Electronic Journal of Differential Equations, Vol. 2019 (2019), No. 81, pp. 1–17.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

NON-PERTURBATIVE WEAK HÖLDER CONTINUITY OF LYAPUNOV EXPONENT OF DISCRETE ANALYTIC JACOBI OPERATORS WITH SKEW-SHIFT MAPPING

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Abstract. In this article, we study the continuity of the Lyapunov exponent of discrete analytic Jacobi operators with the skew-shift mapping. We prove that the Lyapunov exponent is weak Hölder continuous in E for any Diophantine frequency in the large coupling regimes.

1. Introduction

In this article, we study the discrete analytic Jacobi operators on $l^2(\mathbb{Z})$,

$$\begin{aligned}(H_{\underline{x},\omega}\phi)(n) = &-a\big(\pi_1(T^n_{d,\omega}(\underline{x}))\big)\phi(n+1) - \bar{a}\left(\pi_1(T^n_{d,\omega}(\underline{x}))\right)\phi(n-1)\\ &+ \lambda v\left(\pi_1(T^n_{d,\omega}(\underline{x}))\right)\phi(n), \quad n\in\mathbb{Z},\end{aligned} \tag{1.1}$$

where $v:\mathbb{T}\to\mathbb{R}$ is a real analytic function called potential, $a:\mathbb{T}\to\mathbb{C}$ is a complex analytic function and not identically zero, λ is a real positive constant called coupling number, the transformation $T^n_{d,\omega}$ called shew-shift mapping is defined on $\mathbb{T}^d$:

$$T_{d,\omega}(\underline{x}=(x_1,\dots,x_d)) = (x_1+x_2, x_2+x_3,\dots,x_{d-1}+x_d, x_d+\omega), \tag{1.2}$$

and π_j is a projection from $\mathbb{T}^d$ to its j-th coordinate. Their characteristic equations $H_{\underline{x},\omega}\phi = E\phi$ can be expressed as

$$\begin{pmatrix}\phi(n+1)\\ \phi(n)\end{pmatrix} = M\left(T^n_{d,\omega}(\underline{x}),E\right)\begin{pmatrix}\phi(n)\\ \phi(n-1)\end{pmatrix},$$

where

$$M\left(T^n_{d,\omega}(\underline{x}),E\right) = \frac{1}{a(\pi_1(T^{n+1}_{d,\omega}(\underline{x})))}\begin{pmatrix}\lambda v\big(\pi_1(T^n_{d,\omega}(\underline{x}))\big)-E & -\bar{a}\big(\pi_1(T^n_{d,\omega}(\underline{x}))\big)\\ a\big(\pi_1(T^{n+1}_{d,\omega}(\underline{x}))\big) & 0\end{pmatrix}.$$

Then

$$\begin{pmatrix}\phi(n+1)\\ \phi(n)\end{pmatrix} = M_n(\underline{x},E,\omega)\begin{pmatrix}\phi(1)\\ \phi(0)\end{pmatrix},$$

where

$$M_n(\underline{x},E,\omega) = \prod_{j=n-1}^{0} M\left(T^j_{d,\omega}(\underline{x}),E\right) \tag{1.3}$$

2010 *Mathematics Subject Classification.* 37C55, 37F10.
Key words and phrases. Lyapunov exponent; weak Hölder continuity; Jacobi operator; skew-shift mapping.

Submitted May 10, 2018. Published June 12, 2019.

is called the transfer matrix of (1.1). Define

$$L_n(E,\omega) := \frac{1}{n}\int_{\mathbb{T}^d} \log \|M_n(\underline{x},E,\omega)\| d\underline{x}.$$

By the Kingman's subadditive ergodic theorem and the fact that transformation $T_{d,\omega}$ is ergodic on $\mathbb{T}^d$, we have

$$L(E,\omega) := \lim_{n\to\infty} L_n(E,\omega) = \lim_{n\to\infty} \frac{1}{n}\log \|M_n(\underline{x},E,\omega)\|$$

for almost all $\underline{x} \in \mathbb{T}^d$. Simple computations yield

$$\pi_1(T^n_{d,\omega}(\underline{x})) = s_d(n)\omega + s_{d-1}(n)x_d + s_{d-2}(n)x_{d-1} + \dots + s_1(n)x_2 + x_1, \tag{1.4}$$

where

$$s_j(n) = \sum_{x_j=1}^{n-j+1} \sum_{y_{j-1}=1}^{y_j} \cdots \sum_{y_1=1}^{y_2} 1 = \frac{1}{j!}\prod_{m=0}^{j-1}(n-m),\ j = 1,2,\dots,d.$$

Thus, (1.4) satisfies the definition of the transformation studied in [9]. By the results of that reference, the Lyapunov exponent is always positive in the large coupling regimes as follows: There exists $\lambda_0 = \lambda_0(v,a) > 0$ such that if the coupling number $|\lambda| > \lambda_0$, then for any irrational ω, we have

$$L(E,\omega) \ge c\log|\lambda| \quad \text{for all } E,$$

where c is a constant depending only on v and a.

In this article, we consider the continuity of the Lyapunov exponent in E for the Diophantine ω. When we say that $\omega \in (0,1)$ is Diophantine, it means that ω satisfies the Diophantine condition

$$\|n\omega\| \ge \frac{C_\omega}{n(\log n)^\alpha} \quad \text{for all } n \ne 0. \tag{1.5}$$

It is well known that for a fixed $\alpha > 1$, almost every ω satisfies (1.5). Note that if $d = 1$, then the transformation becomes a shift on $\mathbb{T}$:

$$T_{1,\omega}(x) = x + \omega. \tag{1.6}$$

The operators (1.1) with the transformation (1.6) is called analytic quasi-periodic Jacobi operators. In [8], we proved that the Lyapunov exponent of these operators is Hölder continuity in E for any Diophantine ω. But we do not think it is still true when $d \ge 2$. Here we prove the non-perturbative weak Hölder continuity of the Lyapunov exponent as follows.

Theorem 1.1. *Assume $d \ge 2$. There exists $\lambda_0 = \lambda_0(v,a)$ such that if $\lambda > \lambda_0$ then for any Diophantine ω,*

$$|L(E,\omega) - L(E',\omega)| < C_1 \exp\big(-c_2|\log|E-E'||^{1/30}\big), \quad \text{when } E' \to E,$$

where $C_1 = C_1(\lambda)$ and $c_2 = c_2(\lambda,v,a)$.

In [2], the authors also obtained the weak Hölder continuity of the Lyapunov exponent of the following analytic Schrödinger equations with the same mapping:

$$(J_{\underline{x},\omega,\lambda v}\phi)(n) = \phi(n+1) + \phi(n-1) + \lambda v(T^n_{d,\omega}(\underline{x}))\phi(n) = E\phi(n),$$

where $v(\underline{x})$ is an analytic function on $\mathbb{T}^d$. But their result is perturbative: there exist a small constant $\delta > 0$, a large constant $\lambda_0' = \lambda_0'(v,\delta)$ and a set $\Omega_\delta \subset \mathbb{T}$ satisfying $\operatorname{meas}(\mathbb{T}\backslash\Omega_\delta) < \delta$ such that for any $\omega \in \Omega_\delta$ and $\lambda > \lambda_0'$, the continuity of the Lyapunov exponent holds. The first non-perturbative result for the discrete analytic operators is [10]. In that reference, we first proved the positive Lyapunov exponent of the discrete Schrödinger equations with analytic potentials given by a class of transformations as follows:

$$(S_{(x,y),\lambda v}\phi)(n)=\phi(n+1)+\phi(n-1)+\lambda v(\pi_{\mathbb{T}}(T^n(x,y)))\phi(n)=E\phi(n), \tag{1.7}$$

