Electronic Journal of Differential Equations, Vol. 2022 (2022), No. 25, pp. 1–29.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

FRACTIONAL KIRCHHOFF HARDY PROBLEMS WITH WEIGHTED CHOQUARD AND SINGULAR NONLINEARITY

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Abstract. In this article, we study the existence and multiplicity of solutions to the fractional Kirchhoff Hardy problem involving weighted Choquard and singular nonlinearity

$$\begin{gathered} M(\|u\|^2)(-\Delta)^s u-\gamma\frac{u}{|x|^{2s}} \\ =\lambda\frac{l(x)}{u^q}+\frac{1}{|x|^\alpha}\Big(\int_\Omega\frac{r(y)|u(y)|^p}{|y|^\alpha|x-y|^\mu}\,dy\Big)r(x)|u|^{p-2}u \quad \text{in } \Omega, \\ u>0 \text{ in } \Omega, \quad u=0 \text{ in } \mathbb{R}^N\setminus\Omega, \end{gathered}$$

where $\Omega\subseteq\mathbb{R}^N$ is an open bounded domain with smooth boundary containing 0 in its interior, $N>2s$ with $s\in(0,1)$, $0<q<1$, $0<\mu<N$, γ and λ are positive parameters, $\theta\in[1,p)$ with $1<p<2^*_{\mu,s,\alpha}$, where $2^*_{\mu,s,\alpha}$ is the upper critical exponent in the sense of weighted Hardy-Littlewood-Sobolev inequality. Moreover M models a Kirchhoff coefficient, l is a positive weight and r is a sign-changing function. Under the suitable assumption on l and r, we established the existence of two positive solutions to the above problem by Nehari-manifold and fibering map analysis with respect to the parameters.The results obtained here are new even for $s=1$.

1. Introduction

Let $\Omega\subseteq\mathbb{R}^N$ be a bounded domain with smooth boundary containing 0 in its interior, $N>2s$ with $s\in(0,1)$. We consider the following problem with weighted Choquard and singular nonlinearity with weight functions

$$\begin{gathered} M(\|u\|^2)(-\Delta)^s u-\gamma\frac{u}{|x|^{2s}} \\ =\lambda l(x)u^{-q}+\frac{1}{|x|^\alpha}\Big(\int_\Omega\frac{r(y)|u(y)|^p}{|y|^\alpha|x-y|^\mu}\,dy\Big)r(x)|u|^{p-2}u \quad \text{in } \Omega, \\ u>0 \text{ in } \Omega, \quad u=0 \text{ in } \mathbb{R}^N\setminus\Omega, \end{gathered} \tag{1.1}$$

where $M:\mathbb{R}^+_0\to\mathbb{R}^+_0$ is a continuous function, which is defined as

$$M(t)=c+dt^{\theta-1}, \quad \text{with } c>0, d\geq 0,$$

2020 *Mathematics Subject Classification.* 35A15, 35J75, 36B38.
Key words and phrases. Fractional Kirchhoff Hardy operator; singular nonlinearity; weighted Choquard type nonlinearity; Nehari-manifold; fibering map.

Submitted December 30, 2021. Published March 25, 2022.

λ, γ are positive parameters, $q \in (0,1)$, $\mu \in (0,N)$, $\theta \in [1,p)$ with $1 < p < 2^*_{\mu,s,\alpha}$, where $2^*_{\mu,s,\alpha}$ is upper critical exponent in the sense of weighted Hardy-Littlewood-Sobolev inequality. The fractional Laplacian operator $(-\Delta)^s$ is defined, up to a normalization constant, by the Riesz potential as

$$(-\Delta)^s \xi(x) := \int_{\mathbb{R}^N} \frac{2\xi(x) - \xi(x+y) - \xi(x-y)}{|y|^{N+2s}}\, dy$$

where $x \in \mathbb{R}^N$ and $\xi \in C_0^\infty(\mathbb{R}^N)$.

We use the following assumptions on l and r;

(A1) $l : \Omega \to \mathbb{R}$ such that $l > 0$ a.e. in Ω and $l \in L^m(\Omega)$, where $m = \frac{2^*_s}{2^*_s - 1 + q}$ with $2^*_s = \frac{2N}{N-2s}$ is the fractional critical Sobolev exponents.

(A2) $r : \Omega \to \mathbb{R}$ such that $r \in L^{\frac{2^*_s}{2^*_{\mu,s,\alpha} - p}}(\Omega)$ is sign-changing function, where $2^*_{\mu,s,\alpha} = \frac{(2N-2\alpha-\mu)}{(N-2s)}$ is the upper critical exponent in the sense of weighted Hardy-Littlewood-Sobolev inequality.

Problems of the type (1.1) are motivated by the weighted Hardy-Littlewood-Sobolev inequality proved by Stein and Weiss in [25].

Theorem 1.1. *Let $1 < w, t < \infty$, $0 < \mu < N$, $\alpha + \beta \geq 0$, $\alpha + \beta + \mu \leq N$, and $\frac{1}{w} + \frac{1}{t} + \frac{\alpha+\beta+\mu}{N} = 2$. Then there exists a constant $C(\alpha, \beta, \mu, N, t, w)$ independent of f and h such that*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x|^\alpha |x-y|^\mu |y|^\beta}\, dx\, dy \leq C(\alpha, \beta, \mu, N, t, w) \|f\|_w \|h\|_t \tag{1.2}$$

where $1 - \frac{1}{w} - \frac{\mu}{N} < \frac{\alpha}{N} < 1 - \frac{1}{w}$.

Currently, problems on nonlocal operators emerging an attractive research area, specifically the fractional Laplacian operator attracts a lot of interest in nonlinear analysis for references [5, 10, 15, 23]. The fractional Laplace operator $(-\Delta)^s$ is the infinitesimal generator of Lévy stable diffusion process which arises in anomalous diffusions in plasma, flames propagation, chemical reactions in liquids, geophysical fluid dynamics, and American options in finance, for references [2, 12]. There is a large amount of literature on the problems related to the fractional elliptic equations, so we refer here [19, 24] and references therein.

Kirchhoff type problems, which arise in the modeling of various physical phenomena, specifically Kirchhoff in [17] extended the classical D'Alembert wave equation by considering the effects of the changes in the length of string during the vibrations and give a model which is governed by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - M\Big(\int_0^L \big|\frac{\partial u}{\partial x}\big|^2\, dx \Big) \frac{\partial^2 u}{\partial x^2} = 0,$$

where

$$M\Big(\int_0^L \big|\frac{\partial u}{\partial x}\big|^2\, dx \Big) = \Big(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \big|\frac{\partial u}{\partial x}\big|^2\, dx \Big)$$

where ρ, ρ_0, h, E, L are constants. For further study about the existence of solutions for fractional Kirchhoff problems, we refer here [11, 16, 18, 21].

Problem (1.1) is closely related to the nonlocal Choquard equation, which arise in the study of Hartree-Fock theory of one component plasma and in the model of self-gravitating matter. Currently, lots of work has been done on the nonlocal Choquard equation, for the existence of solutions of such type of equations

$$-\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $V(x)$ is an suitable function, I_α is Riesz potential and $p > 1$ is suitable constant, which is studied in [13, 20]. Moroz and Schaftingen in [20] studied the existence, qualitative properties and decay asymptotics of ground states for the problem (1.3) in case of $V = 1$.

In [7], the authors study the critical nonlocal equation with weighted nonlocal term

$$-\Delta u = \frac{1}{|x|^\alpha}\Big(\int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x-y|^\mu |y|^\alpha}\,dy\Big)|u|^{p-2}u \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $0 < \mu < N$, $\alpha \geq 0$, $2\alpha + \mu \leq N$ and $2 - \frac{2\alpha+\mu}{N} < p \leq 2^*_{\alpha,\mu}$ with $2^*_{\alpha,\mu} = \frac{2N-2\alpha-\mu}{N-2}$. The critical exponent $2^*_{\alpha,\mu}$ appears here from the weighted Hardy-Littlewood-Sobolev inequality. Here they prove the existence of a positive ground state solutions for subcritical cases by using Schwarz symmetrization and critical cases by a nonlocal version of the concentration-compactness principle.

To manipulate the Hardy potential in (1.1), we recall the following fractional Hardy inequality

$$\gamma_H \int_\Omega \frac{|\phi(x)|^2}{|x|^{2s}} \leq \iint_{\mathbb{R}^{2N}} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \tag{1.4}$$

for any $\phi \in C_0^\infty(\Omega)$, where γ_H is the sharp constant given in [6, 28] as

$$\gamma_H := 2^{2s}\frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})},$$

where Γ denotes the Gamma function.

Abdellaoui et al. [1] studied the effect of the Hardy potential on the existence and summability of solutions to a class of nonlocal elliptic problems

$$\begin{gathered}(-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} = f(x,u) \quad \text{in } \Omega,\\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,\end{gathered}$$

for the case $f(x,u) = \frac{h(x)}{u^\sigma}$ for $t > 0$. They prove that for $\sigma \geq 1$, $\lambda \leq \Lambda_{N,s}$ and $h \in L^1(\Omega)$, this problem has a positive weak solution, where

$$\Lambda_{N,s} = 2^{2s}\frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}.$$

For the local case ($s = 1$) with $\lambda = 0$, in [3], the authors proved that for all $h \in L^1(\Omega)$, there exists at least one distributional solution.

Fiscella and Mishra [9] studied the following problem for singular and critical nonlinearities with Hardy term

$$\begin{gathered}M\Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)(-\Delta)^s u - \mu\frac{u}{|x|^{2s}} = \lambda f(x)u^{-\gamma} + g(x)u^{2^*_s-1} \quad \text{in } \Omega\\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,\end{gathered}$$

where $\Omega \subset \mathbb{R}^N$, $N > 2s$ with $s \in (0,1)$ and $2^*_s = \frac{2N}{N-2s}$, $M(t) = a + bt^{2\theta-2}$ is a Kirchhoff coefficients where $\theta \in [1, 2^*_s/2)$, f is a positive weight while g is a sign-changing function. Using the Nehari-manifold technique, they showed the existence of two positive solutions for the combination of critical Sobolev and Hardy nonlinearities. Moreover in [8] author studied the above problem with $\mu = 0$ and $f = g = 1$, here author prove the existence of two solutions by using variational methods with an appropriate truncation argument.

Wang et al. [26] studied the Choquard-Kirchhoff type problem

$$L(u) = \lambda f(x)u^{-p} + \Big(\int_{\mathbb{R}^N} \frac{g(y)|u(y)|^q}{|x-y|^\mu}\,dy\Big) g(x)u^{q-1} \quad \text{in } \Omega$$
$$u = 0 \text{ in } \quad \mathbb{R}^N \setminus \Omega,$$

where $L(u) = \big(a + b[u]_{s,p}^{(\theta-1)p}\big)(-\Delta)^s u$, with $s \in (0,1)$, $N \geq 2$, $1 < p < \frac{N}{s}$, $1 < q < p^*_{\mu,s}$ with $a > 0$, $b \geq 0$, $\theta \in [1, 2q)$, $0 < \beta < 1$ and $\mu \in (0, N)$, where $\lambda, \mu > 0$. Under the above assumptions, the authors obtained the existence of two positive non-trivial solutions by using the Nehari-manifold approach. We also cite [14] where the author investigated the existence, non-existence, and multiplicity of positive solutions fractional problem with Hardy potential and singular nonlinearity.

Before stating our main result for problem (1.1), it is worth noting that the novelty of this work is mainly to obtain the multiple positive solutions for the fractional Kirchhoff Hardy problem with weighted Choquard and singular nonlinearity using the Nehari-manifold and fibering map analysis. To the best of our knowledge, there is no article in the literature that deals with the fractional Kirchhoff Hardy problem for subcritical weighted Choquard and singular nonlinearity. In fact, results are even new for the case $s = 1$ and $\alpha = 0$. Because of the nonlocal behavior of the operator, the bounded support of the test function is not preserved which makes the analysis difficult. Also, because of the singular nature of the problem, the associated functional is not differentiable in the sense of Gâteaux. The results obtained here are somehow expected but we show how the results arise out of nature of the Nehari manifold. Now, we state our results.

Theorem 1.2. *Let $N > 2s$ with $s \in (0,1)$, $\theta \in [1, p)$. $c > 0$, $d \geq 0$ and the assumptions* (A1), (A2) *hold. Then* (1.1) *has at least two positive solutions for $\lambda \in (0, \Lambda_*)$.*

In above theorem $\Lambda_* := \min\{\Lambda_1, \Lambda_2\}$, where

$$\Lambda_1 := \Big(\frac{1+q}{2p-2}\Big)^{\frac{1+q}{2p-2}} \Big(\frac{h_{c,\gamma}(2p-2)}{2p+q-1}\Big)^{\frac{(2p+q-1)}{(2p-2)}} S^{\frac{(2p+q-1)}{2(p-1)}} \frac{1}{\|l\|_m} \Big[\frac{1}{C_r(\alpha,\mu,N)}\Big]^{\frac{1+q}{2p-2}}$$

and

$$\Lambda_2 := \frac{S^{\frac{(\theta+1)(2p+q-1)}{2(2p-\theta-1)}}}{[C_r(\alpha,\mu,N)]^{\frac{\theta+q}{2p-\theta-1}}} \Big[\frac{2\sqrt{(1+q)(2\theta+q-1)h_{c,\gamma}d}}{2p+q-1}\Big]^{\frac{\theta+q}{2p-\theta-1}} \times \Big[\frac{2\sqrt{(2p-2)(2p-2\theta)h_{c,\gamma}d}}{(2p+q-1)\|l\|_m}\Big].$$

This article is organized as follows: In section 2, we recall some important results and useful inequalities. In section 3, we introduce the Nehari-manifold structure and the fiber map analysis related (1.1). Also, we show that the associated energy functional is bounded below and coercive. A crucial compactness property of the energy functional and some important Lemmas are proved in section 4. In section

5, we show the existence of a positive solution in $\mathcal{N}^+_{\gamma,\lambda}$ and $\mathcal{N}^-_{\gamma,\lambda}$ respectively, which complete the proof of Theorem 1.2.

To abbreviate notation, we denote $\|\cdot\|_{X_0}$ by $\|\cdot\|$, and $\|\cdot\|_{L^p(\Omega)}$ by $\|\cdot\|_p$.

2. Preliminaries

In this section, we give the variational setting of problem (1.1) and state important results to be used later. Consider the fractional Sobolev space

$$H^s(\Omega)=\Big\{u\in L^2(\Omega):\frac{u(x)-u(y)}{|x-y|^{\frac{N}{2}+s}}\in L^2(Q)\Big\},$$

where $Q=\mathbb{R}^{2N}\setminus(C\Omega\times C\Omega)$ and $C\Omega=\mathbb{R}^N\setminus\Omega$. The norm for the space $H^s(\Omega)$ is

$$\|u\|_X=\Big(\|u\|_2^2+[u]_X^2\Big)^{1/2},$$

where $\|u\|_p$ is the norm for $L^p(\Omega)$ and $[u]_X=\big(\int_Q\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\big)^{1/2}$.

