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A Remark on ∞-harmonic Functions on Riemannian Manifolds *

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Abstract

In this note we prove an equality for ∞-harmonic functions on Riemannian manifolds. As a corollary, there is no non-constant ∞-harmonic function on positively (or negatively) curved manifolds.

1 Introduction

In [1], [2], Aronsson studied solutions of the boundary value problem for the degenerate elliptic equation

$$\sum_{i,j} \nabla_i u \, \nabla_j u \, \nabla_i \nabla_j u = 0 \tag{1}$$

in a bounded subdomain D of $\mathbb{R}^n$ with the boundary condition $u = \varphi$ on ∂D. His motivation is to consider the *absolutely minimizing Lipschitz extension problem*, which means the problem of finding an extension u in $\mathrm{W}^{1,\infty}(D)$ of any given Lipschitz function φ on ∂D satisfying the minimization property

$$\|\nabla u\|_{\mathrm{L}^\infty(U)} \ \leq \ \|\nabla v\|_{\mathrm{L}^\infty(U)}$$

for any open set $U \subset D$ and for $v \in \mathrm{W}^{1,\infty}(U)$ such that $v - u \in \mathrm{W}_0^{1,\infty}(U)$. The equation (1) is the Euler-Lagrange equation of the functional $F_\infty(u) = \|\nabla u\|_{\mathrm{L}^\infty}$ in the following sense. A p-harmonic function u is a solution of

$$\mathrm{div}(\|\nabla u\|^{p-2} \, \nabla u) \ = \ 0 \,, \tag{2}$$

which is the Euler-Lagrange equation of the functional $F_p(u) = \|\nabla u\|_{\mathrm{L}^p}$. Rewrite (2) to read

$$\frac{1}{p-2} \, \|\nabla u\|^2 \, \triangle \, u \ + \ \sum_{i,j} \nabla_i u \, \nabla_j u \, \nabla_i \nabla_j u \ = \ 0 \,.$$

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Formally passing to the limit as p tends to infinity, the Euler-Lagrange equation (2) of the functional F_p converges in some sense to the Euler-Lagrange equation (1) of the functional F_∞. From the point of view by Aronsson, Jensen [6] obtained existence and uniqueness results. (See also Bhattacharya, DiBenedetto and Manfredi [4].) He proved

1. any solution of the absolutely minimizing Lipschitz extension problem is a viscosity solution of (1), and
2. there exists a unique viscosity solution of (1). Any *bounded* such solution is locally Lipschitz continuous.

Aronsson's pioneering papers [1], [2] investigated classical solutions. Recently Evans [5] obtained a Harnack inequality for classical solutions.

The absolutely minimizing Lipschitz extension problem is considered also on subdomains of *Riemannian manifolds* M. Then the associated equation corresponding to (1) is

$$g^{ip}g^{jq}\nabla_i u\,\nabla_j u\,\nabla_p\nabla_q u = 0\,, \tag{3}$$

where g_{ij} (resp. g^{ij}) is the metric of M (resp. the inverse matrix of g_{ij}), and ∇ denotes the Levi-Civita connection of g. (Throughout this note, we use the Einstein summation convention; if the same index appears twice, once as a superscript and once as a subscript, then the index is summed over all possible values.) In this note we are concerned with $\mathrm{W}^{2,2+\varepsilon}_{loc}$-solutions of (3) $(\varepsilon > 0)$. We say that u is a $\mathrm{W}^{2,2+\varepsilon}_{loc}$ *-solution* of (3) in D if the following two conditions hold:

1. u is locally Lipschitz continuous, and
2. $u \in \mathrm{W}^{2,2+\varepsilon}_{loc}(D)$, and u satisfies (3) a.e.,

where $\mathrm{W}^{2,2+\varepsilon}_{loc}(D)$ denotes the Sobolev space of functions whose second derivatives belong to $\mathrm{L}^{2+\varepsilon}_{loc}(D)$. On this general setting, the curvature of M provides an obstruction on existence of nontrivial $\mathrm{W}^{2,2+\varepsilon}_{loc}$-solutions of (3). The purpose of this note is to prove the following equality.

Theorem 1 *Let M be a Riemannian manifold, and let D be a domain in M. Let u be a $\mathrm{W}^{2,2+\varepsilon}_{loc}$-solution of the equation (3) in D. Then*

$$g^{ip}g^{jq}g^{kr}g^{ls}R_{ijkl}\,\nabla_p u\,\nabla_q u\,\nabla_r u\,\nabla_s u \;=\; 0 \qquad \text{a.e. in } D\,, \tag{4}$$

where R_{ijkl} is the Riemannian curvature tensor of M.

