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EXPONENTIAL STABILITY OF SOLUTIONS TO NONLINEAR TIME-VARYING DELAY SYSTEMS OF NEUTRAL TYPE EQUATIONS WITH PERIODIC COEFFICIENTS

INESSA I. MATVEEVA

Abstract. We consider a class of nonlinear time-varying delay systems of neutral type differential equations with periodic coefficients in the linear terms,

$$\begin{aligned}\frac{d}{dt}y(t) &= A(t)y(t) + B(t)y(t-\tau(t)) + C(t)\frac{d}{dt}y(t-\tau(t)) \\ &\quad + F\Big(t, y(t), y(t-\tau(t)), \frac{d}{dt}y(t-\tau(t))\Big),\end{aligned}$$

where $A(t)$, $B(t)$, $C(t)$ are T-periodic matrices, and

$$\|F(t,u,v,w)\| \le q_1\|u\| + q_2\|v\| + q_3\|w\|, \quad q_1, q_2, q_3 \ge 0, \quad t > 0.$$

We obtain conditions for the exponential stability of the zero solution and estimates for the exponential decay of the solutions at infinity.

1. Introduction

There is a large number of works devoted to the study of delay differential equations (see [1, 4, 14, 17, 18, 19, 20, 23, 24, 29] and the bibliography therein). The intense interest in such equations is due to their arise in many applied problems describing the processes whose speeds are defined not only by the present, but also by the previous states (for example, see [15, 25] and the bibliography therein). One of the important problems is that of the exponential stability of solutions to the equations of such kind. Unlike autonomous equations, the exponential stability for nonautonomous equations has been studied less.

We consider nonlinear time-varying delay systems of the form

$$\begin{aligned}\frac{d}{dt}y(t) &= A(t)y(t) + B(t)y(t-\tau(t)) + C(t)\frac{d}{dt}y(t-\tau(t)) \\ &\quad + F\Big(t, y(t), y(t-\tau(t)), \frac{d}{dt}y(t-\tau(t))\Big), \quad t > 0,\end{aligned} \tag{1.1}$$

where $A(t)$, $B(t)$, $C(t)$ are $(n \times n)$ matrices with continuous T-periodic entries; i.e.,

$$A(t+T) \equiv A(t), \quad B(t+T) \equiv B(t), \quad C(t+T) \equiv C(t),$$

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$\tau(t) \in C^1([0,\infty))$ is the time-varying delay,

$$0 < \tau_1 \le \tau(t) \le \tau_2, \quad \tau_3 \le \frac{d}{dt}\tau(t) \le \tau_4 < 1. \tag{1.2}$$

and F is a continuous vector-function mapping $[0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}^n$. We assume that $F(t,u,v,w)$ satisfies the Lipschitz condition with respect to u on every compact set $G \subset [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$,

$$\|F(t,u,v,w)\| \le q_1\|u\| + q_2\|v\| + q_3\|w\|, \quad t \ge 0, \quad u,v,w \in \mathbb{R}^n, \tag{1.3}$$

for some constants q_1, q_2, $q_3 \ge 0$. Hereafter we use the following dot product and vector norm

$$\langle x, z\rangle = \sum_{j=1}^n x_j \bar{z}_j, \quad \|x\| = \sqrt{\langle x, x\rangle}.$$

In this article we continue the study of exponential stability of solutions to delay differential equations presented in [7, 8, 26, 9, 10, 11, 12, 27]. We investigated linear and nonlinear time-delay systems with $C(t) \equiv 0$ in [7, 8, 26], with a constant matrix $C(t) \equiv C$ in [9, 10, 11, 12]. with a T-periodic matrix $C(t)$ in [27]. Conditions for exponential stability of the zero solution were established and estimates of exponential decay of solutions at infinity have been obtained. However, all the mentioned articles consider systems of the form (1.1) with the constant delay $\tau(t) \equiv \tau$. Linear time-varying delay systems of the form (1.1) with $F(t,u,v,w) \equiv 0$ were considered in [28]. This article deals with the initial value problem

$$\begin{gathered}
\frac{d}{dt}y(t) = A(t)y(t) + B(t)y(t-\tau(t)) + C(t)\frac{d}{dt}y(t-\tau(t)) \\
+ F\Big(t, y(t), y(t-\tau(t)), \frac{d}{dt}y(t-\tau(t))\Big), \quad t > 0, \\
y(t) = \varphi(t), \quad t \in [-\tau_2, 0], \\
y(+0) = \varphi(0),
\end{gathered} \tag{1.4}$$

where $\varphi(t) \in C^1[-\tau_2, 0]$ is a given real-valued vector-function. The solution to (1.4) is defined as a continuous function on $[-\tau_2,\infty)$, continuously differentiable on $[-\tau_2,\infty)$ except for points $k\tau_1$, $k = 0,1,2,\dots$, and satisfying (1.1) everywhere on $[0,\infty)$ except for points $k\tau_1$, $k = 0,1,2,\dots$. Our aim is to establish conditions for the exponential stability of the zero solution to (1.1) and to obtain estimates for the decay rate of solutions to (1.4) at infinity, without using any spectral information (like roots of quasipolynomials in the case of constant coefficients).

