

# A global factorization theorem for the ZS-AKNS system \*

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## Abstract

We prove a global Birkhoff factorization theorem for general loops with finite poles in the ZS-AKNS hierarchy (Zakharov-Shabat-Ablowitz-Kaup-Newell-Segur). We use the inverse scattering method.

## 1 Introduction

Factorization theory and Lie algebra decomposition play a key role in algebraic geometry and representation theory associated with integrable systems; see for example [12, 4, 16, 10, 14, 15, 5, 6, 8, 13, 3]. However, many algebraic or geometric decomposition theorems are valid only locally and in general the scattering data properties of their derived solution space can not, or can only, be partially characterized.

Nevertheless, in the case of the ZS-AKNS system (Zakharov-Shabat-Ablowitz-Kaup-Newell-Segur), as defined by  $su(2)$ - or  $su(n)$ -reality condition (i.e., the solution space contained in  $su(2)$  or  $su(n)$ ), Faddeev and Takhtajan [7], Uhlenbeck and Terng [18, 19] derived a global decomposition theorem and characterize full scattering data for solutions.

Using the inverse scattering theory [1, 2, 17], a completely different approach, we obtain a global Birkhoff factorization theorem and find the global group loop action in the  $su(2)$  (reprove the results of [18, 19]) and  $su(1,1)$  cases [20]. More precisely, in the  $su(2)$  case, the scattering theory of [1, 2, 17] immediately implies a global Birkhoff factorization theorem for each renormalized eigenfunction of the ZS-AKNS system (see (2.1) below). We then show a global factorization theorem for general loop in  $D_-$  and characterize  $D_-$ . The  $D_-$  is actually characterized by that each loop in  $D_-$  is a multiple of a certain renormalized eigenfunction. Where the multiple factor is a diagonal matrix in  $D_-$ .

Under this framework, we generalize the methods in [20] and obtain the following generalized theorem.

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**Theorem 1.1** For  $g(z)$  in a dense and open subset of  $D_-$ , there exists an unique global factorization

$$g(z)^{-1}e^{xzJ} = E(x, z)M(x, z)^{-1},$$

for  $\forall x \in \mathbb{R}$ ,  $\forall z \in \mathbb{C} \setminus (\mathbb{R} \cup P)$ , with  $E(x, \cdot) \in G_+$ ,  $M(x, \cdot) \in D_-$ , and  $P$  poles of  $g$ . Where  $G_+ = \{g : \mathbb{C} \rightarrow SL(2, \mathbb{C}), \text{ holomorphic}\}$ , and  $D_-$  consist of those  $g$  satisfying

1.  $g : \mathbb{C} \setminus \mathbb{R} \rightarrow SL(2, \mathbb{C})$ ,  $g$  is meromorphic
2.  $g$  has smooth boundary values  $g_{\pm}$
3.  $g$  tends to 1 at  $\infty$
4.  $g_+g_-^{-1} - I \in \mathcal{S}$  and decays rapidly at  $\infty$
5.  $\sum_{\mathbb{C}^-} \min\{\tilde{a}, \tilde{b}\} - \sum_{\mathbb{C}^+} \min\{\tilde{c}, \tilde{d}\} = 0$

$$\begin{aligned} \sum_{z \in \mathbb{C}^+} \min\{\tilde{a}, \tilde{b}\} - \sum_{z \in \mathbb{C}^-} \min\{\tilde{c}, \tilde{d}\} &= \frac{1}{2\pi} \int_{\mathbb{R}} d \arg(\tilde{A}_- \tilde{D}_+ - \tilde{B}_- \tilde{C}_+) \\ &\quad + \frac{1}{2\pi} \int_{\mathbb{R}} d \arg(\tilde{A}_+ \tilde{D}_- - \tilde{B}_+ \tilde{C}_-), \end{aligned}$$

with  $g = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}$ , and  $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$  as exponents of the leading terms in the power series expansion of  $\tilde{A}, \tilde{B}, \tilde{C}$ , and  $\tilde{D}$ .

**Theorem 1.2** The scattering data for loop  $g$  in Theorem 1.1 can be explicitly computed.

Note that  $D_-$  is not a group owing to condition 5). Therefore, the methods of Terng and Uhlenbeck, loop group factorization theorem [9], are unable to derive the factorization formula in this case.

## 2 Necessary Conditions for Factorization

The ZS-AKNS system has the form

$$\frac{d}{dx} \psi(x, z) = \psi(x, z) (zJ + q(x)),$$

with  $x \in \mathbb{R}$ ,  $z \in \mathbb{C}$ ,  $\psi(x, z) \in SL(n, \mathbb{C})$ , and  $q \in \mathcal{Q}$ . Where  $J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ ,  $\mathcal{Q} = \{q \in \mathcal{S}(\mathbb{R}; M_2(\mathbb{C})) : q_{11} = q_{22} = 0\}$ , the space of  $2 \times 2$  off-diagonal

matrices whose entries belong to the Schwarz class. The associated renormalized eigenfunction  $m(x, z)$  satisfies

$$\begin{aligned} \frac{d}{dx}m(x, z) + z[J, m(x, z)] &= m(x, z)q(x), \\ \lim_{x \rightarrow -\infty} m(x, z) &= I, \end{aligned} \quad (2.1)$$

with  $m(\cdot, z)$  absolutely continuous and bounded. Therefore, denote the scattering data [1, 2, 11] for generic  $q \in \mathcal{Q}$  as  $\{V, z_j, U^\pm\}$ , where

$$m_+(x, \xi) = e^{-x\xi J} V(\xi) e^{x\xi J} m_-(x, \xi), \quad (2.2)$$

$\{z_j\}$  are the totality of poles of  $m$  in  $\mathbb{C}^\pm$ , and  $U^\pm$  are the upper (lower) triangular factors of  $m(x, z) = e^{-xzJ}(1 + U^\pm(z))e^{xzJ}\eta^\pm(x, z)$  in  $\{x \leq 0\} \times \mathbb{C}^\pm$ . Here,  $\eta^\pm(x, z)$  is a solution of (2.1), holomorphic in  $z$  and approaching 1 as  $z \rightarrow \infty$ . Moreover, for generic  $q$ , the scattering data  $\{V, z_j, U^\pm\}$  satisfies:

- Algebraic constraints:  $U^\pm$  are strictly upper (lower) triangular,  $d_k^+(V) \neq 0$ , and  $d_k^-(V) = 1$ , with  $d_k^+(d_k^-(V))$  as upper (lower) principal  $k$ -th minors of  $V$ ,
- Analytic constraints:  $V - I \in \mathcal{S}(\mathbb{R}; M_2(\mathbb{C}))$ ,  $U^\pm$  is rational in  $z \in \mathbb{C}_\pm$ , holomorphic for  $z \in \mathbb{C}_\mp$ , and approaching zero as  $z \rightarrow \infty$ ,
- Topological constraints (winding number constraints):

$$P_j^+ - P_{j+1}^+ + P_j^- - P_{j-1}^- = \frac{1}{2\pi} \int_{-\infty}^{\infty} d \arg \frac{d_j^+ V}{d_{j-1}^- V}(\xi), \quad (2.3)$$

as  $j = 1, 2$ , with  $P_j^+$  being the number of poles in the  $j$ -th column of  $U^+$ , and  $P_j^-$  the number of poles in the  $j$ -th column of  $U^-$ .

**Lemma 2.1** *Suppose that  $m(x, z)$  is a normalized eigenfunction of the ZS-AKNS system (2.1). Let  $P \subset \mathbb{C} \setminus \mathbb{R}$  be the set of poles of  $m(x, \cdot)$ . Then, there exists a global factorization*

$$m(0, z)^{-1} e^{xzJ} = E(x, z) m(x, z)^{-1},$$

for all  $x \in \mathbb{R}$ , all  $z \in \mathbb{C} \setminus (\mathbb{R} \cup P)$ , and  $E(x, \cdot) \in G_+$ .

One can prove this lemma by factorization formula  $m(x, z) = e^{-xzJ}(1 + U^\pm(z))e^{xzJ}\eta^\pm(x, z)$  in  $\{x \leq 0\} \times \mathbb{C}^\pm$ , and (2.2).

In the  $su(2)$  case, to characterize each loop  $g = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}$  in  $D_-$ , we first simplify  $g$  by factoring out a diagonal factor such that all poles of  $\tilde{A}, \tilde{B}$  lie in  $C^-$ , and all of the zeros of  $\tilde{A}$  and  $\tilde{B}$  belong to  $\overline{C}^+$  (see [20, lemma 4.3]). Therefore, it becomes easier to detect the zeros and poles of the diagonal multiple  $\sigma$  such that  $\sigma g$  is some renormalized eigenfunction. Without the  $su(2)$  symmetry, we generalize the factoring out process to the following.

**Definition** We define a transformation  $\tau : SL(2, \mathbb{C}) \rightarrow SL(2, \mathbb{C})$ , that maps

$$g = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} \text{ to } \tau(g) = \begin{pmatrix} \tilde{X} & 0 \\ 0 & \tilde{Y} \end{pmatrix} \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}. \text{ Here}$$

$$\begin{aligned} \tilde{X} &= \frac{(z - t_1^+) \dots (z - t_\alpha^+) (z - \bar{q}_1^-) \dots (z - \bar{q}_\beta^-) (z - \bar{r}_1^-)}{(z - \bar{t}_1^+) \dots (z - \bar{t}_\alpha^+) (z - q_1^-) \dots (z - q_\beta^-) (z - r_1^-)} \\ &\quad \times \frac{\dots (z - \bar{r}_\gamma^-) (z - s_1^+) \dots (z - s_\delta^+)}{\dots (z - r_\gamma^-) (z - \bar{s}_1^+) \dots (z - \bar{s}_\delta^+)}, \\ \tilde{Y} &= \frac{(z - \bar{p}_1^+) \dots (z - \bar{p}_\alpha^+) (z - q_1^-) \dots (z - q_\beta^-) (z - r_1^-)}{(z - p_1^+) \dots (z - p_\alpha^+) (z - \bar{q}_1^-) \dots (z - \bar{q}_\beta^-) (z - \bar{r}_1^-)} \\ &\quad \times \frac{\dots (z - r_\gamma^-) (z - \bar{s}_1^+) \dots (z - \bar{s}_\delta^+)}{\dots (z - \bar{r}_\gamma^-) (z - s_1^+) \dots (z - s_\delta^+)}, \end{aligned}$$

and

$$\begin{aligned} \{t_1^+, \dots, t_\alpha^+\} &= \{ \text{poles of } \tilde{A} \text{ in } \mathbb{C}^+ \} \cup_{\max} \{ \text{poles of } \tilde{B} \text{ in } \mathbb{C}^+ \}, \\ \{q_1^-, \dots, q_\beta^-\} &= \{ \text{zeros of } \tilde{A} \text{ in } \mathbb{C}^- \} \cup_{\max} \{ \text{zeros of } \tilde{B} \text{ in } \mathbb{C}^- \}, \\ \{r_1^-, \dots, r_\gamma^-\} &= \{ \text{poles of } \tilde{C} \text{ in } \mathbb{C}^- \} \cup_{\max} \{ \text{poles of } \tilde{D} \text{ in } \mathbb{C}^- \}, \\ \{s_1^+, \dots, s_\delta^+\} &= \{ \text{zeros of } \tilde{C} \text{ in } \mathbb{C}^+ \} \cup_{\max} \{ \text{zeros of } \tilde{D} \text{ in } \mathbb{C}^+ \}. \end{aligned}$$