where $v(x)$ is a real analytic function on $\mathbb{T}$ and $\pi_{\mathbb{T}}$ is a projection from $\mathbb{Y}\times\mathbb{T}$ to $\mathbb{T}$. Here we denote by $(\mathbb{Y},\mathscr{B},m)$ the probability space and $\mathbb{T}:=\mathbb{R}/\mathbb{Z}$ the torus equipped with its Haar measure, and let the measure preserving transformation $T:\mathbb{T}\times\mathbb{Y}\to\mathbb{T}\times\mathbb{Y}$ have the form

$$T(x,y)=(x+f(y),g(y))\,, \tag{1.8}$$

where $g:\mathbb{Y}\to\mathbb{Y}$ and $f:\mathbb{Y}\to\mathbb{T}$. Choosing $\mathbb{Y}=\mathbb{T}^{d-1}$, $g(\underline{y})=g(y_1,y_2,\dots,y_{d-1})=(y_1+y_2,y_2+y_3,\dots,y_{d-1}+\omega)$ and $f(\underline{y})=f(y_1,y_2,\dots,y_{d-1})=y_1$ makes the transformation (1.8) become to be our skew-shift (1.2). Then we obtained the weak Hölder continuity of the Lyapunov exponent for the equations (1.7). Therefore, the highlight of our paper is that we extend the non-perturbative continuity of the Lyapunov exponent from the discrete analytic Schrödinger operators (1.7) to our discrete analytic Jacobi operators (1.1).

We organize this article as follows. Some definitions and tools are presented in Section 2, which help us obtain the non-perturbative large deviation theorem in Section 3. Applying it with the avalanche principle, we prove the main theorem in the last section.

2. Preliminaries

For simplicity, we assume $d=2$. Then

$$\pi_1(T^n_\omega(x,y))=x+ny+\frac{n(n-1)}{2}\omega,$$

$$\begin{aligned}M_n(x,y,E,\omega)=&\prod_{j=n-1}^{0}\Big[\frac{1}{a\big(x+(j+1)y+\frac{j(j+1)}{2}\omega\big)}\\&\times\begin{pmatrix}\lambda v(x+jy+\frac{j(j-1)}{2}\omega)-E & -\bar{a}\big(x+jy+\frac{j(j-1)}{2}\omega\big)\\ a\big(x+(j+1)y+\frac{j(j+1)}{2}\omega\big) & 0\end{pmatrix}\Big],\end{aligned}$$

$$L_n(E,\omega)=\frac{1}{n}\iint_{\mathbb{T}^2}\log\|M_n(x,y,E,\omega)\|\,dx\,dy$$

$$L(E,\omega)=\lim_{n\to\infty}\frac{1}{n}\iint_{\mathbb{T}^2}\log\|M_n(x,y,E,\omega)\|\,dx\,dy.$$

The methods we use are complex analytic, so we will work with an extension of the operator to a neighbourhood of the real line. We will use the notation

$$\mathbb{H}_y=\{z\in\mathbb{C}:|\Im z|\le y\}.$$

It is known that $v(x)$, $a(x)$ and $\bar{a}(x)$ admit complex analytic extensions $v(z)$, $a(z)$ and $\tilde{a}(z)$ to $\mathbb{H}_\rho$ with $\rho=\rho(v,a)$. Note that the complex analytic extension of $\bar{a}(x)$ should be defined by

$$\tilde{a}(z):=\overline{a(\frac{1}{z})}.$$

Then, the extension of $M_n(x,y,E,\omega)$ is

$$M_n(z,y,E,\omega)=\prod_{j=n-1}^{0}\Big[\frac{1}{a\big(z+(j+1)y+\frac{j(j+1)}{2}\omega\big)} \times\begin{pmatrix}\lambda v\big(z+jy+\frac{j(j-1)}{2}\omega\big)-E & -\tilde a\big(z+jy+\frac{j(j-1)}{2}\omega\big)\\ a\big(z+(j+1)y+\frac{j(j+1)}{2}\omega\big) & 0\end{pmatrix}\Big]. \tag{2.1}$$

In the proof of the main theorem, we will in fact work with the following two matrices associated with M_n:

$$\begin{aligned}M_n^a(z,y,E,\omega)&=\Big(\prod_{j=0}^{n-1}a\big(z+(j+1)y+\frac{j(j+1)}{2}\omega\big)\Big)M_n(z,y,E,\omega)\\&=\prod_{j=n-1}^{0}\begin{pmatrix}\lambda v(z+jy+\frac{j(j-1)}{2}\omega)-E & -\tilde a(z+jy+\frac{j(j-1)}{2}\omega)\\ a\big(z+(j+1)y+\frac{j(j+1)}{2}\omega\big) & 0\end{pmatrix}\end{aligned} \tag{2.2}$$

and

$$M_n^u(z,y,E,\omega)=\frac{M_n(z,y,E,\omega)}{|\det M_n(z,y,E,\omega)|^{1/2}}=\frac{M_n(z,y,E,\omega)}{\big|\prod_{j=0}^{n-1}\frac{\tilde a\big(z+jy+\frac{j(j-1)}{2}\omega\big)}{a\big(z+(j+1)y+\frac{j(j+1)}{2}\omega\big)}\big|^{1/2}}. \tag{2.3}$$

Based on the definitions, it is straightforward to check that

$$\begin{aligned}\log\|M_n(z,y,E,\omega)\|&=\log\|M_n^a(z,y,E,\omega)\|-S_n(z+y,y+\omega,\omega)\\&=\log\|M_n^u(z,y,E,\omega)\|\\&\quad+\frac{1}{2}\left(\tilde S_n(z,y,\omega)-S_n(z+y,y+\omega,\omega)\right),\end{aligned} \tag{2.4}$$

where

$$S_n(z,y\omega)=\sum_{j=0}^{n-1}\log\big|a\big(z+jy+\frac{j(j-1)}{2}\omega\big)\big|, \tag{2.5}$$

$$\tilde S_n(z,y,\omega)=\sum_{j=0}^{n-1}\log\big|\tilde a\big(z+jy+\frac{j(j-1)}{2}\omega\big)\big|. \tag{2.6}$$

Note that $S_n(x,y,E,\omega)=\tilde S_n(x,y,E,\omega)$ when $x\in\mathbb{T}$. We also consider the quantities $L^a(E,\omega)$, $L_n^a(E,\omega)$, $L^u(E,\omega)$ and $L_n^u(E,\omega)$ which are defined analogously. Furthermore, let

$$D=\int_{\mathbb{T}}\log|a(x)|dx.$$

From (2.4) and (2.4), it follows that

$$L(E,\omega)=L^u(E,\omega)=L^a(E,\omega)-D. \tag{2.7}$$

It is well known that $L(E,\omega)$ is C^∞ function on the resolvent set with fixed ω. So we only need to consider $E\in\mathscr{E}$, where

$$\mathscr{E}:=[-2\|a(x)\|_{L^\infty(\mathbb{T})}-\lambda\|v(x)\|_{L^\infty(\mathbb{T})},\ 2\|a(x)\|_{L^\infty(\mathbb{T})}+\lambda\|v(x)\|_{L^\infty(\mathbb{T})}].$$

Simple computations yield that for any irrational ω and $1\le n\in\mathbb{N}$,

$$\sup_{E\in\mathscr{E},x\in\mathbb{H}_\rho}\|M_n^a(x,y,E,\omega)\|\le M_\rho^n, \tag{2.8}$$

where

$$M_\rho := \big(3\|a\|_{L^\infty(\mathbb{H}_\rho)} + 2\lambda\|v\|_{L^\infty(\mathbb{H}_\rho)}\big)\,.$$

The following three lemmas, all from the same reference, are essential to the successful proof of the non-perturbative large deviation theorem in Section 3.

