Electronic Journal of Differential Equations, Vol. 2020 (2020), No. 110, pp. 1–28.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE OF GLOBAL SOLUTIONS AND BLOW-UP OF SOLUTIONS FOR COUPLED SYSTEMS OF FRACTIONAL DIFFUSION EQUATIONS

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Communicated by Jesus Idelfonso Diaz

Abstract. We study the Cauchy problem for a system of semi-linear coupled fractional-diffusion equations with polynomial nonlinearities posed in $\mathbb{R}_+\times\mathbb{R}^N$. Under appropriate conditions on the exponents and the orders of the fractional time derivatives, we present a critical value of the dimension N, for which global solutions with small data exist, otherwise solutions blow-up in finite time. Furthermore, the large time behavior of global solutions is discussed.

1. Introduction

We consider the system

$$\begin{aligned} {}^C D^{\gamma_1}_{0|t} u - \Delta u = f(v), \quad t>0,\ x\in\mathbb{R}^N,\\ {}^C D^{\gamma_2}_{0|t} v - \Delta v = g(u), \quad t>0,\ x\in\mathbb{R}^N, \end{aligned} \tag{1.1}$$

subject to the initial conditions

$$u(0,x)=u_0(x),\quad v(0,x)=v_0(x),\quad x\in\mathbb{R}^N, \tag{1.2}$$

where $0<\gamma_1,\gamma_2<1$, for $0<\alpha<1$, ${}^C D^{\alpha}_{0|t}u$ denotes the Caputo time fractional derivative defined, for an absolutely continuous function u, by

$$\big({}^C D^{\alpha}_{0|t}u\big)(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\partial_t u(s,\cdot)\,ds,\quad 0<\alpha<1,$$

where Δ is the Laplace operator in $\mathbb{R}^N$. The functions $f(v)$ and $g(u)$ are the nonlinear source terms that will be determined later, and u_0, v_0 are given functions.

Before we present our results and comment on them, let us dwell on existing results concerning the limiting case $\gamma_1=\gamma_2=1$. Escobedo and Herrero [7] studied the existence of global solutions, and blowing-up of solutions for the system

$$\begin{aligned} u_t-\Delta u = v^p,\quad t>0,\ x\in\mathbb{R}^N,\ v>0,\\ v_t-\Delta v = u^q,\quad t>0,\ x\in\mathbb{R}^N,\ u>0. \end{aligned} \tag{1.3}$$

2010 *Mathematics Subject Classification.* 35A01, 35R09, 35K10, 45K05.
Key words and phrases. Coupled fractional-diffusion equations; polynomial nonlinearities; global solution; blow-up.
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Submitted May 1, 2019. Published November 2, 2020.

They showed, in particular, that for

$$pq > 1, \quad \frac{N}{2} \le \frac{\max\{p,q\}+1}{pq-1},$$

every nontrivial solution of (1.3) blows-up in a finite time $T^* = T^*(\|u\|_\infty, \|v\|_\infty)$, in the sense that

$$\limsup_{t\to T^*} \|u(t)\|_\infty = \limsup_{t\to T^*} \|v(t)\|_\infty = +\infty.$$

The work [7] has been followed by works of Escobedo and Herrero in a bounded domain, Escobedo and Levine [8] for more general nonlinear forcing terms, Uda [33], Fila, Levine and Uda [9] for differing diffusive coefficients, Lu [18], Lu and Sleeman [17], Mochizuki [22], Mochizuki and Huang [23], Takase and Sleeman [31, 32] , Samarskii et al. [29], and many other authors; see the review papers [4, 1, 24].

Time-fractional differential equations/systems for global or blowing-up solutions have been studied, for example, in [5, 6, 12, 14, 20, 21, 28, 34, 39].

Kirane, Laskri and Tatar [14] studied the more general system

$$\begin{gathered} {}^C D^{\gamma_1}_{0|t} u + (-\Delta)^{\beta/2} u = |v|^p, \quad t > 0,\ x \in \mathbb{R}^N,\ p > 1, \\ {}^C D^{\gamma_2}_{0|t} v + (-\Delta)^{\gamma/2} v = |u|^q, \quad t > 0,\ x \in \mathbb{R}^N,\ q > 1, \end{gathered} \tag{1.4}$$

(for the definition of $(-\Delta)^{\sigma/2}$, $1 \le \sigma \le 2$ see [14]) with nonnegative initial data, and proved the non-existence of global solutions under the condition

$$pq > 1, \quad N \le \max\Big\{ \frac{\frac{\gamma_2}{q} + \gamma_1 - (1 - \frac{1}{pq})}{\frac{\gamma_2}{\gamma q p'} + \frac{\gamma_1}{\beta q'}}, \frac{\frac{\gamma_1}{p} + \gamma_2 - (1 - \frac{1}{pq})}{\frac{\gamma_1}{\beta p q'} + \frac{\gamma_2}{\gamma p'}} \Big\},$$

where $p + p' = pp'$ and $q + q' = qq'$.

Here, we consider problem (1.1)-(1.2) and will give conditions relating the space dimension N with parameters γ_1, γ_2, p, and q for which the solution of (1.1)-(1.2) exists globally in time and satisfies L^∞-decay estimates. We also discuss blowing-up in finite time solutions with initial data having positive average. Our study of the existence of global solutions relies on the semigroup theory, while for the blow-up of solutions result, we use the test function approach due to Zhang [41] and developed by Mitidieri and Pohozaev [24], and used by several authors (see for example [14, 10, 39]). Our result on blowing-up solutions improves the one obtained in [14]. We should mention that to the best of our knowledge there are no global existence and large time behavior results for the time-fractional diffusion system with two different fractional powers. The paper of Zhang et al. [40] does not treat the case of different time fractional operators. Also in [40], the authors do not obtain the decay rate of the solution in the space $L^\infty(\mathbb{R}^N)$.

The rest of this article is organized as follows. In section 2, we present some preliminary lemmas. In section 3, we present the main results of this paper. Finally, section 4 and section 5 are devoted to the proofs of small data global existence and blow-up in finite time of the solutions of problem (1.1)-(1.2).

Throughout this article, C will denote a positive constant. The space $L^p(\mathbb{R}^N)$ $(1 \le p < \infty)$ will be equipped with the usual norm $\|u\|^p_{L^p(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |u(x)|^p dx$. The space $C_0(\mathbb{R}^N)$ denotes the set of all continuous functions decaying to zero at infinity, equipped with Chebychev's norm $\|u\|_\infty$.

2. Preliminaries

The left-sided and right-sided Riemann-Liouville integrals (see [30]), for $\Psi \in L^1(0,T)$, $0<\alpha<1$, are defined as

$$(I^\alpha_{0|t}\Psi)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t \frac{\Psi(\sigma)}{(t-\sigma)^{\alpha-1}}\,d\sigma,\quad (I^\alpha_{t|T}\Psi)(t)=\frac{1}{\Gamma(\alpha)}\int_t^T \frac{\Psi(\sigma)}{(\sigma-t)^{\alpha-1}}\,d\sigma,$$

respectively, Γ stands for the Euler gamma function.

The left-handed and right-handed Riemann-Liouville derivatives (see [30]), for $\Psi \in AC^1([0,T])$, $0<\alpha<1$, are defined as

$$(D^\alpha_{0|t}\Psi)(t)=(\frac{d}{dt}\circ I^{1-\alpha}_{0|t}\Psi)(t),\quad (D^\alpha_{t|T}\Psi)(t)=-(\frac{d}{dt}\circ I^{1-\alpha}_{t|T}\Psi)(t),$$

respectively.

The Caputo fractional derivative for a function $\Psi \in AC^1([0,T])$ is defined by

$$({}^CD^\alpha_{0|t}\Psi)(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\Psi'(\sigma)}{(t-\sigma)^\alpha}d\sigma,$$
$$({}^CD^\alpha_{t|T}\Psi)(t)=-\frac{1}{\Gamma(1-\alpha)}\int_t^T \frac{\Psi'(\sigma)}{(\sigma-t)^\alpha}d\sigma.$$

For $0<\alpha<1$ and $\Psi \in AC^1([0,T])$, we have

$$\big(D^\alpha_{0|t}\Psi\big)(t)=\frac{1}{\Gamma(1-\alpha)}\big[\frac{\Psi(0)}{t^\alpha}+\int_0^t \frac{\Psi'(\sigma)}{(t-\sigma)^\alpha}d\sigma\big],$$

and

$$\big(D^\alpha_{t|T}\Psi\big)(t)=\frac{1}{\Gamma(1-\alpha)}\Big[\frac{\Psi(T)}{(T-t)^\alpha}-\int_t^T \frac{\Psi'(\sigma)}{(\sigma-t)^\alpha}d\sigma\Big]. \tag{2.1}$$

The Caputo derivative is related to the Riemann-Liouville derivative by

$${}^CD^\alpha_{0|t}\Psi(t)=(D^\alpha_{0|t})(\Psi(t)-\Psi(0)),\quad \text{for } \Psi\in AC^1([0,T]).$$

Let $0<\alpha<1$, $f\in AC^1([0,T])$ and $g\in AC^1([0,T])$. Then

$$\int_0^T f(t)(D^\alpha_{0|t}g)(t)\,dt=\int_0^T g(t)({}^CD^\alpha_{t|T}f)(t)\,dt+f(T)(I^{1-\alpha}_{0|T}g)(T).$$

If $f(T)=0$, then

$$\int_0^T f(t)(D^\alpha_{0|t}g)(t)\,dt=\int_0^T g(t)({}^CD^\alpha_{t|T}f)(t)\,dt.$$

For later use, let

$$\varphi(t)=\Big(1-\frac{t}{T}\Big)^l_+,\quad \text{for } t\ge 0,\ l\ge 2.$$

By a direct calculation, we obtain

$${}^CD^\alpha_{t|T}\varphi(t)=\frac{\Gamma(l+1)}{\Gamma(l+1-\alpha)}T^{-\alpha}\big(1-\frac{t}{T}\big)^{l-\alpha}_+,\quad t\ge 0.$$

Now, we present some properties of two special functions. The two parameter Mittag-Leffler function [30] is defined for $z\in\mathbb{C}$ as

$$E_{\alpha,\beta}(z)=\sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+\beta)},\quad \alpha,\beta\in\mathbb{C},\ \Re(\alpha)>0.$$

It satisfies

$$I_{0|t}^{1-\alpha}(t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})) = E_{\alpha,1}(\lambda t^{\alpha}) \quad \text{for } \lambda \in \mathbb{C}, 0 < \alpha < 1.$$

The Wright type function

$$\begin{aligned} \phi_\alpha(z) &= \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(-\alpha k + 1 - \alpha)} \\ &= \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-z)^k \Gamma(\alpha(k+1)) \sin(\pi(k+1)\alpha)}{k!}, \end{aligned}$$

for $0 < \alpha < 1$, is an entire function; it has the following properties:

(a) $\phi_\alpha(\theta) \geq 0$ for $\theta \geq 0$ and $\int_0^{+\infty} \phi_\alpha(\theta) d\theta = 1$;
(b) $\int_0^{+\infty} \phi_\alpha(\theta)\theta^r d\theta = \frac{\Gamma(1+r)}{\Gamma(1+\alpha r)}$ for $r > -1$;
(c) $\int_0^{+\infty} \phi_\alpha(\theta) e^{-z\theta} d\theta = E_{\alpha,1}(-z)$, $z \in \mathbb{C}$;
(d) $\alpha \int_0^{+\infty} \theta \phi_\alpha(\theta) e^{-z\theta} d\theta = E_{\alpha,\alpha}(-z)$, $z \in \mathbb{C}$.