To establish conditions of stability, researchers often use various Lyapunov–Krasovskii functionals (of Lyapunov type functionals) (see the bibliography in [3, 16]). However, not every Lyapunov–Krasovskii functional allows us to obtain estimates characterizing the decay rate of solutions at infinity. In recent years, the research in this direction has been actively developing. Many works are devoted to time-delay systems with constant coefficients, including systems of neutral type (see the bibliography in [21]). In the nonautonomous case, the most studied systems are the systems of the form (1.1), where $C(t)$ is a constant matrix (see bibliography in [17]). There are very few studies of the systems, where the matrix $C(t)$ is not constant [2, 5, 13, 17, 22, 30]; moreover, the authors of these works require that $\|C(t)\| < 1$. A Lyapunov-Krasovskii functional was proposed in [27], which allowed us to obtain the conditions for the exponential stability and the estimates for the solutions to the linear systems of the form (1.1) ($F(t,u,v,w) \equiv 0$) with the constant delay $\tau(t) \equiv \tau$, without any additional restrictions on the norm $\|C(t)\|$. A generalization of this functional was used in [28] for studying the exponential stability of the linear time-varying delay systems of neutral type; i.e. the systems of the form (1.1) with $F(t,u,v,w) \equiv 0$.

In this article we use the functional proposed in [28]. We introduce the necessary notation and formulate the main results in Section 2. Their proofs are given in Section 3.

2. Main results

At first we formulate the result on the exponential stability of the zero solution to (1.1) with $F(t,u,v,w) \equiv 0$:

$$\frac{d}{dt}y(t) = A(t)y(t) + B(t)y(t-\tau(t)) + C(t)\frac{d}{dt}y(t-\tau(t)), \quad t > 0. \tag{2.1}$$

Theorem 2.1 ([28])**.** *Suppose that there are matrices $H(t) \in C^1[0,T]$, $K(s)$, and $L(s)$ in $C^1[0,\tau_2]$:*

$$H(t) = H^*(t), \quad t \in [0,T], \quad H(0) = H(T) > 0, \tag{2.2}$$

$$K(s) = K^*(s) > 0, \quad \frac{d}{ds}K(s) < 0, \quad s \in [0,\tau_2], \tag{2.3}$$

$$L(s) = L^*(s) > 0, \quad \frac{d}{ds}L(s) < 0, \quad s \in [0,\tau_2], \tag{2.4}$$

such that the matrix

$$Q(t) = \begin{pmatrix} Q_{11}(t) & Q_{12}(t) & Q_{13}(t) \\ Q_{12}^*(t) & Q_{22}(t) & Q_{23}(t) \\ Q_{13}^*(t) & Q_{23}^*(t) & Q_{33}(t) \end{pmatrix} \tag{2.5}$$

with entries

$$\begin{gathered} Q_{11}(t) = -\frac{d}{dt}H(t) - H(t)A(t) - A^*(t)H(t) - K(0) - A^*(t)L(0)A(t), \\ Q_{12}(t) = -H(t)B(t) - A^*(t)L(0)B(t), \\ Q_{13}(t) = -H(t)C(t) - A^*(t)L(0)C(t), \\ Q_{22}(t) = (1-\tau_4)K(\tau_2) - B^*(t)L(0)B(t), \\ Q_{23}(t) = -B^*(t)L(0)C(t), \\ Q_{33}(t) = (1-\tau_3)^{-1}L(\tau_2) - C^*(t)L(0)C(t), \end{gathered} \tag{2.6}$$

is positive definite for $t \in [0,T]$. Then the zero solution to (2.1) *is exponentially stable.*

Assuming that the conditions of Theorem 2.1 are satisfied, we establish conditions for the exponential stability of the zero solution to the nonlinear system (1.1). To state our results, we introduce some notation. If the matrix $H(t)$ satisfies the conditions of Theorem 2.1 then

$$\frac{d}{dt}H(t) + H(t)A(t) + A^*(t)H(t) < -K(0) - A^*(t)L(0)A(t);$$

i.e., $H(t)$ is a solution to a special boundary value problem for the Lyapunov differential equation

$$\frac{d}{dt}H + HA(t) + A^*(t)H = -G(t), \quad t \in [0,T],$$

$$H(0) = H(T) > 0,$$

where $G(t)$ is a positive definite Hermitian matrix with continuous entries. In this case, it follows from the results of [6] that $H(t) > 0$ on the whole segment $[0,T]$. Let us extend this matrix T-periodically on the whole semi-axis $\{t \geq 0\}$, keeping the same notation. Using this matrix $H(t)$ together with the matrices $K(s)$, $L(s)$ satisfying the conditions of Theorem 2.1, we introduce the functions