Lemma 2.1 (Weyl-differencing, [5, Lemma 12])**.** *Let $f(x)$ be a polynomial of degree $d \geq 2$:*

$$f(x) = a_0 + a_1 x + \dots + a_d x^d.$$

Then for any $k \geq 1$, it has

$$\Big|\sum_{x=1}^{P} e^{if(x)}\Big|^{2^k} \leq 2^{2^k-1} P^{2^k-(k+1)} \sum_{y_1=0}^{P_1-1}\dots\sum_{y_k=0}^{P_k-1}\Big|\sum_{x=1}^{P_{k+1}} e^{i\triangle_{y_1,\dots,y_k} f(x)}\Big|,$$

where $P_1 = P$ and under $\nu = 1,2\dots,k$, quantities $P_{\nu+1}$ are determined by the equality $P_{\nu+1} = P_\nu - y_\nu$. Here $\triangle_{y_1} f(x)$ denotes the finite difference of a function $f(x)$ with an integer $y_1 > 0$:

$$\triangle_{y_1} f(x) = f(x+y_1) - f(x),$$

and when $k \geq 1$, the finite difference of the k-th order $\triangle_{y_1,\dots,y_k} f(x)$ is determined with the help of the equality

$$\triangle_{y_1,\dots,y_k} f(x) = \triangle_{y_k}[\triangle_{y_1,\dots,y_{k-1}} f(x)].$$

Lemma 2.2 (cite[Lemma 13]K)**.** *Let M and $m_1, m_2, \dots, m_n$ be positive integers. Denote by $\tau_n(M)$ the number of solutions of the equation $m_1 \dots m_n = M$. Then under any $\epsilon (0 < \epsilon \leq 1)$ we have*

$$\tau_n(M) \leq C_n(\epsilon) M^\epsilon,$$

where the constant $C_n(\epsilon)$ depends on n and ϵ only.

Lemma 2.3 ([5, Lemma 14])**.** *Let $P \geq 2$ and*

$$\omega = \frac{p}{q} + \frac{\theta}{q^2}, \quad (p,q) = 1,\ |\theta| \leq 1.$$

Then under any positive integer Q and an arbitrary real β we have

$$\sum_{x=1}^{Q} \min\big(P, \frac{1}{\|\omega x + \beta\|}\big) \leq 4\big(1 + \frac{Q}{q}\big)(P + q\log P).$$

In this article, the reason that we need the functions $a(x)$, $\bar{a}(x)$ and $v(x)$ to be analytic and to define the complex analytic matrix $M_n^a(z,y,E,\omega)$ is to obtain the subharmonic function

$$u_n^a(z,y,E,\omega) = \frac{1}{n}\log\|M_n^a(z,y,E,\omega)\|$$

with fixed $y \in \mathbb{T}, E \in \mathcal{E}$ and irrational ω. The following lemmas show that the subharmonicity is the key to our paper.

Lemma 2.4 (Riesz's representation theorem [4, Lemma 2.1])**.** *Let $u : \Omega \to \mathbb{R}$ be a subharmonic function on a domain $\Omega \subset \mathbb{C}$. Suppose that $\partial\Omega$ consists of finitely many piece-wise C^1 curves. There exists a positive measure μ on Ω such that for any $\Omega_1 \Subset \Omega$ (i.e., Ω_1 is a compactly contained subregion of Ω),*

$$u(z) = \int_{\Omega_1} \log|z-\zeta|\, d\mu(\zeta) + h(z),$$

where h is harmonic on Ω_1 and μ is unique with this property. Moreover, μ and h satisfy the bounds

$$\mu(\Omega_1) \le C(\Omega,\Omega_1)\,(\sup_{\Omega} u - \sup_{\Omega_1} u),$$
$$\|h - \sup_{\Omega_1} u\|_{L^\infty(\Omega_2)} \le C(\Omega,\Omega_1,\Omega_2)\,(\sup_{\Omega} u - \sup_{\Omega_1} u)$$

for any $\Omega_2 \Subset \Omega_1$.

Lemma 2.5 ([1, Corollary 4.7])**.** *Suppose u is subharmonic on $\mathbb{H}_\rho$. Then there exist a constant $C_\rho := C_\rho(\rho)$ such that the Fourier coefficients of u satisfy*

$$|\hat{u}(k)| \le \frac{C_\rho[\mu(\mathbb{H}_{\rho/2}) + \|h\|_{L^\infty(\mathbb{H}_{\frac{\rho}{4}})}]}{|k|}, \quad \forall k \neq 0,$$

where μ and h come from Lemma 2.4.

Lemma 2.6 (BMO Bound [2, Lemma 2.3])**.** *Suppose u is subharmonic on $\mathbb{H}_\rho$ with $\mu(\mathbb{H}_{\rho/2}) + \|h\|_{L^\infty(\mathbb{H}_{\frac{\rho}{4}})} \le C_\rho n$. Furthermore, assume that $u = u_0 + u_1$, where*

$$\|u_0 - <u_0>\|_{L^\infty(\mathbb{T})} \le \epsilon_0 \quad \text{and} \quad \|u_1\|_{L^1(\mathbb{T})} \le \epsilon_1. \tag{2.9}$$

Then for some constant $\tilde{C}_\rho$ depending only on ρ,

$$\|u\|_{BMO(\mathbb{T})} \le \tilde{C}_\rho\Big(\epsilon_0 \log\big(\frac{n}{\epsilon_1}\big) + \sqrt{n\epsilon_1}\Big).$$

3. Non-perturbative large deviation theorem

Recall that with fixed y, E and ω,

$$u_n^a(z,y,E,\omega) = \frac{1}{n}\log\|M_n^a(z,y,E,\omega)\|$$

is a subharmonic function on $z \in \mathbb{H}_\rho$ with the upper bound

$$\sup u_n^a(z) \le \log\big(3\|a\|_{L^\infty(\mathbb{H}_\rho)} + 2\lambda\|v\|_{L^\infty(\mathbb{H}_\rho)}\big).$$

Define

$$L_n^a(y,E,\omega) = \frac{1}{n}\int_{\mathbb{T}} \log\|M_n^a(x,y,E,\omega)\|dx.$$

If

$$\lambda > \lambda_1 := \max\Big\{\frac{\|a\|_{L^\infty(\mathbb{H}_\rho)}}{\|v\|_{L^\infty(\mathbb{H}_\rho)}}, \exp\big(\frac{5\|v\|_{L^\infty(\mathbb{H}_\rho)}}{\kappa}\big)\Big\},$$

then by the definitions and the subadditivity, it implies that for any $E \in \mathscr{E}$, irrational ω and $n \ge 1$,

$$L^a(E,\omega) \le L_n^a(E,\omega) \le \log\big(5\lambda\|v\|_{L^\infty(\mathbb{H}_\rho)}\big) \le (1+\kappa)\log\lambda, \tag{3.1}$$

and for any $y \in \mathbb{T}$,

$$L^a(y,E,\omega) \le L_n^a(y,E,\omega) \le \log\big(5\lambda\|v\|_{L^\infty(\mathbb{H}_\rho)}\big) \le (1+\kappa)\log\lambda. \tag{3.2}$$

By [9, Lemma 1.2], we have that there exists λ_2 such that if $\lambda > \lambda_2$, then

$$L^a(E,\omega) \ge (1-\kappa)\log\lambda \quad \text{and} \quad L^a(y,E,\omega) \ge (1-\kappa)\log\lambda.$$

Above all, if $\lambda > \lambda_0(\kappa) := \max\{\lambda_1,\lambda_2\}$, then

$$(1-\kappa)\log\lambda \le L^a(E,\omega) \le L_n^a(E,\omega) \le (1+\kappa)\log\lambda, \tag{3.3}$$
$$(1-\kappa)\log\lambda \le L^a(y,E,\omega) \le L_n^a(y,E,\omega) \le (1+\kappa)\log\lambda. \tag{3.4}$$