The operator $A = -\Delta$ with domain

$$D(A) = \{u \in C_0(\mathbb{R}^N) : \Delta u \in C_0(\mathbb{R}^N)\},$$

generates, on $C_0(\mathbb{R}^N)$, a semigroup $\{T(t)\}_{t \geq 0}$, where

$$T(t)u_0(x) = \int_{\mathbb{R}^N} G(t, x-y) u_0(y) dy, \quad G(t,x) = \frac{1}{(4\pi t)^{N/2}} e^{-|x|^2/4t};$$

it is analytic and contractive on $L^q(\mathbb{R}^N)$ [3] and, for $t > 0$, $x \in \mathbb{R}^N$, it satisfies

$$\|T(t)u_0\|_{L^p(\mathbb{R}^N)} \leq (4\pi t)^{-\frac{N}{2}(1/q - 1/p)} \|u_0\|_{L^q(\mathbb{R}^N)}, \tag{2.2}$$

for $1 \leq q \leq p \leq +\infty$.

Let the operators $P_\alpha(t)$ and $S_\alpha(t)$ be defined by

$$P_\alpha(t)u_0 = \int_0^\infty \phi_\alpha(\theta) T(t^\alpha \theta) u_0 d\theta, \quad t \geq 0, \ u_0 \in C_0(\mathbb{R}^N), \tag{2.3}$$

$$S_\alpha(t)u_0 = \alpha \int_0^\infty \theta \phi_\alpha(\theta) T(t^\alpha \theta) u_0 d\theta, \quad t \geq 0, \ u_0 \in C_0(\mathbb{R}^N). \tag{2.4}$$

The operators $P_\alpha(t)$ and $S_\alpha(t)$ acting on the space $C_0(\mathbb{R}^N)$ into itself, see [39, Lemma 2.3, Lemma 2.4]

Consider the problem

$$\begin{gathered} {}^C D_{0|t}^{\alpha} u - \Delta u = f(t,x), \quad t > 0, \ x \in \mathbb{R}^N, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^N, \end{gathered} \tag{2.5}$$

where $u_0 \in C_0(\mathbb{R}^N)$ and $f \in L^1((0,T), C_0(\mathbb{R}^N))$. If u is a solution of (2.5), then by [39], it satisfies

$$u(t,x) = P_\alpha(t)u_0(x) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s,x) \, ds.$$

The following lemmas play an important role in obtaining the results of this paper; their proofs are obtained by combining smoothing effect of the heat semigroup property (2.2) with formulas (2.3) and (2.4) (see [39]).

Lemma 2.1. *The operator $\{P_\alpha(t)\}_{t>0}$ has the following properties:*

(a) *If $u_0 \geq 0, u_0 \not\equiv 0$, then $P_\alpha(t)u_0 > 0$ and $\|P_\alpha(t)u_0\|_{L^1(\mathbb{R}^N)} = \|u_0\|_{L^1(\mathbb{R}^N)}$;*

(b) *If* $p \le q \le +\infty$ *and* $1/r = 1/p - 1/q$, $1/r < 2/N$, *then*

$$\|P_\alpha(t)u_0\|_{L^q(\mathbb{R}^N)} \le (4\pi t^\alpha)^{-\frac{N}{2r}} \frac{\Gamma(1-N/(2r))}{\Gamma(1-\alpha N/(2r))}\|u_0\|_{L^p(\mathbb{R}^N)}.$$

Lemma 2.2. *For the operator family* $\{S_\alpha(t)\}_{t>0}$, *we have the following estimates:*

(a) *If* $u_0 \ge 0$ *and* $u_0 \not\equiv 0$, *then* $S_\alpha(t)u_0 > 0$ *and*

$$\|S_\alpha(t)u_0\|_{L^1(\mathbb{R}^N)} = \frac{1}{\Gamma(\alpha)}\|u_0\|_{L^1(\mathbb{R}^N)};$$

(b) *If* $p \le q \le +\infty$ *and* $1/r = 1/p - 1/q$, $1/r < 4/N$, *then*

$$\|S_\alpha(t)u_0\|_{L^q(\mathbb{R}^N)} \le \alpha(4\pi t^\alpha)^{-\frac{N}{2r}} \frac{\Gamma(1-N/(2r))}{\Gamma(1+\alpha-\alpha N/(2r))}\|u_0\|_{L^p(\mathbb{R}^N)}.$$

Lemma 2.3. *Let* $l \ge 1$, *and let the function* $f(t,x)$ *satisfy*

$$\|f(t,\cdot)\|_l \le \begin{cases} C_1, & 0 \le t \le 1,\\ C_2 t^{-\alpha}, & t > 1,\end{cases}$$

for some positive constants C_1, C_2 *and* α. *Then*

$$\|f(t,\cdot)\|_l \le \max\{C_1, C_2\}(1+t)^{-\beta}, \quad \textit{for all } 0 < \beta \le \alpha \textit{ and } t \ge 0.$$

Proof. For $0 \le t \le 1$, we have $\|f(t,\cdot)\|_l \le C_1 \le C_1 2^\alpha (1+t)^{-\alpha}$, so

$$\|f(t,\cdot)\|_l \le K(1+t)^{-\beta},$$

for some positive constant $K > 0$, and for all $0 < \beta \le \alpha$.

When $t \ge 1$, it follows from $\|f(t,\cdot)\|_l \le C_2 t^{-\alpha}$ that there is a constant $K' > 0$, such that $\|f(t,\cdot)\|_l \le K'(1+t)^{-\alpha}$, and so for all $0 < \beta \le \alpha$ and any $t \ge 1$, we have

$$\|f(t,\cdot)\|_l \le K'(1+t)^{-\beta}, \quad \text{for } 0 < \beta \le \alpha.$$

Therefore $\|f(t,\cdot)\|_l \le \max\{K, K'\}(1+t)^{-\beta}$, for all $0 < \beta \le \alpha$ and $t \ge 0$. □

3. Main results

In this section, we state our main result. First, we present the definition of a mild solution of problem (1.1)-(1.2).

Definition 3.1. Let $(u_0, v_0) \in C_0(\mathbb{R}^N) \times C_0(\mathbb{R}^N)$, $0 < \gamma_1, \gamma_2 < 1$, $p, q \ge 1$ and $T > 0$. We say that $(u,v) \in C([0,T]; C_0(\mathbb{R}^N) \times C_0(\mathbb{R}^N))$ a mild solution of system (1.1)-(1.2) if (u,v) satisfies the integral equations

$$\begin{aligned} u(t,x) &= P_{\gamma_1}(t)u_0 + \int_0^t (t-\tau)^{\gamma_1 - 1} S_{\gamma_1}(t-\tau) f(v(\tau,\cdot))\,d\tau,\\ v(t,x) &= P_{\gamma_2}(t)v_0 + \int_0^t (t-\tau)^{\gamma_2 - 1} S_{\gamma_2}(t-\tau) g(u(\tau,\cdot))\,d\tau. \end{aligned} \tag{3.1}$$

Using the results in [39, Theorem 3.2] and [40, Theorem 3.2], the local solvability and uniqueness of (1.1)-(1.2) can be established.

Proposition 3.2 (Existence of a local mild solution)**.** *Given* u_0 *and* v_0 *in* $C_0(\mathbb{R}^N)$, $0 < \gamma_1$, $\gamma_2 < 1$, p, $q \ge 1$, *there exist a maximal time* $T_{\max} > 0$ *and a unique mild solution* $(u,v) \in C([0,T_{\max}]; C_0(\mathbb{R}^N) \times C_0(\mathbb{R}^N))$ *to problem* (1.1)-(1.2), *such that either*

(i) $T_{\max} = \infty$ *(the solution is global), or*

(ii) $T_{\max} < \infty$ *and* $\lim_{t\to T_{\max}}(\|u(t)\|_\infty + \|v(t)\|_\infty) = \infty$ *(the solution blows up in a finite time).*
If, in addition, $u_0 \geq 0$, $v_0 \geq 0$, u_0, $v_0 \not\equiv 0$, *then* $u(t) > 0$, $v(t) > 0$ *and* $u(t) \geq P_{\gamma_1}(t)u_0$, $v(t) \geq P_{\gamma_2}(t)v_0$ *for* $t \in (0, T_{\max})$.

Moreover, if $(u_0, v_0) \in L^1(\mathbb{R}^N) \times L^1(\mathbb{R}^N)$, *then for all* s_1, $s_2 \in (1, +\infty)$, $(u, v) \in C([0, T_{\max}]; L^{s_1}(\mathbb{R}^N) \times L^{s_2}(\mathbb{R}^N))$.

Now, we state the first main result of this section concerning the existence of a global solution and large time behavior of solutions of (1.1)- (1.2).

Theorem 3.3 (Existence of a global mild solution)**.** *Let* $N \geq 1$, *let* $q \geq p \geq 1$, *be such that* $pq > 1$, *let* $(f(v), g(u)) = (\pm|v|^{p-1}v, \pm|u|^{q-1}u)$, *or* $(\pm|v|^p, \pm|u|^q)$, *and let* $0 < \gamma_1 \leq \gamma_2 < 1$. *If*

$$\frac{N}{2} \geq \frac{(\gamma_2 - \gamma_1)pq + q\gamma_2 + \gamma_1}{\gamma_1(pq-1)}, \tag{3.2}$$

then, for

$$\|u_0\|_1 + \|u_0\|_\infty + \|v_0\|_1 + \|v_0\|_\infty \leq \varepsilon_0,$$

with some $\varepsilon_0 > 0$, *there exist* $s_1 > q$, $s_2 > p$ *such that problem* (1.1)-(1.2) *admits a global mild solution with*

$$u \in L^\infty([0,\infty), L^\infty(\mathbb{R}^N)) \cap L^\infty([0,\infty), L^{s_1}(\mathbb{R}^N)),$$
$$v \in L^\infty([0,\infty), L^\infty(\mathbb{R}^N)) \cap L^\infty([0,\infty), L^{s_2}(\mathbb{R}^N)).$$

Furthermore, for all $\delta > 0$,

$$\max\Big\{1 - \frac{(pq-1)}{\gamma_2 q(p+1)}, 1 - \frac{\gamma_1(pq-1)}{\gamma_2(p+1)}, 1 - \frac{(pq-1)}{q+1}\Big\} < \delta < \min\Big\{1, \frac{N(pq-1)}{2q(p+1)}\Big\},$$
$$\|u(t)\|_{s_1} \leq C(t+1)^{-\frac{(1-\delta)(\gamma_1 + p\gamma_2)}{pq-1}}, \|v(t)\|_{s_2} \leq C(t+1)^{-\frac{(1-\delta)(\gamma_2 + q\gamma_1)}{pq-1}}, \quad t \geq 0.$$

If, in addition,

$$\frac{pN}{2s_2} < 1 \text{ and } \frac{qN}{2s_1} < 1,$$

or

$$N > 2, pN/(2s_2) < 1 \text{ and } qN/(2s_1) \geq 1,$$

or

$$N > 2, \quad qN/(2s_1) \geq 1, \quad pN/(2s_2) \geq 1 \text{ and } q \geq p > 1$$

with

$$\max\Big\{\frac{q+1}{pq(p+1)}, \frac{pq-1}{pq(p+1)}, \gamma_2/p, \sqrt{\frac{\gamma_2}{pq}}\Big\} < \gamma_1 \leq \gamma_2 < 1,$$

then $u, v \in L^\infty([0,\infty), L^\infty(\mathbb{R}^N))$,

$$\|u(t)\|_\infty \leq C(t+1)^{-\tilde{\sigma}}, \quad \|v(t)\|_\infty \leq C(t+1)^{-\hat{\sigma}}, \quad t \geq 0,$$

for some constants $\tilde{\sigma} > 0$ *and* $\hat{\sigma} > 0$.