$$\begin{aligned} \beta_1(t) &= 2\|H(t)\| + (2\|A(t)\| + q_1)\|L(0)\|, \\ \beta_2(t) &= (2\|B(t)\| + q_2)\|L(0)\|, \\ \beta_3(t) &= (2\|C(t)\| + q_3)\|L(0)\|, \end{aligned} \tag{2.7}$$

$$\begin{aligned} \alpha_1(t) &= q_1\beta_1(t) + \frac{q_1\beta_2(t) + q_2\beta_1(t)}{2} + \frac{q_1\beta_3(t) + q_3\beta_1(t)}{2}, \\ \alpha_2(t) &= q_2\beta_2(t) + \frac{q_2\beta_1(t) + q_1\beta_2(t)}{2} + \frac{q_2\beta_3(t) + q_3\beta_2(t)}{2}, \\ \alpha_3(t) &= q_3\beta_3(t) + \frac{q_3\beta_1(t) + q_1\beta_3(t)}{2} + \frac{q_3\beta_2(t) + q_2\beta_3(t)}{2}, \end{aligned} \tag{2.8}$$

and the matrix

$$Q^\alpha(t) = Q(t) - \begin{pmatrix} \alpha_1(t)I & 0 & 0 \\ 0 & \alpha_2(t)I & 0 \\ 0 & 0 & \alpha_3(t)I \end{pmatrix}, \tag{2.9}$$

where I is the unit matrix.

Theorem 2.2. *Let the conditions of Theorem 2.1 be satisfied. Suppose that q_1, q_2, q_3 are such that the matrix $Q^\alpha(t)$ is positive definite for $t \in [0,T]$. Then the zero solution to* (1.1) *is exponentially stable.*

Below we present the estimate for the exponential decay rate of the solution to the initial value problem (1.4) as $t \to \infty$. We use the following notation

$$\begin{aligned} V(0,\varphi) &= \langle H(0)\varphi(0), \varphi(0)\rangle + \int_{-\tau(0)}^0 \langle K(-s)\varphi(s), \varphi(s)\rangle ds \\ &\quad + \int_{-\tau(0)}^0 \Big\langle L(-s)\frac{d}{ds}\varphi(s), \frac{d}{ds}\varphi(s)\Big\rangle ds, \end{aligned} \tag{2.10}$$

$$\begin{aligned} P(t) &= Q_{11}(t) - \alpha_1(t)I - \Big[Q_{12}(t) - Q_{13}(t)(Q_{33}(t) - \alpha_3(t)I)^{-1}Q_{23}^*(t)\Big] \\ &\quad \times \Big[Q_{22}(t) - \alpha_2(t)I - Q_{23}(t)(Q_{33}(t) - \alpha_3(t)I)^{-1}Q_{23}^*(t)\Big]^{-1} \\ &\quad \times \Big[Q_{12}(t) - Q_{13}(t)(Q_{33}(t) - \alpha_3(t)I)^{-1}Q_{23}^*(t)\Big]^* \\ &\quad - Q_{13}(t)(Q_{33}(t) - \alpha_3(t)I)^{-1}Q_{13}^*(t), \end{aligned} \tag{2.11}$$

where the matrices $Q_{ij}(t)$ are defined by (2.6). It is not difficult to show that $P(t)$ is positive definite if $Q^\alpha(t)$ in (2.9) is positive definite (see Lemma 3.1). We denote by $p_{\min}(t) > 0$ the minimal eigenvalue of the matrix $P(t)$, and by $h_{\min}(t) > 0$ the minimal eigenvalue of the matrix $H(t)$.

Theorem 2.3. *Suppose that the conditions of Theorem 2.2 are satisfied. Let k, $l > 0$ be maximal numbers such that*

$$\frac{d}{ds}K(s) + kK(s) \leq 0, \quad \frac{d}{ds}L(s) + lL(s) \leq 0, \quad s \in [0, \tau_2]. \tag{2.12}$$

Then the following estimate holds for the solution to (1.4),

$$\|y(t)\| \le \sqrt{\frac{V(0,\varphi)}{h_{\min}(t)}}\exp\Big(-\frac{1}{2}\int_0^t \gamma(\xi)d\xi\Big), \quad t>0, \tag{2.13}$$

where

$$\gamma(t)=\min\Big\{\frac{p_{\min}(t)}{\|H(t)\|},k,l\Big\}>0.$$

The existence of k, $l>0$ in Theorem 2.3 is provided by using conditions (2.3) and (2.4).

3. Proof of the main results

Obviously, the assertion of Theorem 2.2 follows immediately from estimate (2.13). Therefore, it suffices to prove Theorem 2.3.