In this paper we define  $\cup_{\max}$ ,  $\cup_{\min}$ , and  $\subset_{\text{mult}}$  to be

$$\begin{aligned} \underbrace{\{z_1 \dots z_1 \dots z_k \dots z_k\}}_{n_1 \quad n_k} \cup_{\max} \underbrace{\{z_1 \dots z_1 \dots z_k \dots z_k\}}_{m_1 \quad m_k} &= \{ \underbrace{z_1 \dots z_1}_{\max(n_1, m_1)} \dots \underbrace{z_k \dots z_k}_{\max(n_k, m_k)} \}, \\ \underbrace{\{z_1 \dots z_1 \dots z_k \dots z_k\}}_{n_1 \quad n_k} \cup_{\min} \underbrace{\{z_1 \dots z_1 \dots z_k \dots z_k\}}_{m_1 \quad m_k} &= \{ \underbrace{z_1 \dots z_1}_{\min(n_1, m_1)} \dots \underbrace{z_k \dots z_k}_{\min(n_k, m_k)} \}, \\ \underbrace{\{z_1 \dots z_1 \dots z_k \dots z_k\}}_{n_1 \quad n_k} \subset_{\text{mult}} \underbrace{\{z_1 \dots z_1 \dots z_k \dots z_k\}}_{m_1 \quad m_k} &, \text{ if } n_i \leq m_i, \forall i \in \{1, \dots, k\}. \end{aligned}$$

**Theorem 2.2** Suppose that  $g(z) \in SL(2, \mathbb{C})$ , and  $\sigma g = m(0, z)$ , with  $\sigma$  a diagonal matrix, and  $m(x, z)$  as a normalized eigenfunction of (2.1). Then, the formula of  $\sigma$  and scattering data  $\{V, z_j, U^\pm\}$  of  $m(x, z)$ , can be explicitly computed.

The proof of Theorem 2.2 follows from the following lemmas. First, let  $\tau(g) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ ,  $\sigma = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$ ,  $U^+ = \begin{pmatrix} 0 & u_{12} \\ 0 & 0 \end{pmatrix}$ , and  $U^- = \begin{pmatrix} 0 & 0 \\ u_{21} & 0 \end{pmatrix}$ . Then by (2.3), and the factorization property of scattering theory [1, 2, 11], we obtain

$$\text{a) } XY = 1, AD - BC = 1,$$

- b)  $X_-Y_+(A_-D_+ - B_-C_+) = 1$ ,  $4X_+Y_-(A_+D_- - B_+C_-) \neq 0$
- c)  $\frac{C}{X}$ , and  $\frac{D}{X}$  are holomorphic in  $\mathbb{C}^+$ ,  $XA - u_{12}\frac{C}{X}$  is holomorphic in  $\mathbb{C}^+$ ,  
 $XB - u_{12}\frac{D}{X}$  is holomorphic in  $\mathbb{C}^+$ ,
- d)  $XA$  and  $XB$  are holomorphic in  $\mathbb{C}^-$ ,  $\frac{C}{X} - u_{21}XA$  is holomorphic in  $\mathbb{C}^-$   
 $\frac{D}{X} - u_{21}XB$  is holomorphic in  $\mathbb{C}^-$ ,
- e)

$$\begin{aligned} & -(\text{number of poles of } u_{12}) + (\text{number of poles of } u_{21}) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} d \arg(X_+Y_-(A_+D_- - B_+C_-)). \end{aligned} \quad (2.4)$$

**Lemma 2.3** Let  $\{p_1^+, \dots, p_m^+\}$  be poles of  $X$ , and  $\{z_1^-, \dots, z_k^-\}$  zeros of  $X$ . Then,

$$\begin{aligned} m &= k, \\ \{ \text{poles of } A \} \cup_{\max} \{ \text{poles of } B \} &\subset_{\text{mult}} \{z_1^-, \dots, z_k^-\}, \\ \{ \text{poles of } C \} \cup_{\max} \{ \text{poles of } D \} &\subset_{\text{mult}} \{p_1^+, \dots, p_m^+\}, \\ X(z) &= \frac{(z - z_1^-) \dots (z - z_k^-)}{(z - p_1^+) \dots (z - p_m^+)} \exp \left\{ \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\log(A_-D_+ - B_-C_+)}{t - z} dt \right\}. \end{aligned}$$