From now on, we assume that $\lambda > \lambda_0(\kappa)$. The constant κ will be chosen later. Let $\Omega = \mathbb{H}_\rho$, $\Omega_1 = \mathbb{H}_{\rho/2}$ and $\Omega_2 = \mathbb{H}_{\frac{\rho}{4}}$ in Lemma 2.4. Due to this lemma and (3.1), there exists $C_1 = C_1(\lambda, v, a, \rho)$ such that for any $y \in \mathbb{T}$, $E \in \mathscr{E}$ and irrational ω,

$$\mu(\mathbb{H}_{\rho/2}) + \|h\|_{L^\infty(\mathbb{H}_{\frac{\rho}{4}})} \le C_1,$$

where μ and h come from the Riesz's representation of $u_n(z)$. By Lemma 2.5, it implies that

$$|\hat{u}_n^a(k, y, E, \omega)| \le \frac{C}{k}, \quad \forall k \ne 0, \tag{3.5}$$

where the constant $C = C_\rho C_1$ does not depend on y, E or ω. Note that

$$u_n^a(T_\omega^j(x, y), E, \omega) = L_n^a(y, E, \omega) + \sum_{k\ne 0} \hat{u}(k, y, E, \omega) e^{ik(x+jy+\frac{j(j-1)}{2}\omega)}.$$

Then

$$\begin{aligned}
&\Big|\frac{1}{N}\sum_{j=1}^{N} u_n^a(T_\omega^j(x, y), E, \omega) - L_n(y, E, \omega)\Big| \\
&= \Big|\frac{1}{N}\sum_{j=1}^{N}\sum_{k\in\mathbb{Z}\setminus\{0\}} \hat{u}_n^a(k, y, E, \omega) e^{ik(x+jy+\frac{j(j-1)}{2}\omega)}\Big| \\
&\le \frac{1}{N}\Big|\sum_{0<|k|\le K} \hat{u}_n^a(k, y, E, \omega)\sum_{j=1}^{N} e^{ik(jy+\frac{j(j-1)}{2}\omega)}\Big| \\
&\quad + \frac{1}{N}\Big|\sum_{|k|>K} \hat{u}_n^a(k, y, E, \omega)\sum_{j=1}^{N} e^{ik(jy+\frac{j(j-1)}{2}\omega)}\Big| := (a) + (b)
\end{aligned}$$

From (3.5) we have

$$\|(b)\|_2^2 \le \sum_{|k|>K} |\hat{u}_n^a(k, y, E, \omega)|^2 \le C^2 K^{-1}.$$

To estimate (a), recalling Lemma 2.1, we have

$$\begin{aligned}
\Big|\sum_{j=1}^{N} e^{ik(jy+\frac{j(j-1)}{2}\omega)}\Big|^2 &\le 2N + 2\sum_{m=1}^{N-1} \min\left(N-m, 2\|km\omega\|^{-1}\right) \\
&\le C_2 \sum_{m=1}^{N-1} \min\left(N, 2\|km\omega\|^{-1}\right).
\end{aligned}$$

And from the Cauchy inequality,

$$\begin{aligned}
|(a)|^2 &\le N^{-2}\sum_{0<|k|\le K} |\hat{u}_n^a(k, y, E, \omega)|^2 \sum_{0<|k|\le K}\Big(\sum_{j=1}^{N} e^{ik(jy+\frac{j(j-1)}{2}\omega)}\Big)^2 \\
&\le C_3 N^{-2}\sum_{k=1}^{K}\sum_{m=1}^{N-1} \min\left(N, 2\|km\omega\|^{-1}\right) \qquad (3.6) \\
&\le C_\epsilon N^{-2}(KN)^\epsilon \sum_{k=1}^{KN} \min\left(N, 2\|k\omega\|^{-1}\right),
\end{aligned}$$

where the arbitrary small positive parameter $\epsilon > 0$ and the inequality (3.6) come from Lemma 2.2. By Dirichlet's principle there is an integer $1 \le q \le N$ and an integer p so that $gcd(p,q) = 1$ and $\left|\omega - \frac{p}{q}\right| \le \frac{1}{qN}$. Thus from the definition of Diophantine number, one has

$$N \ge q \ge c_\omega \frac{N}{(\log N)^2}.$$

Combining it with Lemma 2.3, we have

$$\begin{aligned}
|(a)|^2 &\le C_\epsilon N^{-2}(KN)^\epsilon \sum_{k=1}^{KN} \min\big(N, 2\|k\omega\|^{-1}\big) \\
&\le 4C_\epsilon N^{-2}(KN)^\epsilon \big(1 + \frac{KN}{q}\big)(N + q\log N) \\
&\le 4C_\epsilon N^{-2}(KN)^\epsilon \Big(N + N\log N + \frac{1}{c_\omega} KN(\log N)^2 + KN\log N\Big) \\
&\le N^{2\epsilon-1} K^{1+2\epsilon},
\end{aligned}$$

for any $N > N_0(\epsilon, \omega)$. Let $K = n^{1/5}$, $N \ge n^{1/3}$ and $\epsilon = 1/16$. Then for any fixed $y \in \mathbb{T}$, $E \in \mathcal{E}$ and Diophantine ω,

$$\begin{gathered}
\operatorname{meas}(\mathscr{B}) \le n^{-2/15}, \\
\Big(\mathscr{B} := \big\{x : \big|\frac{1}{N}\sum_{j=1}^{N} u_n^a(T_\omega^j(x,y), E, \omega) - L_n^a(y, E, \omega)\big| > 2n^{-1/15}\big\}\Big).
\end{gathered} \tag{3.7}$$

Note that the sum of subharmonic functions is also subharmonic. Thus, with fixed y, E and ω, $u(z) := \frac{n}{N}\sum_{j=1}^{N} u_n^a(T_\omega^j(z,y), E, \omega)$ is a subharmonic function with $\mu(\mathbb{H}_{\rho/2}) + \|h\|_{L^\infty(\mathbb{H}_{\frac{\rho}{4}})} \le C_\rho n$. Let

$$u(x) - \langle u(\cdot)\rangle = u_0 + u_1,$$

where $u_0 = 0$ on $\mathscr{B}$ and $u_1 = 0$ on $\mathbb{T}\backslash\mathscr{B}$. Then

$$\|u_0 - \langle u_0\rangle\|_{L^\infty(\mathbb{T})} \le 2n^{14/15} \quad \text{and} \quad \|u_1\|_{L^1(\mathbb{T})} \le n^{13/15}.$$

Applying Lemma 2.6, we have

$$\|u\|_{BMO(\mathbb{T})} \le \tilde{C}_\rho n^{29/30}.$$

Then, compared to (3.7), the well-known John-Nirenberg inequality [7] will give a better deviation estimation as follows: Let f be a function of bounded mean oscillation on $\mathbb{T}$. Then there exist the absolute constants C and c such that for any $\gamma > 0$,

$$\operatorname{meas}\{x \in \mathbb{T} : |f(x) - \langle f\rangle| > \gamma\} \le C\exp\big(-\frac{c\gamma}{\|u\|_{BMO}}\big). \tag{3.8}$$

Applying (3.8) to $u(z)$, we have

$$\operatorname{meas}\{x \in \mathbb{T} : |u(x) - \langle u\rangle| > \gamma\} \le C\exp\Big(-\frac{c\gamma}{C_\rho n^{29/30}}\Big).$$

Choose $\gamma = \frac{1}{40} n \log\lambda$. Then, for any y, E and ω,

$$\begin{aligned}&\operatorname{meas}\big\{x\in\mathbb{T} : |\frac{1}{N}\sum_{j=1}^{N} u_n^a(T_\omega^j(x,y),E,\omega) - L_n^a(y,E,\omega)| > \frac{1}{40}\log\lambda\big\}\\ &\le C\exp\big(-c_\rho n^{1/30}\log\lambda\big).\end{aligned}\tag{3.9}$$

Now we estimate $\frac{1}{N}\sum_{j=1}^{N} u_n^a(T_\omega^j(x,y),E,\omega) - u_n^a(x,y,E,\omega)$. Recall that

$$\begin{aligned}M^a(z,y,E,\omega) &= M^a\left(T_\omega^0(z,y),E,\omega\right)\\ &= \begin{pmatrix}\lambda v\left(\pi_1(T_\omega^0(z,y))\right)-E & -\tilde a\left(\pi_1(T_\omega^0(z,y))\right)\\ a\left(\pi_1(T_\omega(z,y))\right) & 0\end{pmatrix},\end{aligned}$$

and define

$$M_{[m,n]}^a(z,y,E,\omega) = M_{[m,n]}^a\left(T_\omega^0(z,y),E,\omega\right) = \prod_{j=n-1}^{m-1} M^a\left(T_\omega^j(z,y),E,\omega\right).$$