Proof of Theorem 2.3. We follow the scheme from [8]. Let $y(t)$ be the solution to the initial value problem (1.4). Using the matrices $H(t)$, $K(s)$, and $L(s)$ defined in Section 2, we consider the following Lyapunov-Krasovskii functional on the solution

$$\begin{aligned} V(t,y)&=\langle H(t)y(t),y(t)\rangle+\int_{t-\tau(t)}^t\langle K(t-s)y(s),y(s)\rangle ds\\ &\quad+\int_{t-\tau(t)}^t\Big\langle L(t-s)\frac{d}{ds}y(s),\frac{d}{ds}y(s)\Big\rangle ds. \end{aligned}\tag{3.1}$$

Differentiating we obtain

$$\begin{aligned} \frac{d}{dt}V(t,y)&=\Big\langle\frac{d}{dt}H(t)y(t),y(t)\Big\rangle+\Big\langle H(t)\frac{d}{dt}y(t),y(t)\Big\rangle+\Big\langle H(t)y(t),\frac{d}{dt}y(t)\Big\rangle\\ &\quad+\langle K(0)y(t),y(t)\rangle-\Big(1-\frac{d}{dt}\tau(t)\Big)\langle K(\tau(t))y(t-\tau(t)),y(t-\tau(t))\rangle\\ &\quad+\int_{t-\tau(t)}^t\Big\langle\frac{d}{dt}K(t-s)y(s),y(s)\Big\rangle ds+\Big\langle L(0)\frac{d}{dt}y(t),\frac{d}{dt}y(t)\Big\rangle\\ &\quad-\Big(1-\frac{d}{dt}\tau(t)\Big)^{-1}\Big\langle L(\tau(t))\frac{d}{dt}y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t))\Big\rangle\\ &\quad+\int_{t-\tau(t)}^t\Big\langle\frac{d}{dt}L(t-s)\frac{d}{ds}y(s),\frac{d}{ds}y(s)\Big\rangle ds. \end{aligned}$$

We introduce the notation

$$z(t)=A(t)y(t)+B(t)y(t-\tau(t))+C(t)\frac{d}{dt}y(t-\tau(t)).$$

Taking into account that $y(t)$ satisfies (1.1), we have

$$\begin{aligned} \frac{d}{dt}V(t,y)&=\Big\langle\frac{d}{dt}H(t)y(t),y(t)\Big\rangle+\Big\langle H(t)z(t),y(t)\Big\rangle\\ &\quad+\Big\langle H(t)F\Big(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t))\Big),y(t)\Big\rangle\\ &\quad+\Big\langle H(t)y(t),z(t)\Big\rangle+\Big\langle H(t)y(t),F\Big(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t))\Big)\Big\rangle\\ &\quad+\langle K(0)y(t),y(t)\rangle-\Big(1-\frac{d}{dt}\tau(t)\Big)\langle K(\tau(t))y(t-\tau(t)),y(t-\tau(t))\rangle \end{aligned}$$

$$
\begin{aligned}
&+ \int_{t-\tau(t)}^{t} \Big\langle \frac{d}{dt} K(t-s) y(s), y(s) \Big\rangle ds + \big\langle L(0) z(t), z(t) \big\rangle \\
&+ \Big\langle L(0) F\Big(t, y(t), y(t-\tau(t)), \frac{d}{dt} y(t-\tau(t))\Big), z(t) \Big\rangle \\
&+ \Big\langle L(0) z(t), F\Big(t, y(t), y(t-\tau(t)), \frac{d}{dt} y(t-\tau(t))\Big) \Big\rangle \\
&+ \Big\langle L(0) F\Big(t, y(t), y(t-\tau(t)), \frac{d}{dt} y(t-\tau(t))\Big), \\
&F\Big(t, y(t), y(t-\tau(t)), \frac{d}{dt} y(t-\tau(t))\Big) \Big\rangle \\
&- \Big(1 - \frac{d}{dt}\tau(t)\Big)^{-1} \Big\langle L(\tau(t)) \frac{d}{dt} y(t-\tau(t)), \frac{d}{dt} y(t-\tau(t)) \Big\rangle \\
&+ \int_{t-\tau(t)}^{t} \Big\langle \frac{d}{dt} L(t-s) \frac{d}{ds} y(s), \frac{d}{ds} y(s) \Big\rangle ds.
\end{aligned}
$$