**Proof.** Let  $\{z_1^+, \dots, z_h^+\}$  be zeros of  $X$  in  $\mathbb{C}^+$ ,  $\{\hat{p}_1^+, \dots, \hat{p}_m^+\}$  be poles of  $X$  in  $\overline{\mathbb{C}^+}$ ,  $\{\hat{z}_1^-, \dots, \hat{z}_k^-\}$  be zeros of  $X$  in  $\overline{\mathbb{C}^-}$ ,  $\{p_1^-, \dots, p_n^-\}$  be poles of  $X$  in  $\mathbb{C}^-$ . By the first formula of (2.4-c) and (2.4-d), we obtain

$$\begin{aligned} \{ \text{poles of } A \text{ in } \mathbb{C}^- \} \cup_{\max} \{ \text{poles of } B \text{ in } \mathbb{C}^- \} &\subset_{\text{mult}} \{\hat{z}_1^-, \dots, \hat{z}_k^-\}, \\ \{ \text{zeros of } A \text{ in } \mathbb{C}^- \} \cap_{\min} \{ \text{zeros of } B \text{ in } \mathbb{C}^- \} &\supset_{\text{mult}} \{p_1^-, \dots, p_n^-\}, \\ \{ \text{zeros of } C \text{ in } \mathbb{C}^+ \} \cap_{\min} \{ \text{zeros of } D \text{ in } \mathbb{C}^+ \} &\supset_{\text{mult}} \{z_1^+, \dots, z_h^+\}, \\ \{ \text{poles of } C \text{ in } \mathbb{C}^+ \} \cup_{\max} \{ \text{poles of } D \text{ in } \mathbb{C}^+ \} &\subset_{\text{mult}} \{\hat{p}_1^+, \dots, \hat{p}_m^+\}. \end{aligned}$$

Because of the definition of  $\tau$ , we know that all zeros of  $A$  and  $B$  lie in  $\overline{\mathbb{C}^+}$ , and all zeros of  $C$  and  $D$  lie in  $\overline{\mathbb{C}^-}$ . Thus,  $\{p_1^-, \dots, p_n^-\} = \{z_1^+, \dots, z_h^+\} = \phi$ ,  $\{p_1^+, \dots, p_m^+\} = \{\hat{p}_1^+, \dots, \hat{p}_m^+\}$  and  $\{z_1^-, \dots, z_k^-\} = \{\hat{z}_1^-, \dots, \hat{z}_k^-\}$ . Applying (2.4-b) and the Riemann Hilbert theorem, we obtain the formula for  $X$ ,

$$X(z) = z^{m-k} \frac{(z - z_1^-) \dots (z - z_k^-)}{(z - p_1^+) \dots (z - p_m^+)} \exp \left\{ \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\log(A_-D_+ - B_-C_+)}{t - z} dt \right\}.$$

Finally, since  $g$  has smooth boundary values on  $\mathbb{R}$ ,  $\tau(g)$ ,  $g$  and  $m(0, \cdot)$  to 1 as  $z \rightarrow \infty$ , we derive  $m = k$ .  $\square$

**Lemma 2.4** *We have*

$$\begin{aligned} & \{p_1^+, \dots, p_k^+\} \\ & \quad \subset_{\text{mult}} \{poles\ of\ C\} \cup_{\text{max}} \{poles\ of\ D\} \cup_{\text{max}} \{zeros\ of\ A\} \cup_{\text{max}} \{zeros\ of\ B\}, \\ & \{z_1^-, \dots, z_k^-\} \\ & \quad \subset_{\text{mult}} \{poles\ of\ A\} \cup_{\text{max}} \{poles\ of\ B\} \cup_{\text{max}} \{zeros\ of\ C\} \cup_{\text{max}} \{zeros\ of\ D\}. \end{aligned}$$

**Proof.** First, we introduce  $a, b, c, d, \kappa, \hat{a}, \hat{b}, \hat{c}, \hat{d}$ , and  $\hat{\kappa}$  by

$$\begin{aligned} A(z) &= A_a(z_0)(z - z_0)^a + A_{a+1}(z_0)(z - z_0)^{a+1} + \dots, \\ B(z) &= B_b(z_0)(z - z_0)^b + B_{b+1}(z_0)(z - z_0)^{b+1} + \dots, \\ C(z) &= C_c(z_0)(z - z_0)^c + C_{c+1}(z_0)(z - z_0)^{c+1} + \dots, \\ D(z) &= D_d(z_0)(z - z_0)^d + D_{d+1}(z_0)(z - z_0)^{d+1} + \dots, \\ u_{12}(z) &= u_{12,u}(z_0)(z - z_0)^u + u_{12,u+1}(z_0)(z - z_0)^{u+1} + \dots, \\ (z - p_1^+) \dots (z - p_k^+) &= f_\kappa(z_0)(z - z_0)^\kappa + f_{\kappa+1}(z_0)(z - z_0)^{\kappa+1} + \dots, \end{aligned}$$

for  $z_0 \in \mathbb{C}^+$ . And

$$\begin{aligned} A(z) &= \hat{A}_{\hat{a}}(\omega_0)(z - \omega_0)^{\hat{a}} + \hat{A}_{\hat{a}+1}(\omega_0)(z - \omega_0)^{\hat{a}+1} + \dots, \\ B(z) &= \hat{B}_{\hat{b}}(\omega_0)(z - \omega_0)^{\hat{b}} + \hat{B}_{\hat{b}+1}(\omega_0)(z - \omega_0)^{\hat{b}+1} + \dots, \\ C(z) &= \hat{C}_{\hat{c}}(\omega_0)(z - \omega_0)^{\hat{c}} + \hat{C}_{\hat{c}+1}(\omega_0)(z - \omega_0)^{\hat{c}+1} + \dots, \\ D(z) &= \hat{D}_{\hat{d}}(\omega_0)(z - \omega_0)^{\hat{d}} + \hat{D}_{\hat{d}+1}(\omega_0)(z - \omega_0)^{\hat{d}+1} + \dots, \\ u_{21}(z) &= u_{21,\hat{u}}(\omega_0)(z - \omega_0)^{\hat{u}} + u_{21,\hat{u}+1}(\omega_0)(z - \omega_0)^{\hat{u}+1} + \dots, \\ (z - z_1^-) \dots (z - z_k^-) &= \hat{f}_{\hat{\kappa}}(\omega_0)(z - \omega_0)^{\hat{\kappa}} + \hat{f}_{\hat{\kappa}+1}(\omega_0)(z - \omega_0)^{\hat{\kappa}+1} + \dots, \end{aligned}$$

for  $\omega_0 \in \mathbb{C}^-$ .