Note that when $M = \mathbb{R}^n$, $R_{ijkl} \equiv 0$; hence the equality (4) holds automatically in this case. ¿From equality (4), we have $\nabla u = 0$ at any point where the curvature is positive (or negative). So we have:

Corollary 1 *Suppose that the sectional curvature of M is positive (or negative) in D. Then any $\mathrm{W}^{2,2+\varepsilon}_{loc}$-solution of (3) in D is a constant function.*

We mention a related fact on harmonic functions. Let u be a harmonic function on a Riemannian manifold M. Then u is a constant function if one of the following two conditions holds:

1. M is compact (the maximum principle).
2. M is complete and non-compact, the Ricci curvature of M is nonnegative, and u is bounded on M (Yau [7]).

These results need the assumption that u is globally defined on compact or complete manifolds. On the other hand, the above equality (4) holds when an ∞-harmonic function u is defined on a *subdomain* of M; the structure of ∞-Laplacian gives a restriction on local existence of solutions.

The author thinks that our theorem holds without the assumption that solutions belong to the class $\mathrm{W}^{2,2+\varepsilon}_{loc}(D)$, though we use this assumption. Then Aronsson's minimization approach of the Lipschitz extention problem does not seem to work on any positively (or negatively) curved manifold.

2 A Bochner type formula

In this section we prove the following formula of Bochner type.

Lemma 1 *Let u be a C^3_{loc}-solution of (3) on a subdomain D of a Riemannian manifold M. Then the following equality holds.*

$$\begin{aligned} g^{ip}g^{jq}\nabla_i u\,\nabla_j u\,\nabla_p\nabla_q\,\|\nabla u\|^2 \;+\; \frac{1}{2}\,\|\nabla\|\nabla u\|^2\,\|^2 & \\ +2\,g^{ip}g^{jq}g^{kr}g^{ls}R_{ikjl}\,\nabla_p u\,\nabla_q u\,\nabla_r u\,\nabla_s u \;&=\; 0 \qquad \text{in } D\,, \end{aligned} \tag{5}$$

where $\|\nabla u\|^2 = g^{ij}\nabla_i u\nabla_j u$ and $\|\nabla\|\nabla u\|\,\|^2 = g^{ij}\nabla_i\|\nabla u\|\nabla_j\|\nabla u\|$.

Proof. Note $\nabla g_{ij} = \nabla g^{ij} = 0$, since ∇ is the Levi-Civita connection. Applying ∇_r to both sides of (3), we have

$$g^{ip}g^{jq}\nabla_i u\,\nabla_j u\,\nabla_r\nabla_p\nabla_q u \;+\; 2\,g^{ip}g^{jq}\nabla_i u\,\nabla_r\nabla_j u\,\nabla_p\nabla_q u \;=\; 0\,. \tag{6}$$

We see that

$$\begin{aligned} \nabla_p\nabla_q\nabla_r u \;&=\; \nabla_p\nabla_r\nabla_q u \\ &=\; \nabla_r\nabla_p\nabla_q u \;-\; g^{ls}R_{prqs}\nabla_l u \quad \text{(by the Ricci formula)}\,. \end{aligned} \tag{7}$$