Definition 3.4 (Weak solution)**.** Let $u_0, v_0 \in L^\infty_{\rm loc}(\mathbb{R}^N)$, $T > 0$. We say that

$$(u, v) \in L^q((0,T), L^\infty_{\rm loc}(\mathbb{R}^N)) \times L^p((0,T), L^\infty_{\rm loc}(\mathbb{R}^N))$$

is a weak solution of (1.1)-(1.2) if

$$\int_0^T \int_{\mathbb{R}^N} (|v|^p \varphi + u_0 D^{\gamma_1}_{t|T}\varphi)\,dx\,dt = \int_0^T \int_{\mathbb{R}^N} u(-\Delta\varphi)\,dx\,dt + \int_0^T \int_{\mathbb{R}^N} u D^{\gamma_1}_{t|T}\varphi\,dx\,dt,$$

$$\int_0^T \int_{\mathbb{R}^N} (|u|^q \varphi + v_0 D_{t|T}^{\gamma_2}\varphi)\,dx\,dt = \int_0^T \int_{\mathbb{R}^N} v(-\Delta\varphi)\,dx\,dt + \int_0^T \int_{\mathbb{R}^N} v D_{t|T}^{\gamma_2}\varphi\,dx\,dt,$$

for every $\varphi \in C_{t,x}^{1,2}([0,T]\times\mathbb{R}^N)$ such that $\operatorname{supp}_x \varphi \Subset \mathbb{R}^N$ and $\varphi(T,\cdot)=0$.

Similar to the proof in [39], we can easily obtain the following lemma asserting that the mild solution is the weak solution.

Lemma 3.5. *Assume $u_0, v_0 \in C_0(\mathbb{R}^N)$, and let $(u,v) \in C([0,T], C_0(\mathbb{R}^N)\times C_0(\mathbb{R}^N)$ be a mild solution of (1.1)-(1.2), then (u,v) is a weak solution of (1.1)-(1.2).*

Our next result concerns the blowing-up of solutions of (1.1)-(1.2).

Theorem 3.6 (Blow-up of mild solutions)**.** *Let $N \ge 1$, $p > 1$, $q > 1$, $0 < \gamma_1, \gamma_2 < 1$, let $(f(v), g(u)) = (|v|^p, |u|^q)$, let u_0, $v_0 \in C_0(\mathbb{R}^N)$, $u_0 \ge 0$, $v_0 \ge 0$, $u_0 \not\equiv 0$ and $v_0 \not\equiv 0$. If*

$$\frac{N}{2} < \min\Big\{\frac{(p\gamma_2+\gamma_1)}{\gamma_1(pq-1)}, \frac{(q\gamma_1+\gamma_2)}{\gamma_1(pq-1)}, \frac{(pq(\gamma_1-\gamma_2)+q\gamma_1+\gamma_2)}{\gamma_1(pq-1)}, \frac{(p+1)}{pq-1}\Big\}$$

or

$$\frac{N}{2} < \min\Big\{\frac{(q\gamma_1+\gamma_2)}{\gamma_2(pq-1)}, \frac{(p\gamma_2+\gamma_1)}{\gamma_2(pq-1)}, \frac{(pq(\gamma_2-\gamma_1)+p\gamma_2+\gamma_1)}{\gamma_2(pq-1)}, \frac{(q+1)}{pq-1}\Big\},$$

then the mild solution (u,v) of (1.1)-(1.2) blows up in a finite time.

Also if $p=1$ and $1 < q < 1+\frac{2}{N}$, or $1 < p < 1+\frac{2}{N}$ and $q=1$, then the solution blows-up in a finite time.

A result of blowing-up solutions can be obtained via differential inequalities. Let

$$\chi(x) = \Big(\int_{\mathbb{R}^N} e^{-\sqrt{N^2+|x|^2}}\,dx\Big)^{-1} e^{-\sqrt{N^2+|x|^2}}, \quad x \in \mathbb{R}^N,$$

which satisfies

$$\int_{\mathbb{R}^N} \chi(x)\,dx = 1.$$

In the next theorem, we take $f(v)=|v|^p$ and $g(u)=|u|^q$.

Theorem 3.7. *Let $\gamma_1=\gamma_2=\gamma\in(0,1)$, u_0, $v_0 \in C_0(\mathbb{R}^N)$ and u_0, $v_0 \ge 0$. Let $p>1$, $q>1$ such that $p \le q$ and let $(f(v), g(u)) = (|v|^p, |u|^q)$. If*

$$Z_0 := \int_{\mathbb{R}^N} (u_0(x)+v_0(x))\chi(x)dx > 2^{\frac{p}{p-1}},$$

then the solution of problem (1.1)-(1.2) blows-up in a finite time. Moreover, we have estimate of the time blowing up $\bar t_{**} \le \big[\frac{\ln(1-2^p Z_0^{1-p})}{2(1-p)}\Gamma(\gamma+1)\big]^{1/\gamma}$.

The next lemma plays an important role in establishing lower solution for Caputo fractional differential equation.

Lemma 3.8 ([35, Lemma 3.1])**.** *Let $u=u(t)$ is a solution of the ordinary differential equation*

$$\frac{du}{dt} = F(u), \quad u(0)=u_0, \tag{3.3}$$

where F is a function of u such that $F(0)\ge 0$, $F(u)>0$, $F_u(u)\ge 0$ for $u \ge 0$ then $v(t)=u(\bar t)$ is a lower solution of a Caputo fractional differential equation

$$^C D_{0|t}^{\alpha} u(t) = F(u), \quad u(0)=u_0, \tag{3.4}$$

where $\bar{t} = \frac{t^\alpha}{\Gamma(\alpha+1)}$. *That means*

$$ {}^C D^{\alpha}_{0|t} v(t) \le F(v), v(0) \le u_0. $$

4. Proofs of main results

Proof of Theorem 3.3. We proceed in three steps.

Step 1: Global existence for (u, v) in $L^{s_1}(\mathbb{R}^N) \times L^{s_2}(\mathbb{R}^N)$. Since $q \ge p \ge 1$, $pq > 1$, $0 < \gamma_1 \le \gamma_2 < 1$ and

$$ \frac{N}{2} \ge \frac{(\gamma_2 - \gamma_1)pq + q\gamma_2 + \gamma_1}{\gamma_1(pq-1)} > \frac{\gamma_2 q(p+1) - \gamma_1 q(pq-1)}{\gamma_2(pq-1)}, $$

we have

$$ \frac{N(pq-1)}{2q(p+1)} > \frac{\gamma_2(p+1) - \gamma_1(pq-1)}{\gamma_2(p+1)} = 1 - \frac{\gamma_1(pq-1)}{\gamma_2(p+1)}. $$

Note that, from

$$ \frac{N}{2} \ge \frac{(\gamma_2 - \gamma_1)pq + q\gamma_2 + \gamma_1}{\gamma_1(pq-1)} > \frac{q+1}{pq-1}, $$

we obtain

$$ \frac{N(pq-1)}{2q(p+1)} > \frac{q+1}{pq-1} \times \frac{(pq-1)}{q(p+1)} = \frac{q+1}{q(p+1)} > 1 - \frac{(pq-1)}{\gamma_2 q(p+1)}, $$

$$ \frac{N(pq-1)}{2q(p+1)} > \frac{q+1}{q(p+1)} > 1 - \frac{(pq-1)}{q+1}. $$

From these facts, we can choose $\delta > 0$ such that

$$ \begin{aligned} &\max\Big\{1 - \frac{(pq-1)}{\gamma_2 q(p+1)}, 1 - \frac{\gamma_1(pq-1)}{\gamma_2(p+1)}, 1 - \frac{(pq-1)}{q+1}\Big\} \\ &< \delta < \min\Big\{1, \frac{N(pq-1)}{2q(p+1)}\Big\}. \end{aligned} \tag{4.1} $$

We set

$$ \begin{gathered} r_1 = \frac{N\gamma_1(pq-1)}{2[\gamma_1(1+\delta p) + \gamma_2 p(1-\delta)]}, \quad r_2 = \frac{N\gamma_2(pq-1)}{2[\gamma_2(1+\delta q) + \gamma_1 q(1-\delta)]}, \\ \frac{1}{s_1} = \frac{2\delta}{N}\frac{p+1}{pq-1}, \quad \frac{1}{s_2} = \frac{2\delta}{N}\frac{q+1}{pq-1}, \\ \sigma_1 = \frac{(1-\delta)(\gamma_1 + \gamma_2 p)}{pq-1}, \quad \sigma_2 = \frac{(1-\delta)(\gamma_2 + \gamma_1 q)}{pq-1}. \end{gathered} \tag{4.2} $$

Clearly, we have

$$ \begin{gathered} \frac{1}{r_1} = \frac{2}{N\gamma_1}\frac{(1-\delta)(\gamma_1 + \gamma_2 p)}{pq-1} + \frac{2\delta}{N}\frac{(p+1)}{pq-1}, \\ \frac{1}{r_2} = \frac{2}{N\gamma_2}\frac{(1-\delta)(\gamma_2 + \gamma_1 q)}{pq-1} + \frac{2\delta}{N}\frac{(q+1)}{pq-1}. \end{gathered} $$

It is easy to check that $s_1 > q$, $s_2 > p$, $ps_1 > s_2$, $qs_2 > s_1$, $s_1 > r_1 > 1$, $s_2 > r_2 > 1$,

$$ \frac{N}{2}\gamma_1\big(\frac{1}{r_1} - \frac{1}{s_1}\big)q < 1, \quad \frac{N}{2}\gamma_2\big(\frac{1}{r_2} - \frac{1}{s_2}\big)p < 1, \quad \frac{N}{2}\big(\frac{p}{s_2} - \frac{1}{s_1}\big) = \delta = \frac{N}{2}\big(\frac{q}{s_1} - \frac{1}{s_2}\big), $$