We define the function space

$$X_0:=\{u\in H^s(\mathbb{R}^N):u=0\text{ a.e. in }\mathbb{R}^N\setminus\Omega\}$$

which is a Hilbert space with norm

$$\|u\|_{X_0}=\Big(\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{1/2}.$$

The functional space X_0 can be equivalently considered as the closure of $C_0^\infty(\Omega)$ under the norm $\|u\|=[u]_X$. For further studies on X_0, we refer the reader to [22].

Now if $\alpha=\beta$ and $f(x)=|u(x)|^p$, $h(y)=|u(y)|^p$, that is if $w=t$, then by Theorem 1.1,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{r(x)r(y)|u(x)|^p|u(y)|^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy$$

is well defined if $r|u|^p\in L^t(\mathbb{R}^N)$, where $t=\frac{2N}{2N-2\alpha-\mu}$. Thus by fractional Sobolev embedding theorem, we have $2\le tp\le 2_s^*$, which gives that

$$2_{*\mu,s,\alpha}=\frac{2N-2\alpha-\mu}{N}\le p\le\frac{(2N-2\alpha-\mu)}{(N-2s)}=2^*_{\mu,s,\alpha},$$

where $2^*_{\mu,s,\alpha}$ is known as upper critical exponent and $2_{*\mu,s,\alpha}$ is lower critical exponent in the sense of weighted Hardy-Littlewood-Sobolev inequality.

Thus from Theorem 1.1, $r\in L^{\frac{2_s^*}{2^*_{\mu,s,\alpha}-p}}(\mathbb{R}^N)$, and Hölder inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{r(x)r(y)|u(x)|^p|u(y)|^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy&\le C(\alpha,\mu,N)\|r(x)|u|^p\|^2_{\frac{2N}{2N-2\alpha-\mu}}\\&\le C(\alpha,\mu,N)\|r\|^2_{\frac{2_s^*}{2^*_{\mu,s,\alpha}-p}}\|u\|^{2p}_{2_s^*}.\end{aligned}$$

Further, the fractional Sobolev inequality yields

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{r(x)r(y)|u(x)|^p|u(y)|^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy\le C_r(\alpha,\mu,N)S^{-p}\|u\|^{2p},\tag{2.1}$$

where $C_r(\alpha,\mu,N) = C(\alpha,\mu,N)\|r\|^2_{\frac{2^*_s}{2^*_{\mu,s,\alpha}-p}}$, and S denotes the best constant of the fractional Sobolev embedding theorem, is defined as

$$S = \inf_{v\in D^{s,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}}\,dx\,dy}{\left(\int_{\mathbb{R}^N} |v(x)|^{2^*_s}\,dx\right)^{2/2^*_s}} \tag{2.2}$$

where $D^{s,2}(\mathbb{R}^N)$ is the closure of $C_0^\infty(\mathbb{R}^N)$ under the norm $\left(\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\right)^{1/2}$.

Lemma 2.1. *Let $N \geq 3$, $q \in (1,\infty)$ and $\{u_n\}$ is a bounded sequence in $L^q(\mathbb{R}^N)$. If $u_n \to u$ a.e in $\mathbb{R}^N$ as $n \to \infty$, then $u_n \rightharpoonup u$ weakly in $L^q(\mathbb{R}^N)$.*

A proof of the above lemma can be found in [27, Lemma 1.32]. Now we prove the following Brezis-Lieb type Lemma for singular Choquard in case of fractional. The idea of the proof comes from [7, Lemma 2.2] where is done in case of $s = 1$.

Lemma 2.2. *Let $N \geq 3$, $\alpha \geq 0$, $0 < \mu < N$, $2\alpha + \mu \leq N$ and $1 \leq p \leq \frac{2N-2\alpha-\mu}{N-2s}$. If $\{u_n\}$ is a bounded sequence in $L^{\frac{2Np}{2N-2\alpha-\mu}}(\mathbb{R}^N)$ such that $u_n \to u$ a.e. in $\mathbb{R}^N$ as $n \to \infty$. Then*

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)|^p|u_n(y)|^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(u_n-u)(x)|^p|(u_n-u)(y)|^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}dx\,dy \\
&\to \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^p|u(y)|^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy.
\end{aligned}$$

Proof. Since $\{u_n\}$ is a bounded sequence in $L^{\frac{2Np}{2N-2\alpha-\mu}}(\mathbb{R}^N)$ such that $u_n \to u$ a.e. in $\mathbb{R}^N$ as $n \to \infty$, then from the Brézis-Lieb Lemma [4], we know that

$$|u_n - u|^p - |u_n|^p \to |u|^p \quad \text{in } L^{\frac{2N}{2N-2\alpha-\mu}}(\mathbb{R}^N). \tag{2.3}$$

The weighted Hardy-Littlewood-Sobolev inequality (1.2) implies that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{|u_n|^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dy - \int_{\mathbb{R}^N} \frac{|u_n-u|^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dy \\
&\to \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dy \quad \text{in } L^{\frac{2N}{2\alpha+\mu}}(\mathbb{R}^N).
\end{aligned} \tag{2.4}$$

Now we consider

$$\begin{aligned}
&\int_{\mathbb{R}^N} \Big(|x|^{-\mu} * \big(\frac{1}{|x|^\alpha}|u_n|^{2^*_{\mu,s,\alpha}}\big)\Big)\frac{|u_n|^{2^*_{\mu,s,\alpha}}}{|x|^\alpha}\,dx \\
&- \int_{\mathbb{R}^N} \Big(|x|^{-\mu} * \big(\frac{1}{|x|^\alpha}|u_n-u|^{2^*_{\mu,s,\alpha}}\big)\Big)\frac{|u_n-u|^{2^*_{\mu,s,\alpha}}}{|x|^\alpha}\,dx \\
&= \int_{\mathbb{R}^N} \Big(|x|^{-\mu} * \big(\frac{1}{|x|^\alpha}|u_n|^{2^*_{\mu,s,\alpha}} - \frac{1}{|x|^\alpha}|u_n-u|^{2^*_{\mu,s,\alpha}}\big)\Big) \\
&\quad \times \frac{\big(|u_n|^{2^*_{\mu,s,\alpha}} - |u_n-u|^{2^*_{\mu,s,\alpha}}\big)}{|x|^\alpha}dx \\
&\quad + 2\int_{\mathbb{R}^N} \Big(|x|^{-\mu} * \big(\frac{1}{|x|^\alpha}|u_n|^{2^*_{\mu,s,\alpha}} - \frac{1}{|x|^\alpha}|u_n-u|^{2^*_{\mu,s,\alpha}}\big)\Big)\frac{|u_n-u|^{2^*_{\mu,s,\alpha}}}{|x|^\alpha}\,dx.
\end{aligned} \tag{2.5}$$

By Lemma 2.1, we obtain

$$|u_n - u|^{2^*_{\mu,s,\alpha}} \rightharpoonup 0 \quad \text{in } L^{\frac{2N}{2N-2\alpha-\mu}}(\mathbb{R}^N) \tag{2.6}$$

Hence from (2.3), (2.4), (2.5) and (2.6), we obtain the required result. □

Definition 2.3. We say $u \in X_0$ is a weak solution of the problem (1.1), if $l(x)(u)^{-q}\phi \in L^1(\Omega)$ and for any $\phi \in X_0$ it holds

$$\begin{aligned}&\Big(c + d\|u\|^{2\theta-2}\Big)\iint_{\mathbb{R}^{2N}} \frac{\big(u(x)-u(y)\big)\big(\phi(x)-\phi(y)\big)}{|x-y|^{N+2s}}\,dx\,dy - \gamma\int_\Omega \frac{u\phi}{|x|^{2s}}\,dx\\ &- \lambda\int_\Omega l(x)(u)^{-q}\phi dx - \int_\Omega\int_\Omega \frac{r(x)r(y)(u(y))^p(u(x))^{p-1}\phi(x)}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy = 0.\end{aligned}$$

To obtain a positive solution of (1.1), we consider the problem

$$\begin{gathered}M(\|u\|^2)(-\Delta)^s u - \gamma\frac{u^+}{|x|^{2s}}\\ = \lambda l(x)(u^+)^{-q} + \Big(\int_\Omega \frac{r(y)(u^+(y))^p}{|y|^\alpha|x-y|^\mu}\,dy\Big)\frac{r(u^+)^{p-1}}{|x|^\alpha} \quad \text{in } \Omega,\\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^N\setminus\Omega,\end{gathered} \tag{2.7}$$

where $M(t) = c + dt^{\theta-1}$ and $u^+ = \max\{u, 0\}$. Then the function $u \in X_0$, $u > 0$ in Ω is a weak solution of the problem (2.7), if $l(x)(u^+)^{-q}\phi \in L^1(\Omega)$ and for any $\phi \in X_0$ it holds

$$\begin{aligned}&\Big(c + d\|u\|^{2\theta-2}\Big)\iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}\,dx\,dy - \gamma\int_\Omega \frac{u^+\phi}{|x|^{2s}}\,dx\\ &- \lambda\int_\Omega l(x)(u^+)^{-q}\phi dx - \int_\Omega\int_\Omega \frac{r(x)r(y)(u^+(y))^p(u^+(x))^{p-1}\phi(x)}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy = 0.\end{aligned} \tag{2.8}$$

One can easily see that if $u > 0$ is a solution to (2.7), then it is also a solution to (1.1). To find a solution to (2.7), we use a variational approach. So, we define the energy functional $\mathcal{J}_{\gamma,\lambda}\colon X_0 \to \mathbb{R}$ corresponding to problem (2.7), as

$$\begin{aligned}\mathcal{J}_{\gamma,\lambda}(u) := \frac{c}{2}\|u\|^2 + \frac{d}{2\theta}\|u\|^{2\theta} - \frac{\gamma}{2}\|u^+\|_H^2 - \frac{\lambda}{1-q}\int_\Omega l(x)(u^+)^{1-q}\\ - \frac{1}{2p}\int_\Omega\int_\Omega \frac{r(x)r(y)(u^+(x))^p(u^+(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy.\end{aligned}$$

Here we denote

$$\|u\|_H^2 := \int_\Omega \frac{|u(x)|^2}{|x|^{2s}}\,dx \quad \text{for all } u \in X_0.$$

Then the functional $\mathcal{J}_{\gamma,\lambda}$ is well defined and continuous for any $\gamma \in (0, c\gamma_H)$ by the fractional Hardy inequality (1.4). We note that the relation $v^+ \le |v|$ and (1.4) yield

$$c\iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \gamma\int_\Omega \frac{(v^+)^2}{|x|^{2s}}\,dx \ge h_{c,\gamma}\iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \tag{2.9}$$

for any $v \in X_0 \setminus \{0\}$, and $h_{c,\gamma} = c - \frac{\gamma}{\gamma_H} > 0$ for any $\gamma \in (0, c\gamma_H)$.

3. Fibering map analysis

In this section, we show that $\mathcal{N}^{\pm}_{\gamma,\lambda}$ is non-empty and $\mathcal{N}^{0}_{\gamma,\lambda}=\{0\}$. Moreover $\mathcal{J}_{\gamma,\lambda}$ is bounded below and coercive.

Since the energy functional $\mathcal{J}_{\gamma,\lambda}$ is not bounded below on X_0 but is bounded below on appropriate subset $\mathcal{N}_{\gamma,\lambda}$ of X_0. Therefore in order to obtain the existence results, we introduce the Nehari set

$$\begin{aligned}\mathcal{N}_{\gamma,\lambda} &:= \{u\in X_0 : \langle \mathcal{J}_{\gamma,\lambda}(u),u\rangle = 0\}\\ &= \Big\{u\in X_0 : c\|u\|^2 + d\|u\|^{2\theta} - \gamma\|u^+\|_H^2 - \lambda\int_\Omega l(x)(u^+)^{1-q}\,dx\\ &\quad - \int_\Omega\int_\Omega \frac{r(x)r(y)(u^+(x))^p(u^+(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy = 0\Big\}.\end{aligned}$$

Here we define the fiber map $\phi_u\colon \mathbb{R}^+\to\mathbb{R}$ as $\phi_u(t) := \mathcal{J}_{\gamma,\lambda}(tu)$ for all $t>0$. Then we have

$$\begin{aligned}\phi_u(t) &= \frac{c}{2}t^2\|u\|^2 + \frac{d}{2\theta}t^{2\theta}\|u\|^{2\theta} - \frac{\gamma}{2}t^2\|u^+\|_H^2 - \frac{\lambda t^{1-q}}{1-q}\int_\Omega l(x)(u^+)^{1-q}\,dx\\ &\quad - \frac{t^{2p}}{2p}\int_\Omega\int_\Omega \frac{r(x)r(y)(u^+(x))^p(u^+(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy,\end{aligned}$$

so

$$\begin{aligned}\phi_u'(t) &= ct\|u\|^2 + t^{2\theta-1}d\|u\|^{2\theta} - t\gamma\|u^+\|_H^2 - \lambda t^{-q}\int_\Omega l(x)(u^+)^{1-q}\,dx\\ &\quad - t^{2p-1}\int_\Omega\int_\Omega \frac{r(x)r(y)(u^+(x))^p(u^+(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy\end{aligned}$$

and

$$\begin{aligned}\phi_u''(t) &= c\|u\|^2 + d(2\theta-1)t^{2\theta-2}\|u\|^{2\theta} - \gamma\|u^+\|_H^2 + \lambda q t^{-q-1}\int_\Omega l(x)(u^+)^{1-q}\,dx\\ &\quad - (2p-1)t^{2p-2}\int_\Omega\int_\Omega \frac{r(x)r(y)(u^+(x))^p(u^+(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy.\end{aligned}$$

One can easily see that the Nehari-manifold is closely related to the fibering map such that $tu\in\mathcal{N}_{\gamma,\lambda}$ if and only if $\phi_u'(t)=0$. In particular, $u\in\mathcal{N}_{\gamma,\lambda}$ if and only if $\phi_u'(1)=0$. So it is reasonable to split $\mathcal{N}_{\gamma,\lambda}$ into three parts corresponding to local maxima, local minima and point of inflection as

$$\mathcal{N}^{\pm}_{\gamma,\lambda} := \{u\in\mathcal{N}_{\gamma,\lambda} : \phi_u''(1) \gtrless 0\},\quad \mathcal{N}^{0}_{\gamma,\lambda} := \{u\in\mathcal{N}_{\gamma,\lambda} : \phi_u''(1) = 0\}.$$

For the sign-changing function r, we denote

$$\begin{aligned}R^+ &:= \Big\{u\in X_0 : \int_\Omega\int_\Omega \frac{r(x)r(y)(u^+(x))^p(u^+(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy > 0\Big\},\\ R^- &:= \Big\{u\in X_0 : \int_\Omega\int_\Omega \frac{r(x)r(y)(u^+(x))^p(u^+(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy \le 0\Big\}.\end{aligned}$$