We get

$$
\begin{aligned}
&g^{ip}g^{jq}g^{kr}\nabla_i u\,\nabla_j u\,\nabla_p\nabla_k u\,\nabla_q\nabla_r u \qquad (8)\\
&\quad= g^{kr}\frac{1}{2}\,\nabla_k(g^{ip}\nabla_i u\nabla_p u)\;\frac{1}{2}\,\nabla_r(g^{jq}\nabla_j u\nabla_q u)\\
&\quad= \frac{1}{4}\,g^{kr}\nabla_k\|\nabla u\|^2\,\nabla_r\|\nabla u\|^2\\
&\quad= \frac{1}{4}\,\|\nabla\|\nabla u\|^2\,\|^2\,.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&g^{ip}g^{jq}\nabla_i u\,\nabla_j u\,\nabla_p\nabla_q\,\|\nabla u\|^2\\
&= g^{ip}g^{jq}\nabla_i u\,\nabla_j u\,\nabla_p\nabla_q(g^{kr}\nabla_r u\nabla_k u)\\
&= 2\,g^{ip}g^{jq}g^{kr}\nabla_i u\,\nabla_j u\,\nabla_p\nabla_q\nabla_r u\,\nabla_k u\\
&\quad +2\,g^{ip}g^{jq}g^{kr}\nabla_i u\,\nabla_j u\,\nabla_q\nabla_r u\,\nabla_p\nabla_k u\\
&= 2\,g^{ip}g^{jq}g^{kr}\nabla_i u\,\nabla_j u\,\nabla_r\nabla_p\nabla_q u\,\nabla_k u\\
&\quad -2\,g^{ip}g^{jq}g^{kr}\nabla_i u\,\nabla_j u\,g^{ls}R_{prqs}\,\nabla_l u\,\nabla_k u\\
&\quad +2\,g^{ip}g^{jq}g^{kr}\nabla_i u\,\nabla_j u\,\nabla_q\nabla_r u\,\nabla_p\nabla_k u \qquad \text{(by (7))}\\
&= -\,4\,g^{ip}g^{jq}g^{kr}\nabla_i u\,\nabla_k u\,\nabla_r\nabla_j u\,\nabla_p\nabla_q u\\
&\quad -2\,g^{ip}g^{jq}g^{kr}g^{ls}R_{prqs}\,\nabla_i u\,\nabla_j u\,\nabla_k u\,\nabla_l u\\
&\quad +2\,g^{ip}g^{jq}g^{kr}\nabla_i u\,\nabla_j u\,\nabla_q\nabla_r u\,\nabla_p\nabla_k u \qquad \text{(by (6))}\\
&= -\,2\,g^{ip}g^{jq}g^{kr}g^{ls}R_{pqrs}\,\nabla_i u\,\nabla_j u\,\nabla_k u\,\nabla_l u\\
&\quad -2\,g^{ip}g^{jq}g^{kr}\nabla_i u\,\nabla_j u\,\nabla_q\nabla_r u\,\nabla_p\nabla_k u \qquad \text{(by exchange of indices)}\\
&= -\,2\,g^{ip}g^{jq}g^{kr}g^{ls}R_{pqrs}\,\nabla_i u\,\nabla_j u\,\nabla_k u\,\nabla_l u\;-\;\frac{1}{2}\,\|\nabla\|\nabla u\|^2\,\|^2 \qquad \text{(by (8))}\,.
\end{aligned}
$$

3 Proof of Theorem 1 for C^3_{loc}-solutions

Take any $\eta \in C_0^\infty(D)$. Then from (5), we have

$$
\begin{aligned}
&\int_D \eta\,g^{ip}g^{jq}\nabla_i u\,\nabla_j u\,\nabla_p\nabla_q\,\|\nabla u\|^2 \;+\; \frac{1}{2}\int_D \|\nabla\|\nabla u\|^2\,\|^2\,\eta \qquad (9)\\
&\qquad + 2\int_D \eta\,g^{ip}g^{jq}g^{kr}g^{ls}R_{ikjl}\,\nabla_p u\,\nabla_q u\,\nabla_r u\,\nabla_s u \;=\; 0\,.
\end{aligned}
$$

Note here

$$
\begin{aligned}
g^{jq}\nabla_j u\,\nabla_q\|\nabla u\|^2 &= g^{jq}\nabla_j u\,\nabla_q(g^{ip}\nabla_i u\nabla_p u) \qquad (10)\\
&= 2\,g^{ip}g^{jq}\nabla_j u\,\nabla_i u\,\nabla_q\nabla_p u \;=\; 0\,.
\end{aligned}
$$

Using integration by parts, we get

$$\begin{aligned}
&\int_D \eta\, g^{ip} g^{jq} \nabla_i u\, \nabla_j u\, \nabla_p \nabla_q \|\nabla u\|^2 \qquad (11)\\
&= -\int_D g^{ip} g^{jq} \nabla_p \eta\, \nabla_i u\, \nabla_j u\, \nabla_q \|\nabla u\|^2 \\
&\quad -\int_D \eta\, g^{ip} g^{jq} \nabla_p \nabla_i u\, \nabla_j u\, \nabla_q \|\nabla u\|^2 \\
&\quad -\int_D \eta\, g^{ip} g^{jq} \nabla_i u\, \nabla_p \nabla_j u\, \nabla_q \|\nabla u\|^2 \\
&= -\int_D \eta\, g^{ip} g^{jq} \nabla_i u\, \nabla_p \nabla_j u\, \nabla_q \|\nabla u\|^2 \qquad \text{(by (10))} \\
&= -\int_D \eta\, \frac{1}{2} g^{jq} \nabla_j (g^{ip} \nabla_i u\, \nabla_p u)\, \nabla_q \|\nabla u\|^2 \\
&= -\frac{1}{2}\int_D \eta\, g^{jq} \nabla_j \|\nabla u\|^2\, \nabla_q \|\nabla u\|^2 \\
&= -\frac{1}{2}\int_D \|\nabla \|\nabla u\|^2\,\|^2\, \eta\,.
\end{aligned}$$