By (1.2), (2.3), (2.4), we obtain

$$
\begin{aligned}
&\Big(1 - \frac{d}{dt}\tau(t)\Big) \langle K(\tau(t)) y(t-\tau(t)), y(t-\tau(t)) \rangle \\
&\geq (1-\tau_4) \langle K(\tau_2) y(t-\tau(t)), y(t-\tau(t)) \rangle, \\
&\Big(1 - \frac{d}{dt}\tau(t)\Big)^{-1} \Big\langle L(\tau(t)) \frac{d}{dt} y(t-\tau(t)), \frac{d}{dt} y(t-\tau(t)) \Big\rangle \\
&\geq (1-\tau_3)^{-1} \Big\langle L(\tau_2) \frac{d}{dt} y(t-\tau(t)), \frac{d}{dt} y(t-\tau(t)) \Big\rangle.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\frac{d}{dt} V(t,y) \leq & - \Big\langle Q(t) \begin{pmatrix} y(t) \\ y(t-\tau(t)) \\ \frac{d}{dt} y(t-\tau(t)) \end{pmatrix}, \begin{pmatrix} y(t) \\ y(t-\tau(t)) \\ \frac{d}{dt} y(t-\tau(t)) \end{pmatrix} \Big\rangle \\
&+ \Big\langle H(t) F\Big(t, y(t), y(t-\tau(t)), \frac{d}{dt} y(t-\tau(t))\Big), y(t) \Big\rangle \\
&+ \Big\langle H(t) y(t), F\Big(t, y(t), y(t-\tau(t)), \frac{d}{dt} y(t-\tau(t))\Big) \Big\rangle \\
&+ \Big\langle L(0) F\Big(t, y(t), y(t-\tau(t)), \frac{d}{dt} y(t-\tau(t))\Big), z(t) \Big\rangle \\
&+ \Big\langle L(0) z(t), F\Big(t, y(t), y(t-\tau(t)), \frac{d}{dt} y(t-\tau(t))\Big) \Big\rangle \\
&+ \Big\langle L(0) F\Big(t, y(t), y(t-\tau(t)), \frac{d}{dt} y(t-\tau(t))\Big), \\
&F\Big(t, y(t), y(t-\tau(t)), \frac{d}{dt} y(t-\tau(t))\Big) \Big\rangle \\
&+ \int_{t-\tau(t)}^{t} \Big\langle \frac{d}{dt} K(t-s) y(s), y(s) \Big\rangle ds \\
&+ \int_{t-\tau(t)}^{t} \Big\langle \frac{d}{dt} L(t-s) \frac{d}{ds} y(s), \frac{d}{ds} y(s) \Big\rangle ds,
\end{aligned}
$$

where the matrix $Q(t)$ is defined in (2.5).

Consider the group of the summands containing $F\big(t, y(t), y(t-\tau(t)), \frac{d}{dt}y(t-\tau(t))\big)$ and denote them by $W(t)$. Then,

$$\begin{aligned} \frac{d}{dt}V(t,y) &\le -\Big\langle Q(t)\begin{pmatrix} y(t) \\ y(t-\tau(t)) \\ \frac{d}{dt}y(t-\tau(t)) \end{pmatrix}, \begin{pmatrix} y(t) \\ y(t-\tau(t)) \\ \frac{d}{dt}y(t-\tau(t)) \end{pmatrix} \Big\rangle + W(t) \\ &\quad + \int_{t-\tau(t)}^{t} \langle \frac{d}{dt}K(t-s)y(s), y(s)\rangle ds \\ &\quad + \int_{t-\tau(t)}^{t} \langle \frac{d}{dt}L(t-s)\frac{d}{ds}y(s), \frac{d}{ds}y(s)\rangle ds. \end{aligned} \tag{3.2}$$

Obviously,

$$\begin{aligned} &W(t) \\ &\le \Big(2\|H(t)\|\,\|y(t)\| + 2\|L(0)\|\big\|A(t)y(t) + B(t)y(t-\tau(t)) + C(t)\frac{d}{dt}y(t-\tau(t))\big\| \\ &\quad + \|L(0)\|\Big\|F\Big(t, y(t), y(t-\tau(t)), \frac{d}{dt}y(t-\tau(t))\Big)\Big\|\Big) \\ &\quad \times \Big\|F\Big(t, y(t), y(t-\tau(t)), \frac{d}{dt}y(t-\tau(t))\Big)\Big\|. \end{aligned}$$

Using (1.3), we have

$$\begin{aligned} W(t) &\le \Big(\beta_1(t)\|y(t)\| + \beta_2(t)\|y(t-\tau(t))\| + \beta_3(t)\|\frac{d}{dt}y(t-\tau(t))\|\Big) \\ &\quad \times \Big(q_1\|y(t)\| + q_2\|y(t-\tau(t))\| + q_3\|\frac{d}{dt}y(t-\tau(t))\|\Big), \end{aligned}$$

where $\beta_j(t)$, $j = 1,2,3$, are defined in (2.7). Obviously,

$$W(t) \le \alpha_1(t)\|y(t)\|^2 + \alpha_2(t)\|y(t-\tau(t))\|^2 + \alpha_3(t)\|\frac{d}{dt}y(t-\tau(t))\|^2, \tag{3.3}$$

where the functions $\alpha_j(t)$, $j = 1,2,3$, are defined in (2.8). By (3.3), from (3.2) we obtain

$$\begin{aligned} \frac{d}{dt}V(t,y) &\le -\Big\langle Q^{\alpha}(t)\begin{pmatrix} y(t) \\ y(t-\tau(t)) \\ \frac{d}{dt}y(t-\tau(t)) \end{pmatrix} \begin{pmatrix} y(t) \\ y(t-\tau(t)) \\ \frac{d}{dt}y(t-\tau(t)) \end{pmatrix} \Big\rangle \\ &\quad + \int_{t-\tau(t)}^{t} \langle \frac{d}{dt}K(t-s)y(s), y(s)\rangle ds \\ &\quad + \int_{t-\tau(t)}^{t} \langle \frac{d}{dt}L(t-s)\frac{d}{ds}y(s), \frac{d}{ds}y(s)\rangle ds, \end{aligned} \tag{3.4}$$

where the matrix $Q^{\alpha}(t)$ is given in (2.9).