Lemma 3.1. *Let $\gamma\in(0,c\gamma_H)$. Then the following holds:*

(1) *For $u\in R^+$, there exist $\Lambda_1>0$ and a unique $t_{\max} := t_{\max}(u) > 0$, $t^+ = t^+(u) > 0$ and $t^- = t^-(u) > 0$ with $t^+ < t_{\max} < t^-$ such that $t^+u\in\mathcal{N}^+_{\gamma,\lambda}$, $t^-u\in\mathcal{N}^-_{\gamma,\lambda}$, for any $\lambda\in(0,\Lambda_1)$. Also $\mathcal{J}_{\gamma,\lambda}(t^+u) = \min_{0\le t\le t^-}\mathcal{J}_{\gamma,\lambda}(tu)$ and $\mathcal{J}_{\gamma,\lambda}(t^-u) = \max_{t\ge t_{\max}}\mathcal{J}_{\gamma,\lambda}(tu)$.*

(2) *For $u \in R^-$, $\gamma \in (0, c\gamma_H)$ and $\lambda > 0$ there exists a unique $t^* > 0$ such that $t^*u \in N^+_{\gamma,\lambda}$ and $\mathcal{J}_{\gamma,\lambda}(t^*u) = \inf_{t>0} \mathcal{J}_{\gamma,\lambda}(tu)$.*

Proof. For fixed $u \in X_0$, define $\Psi_u \colon \mathbb{R}^+ \to \mathbb{R}$ as

$$\Psi_u(t) := t^{2-2p}[c\|u\|^2 - \gamma\|u^+\|_H^2] + dt^{2\theta-2p}\|u\|^{2\theta} - \lambda t^{1-q-2p} \int_\Omega l(x)(u^+)^{1-q}\,dx. \quad (3.1)$$

We observe that $tu \in \mathcal{N}_{\gamma,\lambda}$ if and only if t is the root of the equation,

$$\Psi_u(t) = \int_\Omega \int_\Omega \frac{r(x)r(y)(u^+(x))^p(u^+(y))^p}{|x|^\alpha |x-y|^\mu |y|^\alpha}\,dx\,dy.$$

Using (3.1), we notice that $\Psi_u(t) \to -\infty$ as $t \to 0^+$ and $\Psi_u(t) \to 0$ as $t \to \infty$. Moreover,

$$\begin{aligned}\Psi_u'(t) &= (2-2p)t^{1-2p}[c\|u\|^2 - \gamma\|u^+\|_H^2] + d(2\theta-2p)t^{2\theta-2p-1}\|u\|^{2\theta}\\ &\quad - \lambda(1-q-2p)t^{-q-2p}\int_\Omega l(x)(u^+)^{1-q}\,dx.\end{aligned}$$

Since, $q < 1 < 2\theta < 2p$, we see that $\lim_{t\to 0^+} \Psi_u^{'}(t) > 0$ and $\lim_{t\to\infty} \Psi_u'(t) < 0$. Then there exists a unique $t_{\max} = t_{\max}(u) > 0$ such that $\Psi_u(t)$ is increasing in $(0, t_{\max})$, decreasing in $(t_{\max}, \infty)$ and $\Psi_u'(t_{\max}) = 0$. Now we estimate, $\Psi_u(t_{\max})$ given as follows

$$\Psi_u(t_{\max}) = \max_{t>0} \Psi_u(t_{\max}) = \max_{t>0}\Big(d\|u\|^{2\theta}t^{2\theta-2p} + \overline{\Psi_u}(t)\Big) > \max_{t>0}\overline{\Psi_u}(t),$$

where

$$\overline{\Psi_u}(t) = [c\|u\|^2 - \gamma\|u^+\|_H^2]t^{2-2p} - \lambda t^{1-q-2p}\int_\Omega l(x)(u^+)^{1-q}\,dx.$$

Now by (1.4), for $\gamma \in (0, c\gamma_H)$, we can calculate

$$\max_{t>0}\overline{\Psi_u}(t) \ge \overline{\phi_u}(t),$$

where

$$\overline{\phi_u}(t) = h_{c,\gamma}\|u\|^2 t^{2-2p} - \lambda t^{1-q-2p}\int_\Omega l(x)(u^+)^{1-q}\,dx.$$

We noticed that

$$\max_{t>0}\overline{\phi_u}(t) = \Big(\frac{q+1}{2p-2}\Big)\Big(\frac{2p-2}{2p+q-1}\Big)^{\frac{2p+q-1}{1+q}} \frac{\big(h_{c,\gamma}\|u\|^2\big)^{\frac{2p+q-1}{1+q}}}{\big(\lambda\int_\Omega l(x)(u^+)^{1-q}\,dx\big)^{\frac{2p-2}{1+q}}}.$$

Now using this, (A1), Sobolev inequality (2.2), and Hölder inequality, we obtain

$$\begin{aligned}&\Psi_u(t_{\max})\\ &\ge \Big(\frac{1+q}{2p-2}\Big)\Big(\frac{h_{c,\gamma}(2p-2)}{2p+q-1}\Big)^{\frac{2p+q-1}{1+q}} \lambda^{\frac{2-2p}{1+q}} (\|l\|_m)^{\frac{2-2p}{1+q}} S^{\frac{(1-q)(2p-2)}{2(1+q)}} \|u\|^{2p} > 0.\end{aligned}$$

Now, according to the behavior of r, we consider two cases

case 1: Let $u \in R^+$. Choose

$$\lambda < \Lambda_1 := \Big(\frac{1+q}{2p-2}\Big)^{\frac{1+q}{2p-2}}\Big(\frac{h_{c,\gamma}(2p-2)}{2p+q-1}\Big)^{\frac{(2p+q-1)}{(2p-2)}} S^{\frac{(2p+q-1)}{2(p-1)}} \frac{1}{\|l\|_m}\Big[\frac{1}{C_r(\alpha,\mu,N)}\Big]^{\frac{1+q}{2p-2}},$$

then there exists a unique $t^+ := t^+(u) < t_{\max}$ and $t^- := t^-(u) > t_{\max}$, such that

$$\Psi_u(t^+) = \int_\Omega \int_\Omega \frac{r(x)r(y)(u^+(x))^p(u^+(y))^p}{|x|^\alpha |x-y|^\mu |y|^\alpha}\,dx\,dy = \Psi_u(t^-),$$

that is $t^+u, t^-u \in \mathcal{N}_{\gamma,\lambda}$. Now using the relation $\phi''_{tu}(1) = t^{2p+1}\Psi'_u(t)$, it follows that $\Psi'_u(t^+) > 0$ and $\Psi'_u(t^-) < 0$, which yield $t^+u \in \mathcal{N}^+_{\gamma,\lambda}$ and $t^-u \in \mathcal{N}^-_{\gamma,\lambda}$. Furthermore, assuming

$$\phi'_u(t) = t^{2p-1}\Big(\Psi_u(t) - \int_\Omega \int_\Omega \frac{r(x)r(y)(u^+(x))^p(u^+(y))^p}{|x|^\alpha |x-y|^\mu |y|^\alpha}\,dx\,dy\Big),$$

we have $\phi'_u(t) < 0$ for all $t \in [0, t^+)$ and $\phi'_u(t) > 0$ for all $t \in (t^+, t^-)$. Therefore,

$$\mathcal{J}_{\gamma,\lambda}(t^+u) = \min_{0\le t\le t^-} \mathcal{J}_{\gamma,\lambda}(tu).$$

Also, $\phi'_u(t) > 0$ for all $t \in [t^+, t^-)$, $\phi'_u(t^-) = 0$ and $\phi'_u(t) < 0$ for all $t \in (t^-, \infty)$ yield that

$$\mathcal{J}_{\gamma,\lambda}(t^-u) = \max_{t\ge t_{\max}} \mathcal{J}_{\gamma,\lambda}(tu).$$

case 2: Let $u \in R^-$. Then using the fact $\Psi_u(t) \to -\infty$ as $t \to 0^+$, there exists a unique $t^* > 0$ such that

$$\Psi_u(t^*) = \int_\Omega \int_\Omega \frac{r(x)r(y)(u^+(x))^p(u^+(y))^p}{|x|^\alpha |x-y|^\mu |y|^\alpha}\,dx\,dy \quad \text{for all } \lambda > 0.$$

As $u \in R^-$, we have $\Psi_u(t^*) < 0$ and $\Psi'_u(t^*) > 0$. Now by doing the similar calculations as in case(1), $\phi''_{tu}(1) = t^{2p+1}\Psi'_u(t)$ with $\Psi'_u(t^*) > 0$, we deduce that $t^*u \in N^+_{\lambda,\gamma}$. This completes the proof. □

Lemma 3.2. *Let $\gamma \in (0, c\gamma_H)$, then there exists $\Lambda_2 > 0$ such that $\mathcal{N}^0_{\gamma,\lambda} = \{0\}$, for all $\lambda \in (0, \Lambda_2)$.*

Proof. Arguing by contradiction argument, we suppose that there exists $u \in \mathcal{N}^0_{\gamma,\lambda} \setminus \{0\}$ for all $\lambda \in (0, \Lambda_2)$. Then we have

$$\begin{aligned} &c\|u\|^2 + d\|u\|^{2\theta} - \gamma\|u^+\|_H^2 - \lambda \int_\Omega l(x)(u^+)^{1-q}\,dx \\ &- \int_\Omega \int_\Omega \frac{r(x)r(y)(u^+(x))^p(u^+(y))^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} = 0, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &c\|u\|^2 + d(2\theta-1)\|u\|^{2\theta} - \gamma\|u^+\|_H^2 + \lambda q \int_\Omega l(x)(u^+)^{1-q}\,dx \\ &- (2p-1)\int_\Omega \int_\Omega \frac{r(x)r(y)(u^+(x))^p(u^+(y))^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} = 0. \end{aligned} \tag{3.3}$$

Now we consider two cases:

Case 1: $\int_\Omega \int_\Omega \frac{r(x)r(y)(u^+(x))^p(u^+(y))^p}{|x|^\alpha |x-y|^\mu |y|^\alpha}\,dx\,dy = 0$. Using this (3.2), (3.3), $2\theta \ge 1 > 1-q$, and $\gamma \in (0, c\gamma_H)$ with (2.9), we obtain

$$\begin{aligned} 0 &= (1+q)[c\|u\|^2 - \gamma\|u^+\|_H^2] + d(2\theta+q-1)\|u\|^{2\theta} \\ &\ge h_{c,\gamma}(1+q)\|u\|^2 + d(2\theta+q-1)\|u\|^{2\theta} > 0, \end{aligned}$$

which gives a contradiction.

Case 2: $\int_\Omega \int_\Omega \frac{r(x)r(y)(u^+(x))^p(u^+(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy \neq 0$. Equations (3.2) and (3.3) yield

$$\begin{aligned} &(2p-2)[c\|u\|^2 - \gamma\|u^+\|_H^2] + d(2p-2\theta)\|u\|^{2\theta} \\ &- \lambda(2p+q-1)\int_\Omega l(x)(u^+)^{1-q}\,dx = 0, \end{aligned} \tag{3.4}$$

$$\begin{aligned} &(1+q)\Big[c\|u\|^2 - \gamma\|u^+\|_H^2\Big] + d(2\theta+q-1)\|u\|^{2\theta} \\ &- (2p+q-1)\int_\Omega\int_\Omega \frac{r(x)r(y)(u^+(x))^p(u^+(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha} = 0. \end{aligned} \tag{3.5}$$

Let us define $\mathcal{E}_{\gamma,\lambda}\colon \mathcal{N}_{\gamma,\lambda} \to \mathbb{R}$ as

$$\begin{aligned} \mathcal{E}_{\gamma,\lambda} := &\frac{(1+q)[c\|u\|^2 - \gamma\|u^+\|_H^2] + d(2\theta+q-1)\|u\|^{2\theta}}{(2p+q-1)} \\ &- \int_\Omega\int_\Omega \frac{r(x)r(y)(u^+(x))^p(u^+(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy. \end{aligned}$$

Then from (3.5), it follows that $\mathcal{E}_{\gamma,\lambda} = 0$ for all $u \in \mathcal{N}^0_{\gamma,\lambda}$. Also using (2.1), (2.9), and $(a+b) \geq 2\sqrt{ab}$, for any $a, b \geq 0$, we obtain

$$\begin{aligned} \mathcal{E}_{\gamma,\lambda} &\geq \frac{2\sqrt{(1+q)(2\theta+q-1)h_{c,\gamma}d}\|u\|^{\theta+1}}{(2p+q-1)} - C_r(\alpha,\mu,N)S^{-p}\|u\|^{2p}, \\ &= \|u\|^{2p}\Big[\frac{2\sqrt{(1+q)(2\theta+q-1)h_{c,\gamma}d}}{(2p+q-1)\|u\|^{2p-\theta-1}} - C_r(\alpha,\mu,N)S^{-p}\Big]. \end{aligned}$$

Now using (2.9), Sobolev inequality (2.2), $A.M \geq G.M$ and Hölder inequality in (3.4), we have

$$\|u\| \leq \Big[\frac{\lambda(2p+q-1)\|l\|_m S^{\frac{-(1-q)}{2}}}{2\sqrt{(2p-2)(2p-2\theta)h_{c,\gamma}d}}\Big]^{\frac{1}{\theta+q}}.$$

Thus,

$$\begin{aligned} \lambda < \Lambda_2 := &\frac{S^{\frac{(\theta+1)(2p+q-1)}{2(2p-\theta-1)}}}{[C_r(\alpha,\mu,N)]^{\frac{\theta+q}{2p-\theta-1}}}\Big[\frac{2\sqrt{(1+q)(2\theta+q-1)h_{c,\gamma}d}}{2p+q-1}\Big]^{\frac{\theta+q}{2p-\theta-1}} \\ &\times \Big[\frac{2\sqrt{(2p-2)(2p-2\theta)h_{c,\gamma}d}}{(2p+q-1)\|l\|_m}\Big]. \end{aligned}$$

Hence $\mathcal{E}_{\gamma,\lambda}(u) > 0$ for all $u \in \mathcal{N}^0_{\gamma,\lambda} \setminus \{0\}$, which gives a contradiction. □

Lemma 3.3. *Let $\gamma \in (0, c\gamma_H)$ and $\lambda \in (0, \Lambda_2)$, then there exists a gap structure in $\mathcal{N}_{\gamma,\lambda}$ such that*

$$\|U\| > A_0 > A_1 > \|u\|,$$

for any $u \in \mathcal{N}^+_{\gamma,\lambda}$, $U \in \mathcal{N}^-_{\gamma,\lambda}$, where

$$\begin{aligned} A_0 &:= \Big[\frac{2\sqrt{(1+q)(2\theta+q-1)h_{c,\gamma}d}}{(2p+q-1)C_r(\alpha,\mu,N)S^{-p}}\Big]^{\frac{1}{2p-\theta-1}}, \\ A_1 &:= \Big[\frac{\lambda(2p+q-1)S^{\frac{-(1-q)}{2}}\|l\|_m}{2\sqrt{(2p-2)(2p-2\theta)h_{c,\gamma}d}}\Big]^{\frac{1}{\theta+q}}. \end{aligned}$$