¿From (9) and (11), we have

$$\int_D \eta\, g^{ip} g^{jq} g^{kr} g^{ls} R_{ikjl}\, \nabla_p u\, \nabla_q u\, \nabla_r u\, \nabla_s u \;=\; 0\,. \qquad (12)$$

Since η is an arbitrary test function in $C_0^\infty(D)$, we have

$$g^{ip} g^{jq} g^{kr} g^{ls} R_{ikjl}\, \nabla_p u\, \nabla_q u\, \nabla_r u\, \nabla_s u \;=\; 0 \quad \text{a.e. in } D\,. \qquad \square$$

4 Proof of Theorem 1

In this section we complete our proof of Theorem 1 using an approximation. For any $W^{2,2+\varepsilon}_{loc}$-solution u of (3), we take an approximating sequence $\{\, u^{(\nu)} \,\}_{\nu=1}^{\infty} \subset C^3_{loc}(D)$ such that for any compact set K in D,

1. $\varphi^{(\nu)} \;:=\; u^{(\nu)} - u$ approaches zero in $W^{2,2+\varepsilon}_{loc}(D)$ as ν tends to infinity, and
2. the Lipschitz constants of $u^{(\nu)}$ ($\nu = 1, 2, ...$) are uniformly bounded on K : hence $\|\nabla u^{(\nu)}\|_{L^\infty(K)}$ and $\|\nabla \varphi^{(\nu)}\|_{L^\infty(K)}$ ($\nu = 1, 2, ...$) are uniformly bounded on K.

Since $u = u^{(\nu)} - \varphi^{(\nu)}$ satisfies (3), we have

$$g^{ip} g^{jq} \nabla_i (u^{(\nu)} - \varphi^{(\nu)})\, \nabla_j (u^{(\nu)} - \varphi^{(\nu)})\, \nabla_p \nabla_q (u^{(\nu)} - \varphi^{(\nu)}) = 0$$

i.e.,

$$g^{ip}g^{jq}\nabla_i u^{(\nu)}\,\nabla_j u^{(\nu)}\,\nabla_p\nabla_q u^{(\nu)} + F(\varphi^{(\nu)},\,u^{(\nu)}) = 0 \tag{13}$$

where

$$\begin{aligned}
&F(\varphi^{(\nu)},\,u^{(\nu)})\\
&\quad= -g^{ip}g^{jq}\nabla_i\varphi^{(\nu)}\,\nabla_j u^{(\nu)}\,\nabla_p\nabla_q u^{(\nu)} - g^{ip}g^{jq}\nabla_i u^{(\nu)}\,\nabla_j\varphi^{(\nu)}\,\nabla_p\nabla_q u^{(\nu)}\\
&\qquad -g^{ip}g^{jq}\nabla_i u^{(\nu)}\,\nabla_j u^{(\nu)}\,\nabla_p\nabla_q\varphi^{(\nu)} + g^{ip}g^{jq}\nabla_i\varphi^{(\nu)}\,\nabla_j\varphi^{(\nu)}\,\nabla_p\nabla_q u^{(\nu)}\\
&\qquad +g^{ip}g^{jq}\nabla_i u^{(\nu)}\,\nabla_j\varphi^{(\nu)}\,\nabla_p\nabla_q\varphi^{(\nu)} + g^{ip}g^{jq}\nabla_i\varphi^{(\nu)}\,\nabla_j u^{(\nu)}\,\nabla_p\nabla_q\varphi^{(\nu)}\\
&\qquad -g^{ip}g^{jq}\nabla_i\varphi^{(\nu)}\,\nabla_j\varphi^{(\nu)}\,\nabla_p\nabla_q\varphi^{(\nu)}\,.
\end{aligned}$$

Let $\psi\in \mathrm{W}_0^{1,1}(D)$. Multiply by $-\nabla_r\psi$ both sides of (13) and use integration by parts, then we have

$$\begin{aligned}
&\int_D \psi\, g^{ip}g^{jq}\nabla_i u^{(\nu)}\,\nabla_j u^{(\nu)}\,\nabla_r\nabla_p\nabla_q u^{(\nu)}\\
&+2\int_D \psi\, g^{ip}g^{jq}\nabla_i u^{(\nu)}\,\nabla_r\nabla_j u^{(\nu)}\,\nabla_p\nabla_q u^{(\nu)}\\
&\qquad -\int_M F(\varphi^{(\nu)},\,u^{(\nu)})\,\nabla_r\psi \;=\; 0\,.
\end{aligned}$$