If  $z_0 \in \{p_1^+, \dots, p_k^+\} \setminus (\{zeros\ of\ A\} \cup_{\text{max}} \{zeros\ of\ B\})$ , then  $a - \kappa < 0$ , and  $b - \kappa < 0$ . Therefore, by (2.4-c) we obtain

$$\begin{aligned} a - \kappa &= u + \kappa + c, \\ b - \kappa &= u + \kappa + d, A_a D_d - B_b C_c = 0. \end{aligned}$$

Then with (2.4-a), we have

$$a + d = b + c < 0.$$

Thus  $c < 0$ ,  $d < 0$ , and the first formula is proved. Additionally, the same argument can be applied to demonstrate the second formula.  $\square$

**Lemma 2.5** *We have*

$$\begin{aligned} \{p_1^+, \dots, p_k^+\}_{\text{mult}} &= \{poles\ of\ C\} \cup_{\text{max}} \{poles\ of\ D\}, \\ \{z_1^-, \dots, z_k^-\}_{\text{mult}} &= \{poles\ of\ A\} \cup_{\text{max}} \{poles\ of\ B\}. \end{aligned}$$

**Proof.** It is sufficient to show that

$$\kappa = -\min\{c, d\} \quad \text{and} \quad \hat{\kappa} = -\min\{\hat{a}, \hat{b}\}.$$

We will prove the first formula of this lemma. The same argument can be adapted for demonstrating the other formula. Without loss of generality, let  $c = \min\{c, d\}$ . Hence,  $\kappa \geq -c$  by Lemma 2.4. Since  $AD - BC = 1$  ((2.4-a)), either of the following cases i)  $a + d = b + c \leq 0$ , ii)  $a + d = 0$ ,  $b + c > 0$ , iii)  $a + d > 0$ ,  $b + c = 0$ .

In case i), we obtain  $\kappa = -c$  by Lemma 2.4 directly. In case ii), we derive  $a - \kappa \leq a + c \leq a + d = 0$ . Thus, either  $a = \kappa$  or  $a < \kappa$ . If  $a = \kappa$ , then  $\kappa = -d \leq -c$ . Hence,  $\kappa = -c$  by Lemma 2.4. If  $a < \kappa$ , then  $a - \kappa = u + \kappa + c$ . This implies  $u + \kappa + d = -\kappa - c \leq 0$ . It suffices to examine the case  $u + \kappa + d = -\kappa - c < 0$ . However, it implies  $a - \kappa = u + \kappa + c$ , and  $b - \kappa = u + \kappa + d$ . That is  $a + d = b + c$ , which is a contradiction.

Case iii) can be proved with the same argument as that of case ii).  $\square$

**Lemma 2.6** *The associated scattering data, denoted by  $\{V, z_j, U^\pm\}$ , in Theorem 2.2 can be explicitly computed in terms of  $g$ .*

**Proof.** It suffices to determine the scattering data in terms of  $\tau(g)$ , that is, in terms of  $A$ ,  $B$ ,  $C$ , and  $D$ . First of all, by (2.2) and (2.3), we obtain

$$\begin{aligned} V(\xi) &= \sigma_+ \tau(g)_+ \tau(g)_-^{-1} \sigma_-^{-1} \\ &= \begin{pmatrix} X_+ Y_- (A_+ D_- - B_+ C_-) & -X_+ X_- (A_+ B_- - A_- B_+) \\ Y_+ Y_- (C_+ D_- - C_- D_+) & 1 \end{pmatrix}. \end{aligned}$$

Let  $z_0 \in \mathbb{C}^+$  be a pole of  $u_{12}$ . If  $z_0 \notin \{p_1^+, \dots, p_k^+\}$ , then we would have  $0 \leq a - u$  and  $u + \kappa + c < 0$ . This contradicts with the second formula of (2.4-c). Therefore, we conclude that  $u_{12}(z) = p(z)(z - p_1^+)^{u_1} \dots (z - p_k^+)^{u_k}$ , with  $u_j \leq 0$ , and  $p(z)$  is a polynomial of degree  $\deg p < -(u_1 + \dots + u_k)$ . Moreover, under similar argument with Lemma 2.5, we can prove that

$$u_i = \min\{a, b\} + \min\{c, d\}. \quad (2.5)$$

Since the degree of polynomial  $p$  is strictly less than  $-(u_1 + \dots + u_k)$ , to determine polynomial  $p$ , it is equivalent to determine the  $u_1 + \dots + u_k$  number of coefficients, of negative terms of expansion

$$u_{12}(z) = u_{i, u_i} (z - p_i^+)^{u_i} + u_{i, u_i+1} (z - p_i^+)^{u_i+1} + \dots + u_{i, -1} (z - p_i^+)^{-1} + \text{h.o.t.}$$

at  $p_i^+$ , for  $p_i^+ \in \{p_1^+, \dots, p_k^+\}$ . Here, h.o.t. denotes the higher order terms. However, the  $u_i$  number of coefficients can be successively computed by equating the  $u_i$  number of negative terms of either the second or third formula of (2.4-c), up to  $a - \kappa < 0$  or  $b - \kappa < 0$ .