Proof. If $u \in \mathcal{N}^+_{\gamma,\lambda} \subset \mathcal{N}_{\gamma,\lambda}$, then

$$\begin{aligned}(2p-2)[c\|u\|^2 - \gamma\|u^+\|_H^2] + d(2p-2\theta)\|u\|^{2\theta}\\ - \lambda(2p+q-1)\int_\Omega l(x)(u^+)^{1-q}dx < 0.\end{aligned} \tag{3.6}$$

Using (A1), (2.2), and Hölder inequality in (3.6), it follows that

$$\begin{aligned}&(2p-2)[c\|u\|^2 - \gamma\|u^+\|_H^2] + d(2p-2\theta)\|u\|^{2\theta}\\ &\le \lambda(2p+q-1)\|l\|_m S^{\frac{-(1-q)}{2}}\|u\|^{1-q}.\end{aligned}$$

This and (2.9) yield

$$(2p-2)h_{c,\gamma}\|u\|^2 + d(2p-2\theta)\|u\|^{2\theta} < \lambda(2p+q-1)\|l\|_m S^{\frac{-(1-q)}{2}}\|u\|^{1-q}. \tag{3.7}$$

At this moment, applying the condition $AM \ge GM$ in (3.7), we have

$$\|u\| < \Big[\frac{\lambda(2p+q-1)S^{\frac{-(1-q)}{2}}\|l\|_m}{2\sqrt{(2p-2)(2p-2\theta)h_{c,\gamma}d}}\Big]^{\frac{1}{\theta+q}} := A_1.$$

Now, if $U \in N^-_{\lambda,\gamma}$ then using (2.1), we have

$$\begin{aligned}&(1+q)[c\|U\|^2 - \gamma\|U^+\|_H^2] + d(2\theta+q-1)\|U\|^{2\theta}\\ &< (2p+q-1)C_r(\alpha,\mu,N)S^{-p}\|U\|^{2p}.\end{aligned}$$

By this, (2.9), and $AM \ge GM$, we obtain

$$2\sqrt{(1+q)(2\theta+q-1)h_{c,\gamma}d}\|U\|^{\theta+1} < (2p+q-1)C_r(\alpha,\mu,N)S^{-p}\|U\|^{2p}.$$

Thus

$$\|U\| > \Big[\frac{2\sqrt{(1+q)(2\theta+q-1)h_{c,\gamma}d}}{(2p+q-1)C_r(\alpha,\mu,N)S^{-p}}\Big]^{\frac{1}{2p-\theta-1}} := A_0.$$

It is easy to check that $A_0 > A_1$ for all $\lambda \in (0,\Lambda_2)$. Thus we conclude that

$$\|U\| > A_0 > A_1 > \|u\| \quad \text{for all } u \in \mathcal{N}^+_{\gamma,\lambda},\ U \in \mathcal{N}^-_{\gamma,\lambda}.$$

This completes the proof. □

Lemma 3.4. *Let $\gamma \in (0, c\gamma_H)$, then $\mathcal{N}^-_{\gamma,\lambda}$ is a closed set in X_0 topology for all $\lambda \in (0,\Lambda_2)$.*

Proof. Assume $\{u_k\}_k$ be any sequence in $\mathcal{N}^-_{\gamma,\lambda}$, such that $u_k \to u$ strongly in X_0. Then $u \in \mathcal{N}^-_{\gamma,\lambda} \cup \{0\}$. Now by Lemma 3.3, we obtain

$$\|u\| = \lim_{k\to\infty}\|u_k\| \ge A_0 > A_1 > 0.$$

Thus the above inequality, yields $u \not\equiv 0$. Hence $u \in \mathcal{N}^-_{\gamma,\lambda}$. □

Lemma 3.5. *Let $u \in \mathcal{N}^\pm_{\gamma,\lambda}$ with $\gamma \in (0, c\gamma_H)$ and $\lambda > 0$. Then there is a $\epsilon > 0$ and a continuous function $\xi\colon B_\epsilon(0) \to \mathbb{R}^+$ such that $\xi(v) > 0$, $\xi(0) = 1$ and $\xi(v)(u+v) \in \mathcal{N}^\pm_{\gamma,\lambda}$ for all $v \in B_\epsilon(0)$, where $B_\epsilon(0) = \{v \in X_0 : \|v\| < \epsilon\}$.*

Proof. Here we only give the proof for the case $u \in \mathcal{N}^+_{\gamma,\lambda}$, while the proof of the case $\mathcal{N}^-_{\gamma,\lambda}$ follows similarly. Define $F\colon X_0 \times \mathbb{R}^+ \to \mathbb{R}$ as

$$\begin{aligned}
F(v,z) := & z^{1+q}\Big(c\|u+v\|^2 - \gamma\|(u+v)^+\|_H^2\Big) + z^{2\theta-1+q} d\|u+v\|^{2\theta} \\
& - \lambda \int_\Omega l(x)\left((u+v)^+\right)^{1-q}\,dx \\
& - z^{2p+q-1}\int_\Omega\int_\Omega \frac{r(x)r(y)\big((u+v)^+(x)\big)^p\big((u+v)^+(y)\big)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy.
\end{aligned}$$

Since $u \in \mathcal{N}^+_{\gamma,\lambda} \subset \mathcal{N}_{\gamma,\lambda}$, it follows that

$$\begin{aligned}
F(0,1) = & c\|u\|^2 + d\|u\|^{2\theta} - \gamma\|u^+\|_H^2 - \lambda\int_\Omega l(x)(u^+)^{1-q}\,dx \\
& - \int_\Omega\int_\Omega \frac{r(x)r(y)(u^+(x))^p(u^+(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy = 0,
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
\frac{\partial F}{\partial z}(0,1) = & (1+q)\big(c\|u\|^2 - \gamma\|u^+\|_H^2\big) + d(2\theta-1+q)\|u\|^{2\theta} \\
& - (2p+q-1)\int_\Omega\int_\Omega \frac{r(x)r(y)(u^+(x))^p(u^+(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy > 0.
\end{aligned} \tag{3.9}$$

Now applying the Implicit function theorem to the function F at $(0,1)$, there exists $\bar\epsilon > 0$ such that for every $v \in X_0$ with $\|v\| < \bar\epsilon$, then $F(v,z) = 0$ has a unique solution $z = \xi(v) > 0$. From (3.8), we have $\xi(0) = 1$. This together with $F(v,\xi(v)) = 0$ for any $v \in X_0$ with $\|v\| < \bar\epsilon$ yields that

$$\begin{aligned}
0 = & \xi(v)^{1+q}\Big(c\|u+v\|^2 - \gamma\|(u+v)^+\|_H^2\Big) + \xi(v)^{2\theta-1+q} d\|u+v\|^{2\theta} \\
& - \lambda\int_\Omega l(x)\big((u+v)^+\big)^{1-q}\,dx \\
& - \xi(v)^{2p+q-1}\int_\Omega\int_\Omega \frac{r(x)r(y)((u+v)^+(x))^p((u+v)^+(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy \\
= & \frac{1}{\xi^{1-q}(v)}\Big(c\|\xi(v)(u+v)\|^2 - \gamma\|\xi(v)(u+v)^+\|_H^2 - \lambda\int_\Omega l(x)\big(\xi(v)(u+v)^+\big)^{1-q}\,dx \\
& + d\|\xi(v)(u+v)\|^{2\theta} \\
& - \int_\Omega\int_\Omega \frac{r(x)r(y)(\xi(v)(u+v)^+(x))^p(\xi(v)(u+v)^+(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy\Big).
\end{aligned}$$

This implies, $\xi(v)(u+v) \in \mathcal{N}_{\gamma,\lambda}$ for any $v \in X_0$ with $\|v\| < \bar\epsilon$. We notice that

$$\begin{aligned}
& \frac{\partial F}{\partial z}\Big|_{(v,\xi(v))} \\
& = \frac{1}{\xi^{2-q}(v)}\Big[(1+q)\Big(c\|\xi(v)(u+v)\|^2 - \gamma\|\xi(v)(u+v)^+\|_H^2\Big) \\
& \quad + (2\theta-1+q)d\|\xi(v)(u+v)\|^{2\theta} \\
& \quad - (2p+q-1)\int_\Omega\int_\Omega \frac{r(x)r(y)(\xi(v)(u+v)^+(x))^p(\xi(v)(u+v)^+(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy\Big].
\end{aligned}$$

Now using (3.9), take $\epsilon > 0$ such that $\epsilon < \bar{\epsilon}$, for any $v \in X_0$ with $\|v\| < \epsilon$, we obtain

$$(1+q)\left(c\|\xi(v)(u+v)\|^2 - \gamma\|\xi(v)(u+v)^+\|_H^2\right) + (2\theta - 1 + q)d\|\xi(v)(u+v)\|^{2\theta}$$
$$- (2p+q-1)\int_\Omega\int_\Omega \frac{r(x)r(y)(\xi(v)(u+v)^+(x))^p(\xi(v)(u+v)^+(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy > 0,$$

that is $\xi(v)(u+v) \in \mathcal{N}^+_{\gamma,\lambda}$ for all $v \in B_\epsilon(0)$. □

Lemma 3.6. *Let $\gamma \in (0, c\gamma_H)$ and $\lambda > 0$, then the functional $\mathcal{J}_{\gamma,\lambda}$ is coercive and bounded from below on $\mathcal{N}_{\gamma,\lambda}$.*

Proof. Let $u \in \mathcal{N}_{\gamma,\lambda}$. Then this together with (2.9), (f_1), (2.2), and Hölder inequality with the condition $2\theta < 2p$, give

$$\begin{aligned}\mathcal{J}_{\gamma,\lambda}(u) &:= \Big(\frac{1}{2} - \frac{1}{2p}\Big)\Big(c\|u\|^2 - \gamma\|u^+\|_H^2\Big) + \Big(\frac{1}{2\theta} - \frac{1}{2p}\Big)d\|u\|^{2\theta} \\ &\quad - \lambda\Big(\frac{1}{1-q} - \frac{1}{2p}\Big)\int_\Omega l(x)(u^+)^{1-q}\,dx \\ &\geq \Big(\frac{1}{2} - \frac{1}{2p}\Big)h_{c,\gamma}\|u\|^2 - \lambda\Big(\frac{1}{1-q} - \frac{1}{2p}\Big)\|l\|_m S^{\frac{-(1-q)}{2}}\|u\|^{1-q},\end{aligned}$$

since $2 > 1 - q$. Hence $\mathcal{J}_{\gamma,\lambda}$ is coercive on $\mathcal{N}_{\gamma,\lambda}$.

let

$$\mathcal{G}(t) := \Big(\frac{1}{2} - \frac{1}{2p}\Big)h_{c,\gamma}t^{\frac{2}{1-q}} - \lambda\Big(\frac{1}{1-q} - \frac{1}{2p}\Big)\|l\|_m S^{\frac{-(1-q)}{2}}t,$$

which has minimum at

$$t_{\min} := \Big(\frac{\lambda(2p+q-1)S^{\frac{-(1-q)}{2}}\|l\|_m}{(2p-2)h_{c,\gamma}}\Big)^{\frac{1-q}{1+q}}.$$

So, we have

$$\mathcal{J}_{\gamma,\lambda}(u) \geq \frac{-(1+q)}{4p(1-q)}\frac{\Big(\lambda(2p+q-1)S^{\frac{-(1-q)}{2}}\|l\|_m\Big)^{\frac{2}{(1+q)}}}{((2p-2)h_{c,\gamma})^{\frac{1-q}{1+q}}} = -C(\text{let}),$$

where $C > 0$. Hence $\mathcal{J}_{\gamma,\lambda}$ is bounded below on $\mathcal{N}_{\gamma,\lambda}$. □

4. A compactness result for $\mathcal{J}_{\gamma,\lambda}$

By Lemma 3.4, $\mathcal{N}^+_{\gamma,\lambda} \cup \{0\}$ and $\mathcal{N}^-_{\gamma,\lambda}$ are two closed sets in X_0 for $\lambda < \Lambda_2$. Hence the Ekeland's Variational principle can be applied to the problem of finding the infimum of $\mathcal{J}_{\gamma,\lambda}$ both on $\mathcal{N}^+_{\gamma,\lambda} \cup \{0\}$ and $\mathcal{N}^-_{\gamma,\lambda}$.

We define

$$m^+_{\gamma,\lambda} := \inf_{u \in \mathcal{N}^+_{\gamma,\lambda}\cup\{0\}} \mathcal{J}_{\gamma,\lambda}(u) \quad \text{and} \quad m^-_{\gamma,\lambda} := \inf_{u \in \mathcal{N}^-_{\gamma,\lambda}} \mathcal{J}_{\gamma,\lambda}(u).$$

By Ekeland's variational principle, we can find a minimizing sequence $\{u_k\} \subset \mathcal{N}^+_{\gamma,\lambda} \cup \{0\}(\mathcal{N}^-_{\gamma,\lambda})$ for $\mathcal{J}_{\gamma,\lambda}$ such that

$$m^\pm_{\gamma,\lambda} < \mathcal{J}_{\gamma,\lambda}(u_k) < m^\pm_{\gamma,\lambda} + \frac{1}{k}, \quad \text{and} \quad \mathcal{J}_{\gamma,\lambda}(u) \geq \mathcal{J}_{\gamma,\lambda}(u_k) + \frac{1}{k}\|u - u_k\|. \tag{4.1}$$

In view of Lemma 3.6, we see that $\{u_k\}_k$ is a bounded sequence in $\mathcal{N}_{\gamma,\lambda}$ with $\|u_k\| \le C_1$ with $C_1 > 0$. Therefore up to a subsequence still denoted by $\{u_k\}$, we may assume that there exists $u_0 \in X_0$ such that

$$u_k \rightharpoonup u_0 \quad \text{weakly in } X_0. \tag{4.2}$$

To prove our main result we need the following Lemmas.