Let $\psi = \eta\, g^{kr}\nabla_k u^{(\nu)}$ and sum them up with respect to r. Then we get

$$\begin{aligned}
&\int_D \eta\, g^{ip}g^{jq}g^{kr}\nabla_i u^{(\nu)}\,\nabla_j u^{(\nu)}\,\nabla_r\nabla_p\nabla_q u^{(\nu)}\,\nabla_k u^{(\nu)}\\
&+2\int_D \eta\, g^{ip}g^{jq}g^{kr}\nabla_i u^{(\nu)}\,\nabla_k u^{(\nu)}\,\nabla_r\nabla_j u^{(\nu)}\,\nabla_p\nabla_q u^{(\nu)}\\
&\qquad -\int_M F(\varphi^{(\nu)},\,u^{(\nu)})\,g^{kr}\nabla_r(\eta\,\nabla_k u^{(\nu)}) \;=\; 0
\end{aligned} \tag{14}$$

We see

$$\begin{aligned}
&\int_D \eta\, g^{ip}g^{jq}\nabla_i u^{(\nu)}\,\nabla_j u^{(\nu)}\,\nabla_p\nabla_q\,\|\nabla u^{(\nu)}\|^2\\
&\quad= \int_D \eta\, g^{ip}g^{jq}\nabla_i u^{(\nu)}\,\nabla_j u^{(\nu)}\,\nabla_p\nabla_q(g^{kr}\nabla_r u\nabla_k u)\\
&\quad= 2\int_D \eta\, g^{ip}g^{jq}g^{kr}\nabla_i u^{(\nu)}\,\nabla_j u^{(\nu)}\,\nabla_p\nabla_q\nabla_r u^{(\nu)}\,\nabla_k u^{(\nu)}\\
&\qquad +2\int_D \eta\, g^{ip}g^{jq}g^{kr}\nabla_i u^{(\nu)}\,\nabla_j u^{(\nu)}\,\nabla_q\nabla_r u^{(\nu)}\,\nabla_p\nabla_k u^{(\nu)}\\
&\quad= 2\int_D \eta\, g^{ip}g^{jq}g^{kr}\nabla_i u^{(\nu)}\,\nabla_j u^{(\nu)}\,\nabla_r\nabla_p\nabla_q u^{(\nu)}\,\nabla_k u^{(\nu)}\\
&\qquad -2\int_D \eta\, g^{ip}g^{jq}g^{kr}\nabla_i u^{(\nu)}\,\nabla_j u^{(\nu)}\,g^{ls}R_{prqs}\,\nabla_l u^{(\nu)}\,\nabla_k u^{(\nu)}
\end{aligned}$$

$$
\begin{aligned}
&+2\int_D \eta\, g^{ip}g^{jq}g^{kr}\nabla_i u^{(\nu)}\,\nabla_j u^{(\nu)}\,\nabla_q\nabla_r u^{(\nu)}\,\nabla_p\nabla_k u^{(\nu)} \qquad \text{(by (7))}\\
= \;& -4\int_D \eta\, g^{ip}g^{jq}g^{kr}\nabla_i u^{(\nu)}\,\nabla_k u^{(\nu)}\,\nabla_r\nabla_j u^{(\nu)}\,\nabla_p\nabla_q u^{(\nu)}\\
&+2\int_D F(\varphi^{(\nu)},\, u^{(\nu)})\, g^{kr}\nabla_k(\eta\,\nabla_r u^{(\nu)})\\
&-2\int_D \eta\, g^{ip}g^{jq}g^{kr}g^{ls}R_{prqs}\,\nabla_i u^{(\nu)}\,\nabla_j u^{(\nu)}\,\nabla_k u^{(\nu)}\,\nabla_l u^{(\nu)}\\
&+2\int_D \eta\, g^{ip}g^{jq}g^{kr}\nabla_i u^{(\nu)}\,\nabla_j u^{(\nu)}\,\nabla_q\nabla_r u^{(\nu)}\,\nabla_p\nabla_k u^{(\nu)} \qquad \text{(by (14))}\\
= \;& -2\int_D \eta\, g^{ip}g^{jq}g^{kr}g^{ls}R_{pqrs}\,\nabla_i u^{(\nu)}\,\nabla_j u^{(\nu)}\,\nabla_k u^{(\nu)}\,\nabla_l u^{(\nu)}\\
&-2\int_D \eta\, g^{ip}g^{jq}g^{kr}\nabla_i u^{(\nu)}\,\nabla_j u^{(\nu)}\,\nabla_q\nabla_r u^{(\nu)}\,\nabla_p\nabla_k u^{(\nu)}\\
&+2\int_D F(\varphi^{(\nu)},\, u^{(\nu)})\, g^{kr}\nabla_k(\eta\,\nabla_r u^{(\nu)}) \qquad \text{(by exchange of indices)}\\
= \;& -2\int_D \eta\, g^{ip}g^{jq}g^{kr}g^{ls}R_{pqrs}\,\nabla_i u^{(\nu)}\,\nabla_j u^{(\nu)}\,\nabla_k u^{(\nu)}\,\nabla_l u^{(\nu)}\\
&-\frac{1}{2}\int_D \|\nabla\|\nabla u^{(\nu)}\|^2\,\|^2 + 2\int_D F(\varphi^{(\nu)},\, u^{(\nu)})\, g^{kr}\nabla_k(\eta\,\nabla_r u^{(\nu)}) \qquad \text{(by (8))}\,.
\end{aligned}
$$