$p\sigma_2 < 1$, and $q\sigma_1 < 1$. From

$$ \frac{pq(\gamma_2 - 1) + 1 + q\gamma_1}{[p\gamma_2 + \gamma_1]q} = \frac{q(p\gamma_2 + \gamma_1) - (pq-1)}{[p\gamma_2 + \gamma_1]q} < 1 - \frac{(pq-1)}{\gamma_2 q(p+1)} < \delta $$

we obtain $\delta > \frac{pq(\gamma_2-1)+q\gamma_1+1}{(\gamma_1+p\gamma_2)q}$ which is equivalent to

$$\big(\gamma_1 - \frac{N}{2}\gamma_1(\frac{p}{s_2} - \frac{1}{s_1}) - p\sigma_2\big)q > -1.$$

In fact, since $\delta = \frac{N}{2}(\frac{p}{s_2} - \frac{1}{s_1})$, the above inequality gives

$$\big(\gamma_1 - \delta\gamma_1 - p\sigma_2\big)q > -1,$$

using definition of $\sigma_2 = \frac{(1-\delta)(\gamma_2+\gamma_1 q)}{pq-1}$, we obtain

$$\Big(\gamma_1 - \delta\gamma_1 - p\frac{(1-\delta)(\gamma_2+\gamma_1 q)}{pq-1}\Big)q > -1,$$

so, we obtain

$$(1-\delta)\Big(\gamma_1 - p\frac{(\gamma_2+\gamma_1 q)}{pq-1}\Big)q > -1.$$

Therefore

$$(1-\delta)\Big(\frac{(pq-1)\gamma_1 - p(\gamma_2+\gamma_1 q)}{pq-1}\Big)q > -1.$$

By simplification, we obtain

$$(\delta-1)\Big(\frac{\gamma_1+p\gamma_2}{pq-1}\Big)q > -1,$$

or

$$\delta\Big(\frac{\gamma_1+p\gamma_2}{pq-1}\Big)q > \Big(\frac{\gamma_1+p\gamma_2}{pq-1}\Big)q - 1.$$

Thus

$$\delta > 1 - \frac{pq-1}{(\gamma_1+p\gamma_2)q} = \frac{pq(\gamma_2-1)+q\gamma_1+1}{(\gamma_1+p\gamma_2)q}.$$

Similarly, we have

$$\Big(\gamma_2 - \frac{N}{2}\gamma_2(\frac{q}{s_1} - \frac{1}{s_2}) - q\sigma_1\Big)p > -1.$$

equivalent to $\delta > \frac{pq(\gamma_1-1)+p\gamma_2+1}{(\gamma_2+q\gamma_1)p}$.

Let $(u_0, v_0) \in C_0(\mathbb{R}^N) \times C_0(\mathbb{R}^N) \cap L^{r_1}(\mathbb{R}^N) \times L^{r_2}(\mathbb{R}^N)$, and let

$$(u, v) \in C([0, T_{\max}); C_0(\mathbb{R}^N) \cap L^{s_1}(\mathbb{R}^N)) \times C([0, T_{\max}); C_0(\mathbb{R}^N) \cap L^{s_2}(\mathbb{R}^N)).$$

For $t \in [0, T_{\max})$, from (3.1), we have

$$\begin{aligned}
\|u(t)\|_{s_1} &\le \|P_{\gamma_1}(t)u_0\|_{s_1} + \Big\| \int_0^t (t-\tau)^{\gamma_1-1} S_{\gamma_1}(t-\tau)|v(\tau)|^p d\tau \Big\|_{s_1} \\
&= \|P_{\gamma_1}(t)u_0\|_{s_1} + \Big[\int_{\mathbb{R}^N} \Big| \int_0^t (t-\tau)^{\gamma_1-1} S_{\gamma_1}(t-\tau)|v(\tau)|^p d\tau \Big|^{s_1} dx \Big]^{1/s_1}.
\end{aligned}$$

We have

$$\begin{aligned}
&\Big[\int_{\mathbb{R}^N} \Big| \int_0^t (t-\tau)^{\gamma_1-1} S_{\gamma_1}(t-\tau)|v(\tau)|^p d\tau \Big|^{s_1} dx \Big]^{1/s_1} \\
&\le \Big[\int_{\mathbb{R}^N} \Big(\int_0^t (t-\tau)^{\gamma_1-1} |S_{\gamma_1}(t-\tau)|v(\tau)|^p d\tau| \Big)^{s_1} dx \Big]^{1/s_1}.
\end{aligned}$$

Using Minkowski's integral inequality, we obtain

$$\Big[\int_{\mathbb{R}^N} \Big(\int_0^t (t-\tau)^{\gamma_1-1} |S_{\gamma_1}(t-\tau)|v(\tau)|^p d\tau| \Big)^{s_1} dx \Big]^{1/s_1}$$

$$\begin{aligned}&\le \int_0^t \Big(\int_{\mathbb{R}^N}(t-\tau)^{s_1(\gamma_1-1)}|S_{\gamma_1}(t-\tau)|v(\tau)|^p|^{s_1}dx\Big)^{1/s_1}d\tau\\&=\int_0^t (t-\tau)^{\gamma_1-1}\Big(\int_{\mathbb{R}^N}|S_{\gamma_1}(t-\tau)|v(\tau)|^p|^{s_1}dx\Big)^{1/s_1}d\tau\\&=\int_0^t (t-\tau)^{\gamma_1-1}\|S_{\gamma_1}(t-\tau)|v(\tau)|^p\|_{s_1}d\tau.\end{aligned}$$

Hence, for $t\in[0,T_{\max})$, we obtain

$$\|u(t)\|_{s_1}\le \|P_{\gamma_1}(t)u_0\|_{s_1}+\int_0^t (t-\tau)^{\gamma_1-1}\|S_{\gamma_1}(t-\tau)|v(\tau)|^p\|_{s_1}d\tau, \tag{4.3}$$

$$\|v(t)\|_{s_2}\le \|P_{\gamma_2}(t)v_0\|_{s_2}+\int_0^t (t-\tau)^{\gamma_2-1}\|S_{\gamma_2}(t-\tau)|u(\tau)|^q\|_{s_2}d\tau. \tag{4.4}$$

Applying lemmas 2.2 and 2.1, we obtain

$$\|u(t)\|_{s_1}\le \|u_0\|_{r_1}t^{-\sigma_1}+C\int_0^t (t-\tau)^{\gamma_1-1}(t-\tau)^{-\frac{N}{2}\gamma_1(\frac{p}{s_2}-\frac{1}{s_1})}\|v(\tau)\|_{s_2}^p d\tau, \tag{4.5}$$

$$\|v(t)\|_{s_2}\le \|v_0\|_{r_2}t^{-\sigma_2}+C\int_0^t (t-\tau)^{\gamma_2-1}(t-\tau)^{-\frac{N}{2}\gamma_2(\frac{q}{s_1}-\frac{1}{s_2})}\|u(\tau)\|_{s_1}^q d\tau. \tag{4.6}$$

By using (4.6) in (4.5), we obtain

$$\begin{aligned}\|u(t)\|_{s_1}\le \|u_0\|_{r_1}t^{-\sigma_1}&+C\int_0^t (t-\tau)^{\gamma_1-1}(t-\tau)^{-\frac{N}{2}\gamma_1(\frac{p}{s_2}-\frac{1}{s_1})}d\tau\\&\times\Big(\|v_0\|_{r_2}t^{-\sigma_2}+C\int_0^t (t-\tau)^{\gamma_2-1}(t-\tau)^{-\frac{N}{2}\gamma_2(\frac{q}{s_1}-\frac{1}{s_2})}\|u\|_{s_1}^q d\tau\Big)^p.\end{aligned}$$

Hence

$$\begin{aligned}\|u(t)\|_{s_1}\le \|u_0\|_{r_1}t^{-\sigma_1}&+C\int_0^t (t-\tau)^{\gamma_1-1-\frac{N}{2}\gamma_1(\frac{p}{s_2}-\frac{1}{s_1})}\tau^{-p\sigma_2}d\tau\|v_0\|_{r_2}^p\\&+C\int_0^t (t-\tau)^{\gamma_1-1-\frac{N}{2}\gamma_1(\frac{p}{s_2}-\frac{1}{s_1})}\tau^{(\gamma_2-\frac{N}{2}\gamma_2(\frac{q}{s_1}-\frac{1}{s_2})-q\sigma_1)p}\\&\times\Big(\tau^{\sigma_1}\|u(\tau)\|_{s_1}\Big)^{pq}d\tau.\end{aligned}\tag{4.7}$$

Multiplying both sides of (4.7) by t^{σ_1}, where $\sigma_1=\frac{(1-\delta)(\gamma_1+\gamma_2p)}{pq-1}$, we find that

$$\begin{aligned}t^{\sigma_1}\|u\|_{s_1}\le \|u_0\|_{r_1}&+Ct^{\sigma_1}\int_0^t (t-\tau)^{\gamma_1-1-\frac{N}{2}\gamma_1(\frac{p}{s_2}-\frac{1}{s_1})}\tau^{-p\sigma_2}d\tau\|v_0\|_{r_2}^p\\&+Ct^{\sigma_1}\int_0^t (t-\tau)^{\gamma_1-1-\frac{N}{2}\gamma_1(\frac{p}{s_2}-\frac{1}{s_1})}\tau^{(\gamma_2-\frac{N}{2}\gamma_2(\frac{q}{s_1}-\frac{1}{s_2})-q\sigma_1)p}\\&\times\Big(\tau^{\sigma_1}\|u\|_{s_1}\Big)^{pq}d\tau.\end{aligned}\tag{4.8}$$

Since $\gamma_1-1-\frac{N}{2}\gamma_1(\frac{p}{s_2}-\frac{1}{s_1})>-1$, and $\big(\gamma_2-\frac{N}{2}\gamma_2(\frac{q}{s_1}-\frac{1}{s_2})-q\sigma_1\big)p>-1$, we have

$$\begin{aligned}&t^{\sigma_1}\|u\|_{s_1}\\&\le \|u_0\|_{r_1}+Ct^{\sigma_1+\gamma_1-\frac{N}{2}\gamma_1(\frac{p}{s_2}-\frac{1}{s_1})-p\sigma_2}\|v_0\|_{r_2}^p\\&\quad+Ct^{\sigma_1+\gamma_1-\frac{N}{2}\gamma_1(\frac{p}{s_2}-\frac{1}{s_1})+(\gamma_2-\frac{N}{2}\gamma_2(\frac{q}{s_1}-\frac{1}{s_2})-q\sigma_1)p}\Big(\sup_{0\le\tau<t}\tau^{\sigma_1}\|u(\tau)\|_{s_1}\Big)^{pq}.\end{aligned}\tag{4.9}$$