Lemma 4.1. *Let $\{u_k\}_k \subset \mathcal{N}^+_{\gamma,\lambda}$ satisfy (4.2) with $\gamma \in (0, c\gamma_H)$ and $\lambda \in (0, \Lambda_1)$, where Λ_1 is given in Lemma 3.1. Then there exists a constant $C_2 > 0$ such that*

(1) *if $\{u_k\} \subset \mathcal{N}^+_{\gamma,\lambda}$ for each $k \in \mathbb{N}$, we have*

$$\begin{aligned}&(1+q)[c\|u_k\|^2 - \gamma\|u_k^+\|_H^2] + d(2\theta+q-1)\|u_k\|^{2\theta}\\&- (2p+q-1)\int_\Omega\int_\Omega \frac{r(x)r(y)(u_k^+(x))^p(u_k^+(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy \ge C_2,\end{aligned}$$

(2) *if $\{u_k\} \subset \mathcal{N}^-_{\gamma,\lambda}$ for each $k \in \mathbb{N}$, we have*

$$\begin{aligned}&(1+q)[c\|u_k\|^2 - \gamma\|u_k^+\|_H^2] + d(2\theta+q-1)\|u_k\|^{2\theta}\\&- (2p+q-1)\int_\Omega\int_\Omega \frac{r(x)r(y)(u_k^+(x))^p(u_k^+(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy \le -C_2.\end{aligned}$$

Proof. We present the proof of (1) only, the proof of (2) follows similarly. Since $\{u_k\}_k \subset \mathcal{N}^+_{\gamma,\lambda}$, it is enough to show that

$$\begin{aligned}&\liminf_{k\to\infty}\big[(2p-2)\big(c\|u_k\|^2 - \gamma\|u_k^+\|_H^2\big) + d(2p-2\theta)\|u_k\|^{2\theta}\big]\\&< \lambda(2p+q-1)\int_\Omega l(x)(u_0^+)^{1-q}dx.\end{aligned}$$

For this, we argue by contradiction, so we assume that

$$\begin{aligned}&\liminf_{k\to\infty}\big[(2p-2)\big(c\|u_k\|^2 - \gamma\|u_k^+\|_H^2\big) + d(2p-2\theta)\|u_k\|^{2\theta}\big]\\&= \lambda(2p+q-1)\int_\Omega l(x)(u_0^+)^{1-q}\,dx.\end{aligned}$$

Since $\{u_k\}_k \in \mathcal{N}^+_{\gamma,\lambda}$, we have

$$(2p-2)\Big(c\|u_k\|^2 - \gamma\|u_k^+\|_H^2\Big) + d(2p-2\theta)\|u_k\|^{2\theta} < \lambda(2p+q-1)\int_\Omega l(x)(u_k^+)^{1-q}dx.$$

Now by $l \in L^{\frac{2_s^*}{2_s^*-1+q}}(\Omega)$ and by Vitali's convergence theorem, we can show that

$$\lim_{k\to\infty}\int_\Omega l(x)(u_k^+)^{1-q}dx = \int_\Omega l(x)(u_0^+)^{1-q}\,dx.$$

Then

$$\begin{aligned}&\liminf_{k\to\infty}\big[(2p-2)\big(c\|u_k\|^2 - \gamma\|u_k^+\|_H^2\big) + d(2p-2\theta)\|u_k\|^{2\theta}\big]\\&\le \limsup_{k\to\infty}\big[(2p-2)\big(c\|u_k\|^2 - \gamma\|u_k^+\|_H^2\big) + d(2p-2\theta)\|u_k\|^{2\theta}\big]\\&\le \lambda(2p+q-1)\int_\Omega l(x)(u_0^+)^{1-q}dx,\end{aligned}$$

which yields

$$\begin{aligned}&\lim_{k\to\infty}\Big[(2p-2)\big(c\|u_k\|^2-\gamma\|u_k^+\|_H^2\big)+d(2p-2\theta)\|u_k\|^{2\theta}\Big]\\&=\lambda(2p+q-1)\int_\Omega l(x)(u_0^+)^{1-q}dx.\end{aligned}\tag{4.3}$$

From (4.3), it is clear that there exists $A>0$ and $A_\gamma>0$ with $h_{c,\gamma}\le A_\gamma\le cA$ for $\gamma\in(0,c\gamma_H)$, by (2.9) such that

$$c\|u_k\|^2-\gamma\|u_k^+\|_H^2\to A_\gamma,\quad \|u_k\|^2\to A\quad\text{as } k\to\infty.$$

Then, we have

$$(2p-2)A_\gamma+d(2p-2\theta)A^\theta=\lambda(2p+q-1)\int_\Omega l(x)(u_0^+)^{1-q}dx,$$

which gives that

$$\lambda\int_\Omega l(x)(u_0^+)^{1-q}\,dx=\frac{(2p-2)A_\gamma}{(2p+q-1)}+\frac{d(2p-2\theta)A^\theta}{(2p+q-1)}.\tag{4.4}$$

By Lemma 3.1, for $\lambda\in(0,\Lambda_1)$, we have

$$\begin{aligned}0\le&\Big(\frac{1+q}{2p-2}\Big)\Big(\frac{2p-2}{2p+q-1}\Big)^{\frac{2p+q-1}{1+q}}(h_{c,\gamma}A)^{\frac{2p+q-1}{1+q}}\Big(\lambda\int_\Omega l(x)(u_0^+)^{1-q}dx\Big)^{\frac{2-2p}{1+q}}\\&-\lim_{k\to\infty}\int_\Omega\int_\Omega\frac{r(x)r(y)(u_k^+(x))^p(u_k^+(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy.\end{aligned}\tag{4.5}$$

Using $\{u_k\}_k\subset\mathcal{N}_{\gamma,\lambda}^+\subset\mathcal{N}_{\gamma,\lambda}$ and (4.4), we obtain

$$\begin{aligned}&\lim_{k\to\infty}\int_\Omega\int_\Omega\frac{r(x)r(y)(u_k^+(x))^p(u_k^+(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy\\&=A_\gamma\Big(\frac{1+q}{2p+q-1}\Big)+dA^\theta\Big(\frac{2\theta+q-1}{2p+q-1}\Big).\end{aligned}$$

Now using (4.3), (4.5) in (4.4), together with $h_{c,\gamma}A\le A_\gamma$, we obtain

$$dA^\theta\Big(\frac{2\theta+q-1}{2p+q-1}\Big)\le 0,$$

which is a contradiction. Thus the proof is complete. □

Fix $\psi\in X_0$ with $\psi\ge0$. Consider the constants $C_1>0$ with $\|u_k\|\le C_1$ and $C_2>0$ given in the Lemma 4.1. Then for $k\in\mathbb{N}$ sufficiently large such that

$$\frac{(1-q)C_1}{k}<C_2.\tag{4.6}$$

By Lemma 3.5, we can extract a sequence of functions $(\xi_k)_k$ such that $\xi_k(0)=1$ and $\xi_k(t\psi)(u_k+t\psi)\in\mathcal{N}_{\gamma,\lambda}^\pm$ for $t>0$ sufficiently small. Using $u_k\in\mathcal{N}_{\gamma,\lambda}$ and $\xi_k(t\psi)(u_k+t\psi)\in\mathcal{N}_{\gamma,\lambda}$, we have

$$\begin{aligned}&c\|u_k\|^2+d\|u_k\|^{2\theta}-\gamma\|u_k^+\|_H^2-\lambda\int_\Omega l(x)(u_k^+)^{1-q}\,dx\\&-\int_\Omega\int_\Omega\frac{r(x)r(y)(u_k^+(x))^p(u_k^+(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy=0\end{aligned}\tag{4.7}$$

and

$$
\begin{aligned}
&\xi_k^2(t\psi)\Big(c\|u_k+t\psi\|^2-\gamma\|(u_k+t\psi)^+\|_H^2\Big)\\
&-\lambda\xi_k^{1-q}(t\psi)\int_\Omega l(x)\left((u_k+t\psi)^+)\right)^{1-q}\,dx+d\xi_k^{2\theta}(t\psi)\|u_k+t\psi\|^2\\
&-\xi_k^{1-q}(t\psi)\int_\Omega\int_\Omega\frac{r(x)r(y)\left((u_k+t\psi)^+(x)\right)^p\left((u_k+t\psi)^+(y)\right)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy=0.
\end{aligned}
\tag{4.8}
$$

Now denote $\xi_k'(0)$ is the derivative of ξ_k at point 0 with $\langle\xi_k'(0),\psi\rangle\in[-\infty,\infty]$ for any $\psi\in X_0$. If it does not exist then $\xi_k'(0)$ can be replace by $p_k(0)=\lim_{k\to\infty}\frac{\xi_k(t\psi)-1}{t_k}$ for some $(t_k)_k$ such that $t_k\to 0$ as $k\to\infty$ and $t_k>0$.

Lemma 4.2. *Let $\lambda\in(0,\Lambda_1)$, $\gamma\in(0,c\gamma_H)$ and let $\{u_k\}_k\subset\mathcal{N}_{\gamma,\lambda}^{\pm}$ satisfying (4.1) and (4.2). Then $\langle\xi_k'(0),\psi\rangle$ is uniformly bounded for every $\psi\in X_0$ with $\psi\geq 0$.*

Proof. Here we prove only for the case $\mathcal{N}_{\gamma,\lambda}^{+}$. The case $\mathcal{N}_{\gamma,\lambda}^{-}$ can be prove in a similar manner. Let $\{u_k\}_k\subset\mathcal{N}_{\gamma,\lambda}^{+}$ and $\xi_k(t\psi)(u_k+t\psi)\in\mathcal{N}_{\gamma,\lambda}^{+}$. Then in view of (4.7) and (4.8), we obtain

$$
\begin{aligned}
0&=[\xi_k^2(t\psi)-1]\Big(c\|u_k+t\psi\|^2-\gamma\|(u_k+t\psi)^+\|_H^2\Big)\\
&\quad+\Big(c\|u_k+t\psi\|^2-\gamma\|(u_k+t\psi)^+\|_H^2\Big)\\
&\quad-\Big(c\|u_k\|^2-\gamma\|u_k^+\|_H^2\Big)+d\Big([\xi_k^{2\theta}(t\psi)-1]\|u_k+t\psi\|^{2\theta}+\|u_k+t\psi\|^{2\theta}-\|u_k\|^{2\theta}\Big)\\
&\quad-\lambda\int_\Omega l(x)\big[\big((u_k+t\psi)^+\big)^{1-q}-\big(u_k^+\big)^{1-q}\big]dx\\
&\quad-\lambda[\xi_k^{1-q}(t\psi)-1]\int_\Omega l(x)\big((u_k+t\psi)^+\big)^{1-q}\,dx\\
&\quad-[\xi_k^2(t\psi)-1]\int_\Omega\int_\Omega\frac{r(x)r(y)(u_k+t\psi)^+(x))^p(u_k+t\psi)^+(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy\\
&\quad-\int_\Omega\int_\Omega r(x)r(y)\Big[\frac{\big((u_k+t\psi)^+(x)\big)^p\big((u_k+t\psi)^+(y)\big)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\\
&\quad-\frac{\big(u_k^+(x)\big)^p\big(u_k^+(y)\big)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\Big]\,dx\,dy.
\end{aligned}
$$

Now dividing the above estimate by $t>0$ and taking the limit $t\to 0^+$, we obtain

$$
\begin{aligned}
0&=\langle\xi_k^{'}(0),\psi\rangle\Big[2\left(c\|u_k\|^2-\gamma\|u_k^+\|_H^2\right)+d2\theta\|u_k\|^{2\theta}\\
&\quad-\lambda(1-\gamma)\int_\Omega l(x)\big((u_k)^+\big)^{1-q}\\
&\quad-2p\int_\Omega\int_\Omega\frac{r(x)r(y)(u_k^+(x))^p(u_k^+(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy\Big]\\
&\quad+\big(2c+2\theta d\|u_k\|^{2\theta-2}\big)\int_\Omega\int_\Omega\frac{(u_k(x)-u_k(y))\left(\psi(x)-\psi(y)\right)}{|x-y|^{N+2s}}\,dx\,dy\\
&\quad-2\gamma\int_\Omega\frac{u_k^+\psi}{|x|^{2s}}\,dx-2p\int_\Omega\int_\Omega\frac{r(x)r(y)(u_k^+(x))^{p-1}\psi(x)(u_k^+(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy.
\end{aligned}
$$

This together with (4.7) yields

$$
\begin{aligned}
0 \le \langle \xi_k'(0), \psi\rangle \Big[&(1+q)\left(c\|u_k\|^2 - \gamma\|u_k^+\|_H^2\right) + d(2\theta+q-1)\|u_k\|^{2\theta} \\
&- (2p-q+1)\int_\Omega\int_\Omega \frac{r(x)r(y)(u_k^+(x))^p(u_k^+(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy\Big] \\
&+ \left(2c + 2\theta d\|u_k\|^{2\theta-2}\right)\int_\Omega\int_\Omega \frac{(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&- 2\gamma\int_\Omega \frac{u_k^+\psi}{|x|^{2s}}\,dx - 2p\int_\Omega\int_\Omega \frac{r(x)r(y)(u_k^+(x))^{p-1}\psi(x)(u_k^+(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy.
\end{aligned}
$$

Then by Lemma 4.1(1) and boundedness of the sequence $\{u_k\}$ in above, we obtain $\langle \xi_k'(0), \psi\rangle$ is bounded from below for any $\psi \in X_0$ with $\psi \ge 0$.

Now we show that $\langle \xi_k'(0), \psi\rangle$ is bounded from above. Arguing by contradiction, we assume that $\langle \xi_k'(0), \psi\rangle = \infty$. Note that

$$
|\xi_k(t\psi) - 1|\|u_k\| + \xi_k(t\psi)\|t\psi\| \ge \|\xi_k(t\psi)(u_k + t\psi) - u_k\| \tag{4.9}
$$

and $\xi_k(t\psi) > \xi_k(0) = 1$ for sufficiently large k. Then by the definition of $\xi_k'(0)$ and (4.1) with $u = \xi_k(t\psi)(u_k + t\psi) \in \mathcal{N}_{\gamma,\lambda}^+$, we obtain

$$
\begin{aligned}
&|\xi_k(t\psi) - 1|\frac{\|u_k\|}{k} + \xi_k(t\psi)\frac{\|t\psi\|}{k} \\
&\ge \frac{1}{k}\|\xi_k(t\psi)(u_k + t\psi) - u_k\| \\
&\ge \mathcal{J}_{\gamma,\lambda}(u_k) - \mathcal{J}_{\gamma,\lambda}\left(\xi_k(t\psi)(u_k + t\psi)\right) \\
&= \Big(\frac{1}{1-q} - \frac{1}{2}\Big)\Big[\Big(c\|(u_k+t\psi)\|^2 - \gamma\|\left((u_k+t\psi)^+\right)\|_H^2\Big) - \Big(c\|u_k\|^2 - \gamma\|u_k^+\|_H^2\Big)\Big] \\
&\quad + \Big(\frac{1}{1-q} - \frac{1}{2\theta}\Big)d\Big(\|u_k+t\psi\|^{2\theta} - \|u_k\|^{2\theta}\Big) \\
&\quad + \Big(\frac{1}{1-q} - \frac{1}{2\theta}\Big)d[\xi_k^{2\theta}(t\psi) - 1]\|u_k+t\psi\|^{2\theta} \\
&\quad + \Big(\frac{1}{1-q} - \frac{1}{2}\Big)[\xi_k^2(t\psi) - 1]\Big(c\|u_k+t\psi\|^2 - \gamma\|(u_k+t\psi)^+\|_H^2\Big) \\
&\quad - \Big(\frac{1}{1-q} - \frac{1}{2p}\Big)[\xi_k^{2p}(t\psi) - 1]\int_\Omega\int_\Omega \frac{r(x)r(y)\left((u_k)^+(x)\right)^p\left((u_k)^+(y)\right)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy \\
&\quad - \Big(\frac{1}{1-q} - \frac{1}{2p}\Big)\xi_k^{2p}(t\psi)\int_\Omega\int_\Omega r(x)r(y)\Big[\frac{\left((u_k+t\psi)^+(x)\right)^p\left((u_k+t\psi)^+(y)\right)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha} \\
&\quad - \frac{\left((u_k)^+(x)\right)^p\left((u_k)^+(y)\right)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\Big]\,dx\,dy.
\end{aligned}
$$