Therefore we obtain an integral form of the Bochner equality for $u^{(\nu)}$:

$$
\begin{aligned}
\int_M \eta\, g^{ip}g^{jq}\nabla_i u^{(\nu)}\,\nabla_j u^{(\nu)}\,\nabla_p\nabla_q\,\|\nabla u^{(\nu)}\|^2 \;+\; \frac{1}{2}\int_M \|\nabla\|\nabla u^{(\nu)}\|^2\,\|^2\,\eta &\\
-2\int_M F(\varphi^{(\nu)},\, u^{(\nu)})\, g^{kr}\nabla_k(\eta\,\nabla_r u^{(\nu)}) & \qquad (15)\\
+2\int_M \eta\, g^{ip}g^{jq}g^{kr}g^{ls}R_{ikjl}\,\nabla_p u^{(\nu)}\,\nabla_q u^{(\nu)}\,\nabla_r u^{(\nu)}\,\nabla_s u^{(\nu)} &\;=\; 0\,.
\end{aligned}
$$

for any $\eta \in C_0^\infty(D)$. Note here

$$
\begin{aligned}
g^{jq}\nabla_j u^{(\nu)}\,\nabla_q\|\nabla u^{(\nu)}\|^2 &= g^{jq}\nabla_j u^{(\nu)}\,\nabla_q(g^{ip}\nabla_i u^{(\nu)}\nabla_p u^{(\nu)}) \qquad (16)\\
&= 2\,g^{ip}g^{jq}\nabla_j u^{(\nu)}\,\nabla_i u^{(\nu)}\,\nabla_q\nabla_p u^{(\nu)}\\
&= -\,2\,F(\varphi^{(\nu)},\, u^{(\nu)}) \qquad \text{(by (13))}\,.
\end{aligned}
$$

Then using integration by parts, we get

$$
\begin{aligned}
&\int_D \eta\, g^{ip}g^{jq}\nabla_i u^{(\nu)}\,\nabla_j u^{(\nu)}\,\nabla_p\nabla_q\,\|\nabla u^{(\nu)}\|^2 \qquad (17)\\
&= \; -\int_D g^{ip}g^{jq}\nabla_p\eta\,\nabla_i u^{(\nu)}\,\nabla_j u^{(\nu)}\,\nabla_q\,\|\nabla u^{(\nu)}\|^2
\end{aligned}
$$

$$
\begin{aligned}
& - \int_D \eta\, g^{ip} g^{jq} \nabla_p \nabla_i u^{(\nu)}\, \nabla_j u^{(\nu)}\, \nabla_q \, \|\nabla u^{(\nu)}\|^2 \\
& - \int_D \eta\, g^{ip} g^{jq} \nabla_i u^{(\nu)}\, \nabla_p \nabla_j u^{(\nu)}\, \nabla_q \, \|\nabla u^{(\nu)}\|^2 \\
= \;& 2 \int_D g^{ip} \nabla_p \eta\, \nabla_i u^{(\nu)}\, F(\varphi^{(\nu)},\, u^{(\nu)}) = 2 \int_D \eta\, g^{ip} \nabla_p \nabla_i u^{(\nu)}\, F(\varphi^{(\nu)},\, u^{(\nu)}) \\
& - \int_D \eta\, g^{ip} g^{jq} \nabla_i u^{(\nu)}\, \nabla_p \nabla_j u^{(\nu)}\, \nabla_q \, \|\nabla u^{(\nu)}\|^2 \qquad \text{(by (16))} \\
= \;& 2 \int_D g^{ip} \nabla_i u^{(\nu)}\, \nabla_p \eta\, F(\varphi^{(\nu)},\, u^{(\nu)}) \;+\; 2 \int_D \eta \, \triangle \, u^{(\nu)}\, F(\varphi^{(\nu)},\, u^{(\nu)}) \\
& - \frac{1}{2} \int_D \|\nabla \|\nabla u^{(\nu)}\| \,\|^2\, \eta \,,
\end{aligned}
$$