For further transformations, we use an auxiliary lemma from matrix theory.

Lemma 3.1. *Let $R(t)$ be a positive definite Hermitian matrix with continuous entries*

$$R(t) = \begin{pmatrix} R_{11}(t) & R_{12}(t) & R_{13}(t) \\ R_{12}^*(t) & R_{22}(t) & R_{23}(t) \\ R_{13}^*(t) & R_{23}^*(t) & R_{33}(t) \end{pmatrix}, \quad t \in [0,T],$$

Then the following representation holds

$$
\begin{aligned}
R(t) = &\begin{pmatrix} I & \widetilde{R}_1(t)\widetilde{R}_2^{-1}(t) & R_{13}(t)R_{33}^{-1}(t) \\ 0 & I & R_{23}(t)R_{33}^{-1}(t) \\ 0 & 0 & I \end{pmatrix} \\
&\times \begin{pmatrix} R_{11}(t) - \widetilde{R}_1(t)\widetilde{R}_2^{-1}(t)\widetilde{R}_1^*(t) - R_{13}(t)R_{33}^{-1}(t)R_{13}^*(t) & 0 & 0 \\ 0 & \widetilde{R}_2(t) & 0 \\ 0 & 0 & R_{33}(t) \end{pmatrix} \\
&\times \begin{pmatrix} I & 0 & 0 \\ \widetilde{R}_2^{-1}(t)\widetilde{R}_1^*(t) & I & 0 \\ R_{33}^{-1}(t)R_{13}^*(t) & R_{33}^{-1}(t)R_{23}^*(t) & I \end{pmatrix},
\end{aligned}
$$

where

$$
\widetilde{R}_1(t) = R_{12}(t) - R_{13}(t)R_{33}^{-1}(t)R_{23}^*(t), \quad \widetilde{R}_2(t) = R_{22}(t) - R_{23}(t)R_{33}^{-1}(t)R_{23}^*(t);
$$

moreover, the matrices

$$
R_{11}(t) - \widetilde{R}_1(t)\widetilde{R}_2^{-1}(t)\widetilde{R}_1^*(t) - R_{13}(t)R_{33}^{-1}(t)R_{13}^*(t), \quad \widetilde{R}_2(t), \quad R_{33}(t)
$$

are positive definite.

By the above lemma, for the matrix $Q^\alpha(t)$ in (2.9), we have

$$
\Big\langle Q^\alpha(t) \begin{pmatrix} y(t) \\ y(t-\tau(t)) \\ \frac{d}{dt}y(t-\tau(t)) \end{pmatrix}, \begin{pmatrix} y(t) \\ y(t-\tau(t)) \\ \frac{d}{dt}y(t-\tau(t)) \end{pmatrix} \Big\rangle \geq \langle P(t)y(t), y(t)\rangle,
$$

where $P(t)$ is the positive definite Hermitian matrix given in (2.11). Then

$$
\langle P(t)y(t), y(t)\rangle \geq p_{\min}(t)\|y(t)\|^2,
$$

where $p_{\min}(t) > 0$ is the minimal eigenvalue of the matrix $P(t)$. Consequently, from (3.4) we obtain

$$
\begin{aligned}
\frac{d}{dt}V(t,y) \leq -\langle p_{\min}(t)y(t), y(t)\rangle + \int_{t-\tau(t)}^t \big\langle \frac{d}{dt}K(t-s)y(s), y(s)\big\rangle ds \\
+ \int_{t-\tau(t)}^t \big\langle \frac{d}{dt}L(t-s)\frac{d}{ds}y(s), \frac{d}{ds}y(s)\big\rangle ds.
\end{aligned}
$$

Clearly,

$$
h_{\min}(t)\|y(t)\|^2 \leq \big\langle H(t)y(t), y(t)\big\rangle \leq \|H(t)\|\,\|y(t)\|^2, \tag{3.5}
$$

where $h_{\min}(t) > 0$ is the minimal eigenvalue of the matrix $H(t)$. Hence,

$$
\begin{aligned}
\frac{d}{dt}V(t,y) \leq -\frac{p_{\min}(t)}{\|H(t)\|}\big\langle H(t)y(t), y(t)\big\rangle + \int_{t-\tau(t)}^t \big\langle \frac{d}{dt}K(t-s)y(s), y(s)\big\rangle ds \\
+ \int_{t-\tau(t)}^t \big\langle \frac{d}{dt}L(t-s)\frac{d}{ds}y(s), \frac{d}{ds}y(s)\big\rangle ds.
\end{aligned}
$$