Dividing above estimate by $t > 0$ and passing the limit $t \to 0$, we obtain

$$
\begin{aligned}
&\langle \xi_k'(0), \psi\rangle\frac{\|u_k\|}{k} + \frac{\|\psi\|}{k} \\
&\ge \frac{1+q}{1-q}\Big(c\iint_{R^{2N}} \frac{(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dx\,dy - \gamma\int_\Omega \frac{u_k^+\psi}{|x|^{2s}}\,dx\Big) \\
&\quad + \frac{1+q}{1-q}\langle \xi_k'(0), \psi\rangle\left(c\|u_k\|^2 - \gamma\|u_k^+\|_H^2\right) + \Big(\frac{2\theta+q-1}{1-q}\Big)d\langle \xi_k'(0), \psi\rangle\|u_k\|^{2\theta}
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(\frac{2\theta+q-1}{1-q}\Big)d\|u_k\|^{2\theta-2}\iint_{\mathbb{R}^{2N}}\frac{(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dx\,dy\\
&-\frac{(2p+q-1)}{(1-q)}\int_\Omega\int_\Omega\frac{r(x)r(y)\big(u_k^+(x)\big)^{p-1}\psi(x)\big(u_k^+(y)\big)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\\
&-\frac{(2p+q-1)}{(1-q)}\langle\xi_k'(0),\psi\rangle\int_\Omega\int_\Omega\frac{r(x)r(y)\left((u_k)^+(x)\right)^p\left((u_k)^+(y)\right)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy.\\
\ge{}&\frac{\langle\xi_k'(0),\psi\rangle}{(1-q)}\Big[(1+q)\Big(c\|u_k\|^2-\gamma\|u_k^+\|_H^2\Big)+(2\theta+q-1)d\|u_k\|^{2\theta}\\
&-(2p+q-1)\int_\Omega\int_\Omega\frac{r(x)r(y)\left((u_k)^+(x)\right)^p\left((u_k)^+(y)\right)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy\Big]\\
&+\Big(\frac{1+q}{1-q}\Big)\Big(c\iint_{\mathbb{R}^{2N}}\frac{(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dx\,dy-\gamma\int_\Omega\frac{u_k^+\psi}{|x|^{2s}}\,dx\Big)\\
&+\Big(\frac{2\theta+q-1}{1-q}\Big)d\|u_k\|^{2\theta-2}\iint_{\mathbb{R}^{2N}}\frac{(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dx\,dy\\
&-\frac{(2p+q-1)}{(1-q)}\int_\Omega\int_\Omega\frac{r(x)r(y)\big(u_k^+(x)\big)^{p-1}\psi(x)\big(u_k^+(y)\big)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\frac{\|\psi\|}{k}\ge{}&\frac{\langle\xi_k'(0),\psi\rangle}{(1-q)}\Big[(1+q)\big(c\|u_k\|^2-\mu\|u_k^+\|_H^2\big)+(2\theta+q-1)d\|u_k\|^{2\theta}\\
&-(2p+q-1)\int_\Omega\int_\Omega\frac{r(x)r(y)\left((u_k)^+(x)\right)^p\left((u_k)^+(y)\right)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy-(1-q)\frac{\|u_k\|}{k}\Big]\\
&+\Big(\frac{1+q}{1-q}\Big)\Big(c\iint_{\mathbb{R}^{2N}}\frac{(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dx\,dy-\gamma\int_\Omega\frac{u_k^+\psi}{|x|^{2s}}\,dx\Big)\\
&+\Big(\frac{2\theta+q-1}{1-q}\Big)d\|u_k\|^{2\theta-2}\iint_{\mathbb{R}^{2N}}\frac{(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dx\,dy\\
&-\frac{(2p+q-1)}{(1-q)}\int_\Omega\int_\Omega\frac{r(x)r(y)\big(u_k^+(x)\big)^{p-1}\psi(x)\big(u_k^+(y)\big)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}.
\end{aligned}
$$

which is impossible under the assumption that $\langle\xi_k'(0),\psi\rangle=\infty$. Now by Lemma 3.6(1), boundedness of $\{u_k\}_k$ and (4.6), we have

$$
\begin{aligned}
&\Big[(1+q)\Big(c\|u_k\|^2-\gamma\|u_k^+\|_H^2\Big)+(2\theta+q-1)d\|u_k\|^{2\theta}\\
&-(2p+q-1)\int_\Omega\int_\Omega\frac{r(x)r(y)\left(u_k^+(x)\right)^p\left((u_k)^+(y)\right)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy-(1-q)\frac{\|u_k\|}{k}\Big]\\
&\ge C_2-(1-q)\frac{C_1}{k}>0.
\end{aligned}
$$

Hence $\langle\xi_k'(0),\psi\rangle$ is uniformly bounded for sufficiently large k with any $\psi\in X_0$ and $\psi\ge 0$. $\square$

Lemma 4.3. *Let $\gamma \in (0, c\gamma_H)$, $\lambda \in (0, \Lambda_1)$ and $\{u_k\}_k \subset \mathcal{N}^{\pm}_{\gamma,\lambda}$ verify (4.1) and (4.2). Then for every $\psi \in X_0$, we have $l(x)(u_k^+)^{-q}\psi \in L^1(\Omega)$, and*

$$\begin{aligned}
&\Big(c + d\|u_k\|^{2\theta-2}\Big) \iint_{\mathbb{R}^{2N}} \frac{(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&- \gamma \int_\Omega \frac{u_k^+\psi}{|x|^{2s}}dx - \lambda \int_\Omega l(x)(u_k^+)^{-q}\psi dx \\
&- \int_\Omega\int_\Omega \frac{r(x)r(y)\big(u_k^+(x)\big)^{p-1}\psi(x)\big(u_k^+(y)\big)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy = H(u_k,\psi) = o(1)
\end{aligned} \tag{4.10}$$

as $k \to \infty$.

Proof. Let $\psi \in X_0$ with $\psi \geq 0$. Then (4.1) and (4.9), we have

$$\begin{aligned}
&[\xi_k(t\psi) - 1]\frac{\|u_k\|}{k} + \xi_k(t\psi)\frac{\|t\psi\|}{k} \\
&\geq \mathcal{J}_{\gamma,\lambda}(u_k) - \mathcal{J}_{\gamma,\lambda}\left(\xi_k(t\psi)(u_k + t\psi)\right) \\
&= -\frac{\big(\xi_k^2(t\psi)-1\big)}{2}\Big(c\|u_k\|^2 - \gamma\|u_k^+\|_H^2\Big) - \frac{\big(\xi_k^2(t\psi)-1\big)}{2\theta}d\|u_k\|^{2\theta} \\
&\quad - \frac{\xi_k^2(t\psi)}{2}\Big[\Big(c\|u_k+t\psi\|^2 - \gamma\|(u_k+t\psi)^+\|_H^2\Big) - \Big(c\|u_k\|^2 - \gamma\|u_k^+\|_H^2\Big)\Big] \\
&\quad - \frac{\xi_k^{2\theta}(t\psi)}{2\theta}d\Big[\|u_k+t\psi\|^{2\theta} - \|u_k\|^{2\theta}\Big] \\
&\quad + \frac{\lambda\big(\xi_k^{1-q}(t\psi)-1\big)}{1-q}\int_\Omega l(x)\left((u_k+t\psi)^+\right)^{1-q}\,dx \\
&\quad + \frac{\lambda}{1-q}\int_\Omega l(x)\big[\left((u_k+t\psi)^+\right)^{1-q} - \left((u_k)^+\right)^{1-q}\big]dx \\
&\quad + \frac{\xi_k^{2p}(t\psi)-1}{2p}\int_\Omega\int_\Omega \frac{r(x)r(y)\left((u_k+t\psi)^+(x)\right)^p\left((u_k+t\psi)^+(y)\right)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy \\
&\quad + \frac{1}{2p}\int_\Omega\int_\Omega \frac{r(x)r(y)}{|x|^\alpha|x-y|^\mu|y|^\alpha}\Big[\left((u_k+t\psi)^+(x)\right)^p\left((u_k+t\psi)^+(y)\right)^p \\
&\quad - \left((u_k)^+(x)\right)^p\left((u_k)^+(y)\right)^p\Big]\,dx\,dy.
\end{aligned}$$

Dividing by $t > 0$ and passing the limit $t \to 0^+$, we obtain

$$\begin{aligned}
&|\langle \xi_k'(0),\psi\rangle|\frac{\|u_k\|}{k} + \frac{\|\psi\|}{k} \\
&\geq -\langle \xi_k'(0),\psi\rangle\Big[\Big(c\|u_k\|^2 - \gamma\|u_k^+\|_H^2\Big) - \lambda\int_\Omega l(x)(u_k^+)^{1-q}dx + d\|u_k\|^{2\theta} \\
&\quad - \int_\Omega\int_\Omega \frac{r(x)r(y)\left((u_k)^+(x)\right)^p\left((u_k)^+(y)\right)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy\Big] \\
&\quad - \Big(c + d\|u_k\|^{2\theta-2}\Big)\iint_{\mathbb{R}^{2N}} \frac{(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dx\,dy + \gamma\int_\Omega \frac{u_k^+\psi}{|x|^{2s}}\,dx \\
&\quad + \int_\Omega\int_\Omega \frac{r(x)r(y)\big(u_k^+(x)\big)^{p-1}\psi(x)\big(u_k^+(y)\big)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy
\end{aligned}$$

$$
\begin{aligned}
&+\liminf_{t\to 0^+}\frac{\lambda}{1-q}\int_\Omega \frac{l(x)\Big[\big((u_k+t\psi)^+\big)^{1-q}-(u_k^+)^{1-q}\Big]}{t}\,dx.\\
&=-\Big(c+d\|u_k\|^{2\theta-2}\Big)\iint_{\mathbb{R}^{2N}}\frac{(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dx\,dy+\gamma\int_\Omega\frac{u_k^+\psi}{|x|^{2s}}\,dx\\
&+\int_\Omega\int_\Omega\frac{r(x)r(y)\big(u_k^+(x)\big)^{p-1}\psi(x)\big(u_k^+(y)\big)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy\\
&+\liminf_{t\to 0^+}\frac{\lambda}{1-q}\int_\Omega\frac{l(x)[\big((u_k+t\psi)^+\big)^{1-q}-(u_k^+)^{1-q}]}{t}\,dx,
\end{aligned}
$$

since $\{u_k\}_k\in\mathcal{N}_{\gamma,\lambda}$. Applying the above inequality, we have

$$
\liminf_{t\to 0^+}\int_\Omega\frac{l(x)[\big((u_k+t\psi)^+\big)^{1-q}-(u_k^+)^{1-q}]}{t}\,dx
$$

is finite. Now, it follows from $l(x)[\big((u_k+t\psi)^+\big)^{1-q}-(u_k^+)^{1-q}]\geq 0$ and $\{u_k\}_k$ is bounded in X_0, Fatou's lemma and Lemma 4.1 that

$$
\begin{aligned}
&\lambda\int_\Omega l(x)(u_k^+)^{-q}\psi dx\\
&\leq\liminf_{t\to 0^+}\frac{\lambda}{1-q}\int_\Omega\frac{l(x)[\big((u_k+t\psi)^+\big)^{1-q}-(u_k^+)^{1-q}]}{t}\,dx\\
&\leq\frac{\langle\xi_k'(0),\psi\rangle\|u_k\|+\|\psi\|}{k}-\int_\Omega\int_\Omega\frac{r(x)r(y)\big(u_k^+(x)\big)^{p-1}\psi(x)\big(u_k^+(y)\big)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy\\
&\quad+\big(c+d\|u_k\|^{2\theta-2}\big)\iint_{\mathbb{R}^{2N}}\frac{(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dx\,dy-\gamma\int_\Omega\frac{u_k^+\psi}{|x|^{2s}}\,dx\\
&\leq\frac{C_1C_3+\|\psi\|}{k}+\Big(c+d\|u_k\|^{2\theta-2}\Big)\iint_{\mathbb{R}^{2N}}\frac{(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dx\,dy\\
&\quad-\gamma\int_\Omega\frac{u_k^+\psi}{|x|^{2s}}\,dx-\int_\Omega\int_\Omega\frac{r(x)r(y)\big(u_k^+(x)\big)^{p-1}\psi(x)\big(u_k^+(y)\big)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy,
\end{aligned}
$$

where $C_3>0$ is given for the boundedness of $\langle\xi_k'(0),\psi\rangle$ and $\|u_k\|\leq C_1$. At the moment, taking limit $k\to\infty$, we obtain

$$
\begin{aligned}
&\Big(c+d\|u_k\|^{2\theta-2}\Big)\iint_{\mathbb{R}^{2N}}\frac{(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dx\,dy\\
&-\gamma\int_\Omega\frac{u_k^+\psi}{|x|^{2s}}\,dx-\int_\Omega l(x)(u_k^+)^{-q}\psi dx\\
&-\int_\Omega\int_\Omega\frac{r(x)r(y)\big(u_k^+(x)\big)^{p-1}\psi(x)\big(u_k^+(y)\big)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy\\
&\geq o(1).
\end{aligned}\tag{4.11}
$$

Next, we prove that the equation (4.11) holds for any $\psi\in X_0$. For this, in (4.11), we choose $\psi=\Psi_\epsilon^+$ for $\epsilon>0$ as test function with $\Psi_\epsilon=u_k^+ +\epsilon\psi$ then as limit

$k \to \infty$, we obtain

$$
\begin{aligned}
o(1) \leq & \Big(c + d\|u_k\|^{2\theta-2}\Big) \iint_{\mathbb{R}^{2N}} \frac{(u_k(x) - u_k(y))\,(\Psi_\epsilon^+(x) - \Psi_\epsilon^+(y))}{|x-y|^{N+2s}}\,dx\,dy \\
& - \gamma \int_\Omega \frac{u_k^+ \Psi_\epsilon^+}{|x|^{2s}}\,dx - \lambda \int_\Omega l(x)(u_k^+)^{-q}\Psi_\epsilon^+ dx \\
& - \int_\Omega \int_\Omega \frac{r(x)r(y)\big(u_k^+(x)\big)^{p-1}\Psi_\epsilon^+\big(u_k^+(y)\big)^p}{|x|^\alpha |x-y|^\mu |y|^\alpha}\,dx\,dy \\
= & \Big(c + d\|u_k\|^{2\theta-2}\Big) \\
& \times \iint_{\mathbb{R}^{2N}} \frac{(u_k(x) - u_k(y))\,((\Psi_\epsilon + \Psi_\epsilon^-)(x) - (\Psi_\epsilon + \Psi_\epsilon^-)(y))}{|x-y|^{N+2s}}\,dx\,dy \\
& - \gamma \int_\Omega \frac{u_k^+(\Psi_\epsilon + \Psi_\epsilon^-)}{|x|^{2s}}\,dx - \lambda \int_\Omega l(x)(u_k^+)^{-q}(\Psi_\epsilon + \Psi_\epsilon^-)dx \\
& - \int_\Omega \int_\Omega \frac{r(x)r(y)\big(u_k^+(x)\big)^{p-1}\big(u_k^+(y)\big)^p(\Psi_\epsilon + \Psi_\epsilon^-)}{|x|^\alpha |x-y|^\mu |y|^\alpha}\,dx\,dy.
\end{aligned} \tag{4.12}
$$