because

$$
g^{ip} \nabla_i u^{(\nu)}\, \nabla_p \nabla_j u^{(\nu)} \;=\; \frac{1}{2}\, \nabla_j (g^{ip} \nabla_i u^{(\nu)}\, \nabla_p u^{(\nu)}) \;=\; \frac{1}{2}\, \nabla_j \|\nabla u^{(\nu)}\|^2 \,.
$$

Then by (15) and (17), we obtain

$$
\begin{aligned}
& 2 \int_D \eta\, g^{ip} g^{jq} g^{kr} g^{ls} R_{ijkl} \nabla_p u^{(\nu)}\, \nabla_q u^{(\nu)}\, \nabla_r u^{(\nu)}\, \nabla_s u^{(\nu)} \\
= \;& -2 \int_D g^{ip} \nabla_i u^{(\nu)}\, \nabla_p \eta\, F(\varphi^{(\nu)},\, u^{(\nu)}) \\
& -2 \int_D \eta \, \triangle \, u^{(\nu)}\, F(\varphi^{(\nu)},\, u^{(\nu)}) \\
& +2 \int_D g^{kr} \nabla_r (\eta\, \nabla_k u^{(\nu)})\, F(\varphi^{(\nu)},\, u^{(\nu)}) \,.
\end{aligned} \tag{18}
$$

Since $\|\nabla u^{(\nu)}\|$ is bounded uniformly on K, we get

$$
\begin{aligned}
& |\, \text{the right hand side of } (18)\, | \\
\leq \;& C \int_K \|F(\varphi^{(\nu)},\, u^{(\nu)})\| + C \int_K \|\nabla\nabla u^{(\nu)}\|\, \|F(\varphi^{(\nu)},\, u^{(\nu)})\| \\
\leq \;& C \int_K \|\nabla \varphi^{(\nu)}\|\, \|\nabla\nabla u^{(\nu)}\| + C \int_K \|\nabla\nabla \varphi^{(\nu)}\| \\
& + C \int_K \|\nabla \varphi^{(\nu)}\|^2\, \|\nabla\nabla u^{(\nu)}\| + C \int_K \|\nabla \varphi^{(\nu)}\|\, \|\nabla\nabla \varphi^{(\nu)}\| \\
& + C \int_K \|\nabla \varphi^{(\nu)}\|^2\, \|\nabla\nabla \varphi^{(\nu)}\| + C \int_K \|\nabla \varphi^{(\nu)}\|\, \|\nabla\nabla u^{(\nu)}\|^2 \\
& + C \int_K \|\nabla\nabla \varphi^{(\nu)}\|\, \|\nabla\nabla u^{(\nu)}\| + C \int_K \|\nabla \varphi^{(\nu)}\|^2\, \|\nabla\nabla u^{(\nu)}\|^2 \\
& + C \int_K \|\nabla \varphi^{(\nu)}\|\, \|\nabla\nabla \varphi^{(\nu)}\|\, \|\nabla\nabla u^{(\nu)}\|
\end{aligned}
$$

$$+ C \int_K \|\nabla\varphi^{(\nu)}\|^2 \, \|\nabla\nabla\varphi^{(\nu)}\| \, \|\nabla\nabla u^{(\nu)}\|$$
$$=: \; \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4 + \mathrm{I}_5 + \mathrm{I}_6 + \mathrm{I}_7 + \mathrm{I}_8 + \mathrm{I}_9 + \mathrm{I}_{10} \,.$$