It is easily seen that $M_n^a(z,y,E,\omega) = M_{[1,n]}^a\left(T_\omega^0(z,y),E,\omega\right)$. From (2.8) and the definition of λ_0, for any y, E and ω, we have

$$\sup_{z\in A_{\rho/2}} \|M^a(z,y,E,\omega)\| < \lambda^{1+\kappa}\tag{3.10}$$

and

$$\begin{aligned}&\sup_{z\in A_{\rho/2}} \|\left(M^a\right)^{-1}(z,y,E,\omega)\|\\ &= \sup_{z\in A_{\rho/2}} \Big\|\frac{1}{\tilde a\left(\pi_1(T_\omega^0(z,y))\right)a\left(\pi_1(T_\omega(z,y))\right)}\\ &\quad\times\begin{pmatrix}0 & -\tilde a\left(\pi_1(T_\omega^0(z,y))\right)\\ a\left(\pi_1(T_\omega(z,y))\right) & \lambda v\left(\pi_1(T_\omega^0(z,y))\right)-E\end{pmatrix}\Big\|\\ &\le \frac{\lambda^{1+\kappa}}{|\tilde a\left(\pi_1(T_\omega^0(z,y))\right)a\left(\pi_1(T_\omega(z,y))\right)|}.\end{aligned}\tag{3.11}$$

Thus, from (3.10), (3.11), the definition of $M_{[m,n]}^a\left(T_\omega^0(z,y),E,\omega\right)$ and

$$\begin{aligned}&\prod_{j=n}^{1} M^a\left(T_\omega^j(z,y)),E,\omega\right)\\ &= M^a\left(T_\omega^n(z,y),E,\omega\right)\prod_{j=n-1}^{0} M^a\left(T_\omega^j(z,y),E,\omega\right)\cdot\left(M^a\right)^{-1}\left(T_\omega^0(z,y),E,\omega\right),\end{aligned}$$

we have

$$\begin{aligned}\log\|M_n^a\left(T_\omega(z,y),E,\omega\right)\| &\le 2(1+\kappa)\log\lambda + \log\|M_n^a\left(T_\omega^0(z,y),E,\omega\right)\|\\ &\quad - \log|\tilde a\left(\pi_1(T_\omega^0(z,y))\right)a\left(\pi_1(T_\omega(z,y))\right)|.\end{aligned}$$

Similarly,

$$\begin{aligned}\log\|M_n^a\left(T_\omega^0(z,y),E,\omega\right)\| &\le 2(1+\kappa)\log\lambda + \log\|M_n^a\left(T_\omega(z,y),E,\omega\right)\|\\ &\quad - \log|\tilde a\left(\pi_1(T_\omega^{n-1}(z,y))\right)a\left(\pi_1(T_\omega^n(z,y))\right)|.\end{aligned}$$

Therefore,

$$-2(1+\kappa)\log\lambda + \log|\tilde a\left(\pi_1(T_\omega^0(z,y))\right)a\left(\pi_1(T_\omega(z,y))\right)|$$

$$\begin{aligned}&\le \log\|M_n^a\left(T_\omega^0(z,y),E,\omega\right)\| - \log\|M_n^a\left(T_\omega(z,y),E,\omega\right)\|\\ &\le 2(1+\kappa)\log\lambda - \log|\tilde a\left(\pi_1(T_\omega^{n-1}(z,y))\right)a\left(\pi_1(T_\omega^{n}(z,y))\right)|.\end{aligned}$$

It is obvious that for any $k\ge 1$,

$$\begin{aligned}&-2k(1+\kappa)\log\lambda + \sum_{m=0}^{k-1}\log|\tilde a\left(\pi_1(T_\omega^{m}(z,y))\right)a\left(\pi_1(T_\omega^{m+1}(z,y))\right)|\\ &\le \log\|M_n^a\left(T_\omega^0(z,y),E,\omega\right)\| - \log\|M_n^a\left(T_\omega^k(z,y),E,\omega\right)\|\\ &\le 2k(1+\kappa)\log\lambda - \sum_{m=0}^{k-1}\log|\tilde a\left(\pi_1(T_\omega^{n+m-1}(z,y))\right)a\left(\pi_1(T_\omega^{n+m}(z,y))\right)|.\end{aligned}$$

It implies that

$$\begin{aligned}&-\frac{2k(1+\kappa)\log\lambda}{Nn} + \sum_{m=0}^{k-1}\frac{1}{Nn}\log|\tilde a\left(\pi_1(T_\omega^{m}(z,y))\right)a\left(\pi_1(T_\omega^{m+1}(z,y))\right)|\\ &\le \frac{1}{N}u_n^a\left(T_\omega^0(z,y),E,\omega\right) - \frac{1}{N}u_n^a\left(T_\omega^k(z,y),E,\omega\right)\\ &\le \frac{2k(1+\kappa)\log\lambda}{Nn} - \sum_{m=0}^{k-1}\frac{1}{Nn}\log|\tilde a\left(\pi_1(T_\omega^{n+m-1}(z,y))\right)a\left(\pi_1(T_\omega^{n+m}(z,y))\right)|,\end{aligned}$$

and

$$\begin{aligned}&-\frac{2N(1+\kappa)\log\lambda}{n} + \sum_{m=0}^{N-1}\frac{N-m}{Nn}\log|\tilde a\left(\pi_1(T_\omega^{m}(x,y))\right)a\left(\pi_1(T_\omega^{m+1}(x,y))\right)|\\ &\le u_n^a(x,y,E,\omega) - \frac{1}{N}\sum_{j=1}^{N}u_n^a\left(T_\omega^j(x,y),E,\omega\right) \qquad (3.12)\\ &\le \frac{2N(1+\kappa)\log\lambda}{n} - \sum_{m=0}^{k-1}\frac{N-m}{Nn}\log|\tilde a\left(\pi_1(T_\omega^{n+m-1}(x,y))\right)a\left(\pi_1(T_\omega^{n+m}(x,y))\right)|.\end{aligned}$$

Recall the Lojasiewicz inequality ([6]): For any analytic function $f(x)$, there exists $\alpha=\alpha(f)$ such that for any $\delta>0$,

$$\operatorname{meas}\{x\in\mathbb{T}:|f(x)|<\delta\}<\delta^\alpha. \qquad (3.13)$$

Define

$$d_j(x,y,\omega)=\det M^a\left(T_\omega^j(x,y),E,\omega\right)=\tilde a\left(\pi_1(T_\omega^j(x,y))\right)a\left(\pi_1(T_\omega^{j+1}(x,y))\right).$$

It is obvious that for any $j\ge 0$, $d_j(x,y,\omega)$ is an analytic function with fixed y and ω. Moreover, the constant $\alpha_j=\alpha_j(d_j)$ in the inequality (3.13) only depends on $\tilde a(x)$ and $a(x)$, not on j, which can be denoted by α_d. Thus, for fixed y and ω,

$$\operatorname{meas}\left(\mathcal{B}_j:=\left\{x\in\mathbb{T}:|d_j(x,y,\omega)|<\exp(-n^\delta)\right\}\right)<\exp(-\alpha_d n^\delta).$$

Thus, if $x\notin\bigcup_{j=0}^N\mathcal{B}_j$, then

$$\sum_{m=0}^{N-1}\frac{N-m}{Nn}\log|\tilde a\left(\pi_1(T_\omega^{m}(x,y))\right)a\left(\pi_1(T_\omega^{m+1}(x,y))\right)|>-\frac{N+1}{2}n^{\delta-1}.$$