We notice that for a.e. $x, y \in \mathbb{R}^N$,

$$
(u_k(x) - u_k(y))\big(u_k^-(x) - u_k^-(y)\big) \leq -|u_k^-(x) - u_k^-(y)|^2, \tag{4.13}
$$

$$
(u_k(x) - u_k(y))\big(u_k^+(x) - u_k^+(y)\big) \leq |u_k(x) - u_k(y)|^2. \tag{4.14}
$$

Applying (4.14), we have

$$
\begin{aligned}
& \iint_{\mathbb{R}^{2N}} \frac{(u_k(x) - u_k(y))\,((\Psi_\epsilon + \Psi_\epsilon^-)(x) - (\Psi_\epsilon + \Psi_\epsilon^-)(y))}{|x-y|^{N+2s}}\,dx\,dy \\
& = \iint_{\mathbb{R}^{2N}} \frac{(u_k(x) - u_k(y))\big(u_k^+(x) - u_k^+(y)\big)}{|x-y|^{N+2s}}\,dx\,dy \\
& \quad + \epsilon \iint_{\mathbb{R}^{2N}} \frac{(u_k(x) - u_k(y))\,(\psi(x) - \psi(y))}{|x-y|^{N+2s}}\,dx\,dy \\
& \quad + \iint_{\mathbb{R}^{2N}} \frac{(u_k(x) - u_k(y))\,(\Psi_\epsilon^-(x) - \Psi_\epsilon^-(y))}{|x-y|^{N+2s}}\,dx\,dy \\
& \leq \iint_{\mathbb{R}^{2N}} \frac{|u_k(x) - u_k(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \\
& \quad + \epsilon \iint_{\mathbb{R}^{2N}} \frac{(u_k(x) - u_k(y))\,(\psi(x) - \psi(y))}{|x-y|^{N+2s}}\,dx\,dy \\
& \quad + \iint_{\mathbb{R}^{2N}} \frac{(u_k(x) - u_k(y))\,(\Psi_\epsilon^-(x) - \Psi_\epsilon^-(y))}{|x-y|^{N+2s}}\,dx\,dy.
\end{aligned} \tag{4.15}
$$

Also,

$$
\begin{aligned}
\int_\Omega \frac{u_k^+(\Psi_\epsilon + \Psi_\epsilon^-)}{|x|^{2s}}\,dx &= \int_\Omega \frac{|u_k^+|^2}{|x|^{2s}}\,dx + \epsilon \int_\Omega \frac{u_k^+\psi}{|x|^{2s}}\,dx + \int_\Omega \frac{u_k^+\Psi_\epsilon^-}{|x|^{2s}}\,dx \\
&\geq \int_\Omega \frac{|u_k^+|^2}{|x|^{2s}}\,dx + \epsilon \int_\Omega \frac{u_k^+\psi}{|x|^{2s}}\,dx + \epsilon \int_{\Omega_\epsilon} \frac{u_k^+\psi}{|x|^{2s}}\,dx,
\end{aligned} \tag{4.16}
$$

where $\Omega_\epsilon = \{x \in \mathbb{R}^N : \Psi_\epsilon \le 0\}$. Now using (4.16), (4.15), and (4.12), yield

$$\begin{aligned}
o(1) \le \Big[&\Big(c + d\|u_k\|^{2\theta-2}\Big) \|u_k\|^2 - \gamma \|u_k^+\|_H^2 - \lambda \int_\Omega l(x)(u_k^+)^{1-q}\,dx \\
&- \int_\Omega\int_\Omega \frac{r(x)r(y)\big(u_k^+(x)\big)^p\big(u_k^+(y)\big)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy\Big] \\
&+ \epsilon\Big[\Big(c + d\|u_k\|^{2\theta-2}\Big)\iint_{\mathbb{R}^{2N}} \frac{(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&- \gamma\int_\Omega \frac{u_k^+\psi}{|x|^{2s}}\,dx - \lambda\int_\Omega l(x)(u_k^+)^{-q}\psi dx \\
&- \int_\Omega\int_\Omega \frac{r(x)r(y)\big(u_k^+(x)\big)^{p-1}\psi(x)\big(u_k^+(y)\big)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy\Big] \\
&+ \Big(c + d\|u_k\|^{2\theta-2}\Big)\iint_{\mathbb{R}^{2N}} \frac{(u_k(x)-u_k(y))\,(\Psi_\epsilon^-(x)-\Psi_\epsilon^-(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&- \epsilon\gamma\int_{\Omega_\epsilon} \frac{u_k^+\psi}{|x|^{2s}}\,dx + \lambda\int_{\Omega_\epsilon} l(x)(u_k^+)^{-q}(u_k^+ + \epsilon\psi)dx \\
&+ \int_\Omega\int_{\Omega_\epsilon} \frac{r(x)r(y)\big(u_k^+(x)\big)^{p-1}(u_k^+ + \epsilon\psi)(x)\big(u_k^+(y)\big)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy.
\end{aligned}$$

Using that $\{u_k\}_k \in \mathcal{N}_{\gamma,\lambda}$, we deduce that

$$\begin{aligned}
o(1) \le \epsilon\Big[&\Big(c + d\|u_k\|^{2\theta-2}\Big)\iint_{\mathbb{R}^{2N}} \frac{(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&- \gamma\int_\Omega \frac{u_k^+\psi}{|x|^{2s}}\,dx - \lambda\int_\Omega l(x)(u_k^+)^{-q}\psi dx \\
&- \int_\Omega\int_\Omega \frac{r(x)r(y)\big(u_k^+(x)\big)^{p-1}\psi(x)\big(u_k^+(y)\big)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy\Big] \\
&+ \big(c + d\|u_k\|^{2\theta-2}\big)\iint_{R^{2N}} \frac{(u_k(x)-u_k(y))\,(\Psi_\epsilon^-(x)-\Psi_\epsilon^-(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&- \epsilon\gamma\int_{\Omega_\epsilon} \frac{u_k^+\psi}{|x|^{2s}}\,dx \\
&+ \int_\Omega\int_{\Omega_\epsilon} \frac{r(x)r(y)\big(u_k^+(x)\big)^{p-1}(u_k^+ + \epsilon\psi)(x)\,\big(u_k^+(y)\big)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy.
\end{aligned} \tag{4.17}$$

Now the symmetry of the fractional kernel and

$$(u_k(x)-u_k(y))\,\big(u_k^+(x)-u_k^+(y)\big) \ge |u_k^+(x)-u_k^+(y)|^2,$$

yield

$$\begin{aligned}
&\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u_k(x)-u_k(y))\,(\Psi_\epsilon^-(x)-\Psi_\epsilon^-(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&= \iint_{\Omega_\epsilon\times\Omega_\epsilon} \frac{(u_k(x)-u_k(y))\,(\Psi_\epsilon^-(x)-\Psi_\epsilon^-(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&\quad + 2\iint_{\Omega_\epsilon\times(\mathbb{R}^N\setminus\Omega_\epsilon)} \frac{(u_k(x)-u_k(y))\,(\Psi_\epsilon^-(x)-\Psi_\epsilon^-(y))}{|x-y|^{N+2s}}\,dx\,dy
\end{aligned}$$

$$
\begin{aligned}
&\leq -\epsilon\Big(\iint_{\Omega_\epsilon\times\Omega_\epsilon}\frac{(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dx\,dy\\
&\quad+2\iint_{\Omega_\epsilon\times(\mathbb{R}^N\setminus\Omega_\epsilon)}\frac{(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dx\,dy\Big)\\
&\leq 2\epsilon\iint_{\Omega_\epsilon\times\mathbb{R}^N}\Big|\frac{(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\Big|\,dx\,dy
\end{aligned}
$$

Now by Hölder inequality and boundedness of $\{u_k\}_k$ in X_0, we have

$$
\begin{aligned}
&\iint_{\Omega_\epsilon\times\mathbb{R}^N}\Big|\frac{(u_k(x)-u_k(y))(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\Big|\,dx\,dy\\
&\leq C\Big(\iint_{\Omega_\epsilon\times\mathbb{R}^N}\Big|\frac{(\psi(x)-\psi(y))}{|x-y|^{(N+2s)/2}}\Big|^2\,dx\,dy\Big)^{1/2},
\end{aligned}
\tag{4.18}
$$

Here $\frac{(\psi(x)-\psi(y))}{|x-y|^{(N+2s)/2}}\in L^2(\mathbb{R}^{2N})$, Now for $\nu>0$ there exists R_ν sufficiently large such that

$$
\iint_{(\operatorname{supp}\psi)\times[\mathbb{R}^N\setminus B_{R_\nu}]}\Big|\frac{(\psi(x)-\psi(y))}{|x-y|^{(N+2s)/2}}\Big|^2\,dx\,dy<\frac{\nu}{2}.
$$

Now by the definition of Ω_ϵ, we have $\Omega_\epsilon\subset\operatorname{supp}\psi$ and $|\Omega_\epsilon\times B_{R_\nu}|\to 0$ as $\epsilon\to 0^+$. So by $\frac{(\psi(x)-\psi(y))}{|x-y|^{(N+2s)/2}}\in L^2(\mathbb{R}^{2N})$, there is a $\delta_\nu>0$ and $\epsilon_\nu>0$ such that for any $\epsilon\in(0,\epsilon_\nu]$, we have $|\Omega_\epsilon\times B_{R_\nu}|<\delta_\nu$ and $\iint_{\Omega_\epsilon\times B_{R_\nu}}\big|\frac{(\psi(x)-\psi(y))}{|x-y|^{(N+2s)/2}}\big|^2\,dx\,dy<\frac{\nu}{2}$. Hence for $\epsilon\in(0,\epsilon_\nu]$, we obtain

$$
\lim_{\epsilon\to 0^+}\iint_{\Omega_\epsilon\times\mathbb{R}^N}\Big|\frac{(\psi(x)-\psi(y))}{|x-y|^{(N+2s)/2}}\Big|^2\,dx\,dy=0. \tag{4.19}
$$

Hence by (4.18),

$$
\lim_{\epsilon\to 0^+}\iint_{\Omega_\epsilon\times\mathbb{R}^N}\Big|\frac{(u_k(x)-u_k(y))(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\Big|\,dx\,dy=0.
$$

Now we claim that

$$
\lim_{\epsilon\to 0}\int_{\Omega_\epsilon}\frac{u_k^+\psi}{|x|^{2s}}\,dx=0. \tag{4.20}
$$

Using that for $x\in\Omega_\epsilon$, we have $u_k^++\epsilon\psi\leq 0$, which imply that $\psi(x)\leq 0$. Hence, by (1.4)

$$
0\leq\Big|\int_{\Omega_\epsilon}\frac{u_k^+\psi}{|x|^{2s}}\,dx\Big|\leq\int_{\Omega_\epsilon}\frac{|u_k^+||\psi|}{|x|^{2s}}\,dx\leq\epsilon\int_{\Omega_\epsilon}\frac{|\psi|^2}{|x|^{2s}}\,dx\leq\epsilon\|\psi\|_H^2\leq\epsilon\frac{\|\psi\|^2}{\mu_H}.
$$

As $\epsilon\to 0$, we prove claim (4.20).

Next, we show that

$$
\lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\int_\Omega\int_{\Omega_\epsilon}\frac{r(x)r(y)\big(u_k^+(x)\big)^{p-1}(u_k^++\epsilon\psi)(x)\big(u_k^+(y)\big)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy=0 \tag{4.21}
$$

For this, consider

$$
\begin{aligned}
&\int_\Omega\int_{\Omega_\epsilon}\frac{r(x)r(y)\big(u_k^+(x)\big)^{p-1}(u_k^++\epsilon\psi)(x)\big(u_k^+(y)\big)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy\\
&\leq\int_\Omega\int_{\Omega_\epsilon}\frac{r(x)r(y)\big(u_k^+(x)\big)^p\big(u_k^+(y)\big)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy
\end{aligned}
$$

$$
\begin{aligned}
&+\epsilon\int_\Omega\int_{\Omega_\epsilon}\frac{r(x)r(y)\big(u_k^+(x)\big)^{p-1}\psi(x)\big(u_k^+(y)\big)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy\\
&\le\int_\Omega\int_{\Omega_\epsilon}\frac{r(x)r(y)\big(u_k^+(x)\big)^{p}\big(u_k^+(y)\big)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy\\
&\quad+\epsilon\Big(\int_\Omega\int_{\Omega_\epsilon}\frac{r(x)r(y)\big(u_k^+(x)\big)^{p-1}\psi(x)\big(u_k^+(y)\big)^{p-1}\psi(y)}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy\Big)^{1/2}\\
&\quad\times\Big(\int_\Omega\int_{\Omega}\frac{r(x)r(y)\big(u_k^+(x)\big)^{p}\big(u_k^+(y)\big)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy\Big)^{1/2}\\
&\le CC_r(\alpha,\mu,N)\Big(\int_{\Omega_\epsilon}\big(u_k^+(x)\big)^{2_s^*}\,dx\Big)^{p/2_s^*}\\
&\quad+C\epsilon C_r(\alpha,\mu,N)\Big(\int_{\Omega_\epsilon}\big([u_k^+(x)]^{p-1}\psi(x)\big)^{2_s^*/p}\,dx\Big)^{p/2_s^*}\\
&\le CC_r(\alpha,\mu,N)\Big(\int_{\Omega_\epsilon}\big(u_k^+(x)\big)^{2_s^*}\,dx\Big)^{p/2_s^*}\\
&\quad+C\epsilon C_r(\alpha,\mu,N)\Big(\int_{\Omega_\epsilon}\big(u_k^+(x)\big)^{2_s^*}\,dx\Big)^{(p-1)/2_s^*}\Big(\int_{\Omega_\epsilon}|\psi(x)|^{2_s^*}\,dx\Big)^{1/2_s^*}\\
&\le CC_r(\alpha,\mu,N)\epsilon^p\Big(\int_{\Omega_\epsilon}|\psi(x)|^{2_s^*}\,dx\Big)^{p/2_s^*}+\tilde{C}\epsilon C_r(\alpha,\mu,N)\epsilon^p\Big(\int_{\Omega_\epsilon}|\psi(x)|^{2_s^*}\,dx\Big)^{p/2_s^*}.
\end{aligned}
$$

Dividing the above estimate by ϵ and using $|\Omega_\epsilon|\to 0$ as $\epsilon\to 0^+$, we can easily deduce (4.21).