Since $\varphi^{(\nu)}$ converges to in $\mathrm{W}^{2,2+\varepsilon}(K)$ as ν tends to infinity, $\nabla\varphi^{(\nu)}$ and $\nabla\nabla\varphi^{(\nu)}$ approaches zero in $\mathrm{L}^2(K)$. Then

$$\mathrm{I}_1 \le C \left\{ \int_K \|\nabla\varphi^{(\nu)}\|^2 \right\}^{1/2} \left\{ \int_K \|\nabla\nabla u^{(\nu)}\|^2 \right\}^{1/2} \to 0 \,,$$

$$\mathrm{I}_2 \le C \left\{ \int_K \|\nabla\nabla\varphi^{(\nu)}\|^2 \right\}^{1/2} \left\{ \int_K 1^2 \right\}^{1/2} \to 0 \,,$$

$$\mathrm{I}_3 \le C \sup_K \|\nabla\varphi^{(\nu)}\| \left\{ \int_K \|\nabla\varphi^{(\nu)}\|^2 \right\}^{1/2} \left\{ \int_K \|\nabla\nabla u^{(\nu)}\|^2 \right\}^{1/2} \to 0 \,,$$

$$\mathrm{I}_4 \le C \left\{ \int_K \|\nabla\varphi^{(\nu)}\|^2 \right\}^{1/2} \left\{ \int_K \|\nabla\nabla\varphi^{(\nu)}\|^2 \right\}^{1/2} \to 0 \,,$$

$$\mathrm{I}_5 \le C \sup_K \|\nabla\varphi^{(\nu)}\| \left\{ \int_K \|\nabla\varphi^{(\nu)}\|^2 \right\}^{1/2} \left\{ \int_K \|\nabla\nabla\varphi^{(\nu)}\|^2 \right\}^{1/2} \to 0 \,,$$

$$\mathrm{I}_7 \le C \left\{ \int_K \|\nabla\nabla\varphi^{(\nu)}\|^2 \right\}^{1/2} \left\{ \int_K \|\nabla\nabla u^{(\nu)}\|^2 \right\}^{1/2} \to 0 \,,$$

$$\mathrm{I}_9 \le C \sup_K \|\nabla\varphi^{(\nu)}\| \left\{ \int_K \|\nabla\nabla\varphi^{(\nu)}\|^2 \right\}^{1/2} \left\{ \int_K \|\nabla\nabla u^{(\nu)}\|^2 \right\}^{\frac{1}{2}} \to 0 \,,$$

$$\mathrm{I}_{10} \le C \sup_K \|\nabla\varphi^{(\nu)}\|^2 \left\{ \int_K \|\nabla\nabla\varphi^{(\nu)}\|^2 \right\}^{1/2} \left\{ \int_K \|\nabla\nabla u^{(\nu)}\|^2 \right\}^{1/2} \to 0 \,.$$

Furthermore, since $\varphi^{(\nu)}$ converges to zero in $\mathrm{W}^{2,2+\varepsilon}(K)$, we have

$$\mathrm{I}_6 \le C \sup_K \|\nabla\varphi^{(\nu)}\|^{1-\varepsilon} \left\{ \int_K \|\nabla\varphi^{(\nu)}\|^{2+\varepsilon} \right\}^{\varepsilon/(2+\varepsilon)} \left\{ \int_K \|\nabla\nabla u^{(\nu)}\|^{2+\varepsilon} \right\}^{2/(2+\varepsilon)} \to 0 \,,$$

$$\mathrm{I}_8 \le C \sup_K \|\nabla\varphi^{(\nu)}\|^{2-\varepsilon} \left\{ \int_K \|\nabla\varphi^{(\nu)}\|^{2+\varepsilon} \right\}^{\varepsilon/(2+\varepsilon)} \left\{ \int_K \|\nabla\nabla u^{(\nu)}\|^{2+\varepsilon} \right\}^{2/(2+\varepsilon)} \to 0 \,.$$

Thus the right hand side of (18) converges to zero as ν tends to infinity. Then, letting ν go to infinity in (18), we have (12). This completes the proof. □

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In this article, Theorem 1 follows from general properties of the Riemannian curvature tensor, and Corollary 1 is incorrect. The Bochner formula does not seem to work in this situation.

Lemma 1 can be used in proving the following Liouville theorem for C^3-solutions.

Theorem A. *Let M be a complete noncompact Riemannian manifold of non-negative (sectional) curvature. Let u be a bounded ∞-harmonic function of C^3-class on M. Then u is a constant function.*

The curvature assumption in Theorem A is necessary only for applying the Hessian comparison theorem in the proof (Here we use the operator $Q^{ij} = g^{ip}g^{jq}\nabla_p\nabla_q$). Theorem A also follows from arguments in [Cheng, S.Y., *Liouville theorem for harmonic maps*, Proc. Symp. Pure Math. 36(1980), 147-151]. See also the article [Hong, N.C., *Liouville theorems for exponentially harmonic functions on Riemannian manifolds*, Manuscripta Math. 77(1992), 41-46].

Sincerely yours,
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