Note that

$$
\begin{gathered}
\sigma_1 = \frac{N}{2}\gamma_1(\frac{1}{r_1} - \frac{1}{s_1}),\\
\sigma_1 + \gamma_1 - \frac{N}{2}\gamma_1(\frac{p}{s_2} - \frac{1}{s_1}) - p\sigma_2 = 0,\\
\sigma_1 + \gamma_1 - \frac{N}{2}\gamma_1(\frac{p}{s_2} - \frac{1}{s_1}) + \Big(\gamma_2 - \frac{N}{2}\gamma_2(\frac{q}{s_1} - \frac{1}{s_2}) - q\sigma_1\Big)p = 0,\\
\sigma_1 + \gamma_1 - \gamma_1\delta + (\gamma_2 - \gamma_2\delta - q\sigma_1)p = 0.
\end{gathered}
$$

Defining $h(t) = \sup_{0\leq\tau\leq t} \tau^{\sigma_1}\|u(\tau)\|_{s_1}$, $t \in [0, T_{\max})$, we deduce from (4.8) that

$$
h(t) \leq C(\|u_0\|_{r_1} + \|v_0\|_{r_2}^p + h(t)^{pq}) \tag{4.10}
$$

for all $t \in (0, T_{\max})$. Here C is independent of t. Set

$$
A := \|u_0\|_{r_1} + \|v_0\|_{r_2}^p.
$$

Then, it follows by a continuity argument that for sufficiently small u_0 and v_0 such that $A < (2C)^{\frac{pq}{1-pq}}$ that

$$
h(t) \leq 2CA, \quad \text{for all } t \in [0, T_{\max}). \tag{4.11}
$$

Otherwise, there exists $t_0 \in (0, T_{\max})$ such that $h(t_0) > 2CA$; by the intermediate value theorem, since h is continuous and $h(0) = 0$, there exists $t_1 \in (0, t_0)$ such that

$$
h(t_1) = 2CA. \tag{4.12}
$$

Using (4.12) in (4.10), we obtain

$$
h(t_1) \leq C(A + h(t_1)^{pq}) = \Big(\frac{h(t_1)}{2} + Ch(t_1)^{pq}\Big),
$$

from which, we infer

$$
\frac{h(t_1)}{2} \leq Ch(t_1)^{pq}, \tag{4.13}
$$

using (4.12) in (4.13), we obtain $CA \leq C(2CA)^{pq}$, so

$$
A \leq (2CA)^{pq} = (2C)^{pq}A^{pq},
$$

then it yields $(2C)^{-pq} \leq A^{pq-1}$, which is equivalent to

$$
A \geq (2C)^{\frac{pq}{1-pq}}.
$$

This contradicts the choice of A. It then follows that $h(t)$ remains bounded in all time $t > 0$ provided that $\|u_0\|_{r_1}$ and $\|v_0\|_{r_2}$ are small. Therefore

$$
t^{\sigma_1}\|u(t)\|_{s_1} \leq C, \quad \text{for all } t > 0. \tag{4.14}
$$

Similarly, we obtain

$$
t^{\sigma_2}\|v(t)\|_{s_2} \leq C, \quad \text{for all } t > 0. \tag{4.15}
$$

Step 2: L^∞-global existence estimates of (u, v) in $L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$. Let s_1, s_2 be the same as in (4.2). Since $p \leq q$, we have

$$
\frac{Np}{2s_2} \leq \frac{Nq}{2s_1}.
$$

We further assume for some $\xi > q$, $w > p$, $k_1 > 0$, $k_2 > 0$ that $u(t) \in L^w(\mathbb{R}^N)$, $v(t) \in L^\xi(\mathbb{R}^N)$ and

$$
\|u(t)\|_w \leq C(1 + t^{k_1}), \quad \|v(t)\|_\xi \leq C(1 + t^{k_2}) \quad \text{for every } t \in [0, T_{\max}). \tag{4.16}
$$

Then, by applying Lemmas 2.1 and 2.2 again to (3.1), we obtain

$$\|u(t)\|_\infty \le \|P_{\gamma_1}(t)u_0\|_\infty + \int_0^t (t-\tau)^{\gamma_1-1-\frac{N\gamma_1 p}{2\xi}} \|v(\tau)\|_\xi^p d\tau, \tag{4.17}$$

$$\|v(t)\|_\infty \le \|P_{\gamma_2}(t)v_0\|_\infty + \int_0^t (t-\tau)^{\gamma_2-1-\frac{N\gamma_2 q}{2w}} \|u(\tau)\|_w^q d\tau, \tag{4.18}$$

for all $t \in [0, T_{\max})$. Now, if one can find ξ and w such that

$$\frac{Np}{2\xi} < 1 \quad \text{or} \quad \frac{Nq}{2w} < 1, \tag{4.19}$$

then the L^∞-estimates of (u,v) can be obtained. In fact, if $\frac{Np}{2\xi} < 1$, in view of (4.16), from (4.17) we have

$$\begin{aligned} \|u(t)\|_\infty &\le \|P_{\gamma_1}(t)u_0\|_\infty + C \max_{\tau\in[0,t]} \|v(\tau)\|_\xi^p t^{(1-\frac{Np}{2\xi})\gamma_1} \\ &\le C(1 + t^{(1-\frac{Np}{2\xi})\gamma_1 + pk_2}), \end{aligned} \tag{4.20}$$

and by taking $w = \infty$ in (4.18), we obtain

$$\begin{aligned} \|v(t)\|_\infty &\le \|P_{\gamma_2}(t)v_0\|_\infty + \int_0^t (t-\tau)^{\gamma_2-1} \|u(\tau)\|_\infty^q d\tau \\ &\le \|P_{\gamma_2}(t)v_0\|_\infty + \int_0^t (t-\tau)^{\gamma_2-1} \Big(1 + t^{(1-\frac{Np}{2\xi})\gamma_1+pk_2}\Big)^q d\tau \\ &\le C\Big(1 + t^{\gamma_2 + [(1-\frac{Np}{2\xi})\gamma_1 + pk_2]q}\Big). \end{aligned} \tag{4.21}$$

These estimates show that $T_{\max} = \infty$ and

$$u, v \in L^\infty_{\rm loc}([0,\infty); L^\infty(\mathbb{R}^N)). \tag{4.22}$$

In a similar way, we can deal with the case $\frac{Nq}{2w} < 1$.

To find such appropriate ξ and w, we note that if $\frac{Nq}{2s_1} < 1$ or $\frac{Np}{2s_2} < 1$, then (4.20) and (4.21) hold by taking $\xi = s_1$ or $w = s_2$. This is certainly the case if $N \le 2$ as $s_1 > q$ and $s_2 > p$.

Thus it remains to deal with the case $N > 2$, $\frac{Nq}{2s_1} \ge 1$ and $\frac{Np}{2s_2} \ge 1$. We will do this via an iterative process. Define $s_1' = s_1$, $s_1'' = s_2$. Since $s_1' > q$ and $s_1'' > p$, using the Hölder inequality and lemmas 2.1,2.2, we obtain from (4.3) and (4.4) that

$$\|u(t)\|_{s_2'} \le \|P_{\gamma_1}(t)u_0\|_{s_2'} + \int_0^t (t-\tau)^{\gamma_1-1-\frac{N\gamma_1}{2}(\frac{p}{s_2''} - \frac{1}{s_2'})} \|v(\tau)\|_{s_1''}^p d\tau,$$

$$\|v(t)\|_{s_2''} \le \|P_{\gamma_2}(t)v_0\|_{s_2''} + \int_0^t (t-\tau)^{\gamma_2-1-\frac{N\gamma_2}{2}(\frac{q}{s_1'} - \frac{1}{s_2''})} \|u(\tau)\|_{s_1'}^q d\tau,$$

where s_2' and s_2'' are such that

$$\frac{N}{2}\Big(\frac{p}{s_1''} - \frac{1}{s_2'}\Big) < 1, \quad \frac{N}{2}\Big(\frac{q}{s_1'} - \frac{1}{s_2''}\Big) < 1.$$

This can be verified by taking

$$\frac{1}{s_2'} = \frac{p}{s_1''} - \frac{2}{N} + \eta, \quad \frac{1}{s_2''} = \frac{q}{s_1'} - \frac{2}{N} + \eta,$$

where $0 < \eta < 2(1-\delta)/N$. Observe that

$$\frac{1}{s'_1} - \frac{1}{s'_2} = \frac{2}{N}(1-\delta) - \eta > 0, \quad \frac{1}{s''_1} - \frac{1}{s''_2} = \frac{2}{N}(1-\delta) - \eta > 0, \tag{4.23}$$

and hence $s'_2 > s'_1 > q$ and $s''_2 > s''_1 > p$.

Next, we define the sequences $\{s'_i\}_{i\geq 1}$ and $\{s''_i\}_{i\geq 1}$ iteratively as follows

$$\frac{1}{s'_i} = \frac{p}{s''_{i-1}} - \frac{2}{N} + \eta, \quad \frac{1}{s''_i} = \frac{q}{s'_{i-1}} - \frac{2}{N} + \eta, \quad i \geq 3. \tag{4.24}$$

Then

$$\begin{gathered}
\frac{1}{s'_i} - \frac{1}{s'_{i+1}} = p\Big(\frac{1}{s''_{i-1}} - \frac{1}{s''_i}\Big) = pq\Big(\frac{1}{s'_{i-2}} - \frac{1}{s'_{i-1}}\Big),\\
\frac{1}{s''_i} - \frac{1}{s''_{i+1}} = q\Big(\frac{1}{s'_{i-1}} - \frac{1}{s'_i}\Big) = pq\Big(\frac{1}{s''_{i-2}} - \frac{1}{s''_{i-1}}\Big).
\end{gathered}$$

Since $pq > 1$, in view of (4.23), we obtain

$$\frac{1}{s'_i} > \frac{1}{s'_{i+1}}, \quad \frac{1}{s''_i} > \frac{1}{s''_{i+1}}, \quad i \geq 1 \tag{4.25}$$

$$\lim_{i\to+\infty}\Big(\frac{1}{s'_i} - \frac{1}{s'_{i+1}}\Big) = \lim_{i\to+\infty}(\frac{1}{s''_i} - \frac{1}{s''_{i+1}}) = +\infty. \tag{4.26}$$

Now, we ensure that there exists i_0 such that

$$\frac{p}{s''_{i_0}} < \frac{2}{N} \quad \text{or} \quad \frac{q}{s'_{i_0}} < \frac{2}{N}. \tag{4.27}$$

In fact, if (4.27) is not true, that is $\frac{p}{s''_i} \geq \frac{2}{N}$ and $\frac{q}{s'_i} \geq \frac{2}{N}$ for all $i \geq 1$.

Then, by (4.24), we see that $s'_i > 0$, $s''_i > 0$ for all $i \geq 1$ and hence, by (4.25),

$$q < s'_1 < \dots < s'_i < \dots, \quad p < s''_1 < \dots < s''_i < \dots .$$

Therefore

$$\Big|\frac{1}{s'_i} - \frac{1}{s'_{i+1}}\Big| \leq \frac{1}{s'_i} + \frac{1}{s'_{i+1}} < \frac{2}{q} < 2, \quad \text{for all } i \geq 1,$$

which contradicts (4.26).