Similarly we can determine  $u_{21}$  using the second formula of Lemmas 2.3-2.5 and adapting the above argument. Note that

$$\hat{u}_i = \min\{\hat{a}, \hat{b}\} + \min\{\hat{c}, \hat{d}\} \quad (2.5')$$

which completes the proof of this lemma.  $\square$

Now the proof of Theorem 2.2 follows from Lemmas 2.3-2.6. Next we use condition (2.4-e) to characterize  $g$  in Theorem 2.2, through the the following two lemmas.

**Lemma 2.7** *We have*

$$\begin{aligned} & \frac{1}{2\pi} \int_{\mathbb{R}} d \arg(A_- D_+ - B_- C_+) + \frac{1}{2\pi} \int_{\mathbb{R}} d \arg(A_+ D_- - B_+ C_-) \\ &= \#(\{\text{zeros of } A\} \cap_{\min} \{\text{zeros of } B\}) - \#(\{\text{zeros of } C\} \cap_{\min} \{\text{zeros of } D\}). \end{aligned}$$

**Proof.** Since  $AD - BC = 1$ , we have  $\min\{a, b\} + \min\{c, d\} \leq 0$  in  $\mathbb{C}^+$ , and  $\min\{\hat{a}, \hat{b}\} + \min\{\hat{c}, \hat{d}\} \leq 0$  in  $\mathbb{C}^-$ . Therefore, by Lemma 2.6, (2.5) and (2.5'), we obtain

$$\begin{aligned} \#(\text{pole of } u_{12}) &= -(u_1 + \cdots + u_m) = - \sum_{z \in \mathbb{C}^+} (\min\{a, b\} + \min\{c, d\}), \\ \#(\text{pole of } u_{21}) &= -(\hat{u}_1 + \cdots + \hat{u}_k) = - \sum_{z \in \mathbb{C}^-} (\min\{\hat{a}, \hat{b}\} + \min\{\hat{c}, \hat{d}\}). \end{aligned}$$

Using conditions (2.4-b), (2.4-e), and the above formula,

$$\begin{aligned} & \frac{1}{2\pi} \int_{\mathbb{R}} d \arg(A_- D_+ - B_- C_+) + \frac{1}{2\pi} \int_{\mathbb{R}} d \arg(A_+ D_- - B_+ C_-) \\ &= \#(\{\text{zeros of } A\} \cap_{\min} \{\text{zeros of } B\}) + \#(\{\text{poles of } A\} \cup_{\max} \{\text{poles of } B\}) \\ & \quad - \#(\{\text{zeros of } C\} \cap_{\min} \{\text{zeros of } D\}) - \#(\{\text{poles of } C\} \cup_{\max} \{\text{poles of } D\}). \end{aligned}$$

The lemma is proved by noting that

$$\#(\{\text{poles of } A\} \cup_{\max} \{\text{poles of } B\}) = \#(\{\text{poles of } C\} \cup_{\max} \{\text{poles of } D\}) = k.$$

cf. Lemmas 2.3 and 2.5.  $\square$

**Lemma 2.8** *The loop  $g$  in Theorem 2.2 satisfies condition 5) of Theorem 1.1.*

**Proof.** By the definition of  $\tau$  and  $\tilde{A}\tilde{D} - \tilde{B}\tilde{C} = 1$ , we obtain

$$\begin{aligned} \{\text{zeros of } A\} \cap_{\min} \{\text{zeros of } B\} &= \{\text{zeros of } \tilde{X}\tilde{A} \text{ in } \mathbb{C}^+\} \cap_{\min} \{\text{zeros of } \tilde{X}\tilde{B} \text{ in } \mathbb{C}^+\} \\ &= \alpha + \beta + \gamma + \delta + \sum_{\mathbb{C}^+} \min\{\tilde{a}, \tilde{b}\}, \\ \{\text{zeros of } C\} \cap_{\min} \{\text{zeros of } D\} &= \{\text{zeros of } \tilde{Y}\tilde{C} \text{ in } \mathbb{C}^-\} \cap_{\min} \{\text{zeros of } \tilde{Y}\tilde{D} \text{ in } \mathbb{C}^-\} \\ &= \alpha + \beta + \gamma + \delta + \sum_{\mathbb{C}^-} \min\{\tilde{c}, \tilde{d}\}. \end{aligned} \tag{2.6}$$



Thus, we prove that  $g$  satisfies the second formula of 5) of Theorem 1.1. Moreover, note that

$$\begin{aligned} \{\text{poles of } A\} \cup_{\max} \{\text{poles of } B\} &= \{\text{poles of } \tilde{X}\tilde{A} \text{ in } \mathbb{C}^-\} \cup_{\max} \{\text{poles of } \tilde{X}\tilde{B} \text{ in } \mathbb{C}^-\} \\ &= \alpha + \beta + \gamma + \delta - \sum_{\mathbb{C}^-} \min\{\tilde{a}, \tilde{b}\}, \\ \{\text{poles of } C\} \cup_{\max} \{\text{poles of } D\} &= \{\text{poles of } \tilde{Y}\tilde{C} \text{ in } \mathbb{C}^+\} \cup_{\max} \{\text{poles of } \tilde{Y}\tilde{D} \text{ in } \mathbb{C}^+\} \\ &= \alpha + \beta + \gamma + \delta - \sum_{\mathbb{C}^+} \min\{\tilde{c}, \tilde{d}\}. \end{aligned} \tag{2.6'}$$

Thereby, the first formula of condition 5) is proved by Lemmas 2.3 or 2.5.  $\square$