Similarly, if $x \notin \bigcup_{j=n}^{n+N-1} \mathcal{B}_j$, then

$$-\sum_{m=0}^{k-1} \frac{N-m}{Nn} \log |\tilde{a}\left(\pi_1(T_\omega^{n+m-1}(x,y))\right) a\left(\pi_1(T_\omega^{n+m}(x,y))\right)| < \frac{N+1}{2} n^{\delta-1}.$$

Combining this with (3.12), we have that there exists a set $\mathcal{B}'$ satisfying $\operatorname{meas}(\mathcal{B}') < 2N \exp(-\alpha_d n^\delta)$ such that if $x \notin \mathcal{B}'$, then

$$\big|u_n^a(x,y,E,\omega) - \frac{1}{N}\sum_{j=1}^{N} u_n^a(T_\omega^j(x,y),E,\omega)\big| < \frac{2N(1+\kappa)\log\lambda}{n} + \frac{N+1}{2} n^{\delta-1}.$$

Recall that for any $N \ge n^{1/3}$,

$$\begin{aligned}
&\operatorname{meas}\big\{x\in\mathbb{T} : \big|\frac{1}{N}\sum_{j=1}^{N} u_n^a(T_\omega^j(x,y),E,\omega) - L_n^a(y,E,\omega)\big| > \frac{1}{40}\log\lambda\big\}\\
&\le C\exp\big(-c_\rho n^{1/30}\log\lambda\big).
\end{aligned}$$

Choose $N = n^{1/3}$ and $\delta = \frac{1}{3}$. We obtain that there exists $n_0 = n_0(\lambda,\alpha_d)$ such that for any $n > n_0$,

$$\begin{aligned}
&\operatorname{meas}\{x\in\mathbb{T} : |u_n^a(x,y,E,\omega) - L_n^a(y,E,\omega)| > \frac{1}{30}\log\lambda\}\\
&\le C\exp\big(-2c_\rho n^{1/30}\log\lambda\big).
\end{aligned} \tag{3.14}$$

Finally, we obtain a similar estimation for $u_n^u(z,y,E,\omega) := \frac{1}{n}\log\|M_n^u(z,y,E,\omega)\|$. Then using the key lemma, called large deviation theorem, to prove the main theorem in Section 4. Note that for fixed y, E and ω, $u_n^u(z,y,E,\omega)$ is not a subharmonic function, but satisfies the assumption of the avalanche principle in the next section. By (2.4) and (2.4), we have

$$u_n^u(z,y,E,\omega) + \frac{1}{2n}\Big(\tilde{S}_n(x,y,\omega) + S_n(x+y,y+\omega,\omega)\Big) = u_n^a(x,y,E,\omega). \tag{3.15}$$

Recall (2.5), (2.6) and the definitions of $\tilde{S}_n(x,y,\omega)$ and $S_n(x,y,\omega)$. By the subharmonicity of $\log|a(z)|$ and $\log|\tilde{a}(z)|$, the estimation (3.9) also holds for $\log|a(z)|$ and $\log|\tilde{a}(z)|$, i.e.,for any fixed $y\in\mathbb{T}$ and Diophantine ω,

$$\operatorname{meas}\big\{x\in\mathbb{T} : |\frac{1}{n}S_n(x,y,\omega) - D| > \frac{1}{40}\log\lambda\big\} \le C\exp\big(-c_\rho n^{1/30}\log\lambda\big), \tag{3.16}$$

$$\operatorname{meas}\{x\in\mathbb{T} : |\frac{1}{n}\tilde{S}_n(x,y,\omega) - D| > \frac{1}{40}\log\lambda\} \le C\exp\big(-c_\rho n^{1/30}\log\lambda\big). \tag{3.17}$$

Combining (2.7), (3.14), (3.15), (3.16) and (3.17), we have that

$$\begin{aligned}
&\operatorname{meas}\big\{x\in\mathbb{T} : |u_n^u(x,y,E,\omega) - L_n^u(y,E,\omega)| > \frac{3}{40}\log\lambda\big\}\\
&\le 3C\exp\Big(-c_\rho n^{1/30}\log\lambda\Big),
\end{aligned} \tag{3.18}$$

where

$$\begin{aligned}
L_n^u(y,E,\omega) &= \int_{\mathbb{T}} \frac{1}{n}\log\|M_n^u(x,y,E,\omega)\|dx = \frac{1}{n}\int_{\mathbb{T}}\log\|M_n^a(x,y,E,\omega)\|dx - D\\
&= L_n^a(y,E,\omega) - D = \int_{\mathbb{T}}\frac{1}{n}\log\|M_n(x,y,E,\omega)\|dx = L_n(y.E,\omega).
\end{aligned}$$

Choose $\kappa = 1/100$ in $\lambda_0(\kappa)$. By (3.4), if $\lambda > \lambda_0\left(\frac{1}{100}\right)$, then

$$\frac{99}{100}\log\lambda \le L^a(y,E,\omega) \le L_n^a(y,E,\omega) \le \frac{101}{100}\log\lambda.$$

Recall that

$$D = \int_{\mathbb{T}} \log|a(x)|dx.$$

Therefore, there exists $\lambda_3 = \exp(100|D|)$ such that for any $\lambda > \lambda_3$,

$$|D| < \frac{1}{100}\log\lambda.$$

Redefine $\lambda_0 = \max\left\{\lambda_0\left(\frac{1}{100}\right), \lambda_3\right\}$. So, if $\lambda > \lambda_0$, then

$$\frac{49}{50}\log\lambda \le L_n(y,E,\omega) = L_n^u(y,E,\omega) \le \frac{51}{50}\log\lambda,$$

$$|L_n(y,E,\omega) - L_n^a(y,E,\omega)| = |L_n^u(y,E,\omega) - L_n^a(y,E,\omega)| \le \frac{1}{100}\log\lambda,$$

$$|L_n(y,E,\omega) - L_n(E,\omega)| < \frac{3}{100}\log\lambda.$$

Replacing $L_n^u(y,E,\omega)$ by $L_n(E,\omega)$ in (3.18), we obtain the following large deviation theorem for u_n^u.

Lemma 3.1. *Assume $\lambda > \lambda_0$. Then there exists $n_0 = n_0(\lambda,\alpha_d)$ such that for any $n > n_0$,*

$$\operatorname{meas}\left\{(x,y)\in\mathbb{T}^2 : |u_n^u(x,y,E,\omega) - L_n(E,\omega)| > \frac{1}{9}\log\lambda\right\} \le \exp\left(-c_\rho^u n^{1/30}\log\lambda\right).$$

4. Proof of main Theorem

In this section, we will apply the following lemma, called the avalanche principle, to prove the main theorem.

Proposition 4.1. *Let $A_1,\dots,A_m$ be a sequence of 2×2-matrices whose determinants satisfy*

$$\max_{1\le j\le m}|\det A_j| \le 1.$$

Suppose that

$$\min_{1\le j\le m}\|A_j\| \ge \mu > m, \tag{4.1}$$

$$\max_{1\le j<n}[\log\|A_{j+1}\| + \log\|A_j\| - \log\|A_{j+1}A_j\|] < \frac{1}{2}\log\mu\,. \tag{4.2}$$

Then

$$\Big|\log\|A_m\cdot\ldots\cdot A_1\| + \sum_{j=2}^{m-1}\log\|A_j\| - \sum_{j=1}^{m-1}\log\|A_{j+1}A_j\|\Big| < C\frac{m}{\mu}$$

with some absolute constant C.

For a proof of the above proposition see [3].