Let i_0 be the smallest number that satisfies (4.27). We note that $i_0 \geq 2$. Without loss of generality, we assume that

$$\begin{gathered}
\frac{p}{s''_{i_0}} < \frac{2}{N}, \quad \frac{p}{s''_i} \geq \frac{2}{N} \quad \text{for } 1 \leq i \leq i_0 - 1,\\
\frac{q}{s'_i} \geq \frac{2}{N} \quad \text{for } 1 \leq i \leq i_0.
\end{gathered} \tag{4.28}$$

Thus (4.24) yields

$$s'_i > 0 \text{ for } 1 \leq i \leq i_0, \quad s''_i > 0 \text{ for } 1 \leq i \leq i_0 + 1,$$

which together with (4.25) leads to

$$q < \dots < s'_{i_0-1} < s'_{i_0} \quad p < \dots < s''_{i_0} < s''_{i_0+1}.$$

Now, from (4.24), for all $i \geq 2$ we have

$$\frac{N}{2}\Big(\frac{p}{s''_{i-1}} - \frac{1}{s'_i}\Big) = 1 - \frac{N}{2}\eta = \frac{N}{2}\Big(\frac{q}{s'_{i-1}} - \frac{1}{s''_i}\Big).$$

Now, let us deal with the boundedness of $(u(t), v(t))$ in $L^{s'_i}(\mathbb{R}^N) \times L^{s''_i}(\mathbb{R}^N)$.

By using Lemmas 2.1 and 2.2, it follows from (3.1) inductively that, for all $2 \le i \le i_0$ and for all $t \in (0, T_{\max})$,

$$\begin{aligned} \|u(t)\|_{s'_i} &\le \|P_{\gamma_1}(t)u_0\|_{s'_i} + C\int_0^t (t-\tau)^{\gamma_1 - 1 - \frac{N\gamma_1}{2}(\frac{p}{s''_{i-1}} - \frac{1}{s'_i})} \|v(\tau)\|^p_{s''_{i-1}} d\tau \\ &\le C\|u_0\|_{s'_i} + C\int_0^t (t-\tau)^{\gamma_1 - 1 - \gamma_1(1-\frac{N\eta}{2})} \|v(\tau)\|^p_{s''_{i-1}} d\tau, \end{aligned} \tag{4.29}$$

and

$$\begin{aligned} \|v(t)\|_{s''_i} &\le \|P_{\gamma_2}(t)v_0\|_{s''_i} + C\int_0^t (t-\tau)^{\gamma_2 - 1 + \frac{N\gamma_2}{2}(\frac{q}{s'_{i-1}} - \frac{1}{s''_i})} \|u(\tau)\|^q_{s'_{i-1}} d\tau \\ &\le C\|v_0\|_{s''_i} + C\int_0^t (t-\tau)^{\gamma_2 - 1 - \gamma_2(1-\frac{N\eta}{2})} \|u(\tau)\|^q_{s'_{i-1}} d\tau, \end{aligned} \tag{4.30}$$

for $2 \le i \le i_0 + 1$ and $t \in (0, T_{\max})$.

From the Hölder inequality, we have

$$\begin{aligned} \|P_{\gamma_1}(t)u_0\|_{s'_i} &\le \|u_0\|_{s'_i} \le \|u_0\|_{s_1}^{\frac{s_1}{s'_i}} \|u_0\|_\infty^{1-\frac{s_1}{s'_i}} < \infty, \\ \|P_{\gamma_2}(t)v_0\|_{s''_i} &\le \|v_0\|_{s''_i} \le \|v_0\|_{s_1}^{\frac{s_1}{s''_i}} \|v_0\|_\infty^{1-\frac{s_1}{s''_i}} < \infty, \end{aligned} \tag{4.31}$$

since $u_0 \in L^{s_1} \cap L^\infty$ and $v_0 \in L^{s_2} \cap L^\infty$. From (4.29), (4.30) and (4.31) it follows that

$$\begin{gathered} u(t) \in L^{s'_i}(\mathbb{R}^N), \quad \|u(t)\|_{s'_i} \le C(1 + t^{a_i}), \quad 1 \le \forall i \le i_0, \\ v(t) \in L^{s''_i}(\mathbb{R}^N), \quad \|v(t)\|_{s''_i} \le C(1 + t^{b_i}), \quad 1 \le \forall i \le i_0 + 1, \end{gathered} \tag{4.32}$$

for all $t \in (0, T_{\max})$ and for some positive constants a_i, b_i. Since $\frac{Np}{2s_{i''_0}} < 1$, taking $s_2 = s''_{i_0}$, (4.19) holds; hence $T_{\max} = +\infty$ and (4.22) holds.

Step 3: First, we show the following decay estimates

$$\|u(t)\|_{s_1} \le C(t+1)^{-\sigma_1}, \quad \|v(t)\|_{s_2} \le C(t+1)^{-\sigma_2}, \quad \text{for } t \ge 0,$$

where s_1 and s_2 are given by (4.2).

According to Lemma 2.3, it suffices to prove that $\|u(t)\|_{s_1} \le C, \|v(t)\|_{s_2} \le C$, for all $t \in [0, 1]$.

To do this, we need to show that

$$\|u(t)\|_\infty \le C, \quad \|v(t)\|_\infty \le C, \quad \text{for } t \in [0, 1]. \tag{4.33}$$

In fact, by applying Lemmas 2.1 and 2.2 to (3.1) we see that

$$\begin{aligned} \|u(s)\|_\infty &\le \|P_{\gamma_1}(s)u_0\|_\infty + \int_0^s (s-\tau)^{\gamma_1 - 1} \|v(\tau)\|^p_\infty d\tau \\ &\le \|u_0\|_\infty + \int_0^s (s-\tau)^{\gamma_1 - 1} \|v(\tau)\|^p_\infty d\tau, \\ \|v(s)\|_\infty &\le \|P_{\gamma_2}(s)v_0\|_\infty + \int_0^s (s-\tau)^{\gamma_2 - 1} \|u(\tau)\|^q_\infty d\tau \\ &\le \|v_0\|_\infty + \int_0^s (s-\tau)^{\gamma_2 - 1} \|u(\tau)\|^q_\infty d\tau, \end{aligned}$$

for $0 \le s \le t$. For $0 \le s \le t \le 1$, the two inequalities above give

$$\begin{aligned}\sup_{0\le s\le t}\|u(s)\|_\infty &\le \|u_0\|_\infty + \frac{1}{\gamma_1}\Big(\sup_{0\le\tau\le t}\|v(\tau)\|_\infty\Big)^p t^{\gamma_1}\\ &\le \|u_0\|_\infty + \frac{1}{\gamma_1}(\sup_{0\le\tau\le t}\|v(\tau)\|_\infty)^p,\\ \sup_{0\le s\le t}\|v(s)\|_\infty &\le \|v_0\|_\infty + \frac{1}{\gamma_2}\Big(\sup_{0\le\tau\le t}\|u(\tau)\|_\infty\Big)^q t^{\gamma_2}\\ &\le \|v_0\|_\infty + \frac{1}{\gamma_2}(\sup_{0\le\tau\le t}\|u(\tau)\|_\infty)^q.\end{aligned}$$

Using the second inequality into first inequality, it yields that

$$\begin{aligned}\sup_{0\le s\le t}\|u(s)\|_\infty &\le \|u_0\|_\infty + \frac{1}{\gamma_1}\Big(\|v_0\|_\infty + \frac{1}{\gamma_2}\Big(\sup_{0\le\tau\le t}\|u(\tau)\|_\infty\Big)^q\Big)^p\\ &\le C\Big(\|u_0\|_\infty + \|v_0\|_\infty^p + \Big(\sup_{0\le\tau\le t}\|u(\tau)\|_\infty\Big)^{pq}\Big)\end{aligned}$$

So, arguing as in the first step by setting $h(t) = \sup_{0\le s\le t}\|u(s)\|_\infty$ and $A = \|u_0\|_\infty + \|v_0\|_\infty^p$, we obtain

$$h(t) \le A + Ch^{pq}(t), \quad \text{for all } t \le 1,$$

which implies (4.33) for A small since $pq > 1$.

We see from (4.3) and Lemmas 2.1, 2.2 that

$$\|u(t)\|_{s_1} \le C\|u_0\|_{s_1} + C\int_0^t (t-\tau)^{\gamma_1-1}\||v(\tau)|^p\|_{s_1}d\tau,$$

where s_1 given explicitly by (4.2). Therefore

$$\|u(t)\|_{s_1} = C\|u_0\|_{s_1} + C\int_0^t (t-\tau)^{\gamma_1-1}\|v(\tau)\|_{ps_1}^p d\tau.$$

By the interpolation inequality $\|v(\tau)\|_{ps_1}^p \le \|v(\tau)\|_{s_2}^{\frac{s_2}{s_1}}\|v(\tau)\|_\infty^{p(1-\frac{s_2}{ps_1})}$, we obtain

$$\|u(t)\|_{s_1} = C\|u_0\|_{s_1} + C\sup_{\tau\in(0,t)}\|v(\tau)\|_\infty^{p(1-\frac{s_2}{ps_1})}\int_0^t (t-\tau)^{\gamma_1-1}\|v(\tau)\|_{s_2}^{\frac{s_2}{s_1}}d\tau. \tag{4.34}$$

Now, using (4.15) and (4.33) in (4.34), we obtain

$$\|u(t)\|_{s_1} \le C\|u_0\|_{s_1} + C\int_0^t (t-\tau)^{\gamma_1-1}\tau^{-\frac{s_2}{s_1}\sigma_2}d\tau,$$

provided that $\frac{s_2}{s_1}\sigma_2 < 1$. On the other hand, since $s_1$and s_2 satisfy

$$\frac{s_1}{s_2}\sigma_1 = \frac{(1-\delta)(p\gamma_2+\gamma_1)s_1}{(pq-1)s_2} \le \gamma_2, \quad \frac{s_2}{s_1}\sigma_2 = \frac{(1-\delta)(q\gamma_1+\gamma_2)s_2}{(pq-1)s_1} \le \gamma_1,$$

we obtain $\gamma_1 - \frac{s_2}{s_1}\sigma_2 \ge 0$ and consequently

$$\|u(t)\|_{s_1} \le C\|u_0\|_{s_1} + Ct^{\gamma_1-\frac{s_2}{s_1}\sigma_2} \le C, \quad \text{for all } t\in[0,1].$$

Analogously,

$$\|v(t)\|_{s_2} \le C \quad \text{for all } t\in[0,1].$$

From (4.14), (4.15), (4.33) and Lemma 2.3, we conclude that

$$\|u(t)\|_{s_1} \le C(t+1)^{-\frac{(1-\delta)(\gamma_1+p\gamma_2)}{pq-1}}, \quad \|v(t)\|_{s_2} \le C(t+1)^{-\frac{(1-\delta)(\gamma_2+q\gamma_1)}{pq-1}}, \tag{4.35}$$

for all $t \geq 0$.