Using the condition (2.12), we arrive at

$$
\begin{aligned}
\frac{d}{dt}V(t,y) \leq -\frac{p_{\min}(t)}{\|H(t)\|}\big\langle H(t)y(t), y(t)\big\rangle - k\int_{t-\tau(t)}^t \big\langle K(t-s)y(s), y(s)\big\rangle ds \\
- l\int_{t-\tau(t)}^t \big\langle L(t-s)\frac{d}{ds}y(s), \frac{d}{ds}y(s)\big\rangle ds.
\end{aligned}
$$

By the definition of the functional in (3.1), we have

$$\frac{d}{dt}V(t,y) \le -\gamma(t)V(t,y),$$

where $\gamma(t) = \min\big\{\frac{p_{\min}(t)}{\|H(t)\|}, k, l\big\}$. From this differential inequality, we obtain the estimate

$$V(t,y) \le V(0,\varphi)\exp\Big(-\int_0^t \gamma(\xi)d\xi\Big),$$

where $V(0,\varphi)$ is defined by (2.10). Using (3.5) and taking into account the definition of the functional (3.1), we infer

$$\|y(t)\|^2 \le \frac{1}{h_{\min}(t)}\langle H(t)y(t), y(t)\rangle \le \frac{V(t,y)}{h_{\min}(t)} \le \frac{V(0,\varphi)}{h_{\min}(t)}\exp\Big(-\int_0^t \gamma(\xi)d\xi\Big).$$

Hence, we have the required inequality (2.13). This completes the proof. □

Repeating the reasoning of the proof of Theorem 2.3 for $F(t,u,v,w) \equiv 0$, we arrive immediately to the statement of Theorem 2.1.

Theorem 3.2. *Suppose that there are matrices $H(t) \in C^1[0,T]$, $K(s)$, $L(s) \in C^1[0,\tau_2]$ satisfying* (2.2)–(2.4) *and such that $P(t) > 0$,*

$$Q_{22}(t) - \alpha_2(t)I - Q_{23}(t)(Q_{33}(t) - \alpha_3(t)I)^{-1}Q_{23}^*(t) > 0,$$

and $Q_{33}(t) - \alpha_3(t)I > 0$ for $t \in [0,T]$. Then the zero solution to (1.1) *is exponentially stable.*

Proof. By Lemma 3.1, the matrix $Q^\alpha(t)$ in (2.9) is positive definite if and only if the matrices $P(t)$, $Q_{22}(t)-\alpha_2(t)I-Q_{23}(t)(Q_{33}(t)-\alpha_3(t)I)^{-1}Q_{23}^*(t)$, and $Q_{33}(t)-\alpha_3(t)I$ are positive definite. □

Remark 3.3. The results obtained above give us the assertions on the robust stability for (2.1). Indeed, consider uncertain systems of the form

$$\begin{aligned}\frac{d}{dt}y(t) &= A(t)y(t) + B(t)y(t-\tau(t)) + C(t)\frac{d}{dt}y(t-\tau(t)) \\ &\quad + \Delta A(t)y(t) + \Delta B(t)y(t-\tau(t)) + \Delta C(t)\frac{d}{dt}y(t-\tau(t)),\end{aligned} \tag{3.6}$$

where $\Delta A(t)$, $\Delta B(t)$, and $\Delta C(t)$ are unknown $(n \times n)$ matrices such that

$$\|\Delta A(t)\| \le q_1, \quad \|\Delta B(t)\| \le q_2, \quad \|\Delta C(t)\| \le q_3\,.$$

Obviously, in this case the vector-function

$$F(t,u,v,w) = \Delta A(t)u + \Delta B(t)v + \Delta C(t)w$$

satisfies (1.3). Then Theorem 2.2 gives us the conditions of the robust exponential stability for (2.1). From Theorems 2.3 we have the estimates of the exponential decay of solutions to (3.6).

4. Examples

Consider the following time-delay equation of the form (1.1),

$$\frac{d}{dt}y(t) = (0.1\cos t - 2)y(t) - 0.1y(t-\tau(t)) + q\cos\Big(\frac{d}{dt}y(t-\tau(t))\Big)\frac{d}{dt}y(t-\tau(t)), \quad (4.1)$$

where $\tau(t) = 0.5\sin t + 1$. Obviously, $\tau(t)$ satisfies (1.2) with $\tau_1 = 0.5$, $\tau_2 = 1.5$, $\tau_3 = -0.5$, and $\tau_4 = 0.5$. The function $F(t,u,v,w) = q\cos(w)w$ satisfies (1.3) with $q_1 = q_2 = 0$, $q_3 = q$.

