*, J. Math. Pures Appl.; 85 (2006) 465–492.
[18] G. Talenti; *Linear Elliptic P.D.E.'s: Level sets, rearrangements and a priori estimates of solutions*, Boll. Unione Mat. Ital., B (6) 4 (1985) 917–949.

Aharrouch Benali
Sidi Mohamed Ben Abdellah University, Faculty of Sciences Dhar El Mahraz, Laboratory LAMA, Department of Mathematics, P.O. Box 1796 Atlas Fez, Morocco
Email address: bnaliaharrouch@gmail.com

Bennouna Jaouad
Sidi Mohamed Ben Abdellah University, Faculty of Sciences Dhar El Mahraz, Laboratory LAMA, Department of Mathematics, P.O. Box 1796 Atlas Fez, Morocco
Email address: jbennouna@hotmail.com