Next, we derive L^∞-decay estimates. Let

$$\sigma_1 = \frac{(1-\delta)(p\gamma_2+\gamma_1)}{(pq-1)}, \quad \sigma_2 = \frac{(1-\delta)(q\gamma_1+\gamma_2)}{(pq-1)}.$$

If $\frac{pN}{2s_2} < 1$, by taking $\xi = s_2$ in (4.18) and using (4.35), we obtain

$$\|u(t)\|_\infty \leq Ct^{-\frac{N}{2}\gamma_1}\|u_0\|_1 + C\int_0^t (t-\tau)^{\gamma_1 - 1 - \frac{N\gamma_1}{2}\frac{p}{s_2}} \tau^{-p\sigma_2} d\tau \tag{4.36}$$

and

$$p\sigma_2 < 1, \quad \gamma_1 - \frac{N\gamma_1}{2}\frac{p}{s_2} - p\sigma_2 = -\frac{[\gamma_1 + \gamma_1 p\delta + (1-\delta)p\gamma_2]}{pq-1}.$$

On the other hand, we have

$$\frac{\gamma_1 + \gamma_1 p\delta + p\gamma_2(1-\delta)}{pq-1} < \frac{\gamma_1 + p\gamma_2}{pq-1} \leq \frac{N}{2}\gamma_1. \tag{4.37}$$

Then, it follows from (4.36), (4.37) and lemma 2.3 that

$$\|u(t)\|_\infty \leq Ct^{-\frac{N}{2}\gamma_1} + Ct^{-\frac{[\gamma_1+\gamma_1 p\delta+(1-\delta)p\gamma_2]}{pq-1}}.$$

Thus

$$\|u(t)\|_\infty \leq C(1+t)^{-\frac{[\gamma_1+\gamma_1 p\delta+(1-\delta)p\gamma_2]}{pq-1}}. \tag{4.38}$$

for all $t \geq 0$.

Similarly, if $\frac{qN}{2s_1} < 1$, then one can find that

$$\|v(t)\|_\infty \leq C(1+t)^{-\frac{[\gamma_2+\gamma_2 q\delta+(1-\delta)q\gamma_1]}{pq-1}}, \quad \text{for } t \geq 0. \tag{4.39}$$

At the same time, (4.38) holds as $pN/(2s_2) \leq qN/(2s_1)$.

In particular, if $pq > q+2$, and $\gamma_1 q^2 > 2q+1$, we can choose

$$\delta > \max\big\{1 - \frac{(pq-1)}{\gamma_2 q(p+1)}, 1 - \frac{\gamma_1(pq-1)}{\gamma_2(p+1)}\big\}$$

and $\delta \approx \max\{1 - \frac{(pq-1)}{\gamma_2 q(p+1)}, 1 - \frac{\gamma_1(pq-1)}{\gamma_2(p+1)}\}$ such that $qN/(2s_1) < 1$. Therefore, estimates (4.38) and (4.39) hold.

It is useful to note that $N \leq 2$ implies $pN/(2s_2) < 1$ and $qN/(2s_1) < 1$ implies $pq > 2+q$.

It remains to consider the following two cases:

(1) The case $N > 2$, $\frac{Np}{2s_2} < 1$ and $\frac{Nq}{2s_1} \geq 1$. Let

$$\sigma' = \frac{\gamma_1 + \gamma_1 p\delta + (1-\delta)p\gamma_2}{pq-1}.$$

For a positive μ such that $\mu < \min\{\sigma', \sigma_1\}$ and $q\mu < 1$, since $N > 2$ and $q > 1$, we can choose $k > 0$ such that $k > \frac{qN}{2}$ and $q\mu + \frac{qN\gamma_2}{2k} > \gamma_2$. Since $s_1 \leq qN/2$, we have $k > s_1$.

Using the interpolation inequality

$$\|u(t)\|_k \leq \|u(t)\|_\infty^{(k-s_1)/k}\|u(t)\|_{s_1}^{s_1/k} \leq Ct^{-\sigma'(k-s_1)/k}t^{-\sigma_1 s_1/k} \quad \text{for all } t > 0,$$

it follows from (4.14) and (4.38) that

$$\|u(t)\|_k \leq Ct^{-\mu} \quad \text{for all } t > 0.$$

Whereupon,

$$\begin{aligned}\|v(t)\|_\infty &\le \|P_{\gamma_2}(t)v_0\|_\infty + C\int_0^t (t-\tau)^{\gamma_2-1-\frac{Nq}{2k}\gamma_2}\|u(\tau)\|_k^q d\tau\\ &\le Ct^{-\frac{N}{2}\gamma_2}\|v_0\|_1 + C\int_0^t (t-\tau)^{\gamma_2-1-\frac{N\gamma_2 q}{2k}}\tau^{-q\mu}d\tau\\ &\le C(t^{-\frac{N}{2}\gamma_2}+t^{\gamma_2-\frac{N\gamma_2 q}{2k}-q\mu})\\ &\le Ct^{-\alpha},\end{aligned} \tag{4.40}$$

for all $t>0$, where $\alpha=\min\{\frac{N}{2}\gamma_2, -\gamma_2+\frac{N\gamma_2 q}{2k}+q\mu\}>0$. From (4.33) and (4.40), we infer that

$$\|v(t)\|_\infty \le C(1+t)^{-\alpha} \quad \text{for all } t\ge 0.$$

We remark that, in the particular case $p=1$, $q>3$ and $q^2>\max\{4\gamma_2 q+1, \frac{4\gamma_2+\gamma_1}{\gamma_1}\}$, we can choose

$$\begin{aligned}\delta &> \max\Big\{1-\frac{(pq-1)}{\gamma_2 q(p+1)}, 1-\frac{\gamma_1(pq-1)}{\gamma_2(p+1)}, 1-\frac{(pq-1)}{q+1}\Big\}\\ &= \max\Big\{1-\frac{(q-1)}{2\gamma_2 q}, 1-\frac{\gamma_1(q-1)}{2\gamma_2}, 1-\frac{(q-1)}{q+1}\Big\}\end{aligned}$$

and $\delta\approx\max\{1-\frac{(q-1)}{2\gamma_2 q}, 1-\frac{\gamma_1(q-1)}{2\gamma_2}, 1-\frac{(q-1)}{q+1}\}$ such that $N/(2s_2)<1$. Therefore, we have the estimate (4.38).

(2) The case: $N>2$, $qN/(2s_1)\ge 1$, $pN/(2s_2)\ge 1$, $q\ge p>1$, and $\gamma_1\le\gamma_2$. It needs a careful handling and we need to restrict further the choice of δ. From $\max\{\frac{q+1}{pq(p+1)}, \frac{pq-1}{pq(p+1)}, \gamma_2/p, \sqrt{\frac{\gamma_2}{pq}}\}<\gamma_1\le\gamma_2<1$ and $pq>1$, we obtain

$$\begin{aligned}&\max\Big\{1-\frac{(pq-1)}{\gamma_2 q(p+1)}, 1-\frac{\gamma_1(pq-1)}{\gamma_2(p+1)}, 1-\frac{(pq-1)}{q+1}\Big\}\\ &< \min\Big\{1-\frac{(pq-1)}{\gamma_1 pq(p+1)}, \frac{N(pq-1)}{2q(p+1)}\Big\}.\end{aligned}$$

So, we select $\delta>0$ such that

$$\begin{aligned}&\max\Big\{1-\frac{(pq-1)}{\gamma_2 q(p+1)}, 1-\frac{\gamma_1(pq-1)}{\gamma_2(p+1)}, 1-\frac{(pq-1)}{q+1}\Big\}\\ &< \delta < \min\Big\{\frac{N(pq-1)}{2q(p+1)}, 1-\frac{(pq-1)}{\gamma_1 pq(p+1)}\Big\}.\end{aligned}$$

We get immediately $p\sigma_2>1/q$ and $q\sigma_1>1/p$. Further, we notice that there exist $\varepsilon\in(0,1)$ and $\beta<1$ close to 1 such that

$$p\sigma_2-\varepsilon>1/q,\quad q\sigma_1-\varepsilon>1/p,\quad 1/p<\beta-\varepsilon,\quad 1/q<\beta-\varepsilon. \tag{4.41}$$

By taking $\eta=2\varepsilon(1-\delta)/N$, we find the integer i_0 as in the step 2, and, without loss of generality, we assume that (4.28) holds. We choose β in addition to (4.41) satisfying

$$\gamma_1<\gamma_1\frac{pN}{2s''_{i_0}}+\gamma_2\beta,\quad \text{since } \gamma_2\ge\gamma_1. \tag{4.42}$$

As

$$\delta<\frac{N(pq-1)}{2(p+1)q}\le\frac{N(pq-1)}{2(q+1)p},$$

and $\beta<1$, we have

$$\beta+\frac{(p+1)q\delta}{(pq-1)}<1+\frac{N}{2},\quad \beta+\frac{(q+1)p\delta}{(pq-1)}<1+\frac{N}{2}. \tag{4.43}$$

For $2\le i\le i_0-1$, define r'_{i+1} and r''_{i+1} inductively as follows:

$$\begin{gathered}
\frac{1}{r'_2}=\frac{1}{s'_2}+\frac{2}{N}[p\sigma_2-\varepsilon(1-\delta)],\quad \frac{1}{r''_2}=\frac{1}{s''_2}+\frac{2}{N}[q\sigma_1-\varepsilon(1-\delta)],\\
\frac{1}{r'_{i+1}}=\frac{1}{s'_{i+1}}+\frac{2}{N}[\beta-\varepsilon(1-\delta)],\quad \frac{1}{r''_{i+1}}=\frac{1}{s''_{i+1}}+\frac{2}{N}[\beta-\varepsilon(1-\delta)].
\end{gathered}$$

It is clear that $r'_i,\, r''_i>0$ and $r'_i<s'_i,\, r''_i<s''_i$ for all $2\le i\le i_0$. A simple calculation shows that $r'_i,\, r''_i>1$. As s'_i and s''_i are increasing in i for $1\le i\le i_0$; we have

$$\begin{aligned}
\frac{1}{r'_{i+1}}&<\frac{1}{s'_2}+\frac{2}{N}[\beta-\varepsilon(1-\delta)]\\
&=\frac{p}{s''_1}-\frac{2}{N}+\frac{2}{N}\varepsilon(1-\delta)+\frac{2}{N}[\beta-\varepsilon(1-\delta)]\\
&=\frac{2}{N}\Big(\frac{p(q+1)\delta}{pq-1}+\beta-1\Big)<1,
\end{aligned}$$

from (4.43); therefore $r'_{i+1}>1$. Similarly, we can check that $r''_{i+1}>1$.