Lemma 4.2. *Let $\lambda > \lambda_0\left(\frac{1}{100}\right)$, $N_0 > n_0(\lambda,\alpha_d)$ and*

$$N_1N_0^{-1} = m \sim \exp\Big(\frac{c_\rho^u}{3}N_0^{1/30}\log\lambda\Big).$$

Then we have

$$|L_{N_1}(E,\omega)+L_{N_0}(E,\omega)-2L_{2N_0}(E,\omega)| < \log\lambda \exp\big(-\frac{c_\rho^u}{4}N_0^{1/30}\log\lambda\big).$$

Proof. Note that for any $n>n_0$,

$$\frac{49}{50}\log\lambda \le L(E,\omega)\le L_{2n}(E,\omega)\le L_n(E,\omega) < \frac{51}{50}\log\lambda. \tag{4.3}$$

Then, we will use the avalanche principle on $A_j(x,y,E,\omega)=M^u_{N_0}(T_\omega^{jN_0}(x,y),E,\omega)$ with (x,y) restricted to the set $\Lambda\subset\mathbb{T}^2$ defined by $2m-1$ conditions:

$$|\frac{1}{N_0}\log\|A_j(x,y,E,\omega)\| - L_{N_0}(E,\omega)| < \frac{1}{9}\log\lambda, \quad \forall 0\le j\le m-1, \tag{4.4}$$

$$\big|\frac{1}{2N_0}\log\|A_{j+1}A_j(x,y,E,\omega)\| - L_{2N_0}(E,\omega)\big| < \frac{1}{9}\log\lambda, \quad \forall 0\le j\le m-2. \tag{4.5}$$

By Lemma 3.1,

$$\mathrm{meas}\{\mathbb{T}^2\backslash\Lambda\} < (2m-1)\exp\Big(-c_\rho^u N_0^{1/30}\log\lambda\Big) < \exp\Big(-\frac{c_\rho^u}{3}N_0^{1/30}\log\lambda\Big).$$

By (4.4), (4.5) and (4.3), for each $A_j(x,y)$ with $(x,y)\in\Lambda$, we have

$$\|A_j\|\ge\exp\Big(\frac{6}{7}\log\lambda N_0\Big),$$

and

$$\begin{aligned}
&\big|\log\|A_j\|+\log\|A_{j+1}\|-\log\|A_{j+1}A_j\|\,\big|\\
&\le \big|\log\|A_j\|-N_0L_{N_0}\big| + \big|\log\|A_{j+1}\|-N_0L_{N_0}\big|\\
&\quad + \big|2N_0L_{N_0}-2N_0L_{2N_0}\big| + \big|2N_0L_{2N_0}-\log\|A_{j+1}A_j\|\big|\\
&< \frac{1}{3}N_0\log\lambda + \frac{2}{25}N_0\log\lambda < \frac{3}{7}N_0\log\lambda.
\end{aligned}$$

This implies that all assumptions of the avalanche principle have been satisfied with the setting $\mu=\exp(\frac{6}{7}N_0\log\lambda)$. Thus,

$$\big|\log\|\prod_{j=m}^{1}A_j\| + \sum_{j=2}^{m-1}\log\|A_j\| - \sum_{j=1}^{m-1}\log\|A_{j+1}A_j\|\big| \le C\frac{m}{\mu} < \exp\big(-\frac{6}{7}N_0\log\lambda\big).$$

Intergrading on Λ, we have

$$\begin{aligned}
&\Big|\iint_\Lambda \log\|M^u_{N_1}(x,y,E,\omega)\|\,dx\,dy + \sum_{j=1}^{m-2}\iint_\Lambda \log\|M^u_{N_0}\big(T_\omega^{jN_0}(x,y),E,\omega\big)\|\,dx\,dy\\
&- \sum_{j=0}^{m-2}\iint_\Lambda \log\|M^u_{2N_0}\big(T_\omega^{jN_0}(x,y),E,\omega\big)\|\,dx\,dy\Big|\\
&< \exp\big(-\frac{6}{7}N_0\log\lambda\big).
\end{aligned}$$

We want to replace integration over Λ by integration over $\mathbb{T}^2$. Then (2.4) and (2.4) imply

$$\log\|M^u_{N_1}(x,y,E,\omega)\| + \sum_{j=1}^{m-2}\log\|M^u_{N_0}\big(T_\omega^{jN_0}(x,y),E\big)\|$$

$$
\begin{aligned}
&- \sum_{j=0}^{m-2} \log \| M^u_{2N_0}\big(T^{jN_0}_\omega(x,y),E\big)\| \\
&= \log\|M^a_{N_1}(x,y,E,\omega)\| + \sum_{j=1}^{m-2} \log\|M^a_{N_0}\big(T^{jN_0}_\omega(x,y),E\big)\| \\
&\quad - \sum_{j=0}^{m-2} \log \| M^a_{2N_0}\big(T^{jN_0}_\omega(x,y),E\big)\|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\Big| \iint_{\mathbb{T}^2\setminus\Lambda} \log\|M^u_{N_1}(x,y,E,\omega)\|\,dx\,dy \\
&+ \sum_{j=1}^{m-2} \iint_{\mathbb{T}^2\setminus\Lambda} \log\|M^u_{N_0}\big(T^{jN_0}_\omega(x,y),E,\omega\big)\|\,dx\,dy \\
&- \sum_{j=0}^{m-2} \iint_{\mathbb{T}^2\setminus\Lambda} \log\|M^u_{2N_0}\big(T^{jN_0}_\omega(x,y),E,\omega\big)\|\,dx\,dy\Big| \\
&< 4N_1 \frac{51}{50}\log\lambda \exp\big(-\frac{c^u_\rho}{3} n^{1/30}\log\lambda\big).
\end{aligned}
$$

In summary,

$$
\begin{aligned}
&|L_{N_1}(E,\omega) + L_{N_0}(E,\omega) - 2L_{2N_0}(E,\omega)| \\
&\le |L_{N_1}(E,\omega) + \frac{m-2}{m} L_{N_0}(E,\omega) - \frac{2(m-1)}{m} L_{2N_0}(E,\omega)| \\
&\quad + \frac{2}{m}\left(L_{N_0}(E,\omega) + L_{2N_0}(E,\omega)\right) \\
&\le \frac{1}{N_1}\exp\big(-\frac{6}{7}N_0\log\lambda\big) + \frac{408}{50}\log\lambda\exp\big(-\frac{c^u_\rho}{3}N_0^{1/30}\log\lambda\big) \\
&\le \log\lambda\exp\big(-\frac{c^u_\rho}{4}N_0^{1/30}\log\lambda\big).
\end{aligned}
$$

□

Similarly,

$$
|L_{2N_1}(E,\omega) + L_{N_0}(E,\omega) - 2L_{2N_0}(E,\omega)| \le \log\lambda\exp\big(-\frac{c^u_\rho}{4}N_0^{1/30}\log\lambda\big).
$$

Thus,

$$
|L_{2N_1}(E,\omega) - L_{N_1}(E,\omega)| < 2\exp\big(-\frac{c^u_\rho}{4}N_0^{1/30}\log\lambda\big).
$$

In general, setting $N_{s+1} = m_s N_s$ with $m_s \sim \exp\Big(\frac{c^u_\rho}{3}N_s^{1/30}\log\lambda\Big)$, we have

$$
\begin{gathered}
|L_{N_{s+1}}(E,\omega) + L_{N_s}(E,\omega) - 2L_{2N_s}(E,\omega)| < \log\lambda\exp\big(-\frac{c^u_\rho}{4}N_s^{1/30}\log\lambda\big), \\
|L_{2N_{s+1}}(E,\omega) - L_{N_{s+1}}(E,\omega)| < 2\log\lambda\exp\big(-\frac{c^u_\rho}{4}N_s^{1/30}\log\lambda\big),
\end{gathered}
\tag{4.6}
$$

and

$$\begin{aligned}
&|L_{N_{s+1}}(E,\omega) - L_{N_s}(E,\omega)| \\
&\le |L_{N_{s+1}}(E,\omega) + L_{N_s}(E,\omega) - 2L_{2N_s}(E,\omega)| + |L_{N_s}(E,\omega) - L_{2N_s}(E,\omega)| \\
&< 3\log\lambda \exp\big(-\frac{c_\rho^u}{4}N_s^{1/30}\log\lambda\big).
\end{aligned} \tag{4.7}$$