### 3 Proof Theorems 1.1 and 1.2

**Step 1. Extraction of Scattering Data** For loop  $g(z) \in D_-$ , we denote

$$\begin{aligned} \tau(g) &= \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \sigma = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}, \\ U^+ &= \begin{pmatrix} 0 & u_{12} \\ 0 & 0 \end{pmatrix}, \quad U^- = \begin{pmatrix} 0 & 0 \\ u_{21} & 0 \end{pmatrix}. \end{aligned}$$

Therefore (2.6') and condition 5) of  $D_-$  implies the existence of a  $k$ , such that

$$\begin{aligned} \{p_1^+, \dots, p_k^+\}_{\text{mult}} &= \{\text{poles of } C\} \cup_{\max} \{\text{poles of } D\}, \\ \{z_1^-, \dots, z_k^-\}_{\text{mult}} &= \{\text{poles of } A\} \cup_{\max} \{\text{poles of } B\}. \end{aligned}$$

Set

$$\begin{aligned} X(z) &= \frac{(z - z_1^-) \dots (z - z_k^-)}{(z - p_1^+) \dots (z - p_k^+)} \exp\left\{ \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\log(A_- D_+ - B_- C_+)}{t - z} dt \right\}, \\ Y(z) &= \frac{(z - p_1^+) \dots (z - p_k^+)}{(z - z_1^-) \dots (z - z_k^-)} \exp\left\{ -\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\log(A_- D_+ - B_- C_+)}{t - z} dt \right\}. \end{aligned}$$

Let

$$\begin{aligned} V(\xi) &= \begin{pmatrix} X_+ Y_- (A_+ D_- - B_+ C_-) & -X_+ X_- (A_+ B_- - A_- B_+) \\ Y_+ Y_- (C_+ D_- - C_- D_+) & X_- Y_+ (A_- D_+ - B_- C_+) \end{pmatrix} \\ &= \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} = \sigma_+ \tau(g)_+ \tau(g)_-^{-1} \sigma_-^{-1}. \end{aligned}$$

By the formula of  $X$ , and  $Y$ , and condition 3)-5) of  $D_-$ , one can verify that  $V - I \in \mathcal{S}(\mathbb{R}, SL(2, \mathbb{C}))$ ,  $V_{11} \neq 0$ , and  $V_{22} = 1$ .

Now define

$$u_{12}(z) = p(z)(z - p_1^+)^{u_1} \dots (z - p_k^+)^{u_k},$$

$$u_{21}(z) = \hat{p}(z)(z - z_1^-)^{\hat{u}_1} \dots (z - z_k^-)^{\hat{u}_k},$$

with  $0 \geq u_i = \min\{a, b\} + \min\{c, d\}$ ,  $0 \geq \hat{u}_i = \min\{\hat{a}, \hat{b}\} + \min\{\hat{c}, \hat{d}\}$ ,  $a, b, c, d$  as exponents of the leading terms in the power series expansion of  $A, B, C, D$  at points in  $\mathbf{C}^+$ ,  $\hat{a}, \hat{b}, \hat{c}, \hat{d}$  as exponents of the leading terms in the power series expansion of  $A, B, C, D$  at points in  $\mathbf{C}^-$ , and  $p(z), \hat{p}(z)$  as polynomials of degrees strictly less than  $-(u_1 + \dots + u_k)$ ,  $-(\hat{u}_1 + \dots + \hat{u}_k)$  respectively. Note that it implies either  $XA$  or  $XB$  has a pole at  $p_i^+$  if  $u_i < 0$ . At these points, we can determine the  $u_i$  number of negative terms of

$$u_{12}(z) = u_{i,u_i}(z - p_i^+)^{u_i} + u_{i,u_i+1}(z - p_i^+)^{u_i+1} + \dots + u_{i,-1}(z - p_i^+)^{-1} + \text{h.o.t.}$$

at  $p_i^+$ , in order for both  $XA - u_{12}C/X$  and  $XB - u_{12}D/X$  to be holomorphic in  $\mathbf{C}^+$ . Thus,  $u_{12}$  is uniquely defined. Respectively, we can define  $\hat{p}$ , and hence  $u_{21}$ , by asserting the  $\hat{u}_i$  number of negative terms of

$$u_{21}(z) = \hat{u}_{i,\hat{u}_i}(z - z_i^-)^{\hat{u}_i} + \hat{u}_{i,\hat{u}_i+1}(z - z_i^-)^{\hat{u}_i+1} + \dots + \hat{u}_{i,-1}(z - z_i^-)^{-1} + \text{h.o.t.}$$

in order for both  $C/X - u_{21}XA$  and  $D/X - u_{21}XB$  to be holomorphic in  $\mathbf{C}^-$ .

One can verify the topological constraint (2.3-c) of  $\{V, U^\pm\}$  by:

$$\begin{aligned} & -\#(\text{poles of } u_{12}) + \#(\text{poles of } u_{21}) \\ &= (u_1 + \dots + u_k) - (\hat{u}_1 + \dots + \hat{u}_k) \\ &= \sum_{z \in \mathbf{C}^+} (\min\{a, b\} + \min\{c, d\}) - \sum_{z \in \mathbf{C}^-} (\min\{\hat{a}, \hat{b}\} + \min\{\hat{c}, \hat{d}\}) \\ &= \#(\{\text{zeros of } A\} \cap_{\min} \{\text{zeros of } B\}) + \#(\{\text{poles of } A\} \cup_{\max} \{\text{poles of } B\}) \\ &\quad - \#(\{\text{zeros of } C\} \cap_{\min} \{\text{zeros of } D\}) - \#(\{\text{poles of } C\} \cup_{\max} \{\text{poles of } D\}) \end{aligned}$$