At first we consider the linear case ($F(t,u,v,w) \equiv 0$); i.e. $q = 0$. We choose the functions $H(t)$, $K(s)$, and $L(s)$ as follows

$$H(t) = 0.5 - 0.1\sin t, \quad K(s) = 0.27e^{-1.65s}, \quad L(s) = 0.001e^{-1.65s}.$$

Obviously, these functions satisfy (2.2)–(2.4), and (2.12) with $k = l = 1.65$. In this case the matrix $Q(t)$ has entires

$$Q_{11}(t) = 1.73 - 0.4\sin t + 0.02\sin t\cos t - 0.001(2 - 0.1\cos t)^2,$$
$$Q_{12}(t) = 0.0498 - 0.01\sin t + 0.00001\cos t, \quad Q_{13}(t) = 0,$$
$$Q_{22}(t) = 0.135e^{-2.475} - 0.00001, \quad Q_{23}(t) = 0, \quad Q_{33}(t) = \frac{0.002}{3}e^{-2.475}.$$

It is not difficult to verify that $Q(t)$ is positive definite for $t \in [0, 2\pi]$. Then, by Theorem 2.1, the zero solution to (4.1) with $q = 0$ is exponentially stable. By Theorem 2.3, to estimate the decay rate of solutions to (4.1), we need to calculate (for $q = 0$) the functions $P(t)$ and $\gamma(t) = \min\{P(t)/\|H(t)\|, 1.65\}$. In our case

$$P(t) = Q_{11}(t) - Q_{12}^2(t)(Q_{22}(t))^{-1}, \quad \|H(t)\| = |0.5 - 0.1\sin t|.$$

It is not difficult to show that $P(t) \geq 0.99047$ and that $\|H(t)\| \leq 0.6$. Therefore, $P(t)/\|H(t)\| \geq 1.65078$ and $\gamma(t) = 1.65$. By (2.13), we have the estimate

$$\|y(t)\| \leq ce^{-0.825t}, \quad c > 0,$$

for the solutions to (4.1) with $q = 0$.

We now consider (4.1) with $q = 0.1$. We choose the functions $H(t)$, $K(s)$, and $L(s)$ as follows

$$H(t) = 0.5 - 0.1\sin t, \quad K(s) = 0.06e^{-0.27s}, \quad L(s) = 0.28e^{-0.27s}.$$

Obviously, these functions satisfy (2.2)–(2.4), and (2.12) with $k = l = 0.27$. In this case the entries of the matrix $Q(t)$ has entries

$$Q_{11}(t) = 1.94 - 0.4\sin t + 0.02\sin t\cos t - 0.28(2 - 0.1\cos t)^2,$$
$$Q_{12}(t) = -0.006 - 0.01\sin t + 0.0028\cos t, \quad Q_{13}(t) = 0,$$
$$Q_{22}(t) = 0.03e^{-0.405} - 0.0028, \quad Q_{23}(t) = 0, \quad Q_{33}(t) = \frac{0.56}{3}e^{-0.405},$$

and

$$\beta_1(t) = |1 - 0.2\sin t| + |1.12 - 0.056\cos t|, \quad \beta_2(t) = 0.056, \quad \beta_3(t) = 0.028,$$
$$\alpha_1(t) = |0.05 - 0.01\sin t| + |0.056 - 0.0028\cos t|, \quad \alpha_2(t) = 0.0028,$$
$$\alpha_3(t) = |0.05 - 0.01\sin t| + |0.056 - 0.0028\cos t| + 0.0056.$$

It is not difficult to verify that $Q^\alpha(t)$ defined by (2.9) is positive definite for $t \in [0, 2\pi]$. Then, by Theorem 2.2, the zero solution to (4.1) with $q = 0.1$

is exponentially stable. By Theorem 2.3, to estimate the decay rate of solutions to (4.1), we need to calculate (for $q = 0.1$) the functions $P(t)$ and $\gamma(t) = \min\{P(t)/\|H(t)\|, 0.27\}$. In our case

$$P(t) = Q_{11}(t) - \alpha_1(t) - Q_{12}^2(t)(Q_{22}(t) - \alpha_2(t))^{-1}.$$

It is not difficult to show that $P(t) \geq 0.16319$. Consequently, $P(t)/|H(t)\| \geq 0.27198$ and $\gamma(t) = 0.27$. Taking into account (2.13), we have the estimate

$$\|y(t)\| \leq ce^{-0.135t}, \quad c > 0,$$

for the solutions to (4.1) with $q = 0.1$.

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Inessa I. Matveeva
Laboratory of Differential and Difference Equations, Sobolev Institute of Mathematics, 4, Acad. Koptyug avenue, Novosibirsk 630090, Russia.
Department of Mechanics and Mathematics, Novosibirsk State University, 2, Pirogov street, Novosibirsk 630090, Russia
Email address: matveeva@math.nsc.ru