Combining Lemma 4.2, (4.6) and (4.7), we have

$$\begin{aligned}
&|L(E,\omega) + L_{N_0}(E,\omega) - 2L_{2N_0}(E,\omega)| \\
&= |\sum_{j\ge 1}\big(L_{N_{j+1}}(E,\omega) - L_{N_i}(E,\omega)\big) + L_{N_1}(E,\omega) + L_{N_0}(E,\omega) - 2L_{2N_0}(E,\omega)| \\
&\le \sum_{j\ge 1}|L_{N_{j+1}}(E,\omega) - L_{N_j}(E,\omega)| + |L_{N_1}(E,\omega) + L_{N_0}(E,\omega) - 2L_{2N_0}(E,\omega)| \\
&< \sum_{j\ge 1}3\log\lambda\exp\big(-\frac{c_\rho^u}{4}N_j^{1/30}\log\lambda\big) + \log\lambda\exp\big(-\frac{c_\rho^u}{4}N_0^{1/30}\log\lambda\big) \\
&< \log\lambda\exp\big(-\frac{c_\rho^u}{5}N_0^{1/30}\log\lambda\big).
\end{aligned} \tag{4.8}$$

On the other hand, we need to estimate $|L_n(E,\omega) - L_n(E',\omega)|$. Note that

$$\begin{aligned}
&\big|\|M_n^a(x,y,E,\omega)\| - \|M_n^a(x,y,E',\omega)\|\big| \\
&\le \|M_n^a(x,y,E,\omega) - M_n^a(x,y,E',\omega)\| \\
&\le \sum_{j=0}^{n-1}\Big(\|M^a\left(T_\omega^{n-1}(x,y),E,\omega\right)\times\dots\times M^a\left(T_\omega^{j+1}(x,y),E,\omega\right)\| \\
&\quad \times\|M^a\left(T_\omega^{j}(x,y),E,\omega\right) - M^a\left(T_\omega^{j}(x,y),E',\omega\right)\| \\
&\quad \times\|M^a\left(T_\omega^{j-1}(x,y),E',\omega\right)\times\dots\times M^a\left(T_\omega^{0}(x,y),E',\omega\right)\|\Big) \\
&\le ne^{\frac{51}{50}(n-1)\log\lambda}|E-E'|.
\end{aligned}$$

Then (2.2) and (2.3) imply that

$$\begin{aligned}
&\big|\|M_n^u(x,y,E,\omega)\| - \|M_n^u(x,y,E',\omega)\|\big| \\
&= \frac{\big|\|M_n^a(x,E,\omega)\| - \|M_n^a(x,E',\omega)\|\big|}{|\prod_{j=0}^{n-1}\tilde a\big(z+jy+\frac{j(j-1)}{2}\omega\big)a\big(z+(j+1)y+\frac{j(j+1)}{2}\omega\big)|^{1/2}} \\
&\le ne^{\frac{51}{50}(n-1)\log\lambda}\exp\Big(-\frac12\big(\tilde S_n(x,y,\omega) + S_n(x+y,y+\omega,\omega)\big)\Big)|E-E'|.
\end{aligned}$$

By (3.16) and (3.17) there exists a set $\mathcal{B}'' \subset \mathbb{T}^2$ satisfying

$$\operatorname{meas}\mathcal{B}'' < 2C\exp\big(-c_\rho n^{1/30}\log\lambda\big)$$

such that for any $(x,y)\notin\mathcal{B}''$,

$$S_n(x,y,\omega), S_n(x+y,y+\omega,\omega) \ge nD - \frac{1}{40}n\log\lambda \ge -\frac{1}{20}n\log\lambda.$$

Thus, for any $(x,y)\notin\mathcal{B}''$,

$$\big|\|M_n^u(x,y,E,\omega)\| - \|M_n^u(x,y,E',\omega)\|\big| \le ne^{\frac{6}{5}n\log\lambda}|E-E'|.$$

This implies

$$\begin{aligned}
&|\log\|M_n^u(x,y,E,\omega)\|-\log\|M_n^u(x,y,E',\omega)\||\\
&=\log\Big(1+\frac{\big|\|M_n^u(x,y,E,\omega)\|-\|M_n^u(x,y,E',\omega)\|\big|}{\|M_n^u(x,y,E,\omega)\|}\Big)\\
&\le\Big|\frac{\|M_n^u(x,y,E,\omega)\|-\|M_n^u(x,y,E',\omega)\|}{\|M_n^u(x,y,E,\omega)\|}\Big|\\
&\le\big|\|M_n^u(x,y,E,\omega)\|-\|M_n^u(x,y,E',\omega)\|\big|\\
&\le ne^{\frac{6}{5}n\log\lambda}|E-E'|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&|L_n(E,\omega)-L_n(E',\omega)|\\
&=\iint_{\mathbb{T}\setminus\mathcal{B}''}\frac{1}{n}\big|\log\|M_n^u(x,y,E,\omega)\|-\log\|M_n^u(x,y,E',\omega)\|\big|\,dx\,dy\\
&\quad+\iint_{\mathcal{B}''}\frac{1}{n}\,|\log\|M_n^u(x,y,E,\omega)\|-\log\|M_n^u(x,y,E',\omega)\||\,dx\,dy\\
&\le e^{\frac{6}{5}n\log\lambda}|E-E'|+\iint_{\mathcal{B}''}\frac{1}{n}\,|\log\|M_n^a(x,y,E,\omega)\|-\log\|M_n^a(x,y,E',\omega)\||\,dx\,dy\\
&\le e^{\frac{6}{5}n\log\lambda}|E-E'|+\frac{202C}{100}\log\lambda\exp\big(-c_\rho n^{1/30}\log\lambda\big)
\end{aligned}$$

Now we can give the proof of the main theorem.

Proof of Theorem 1.1.

$$\begin{aligned}
&|L(E,\omega)-L(E',\omega)|\\
&\le|L(E,\omega)+L_{N_0}(E,\omega)-2L_{2N_0}(E,\omega)|+|L(E',\omega)+L_{N_0}(E',\omega)-2L_{2N_0}(E',\omega)|\\
&\quad+|L_{N_0}(E,\omega)-L_{N_0}(E',\omega)|+2|L_{2N_0}(E,\omega)-L_{2N_0}(E',\omega)|\\
&\le 2\log\lambda\exp\big(-\frac{c_\rho^u}{5}N_0^{1/30}\log\lambda\big)+e^{\frac{6}{5}N_0\log\lambda}|E-E'|\\
&\quad+\frac{202C}{100}\log\lambda\exp\Big(-c_\rho N_0^{1/30}\log\lambda\Big)\\
&\quad+2e^{\frac{12}{5}N_0\log\lambda}|E-E'|+\frac{404C}{100}\log\lambda\exp\big(-2c_\rho N_0^{1/30}\log\lambda\big)\\
&\le 10C\log\lambda\exp\big(-2c_\rho N_0^{1/30}\log\lambda\big)+3e^{\frac{12}{5}N_0\log\lambda}|E-E'|.
\end{aligned}$$

If $E'\to E$, then there exists large N_0 satisfying

$$\begin{aligned}
&\frac{10}{3}C\log\lambda\exp(-3N_0\log\lambda)<|E-E'|\\
&\qquad<\frac{10}{3}C\log\lambda\exp\Big(-2c_\rho N_0^{1/30}\log\lambda-\frac{12}{5}N_0\log\lambda\Big).
\end{aligned}$$

Then

$$\log|E-E'|>\log\big(\frac{10}{3}C\log\lambda\big)-3N_0\log\lambda>-4N_0\log\lambda.$$

Therefore,

$$\begin{aligned}
|L(E,\omega)-L(E',\omega)|&\le 20C\log\lambda\exp\big(-2c_\rho N_0^{1/30}\log\lambda\big)\\
&\le 20C\log\lambda\exp\big(-2c_\rho|\log|E-E'||^{1/30}\log\lambda\big).
\end{aligned}$$

□

Acknowledgments. This research was supported by the National Nature Science Foundation of China (Grant 11401166), by the Fundamental Research Funds for the Central Universities(Grant 2017B17214), and by the China Postdoctoral Science Foundation (Grant 2019M650094).

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