By (2.6), (2.6'), condition 5) in  $D_-$ , and  $V_{22} = 1$ , result in

$$\begin{aligned} & \sum_{z \in \mathbf{C}^+} (\min\{\tilde{a}, \tilde{b}\} + \min\{\tilde{c}, \tilde{d}\}) - \sum_{z \in \mathbf{C}^-} (\min\{\tilde{a}, \tilde{b}\} + \min\{\tilde{c}, \tilde{d}\}) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} d \arg(\tilde{A}_- \tilde{D}_+ - \tilde{B}_- \tilde{C}_+) + \frac{1}{2\pi} \int_{\mathbb{R}} d \arg(\tilde{A}_+ \tilde{D}_- - \tilde{B}_+ \tilde{C}_-) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} d \arg \tilde{X}_- \tilde{Y}_+(\tilde{A}_- \tilde{D}_+ - \tilde{B}_- \tilde{C}_+) + \frac{1}{2\pi} \int_{\mathbb{R}} d \arg \tilde{X}_+ \tilde{Y}_-(\tilde{A}_+ \tilde{D}_- - \tilde{B}_+ \tilde{C}_-) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} d \arg(A_- D_+ - B_- C_+) + \frac{1}{2\pi} \int_{\mathbb{R}} d \arg(A_+ D_- - B_+ C_-) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} d \arg X_- Y_+(A_- D_+ - B_- C_+) + \frac{1}{2\pi} \int_{\mathbb{R}} d \arg X_+ Y_-(A_+ D_- - B_+ C_-) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} d \arg V_{11}. \end{aligned}$$

In summary, we have defined  $V, U^\pm$ , and proved that  $\{V, U^\pm\}$  satisfies (2.3), to become a formal scattering data [1, 2]. Therefore, the inverse scattering theory [1, 2] implies generically, the existence of a normalized eigenfunction, denoted as  $m(x, z)$ , such that its associated scattering data is  $\{V, U^\pm\}$ .

**Step 2. Proof of  $\sigma\tau(g) = m(0, x)$**  By the results of *Step 1*, there exists  $\eta^\pm(z)$ , holomorphic in  $\mathbb{C}^\pm$ , such that

$$\sigma\tau(g) = \begin{pmatrix} 1 & u_{12} \\ 0 & 1 \end{pmatrix} \eta^+ \text{ in } \mathbb{C}^+, \quad \sigma\tau(g) = \begin{pmatrix} 1 & 0 \\ u_{21} & 1 \end{pmatrix} \eta^- \text{ in } \mathbb{C}^-.$$

Thereby,

$$\eta_+ \eta_-^{-1} = \begin{pmatrix} 1 & -u_{12} \\ 0 & 1 \end{pmatrix} \sigma_+ \tau(g)_+ \tau(g)_-^{-1} \sigma_-^{-1} \begin{pmatrix} 1 & 0 \\ u_{21} & 1 \end{pmatrix}. \quad (3.1)$$

On the other hand, by inverse scattering theory, there exists  $\tilde{\eta}^\pm(z)$ , holomorphic in  $\mathbb{C}^\pm$ , such that

$$m(0, z) = \begin{pmatrix} 1 & u_{12} \\ 0 & 1 \end{pmatrix} \tilde{\eta}^+ \text{ in } \mathbb{C}^+, \quad m(0, z) = \begin{pmatrix} 1 & 0 \\ u_{21} & 1 \end{pmatrix} \tilde{\eta}^- \text{ in } \mathbb{C}^-.$$

Hence,

$$\tilde{\eta}_+ \tilde{\eta}_-^{-1} = \begin{pmatrix} 1 & -u_{12} \\ 0 & 1 \end{pmatrix} V \begin{pmatrix} 1 & 0 \\ u_{21} & 1 \end{pmatrix}. \quad (3.1')$$

Therefore defining  $\eta(z) = \eta^\pm(z)$ , and  $\tilde{\eta}(z) = \tilde{\eta}^\pm(z)$  when  $z \in \mathbb{C}^\pm$ , and applying (3.1), (3.1'),  $\eta \rightarrow 1, \tilde{\eta} \rightarrow 1$  as  $z \rightarrow \infty$ , and the holomorphicity of  $\eta^\pm, \tilde{\eta}^\pm$  in  $\mathbb{C}^\pm$ , we obtain  $\eta = \tilde{\eta}$ . Hence,  $\sigma\tau(g) = m(0, x)$ .

**Step 3. Proof of factorization of  $g$**  Applying the results of *Step 2* and Lemma 2.1, we derive

$$\begin{aligned} g(z)^{-1} e^{xzJ} &= \tau(g)^{-1} e^{xzJ} \begin{pmatrix} \tilde{X} & 0 \\ 0 & \tilde{Y} \end{pmatrix} \\ &= m(0, z)^{-1} e^{xzJ} \sigma \begin{pmatrix} \tilde{X} & 0 \\ 0 & \tilde{Y} \end{pmatrix} \\ &= E_{\tau(g)}(x, z) m(x, z)^{-1} \sigma \begin{pmatrix} \tilde{X} & 0 \\ 0 & \tilde{Y} \end{pmatrix} \\ &= E_{\tau(g)}(x, z) M_g(x, z)^{-1}, \end{aligned}$$

for all  $x \in \mathbb{R}$ , all  $z \in \mathbb{C} \setminus (\mathbb{R} \cup P)$ , with  $E(x, \cdot) \in G_+$ , and  $P$  as poles of  $g$ . With  $E_{\tau(g)}(0, z) = 1$ , and the property that integer-valued continuous functions are constant, it readily demonstrates that  $M(x, \cdot) \in D_-$ .  $\square$

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