Now dividing (4.17) by ϵ together with (4.19), (4.20), (4.21), and $|\Omega_\epsilon|\to 0$ as $\epsilon\to 0^+$, we deduce that

$$
\begin{aligned}
o(1)\le{}&\Big(c+d\|u_k\|^{2\theta-2}\Big)\iint_{\mathbb{R}^{2N}}\frac{(u_k(x)-u_k(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dx\,dy\\
&-\gamma\int_\Omega\frac{u_k^+\psi}{|x|^{2s}}\,dx-\lambda\int_\Omega l(x)(u_k^+)^{-q}\psi dx\\
&-\int_\Omega\int_\Omega\frac{r(x)r(y)\big(u_k^+(x)\big)^{p-1}\psi(x)\big(u_k^+(y)\big)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy.
\end{aligned}
$$

which prove (4.11). In consonance with the arbitrariness of ψ, we derive that (4.10) holds for any $\psi\in X_0$. □

Lemma 4.4. *Let $\lambda\in(0,\Lambda_1)$, $\gamma\in(0,c\gamma_H)$ and $\{u_k\}_k\subset\mathcal{N}^{\pm}_{\gamma,\lambda}$ with $\mathcal{J}_{\gamma,\lambda}(u_k)\to c$, then the sequence $\{u_k\}_k$ contains a subsequence strongly convergent to u_0 in X_0.*

Proof. By (4.2), $\{u_k\}_k$ and $\{u_k^-\}_k$ are both bounded in X_0. Taking $\psi=u_k^-$ as $k\to\infty$ in (4.10), we have

$$
\lim_{k\to\infty}\Big(c+d\|u_k\|^{2\theta-2}\Big)\iint_{\mathbb{R}^{2N}}\frac{(u_k(x)-u_k(y))\,\big(u_k^-(x)-u_k^-(y)\big)}{|x-y|^{N+2s}}\,dx\,dy=0.
$$

which together with $c>0$ yields that $\|u_k^-\|\to 0$ as $k\to\infty$. So, we can assume that $\{u_k\}_k$ is a sequence of non-negative functions. Furthermore by Lemma 2.1

and (1.4), we can extract a subsequence still denote by $\{u_k\}_k$ such that

$$\begin{aligned} u_k &\rightharpoonup u_0 \quad \text{weakly in } L^{2_s^*}(\Omega), \quad \|u_k\| \to v \\ u_k &\to u_0 \quad \text{in } L^p(\Omega) \quad \text{for any } p \in (1, 2_s^*) \\ u_k &\rightharpoonup u_0 \quad \text{in } L^{2_s^*}(\Omega, |x|^{-2s}), \quad \|u_k\|_H \to l \\ u_k &\to u_0 \quad \text{a.e. in } \Omega \quad u_k \le h \text{ a. e. in } \Omega, \end{aligned} \tag{4.22}$$

as $k \to \infty$ with $h \in L^p(\Omega)$ for a fixed $p \in [1, 2_s^*)$.

Now if $v = 0$ in (4.22), then we can easily see that $u_k \to 0$ strongly in X_0 as $k \to \infty$. Hence we assume $v > 0$. Now by Brézis-Lieb Lemma [4] and Lemma 2.2, we obtain

$$\|u_k\|^2 = \|u_k - u_0\|^2 + \|u_0\|^2 + o(1), \tag{4.23}$$

$$\|u_k\|_H^2 = \|u_k - u_0\|_H^2 + \|u_0\|_H^2 + o(1), \tag{4.24}$$

$$\begin{aligned} &\int_\Omega \int_\Omega \frac{r(x)r(y)(u_k(x))^p(u_k(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy \\ &= \int_\Omega \int_\Omega \frac{r(x)r(y)((u_k-u_0)(x))^p(u_k-u_0)(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy \\ &\quad + \int_\Omega \int_\Omega \frac{r(x)r(y)(u_0(x))^p(u_0(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy + o(1). \end{aligned} \tag{4.25}$$

Now by (4.23), (4.24) and (4.25), we have

$$\begin{aligned} o(1) &= \left(c + d\|u_k\|^{2\theta-2}\right) \iint_{R^{2N}} \frac{(u_k(x)-u_k(y))\left((u_k-u_0)(x)-(u_k-u_0)(y)\right)}{|x-y|^{N+2s}}\,dx\,dy \\ &\quad - \gamma \int_\Omega \frac{u_k(u_k-u_0)}{|x|^{2s}}\,dx - \lambda \int_\Omega l(x)(u_k)^{-q}(u_k-u_0)dx \\ &\quad - \int_\Omega \int_\Omega \frac{r(x)r(y)(u_k(x))^{p-1}(u_k(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}(u_k-u_0)\,dx\,dy \\ &= \left(c + dv^{2\theta-2}\right)\left(v^2 - \|u_0\|^2\right) - \gamma\left(\|u_k\|_H^2 - \|u_0\|_H^2\right) \\ &\quad - \lambda \int_\Omega l(x)(u_k)^{-q}(u_k-u_0)dx - \int_\Omega \int_\Omega \frac{r(x)r(y)(u_k(x))^p((u_k(y))^p}{|x-y|^{N+2s}}\,dx\,dy \\ &\quad + \int_\Omega \int_\Omega \frac{r(x)r(y)(u_0(x))^p(u_0(y))^p}{|x-y|^{N+2s}}\,dx\,dy + o(1) \\ &= \left(c + dv^{2\theta-2}\right)\|u_k-u_0\|^2 - \gamma\|u_k-u_0\|_H^2 - \lambda \int_\Omega l(x)(u_k)^{-q}(u_k-u_0)dx \\ &\quad - \int_\Omega \int_\Omega \frac{r(x)r(y)((u_k(x)-u_0(x)))^p((u_k(y)-u_0(y)))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy + o(1). \end{aligned}$$

Taking limit as $k \to \infty$, we have

$$\begin{aligned} &\left(c + dv^{2\theta-2}\right)\lim_{k\to\infty}\|u_k-u_0\|^2 - \gamma \lim_{k\to\infty}\|u_k-u_0\|_H^2 \\ &= \lambda \lim_{k\to\infty}\int_\Omega l(x)(u_k)^{-q}(u_k-u_0)dx \\ &\quad + \lim_{k\to\infty}\int_\Omega \int_\Omega \frac{r(x)r(y)((u_k(x)-u_0(x)))^p((u_k(y)-u_0(y)))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy. \end{aligned} \tag{4.26}$$

Since $q \in (0,1)$ and $l \in L^{\frac{2_s^*}{2_s^*-1+q}}(\Omega)$, we conclude from Vitali's convergence theorem that

$$\lim_{k\to\infty} \int_\Omega l(x)(u_k^+)^{1-q}\,dx = \int_\Omega l(x)(u_0^+)^{1-q}\,dx$$

In view of Lemma 4.3, we obtain $l(x)u_k^{-q}u_0 \in L^1(\Omega)$ for each $k \in \mathbb{N}$. It follows from Fatou's lemma that

$$\int_\Omega l(x)u_0^{1-q}\,dx \le \liminf_{k\to\infty} \int_\Omega l(x)u_k^{-q}u_0 dx.$$

Now for $r \in L^{\frac{2_s^*}{2^*_{\mu,s,\alpha}-p}}(\Omega)$ using Vitali's convergence theorem, one can noticed that

$$r(x)|u_k - u_0|^p \to 0 \quad \text{strongly in} \quad L^{2^*_{\mu,s,\alpha}}(\Omega).$$

It follows from (2.1) that

$$\lim_{k\to\infty} \int_\Omega \int_\Omega \frac{r(x)r(y)((u_k(x)-u_0(x)))^p((u_k(y)-u_0(y)))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy = 0. \tag{4.27}$$

Using (4.27) in (4.26), we have

$$\begin{aligned}
0 &\ge \left(c + dv^{2\theta-2}\right) \lim_{k\to\infty} \|u_k - u_0\|^2 - \gamma \lim_{k\to\infty} \|u_k - u_0\|_H^2 \\
&= c\left[\lim_{k\to\infty} \|u_k - u_0\|^2 - \frac{\gamma}{c} \lim_{k\to\infty} \|u_k - u_0\|_H^2\right] + dv^{2\theta-2} \lim_{k\to\infty} \|u_k - u_0\|^2.
\end{aligned}$$

This and (2.9) yield

$$0 \ge h_{c,\gamma} \lim_{k\to\infty} \|u_k - u_0\|^2 + dv^{2\theta-2} \lim_{k\to\infty} \|u_k - u_0\|^2$$

which implies that

$$0 \ge \lim_{k\to\infty} \|u_k - u_0\|^2 \left[h_{c,\gamma} + dv^{2\theta-2}\right] > 0,$$

a contradiction. Hence $v = 0$. Thus $u_k \to u_0$ strongly in X_0. ☐

5. Proof of the main Theorem

In this section we prove the existence of solutions in $\mathcal{N}^+_{\gamma,\lambda}$ and $\mathcal{N}^-_{\gamma,\lambda}$.

Theorem 5.1. *Let $0 < \lambda < \Lambda_* = \min(\Lambda_1, \Lambda_2)$. Assume f and g satisfies* (A1) *and* (A2) *respectively. Then the problem* (2.7) *has a positive solution in $\mathcal{N}^+_{\gamma,\lambda}$.*

Proof. We first show that $m^+_{\gamma,\lambda} = \inf_{u\in\mathcal{N}^+_{\gamma,\lambda}} \mathcal{J}_{\gamma,\lambda}(u) < 0$. Since $u \in \mathcal{N}^+_{\gamma,\lambda} \subset \mathcal{N}_{\gamma,\lambda}$ we have

$$\begin{aligned}
\mathcal{J}_{\gamma,\lambda}(u) &= \Big(\frac{1}{2} - \frac{1}{1-q}\Big)[c\|u\|^2 - \gamma\|u^+\|_H^2] + \Big(\frac{1}{2\theta} - \frac{1}{1-q}\Big)d\|u\|^{2\theta} \\
&\quad - \Big(\frac{1}{2p} - \frac{1}{1-q}\Big) \int_\Omega \int_\Omega \frac{r(x)r(y)(u^+(x))^p(u^+(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy \\
&= -\frac{1}{2p(1-q)}\Big[(1+q)[c\|u\|^2 - \gamma\|u^+\|_H^2] + (2\theta+q-1)d\|u\|^{2\theta} \\
&\quad - (2p+q-1)\int_\Omega \int_\Omega \frac{r(x)r(y)(u^+(x))^p(u^+(y))^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy\Big] < 0,
\end{aligned}$$

since $u \in \mathcal{N}^+_{\gamma,\lambda}$. Hence $m^+_{\gamma,\lambda} < 0$.

Now for fix $\lambda < \Lambda_* = \min(\Lambda_1, \Lambda_2)$. Then by Ekeland's variational principle and Lemma 3.2, there exists a minimizing sequence $\{u_k\}_k \subset \mathcal{N}^+_{\gamma,\lambda} \cup \{0\}$, satisfying (4.1) and (4.2). Hence $\mathcal{J}_{\gamma,\lambda}(u_k) \to m^+_{\gamma,\lambda} < 0$ as $n \to \infty$, which gives that $\{u_k\}_k \subset \mathcal{N}^+_{\gamma,\lambda}$. Subsequently using Lemma 4.4 with $c = m^+_{\gamma,\lambda}$, we know that $u_k \to u_0$ in X_0, upto a subsequence. Moreover, by Lemma 4.1 and (3.9), we have

$$\begin{aligned}&(1+q)\Big[c\|u_0\|^2 - \gamma\|u_0\|_H^2\Big] + d(2\theta+q-1)\|u_0\|^{2\theta}\\&- (2p+q-1)\int_\Omega\int_\Omega \frac{r(x)r(y)\big(u_0^+(x)\big)^p\big(u_0^+(y)\big)^p}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,dx\,dy > 0,\end{aligned}$$

which implies that $u_0 \in \mathcal{N}^+_{\gamma,\lambda}$, and $m^+_{\gamma,\lambda}$ is attained at u_0 by $\mathcal{J}_{\gamma,\lambda}$ is continuous on X_0. Take limit as $k \to \infty$ together with Fatou's lemma in (4.10) , we obtain $H(u_0,\psi) \geq 0$[where H is defined in (4.10)] for $\psi \in X_0$ with $\psi \geq 0$.

Now letting a test function $\psi = \Psi_\epsilon^+$ with $\Psi_\epsilon = u_0^+ + \epsilon\psi$ and $\psi \in X_0$. Following the same process for u_0 in place of u_k from (4.10) to (4.21), we know that $H(u_0,\psi) \geq 0$ for $\psi \in X_0$, which produce that $\lambda l(x)(u_0^+)^{-q}\psi \in L^1(\Omega)$ and $u_0 \in \mathcal{N}^+_{\gamma,\lambda}$. So accordingly Lemma 3.2, $u_0 \neq 0$. Furthermore, by (2.8) with $\psi = u_0^-$ together with (4.13), we obtain $\|u_0^-\| = 0$. Hence u_0 is a positive solution of (1.1). □

Theorem 5.2. *Let* $0 < \lambda < \Lambda_* = \min(\Lambda_1, \Lambda_2)$. *Assume* l *satisfies* (A1) *and* r *satisfies* (A2). *Then problem* (2.7) *has a positive solution in* $\mathcal{N}^-_{\gamma,\lambda}$.

Proof. Since $\mathcal{N}^-_{\gamma,\lambda}$ is a closed set in X_0, we can extract a minimizing sequence $\{U_k\}_k \subset \mathcal{N}^-_{\gamma,\lambda}$ satisfying the Ekeland's variational principle for $\inf_{u\in\mathcal{N}^-_{\gamma,\lambda}} \mathcal{J}_{\gamma,\lambda}(u)$, since $\{U_k\}_k$ is bounded in X_0, then suppose $\{U_k\}_k \rightharpoonup U_0$ in X_0, Now by Lemma 4.4, $\{U_k\}_k \to U_0$ in X_0 up to a subsequence because $\mathcal{N}^-_{\gamma,\lambda}$ is closed then $U_0 \in \mathcal{N}^-_{\gamma,\lambda}$ with $\mathcal{J}_{\gamma,\lambda}(U_0) = m^-_{\gamma,\lambda}$, Now repeating the same argument as in Theorem 5.1, U_0 verify $H(U_0,\psi) \geq 0$ (for H one can see (4.10)), so that $\lambda l(x)(U_0^+)^{-q}\psi \in L^1(\Omega)$ for any $\psi \in X_0$ and U_0 in $\mathcal{N}^-_{\gamma,\lambda}$. Binding this with Lemma 3.2, we obtain U_0 is a positive solution to the problem (1.1). □

Proof of Theorem 1.2. By above Theorems 5.1 and 5.2, we can see that problem (1.1) admits two positive solutions u_0 and U_0, since $\mathcal{N}^+_{\gamma,\lambda} \cap \mathcal{N}^-_{\gamma,\lambda} = \emptyset$. Hence these solutions are distinct. □

Acknowledgements. The author would like to thank the Science and Engineering Research Board, Department of Science and Technology, Government of India for the financial support under the grants ECR/2017/002651.

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