From (4.22) and (4.32), we see that there exists a positive constant C such that

$$\|u(t)\|_\infty,\ \|v(t)\|_\infty,\ \|u(t)\|_{k_1},\ \|v(t)\|_{k_2}\le C\quad \text{for } 0\le t\le 1, \tag{4.44}$$

for all $s'_1\le k_1\le s'_{i_0}$, $s''_1\le k_2\le s''_{i_0}$. Furthermore, since $1-\eta N/2=1-\varepsilon(1-\delta)$ and $p\sigma_2<1$, using (4.29), (4.30) with the help of (4.14) and (4.15), we arrive at the estimate

$$\begin{aligned}
\|u(t)\|_{s'_2}&\le\|P_{\gamma_1}(t)u_0\|_{s'_2}+C\int_0^t(t-\tau)^{\gamma_1-1-\gamma_1(1-\varepsilon(1-\delta))}\|u(\tau)\|^p_{s''_2}d\tau\\
&\le Ct^{-\frac{N}{2}\gamma_1(\frac{1}{r'_2}-\frac{1}{s'_2})}\|u_0\|_{s'_2}+C\int_0^t(t-\tau)^{\gamma_1-1-\gamma_1(1-\varepsilon(1-\delta))}\tau^{-p\sigma_2}d\tau\\
&\le Ct^{-\gamma_1(p\sigma_2-\varepsilon(1-\delta))}\|u_0\|_{s'_2}+C\int_0^t(t-\tau)^{\gamma_1-1-\gamma_1(1-\varepsilon(1-\delta))}\tau^{-p\sigma_2}d\tau\\
&\le Ct^{-\gamma_1(p\sigma_2-\varepsilon(1-\delta))}\quad \text{for all } t>0.
\end{aligned}$$

Similarly,

$$\|v(t)\|_{s''_2}\le Ct^{-\gamma_2(p\sigma_1-\varepsilon(1-\delta))}\quad \text{for all } t>0.$$

In view of (4.41) and $\beta<1$, we conclude, thanks to lemma 2.3, that

$$\|u(t)\|_{s'_2}\le Ct^{-\gamma_1\beta/q},\quad \|v(t)\|_{s''_2}\le Ct^{-\gamma_2\beta/p}\quad \text{for all } t>0.$$

An iterative argument gives

$$\begin{aligned}
\|u(t)\|_{s'_{i_0}}&\le Ct^{-\gamma_1(\beta-\varepsilon(1-\delta))}\le Ct^{-\gamma_1\beta/q}\quad \text{for all } t>0,\\
\|v(t)\|_{s''_{i_0}}&\le Ct^{-\gamma_2(\beta-\varepsilon(1-\delta))}\le Ct^{-\gamma_2\beta/p}\quad \text{for all } t>0.
\end{aligned}$$

Therefore, by (4.17) and (4.18), we have

$$\|u(t)\|_\infty\le Ct^{-\frac{N}{2}\gamma_1}\|u_0\|_1+C\int_0^t(t-\tau)^{\gamma_1-1-\gamma_1\frac{pN}{2s''_{i_0}}}\|v(\tau)\|^p_{s''_{i_0}}d\tau$$

$$
\begin{aligned}
&\le Ct^{-\frac{N}{2}\gamma_1}\|u_0\|_1 + C\int_0^t (t-\tau)^{\gamma_1-1-\gamma_1\frac{pN}{2s''_{i_0}}}\tau^{-\gamma_2\beta}d\tau \\
&\le C\Big(t^{-\frac{N}{2}\gamma_1} + t^{\gamma_1-\gamma_1\frac{pN}{2s''_{i_0}}-\gamma_2\beta}\Big) \le Ct^{-\tilde{\sigma}},
\end{aligned}
$$

where $\sigma'' = \min\{\frac{N}{2}\gamma_1, \gamma_1\frac{pN}{2s''_{i_0}} - \gamma_1 + \gamma_2\beta\} > 0$ from (4.42).

Since $\frac{Nq}{2s_1} \ge 1$, using similar arguments as for the case $\frac{Np}{2s_2} < 1$ and $\frac{Nq}{2s_1} \ge 1$, we obtain $\|v(t)\|_\infty \le Ct^{-\sigma''_1}$ for some $\sigma''_1 > 0$ and for every $t > 0$. This completes the proof. □

Remark 4.1. In the particular case $N > 2$, $qN/(2s_1) \ge 1$, $pN/(2s_2) \ge 1$, $q > p = 1$ and $q^2 \le 4\gamma_1 q + 1$, using the above method, we obtain

$$
\|u(t)\|_\infty \le Ct^{-\sigma''}, \quad t > 0,
$$

where $\sigma'' = \min\{\frac{N}{2}\gamma_1, \frac{pN}{2s''_{i_0}}\gamma_1 - \gamma_1 + \gamma_2(\beta - \varepsilon(1-\delta))p\}$. Here, $\varepsilon > 0$ can be arbitrarily small, and β can be arbitrarily close to 1. However, since s''_{i_0} depends on ε and s''_{i_0} is decreasing in ε, it is not clear that σ'' is positive.

Proof of Theorem 3.6.
Case: $p > 1, q > 1$. The proof proceeds by contradiction. Suppose that (u, v) is a nontrivial solution of (1.1) which exists globally in time. We make the judicious choice

$$
\varphi(t,x) = \varphi_1(t)\varphi_2(x) = \varphi_1\big(\frac{t}{T^\lambda}\big)\Phi^l\big(\frac{|x|}{T^2}\big),
$$

where $\Phi \in C_0^\infty(\mathbb{R})$, $0 \le \Phi(z) \le 1$ is such that

$$
\Phi(z) = \begin{cases} 1 & \text{if } |z| \le 1, \\ 0 & \text{if } |z| > 2, \end{cases} \qquad \varphi_1(t) = \begin{cases} (1-\frac{t}{T^\lambda})^l & \text{if } t \le T^\lambda, \\ 0 & \text{if } t > T^\lambda, \end{cases}
$$

where $l > \max\{1, \frac{q}{q-1}\gamma_1 - 1, \frac{p}{p-1}\gamma_2 - 1\}$. We denote by $Q_{T^\lambda} := \mathbb{R}^N \times [0, T^\lambda]$.

From Definition 3.4, of the weak solution, we have

$$
\begin{aligned}
&\int_{Q_{T^\lambda}} |v|^p\varphi_2(x)\varphi_1(t)dx\,dt + T^{\lambda(1-\gamma_1)}\int_{\mathbb{R}^N} u_0(x)\varphi_2(x)dx \\
&= \int_{Q_{T^\lambda}} \varphi_2(x)uD^{\gamma_1}_{t|T^\lambda}\varphi_1(t)dx\,dt - \int_{Q_{T^\lambda}} \Delta\varphi_2(x)\varphi_1(t)u\,dx\,dt,
\end{aligned} \tag{4.45}
$$

$$
\begin{aligned}
&\int_{Q_{T^\lambda}} |u|^q\varphi_2(x)\varphi_1(t)dx\,dt + T^{\lambda(1-\gamma_2)}\int_{\mathbb{R}^N} v_0\varphi_2(x)dx \\
&= \int_{Q_{T^\lambda}} \varphi_2(x)uD^{\gamma_2}_{t|T^\lambda}\varphi_1(t)dx\,dt - \int_{Q_{T^\lambda}} \varphi_1(t)\Delta\varphi_2(x)u\,dx\,dt.
\end{aligned} \tag{4.46}
$$

Using Hölder's inequality with exponents q and q' $(q + q' = qq')$, to the right-hand sides of (4.45) and (4.46), we obtain

$$
\begin{aligned}
&\int_{Q_{T^\lambda}} u\varphi_2(x)D^{\gamma_1}_{t|T^\lambda}\varphi_1(t)dx\,dt \\
&= \int_{Q_{T^\lambda}} u|\varphi_1(t)|^{1/q}|\varphi_2(x)|^{1-\frac{1}{q}+\frac{1}{q}}|\varphi_1(t)|^{-1/q}D^{\gamma_1}_{t|T^\lambda}\varphi_1(t)dx\,dt
\end{aligned}
$$

$$
\begin{aligned}
&\le \Big(\int_{Q_{T^{\lambda_1}}} |D^{\gamma_1}_{t|T^\lambda}\varphi_1(t)|^{q'}|\varphi_1(t)|^{-q'/q}|\varphi_2(x)|^{(1-\frac{1}{q})q'}dx\,dt\Big)^{1/q'}\\
&\quad\times\Big(\int_{Q_{T^\lambda}}|u|^q\varphi_1\varphi_2dx\,dt\Big)^{1/q},
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{Q_{T^\lambda}} u|\Delta\varphi_2(x)|\varphi_1(t)dx\,dt\\
&\le \Big(\int_{\mathbb{R}^N}|\Delta\varphi_2(x)|^{q'}|\varphi_2(x)|^{-q'/q}dx\int_0^{T^\lambda}|\varphi_1(t)|^{(1-\frac{1}{q})q'}dt\Big)^{1/q'}\\
&\quad\times\Big(\int_{Q_{T^\lambda}}|u|^q\varphi_1\varphi_2dx\,dt\Big)^{1/q}.
\end{aligned}
$$

Setting

$$
\begin{aligned}
\mathcal{A}(\sigma,\kappa,\kappa') &= \Big(\int_{Q_{T^{\lambda_1}}}|D^{\sigma}_{t|T^\lambda}\varphi_1(t)|^{\kappa'}|\varphi_1(t)|^{-\frac{\kappa'}{\kappa}}|\varphi_2(x)|^{(1-\frac{1}{\kappa})\kappa'}dx\,dt\Big)^{1/\kappa'},\\
\mathcal{B}(\kappa,\kappa') &= \Big(\int_{Q_{T^{\lambda_1}}}|\Delta\varphi_2(x)|^{\kappa'}|\varphi_2(x)|^{-\frac{\kappa'}{\kappa}}|\varphi_1(t)|^{(1-\frac{1}{\kappa})\kappa'}dx\,dt\Big)^{1/\kappa'},
\end{aligned}
$$

and gathering the above estimates, we obtain

$$
\begin{aligned}
&\int_{Q_{T^\lambda}}|v|^p\varphi_1(t)\varphi_2(x)dx\,dt + T^{\lambda(1-\gamma_1)}\int_{\mathbb{R}^N}u_0\varphi_2(x)dx\\
&\le \mathcal{A}(\gamma_1,q,q')\Big(\int_{Q_{T^\lambda}}|u|^q\varphi_1(t)\varphi_2dx\,dt\Big)^{1/q}\\
&\quad+\mathcal{B}(q,q')\Big(\int_{Q_{T^\lambda}}|u|^q\varphi_1(t)\varphi_2dx\,dt\Big)^{1/q}.
\end{aligned}
\tag{4.47}
$$

Similarly, we obtain

$$
\begin{aligned}
&\int_{Q_{T^\lambda}}|u|^q\varphi_2(x)\varphi_1(t)dtdx + T^{\lambda(1-\gamma_2)}\int_{\mathbb{R}^N}v_0\varphi_2(x)dx\\
&\le \mathcal{A}(\gamma_2,p,p')\Big(\int_{Q_{T^\lambda}}|v|^p\varphi_1\varphi_2dx\,dt\Big)^{1/p}+\mathcal{B}(p,p')\Big(\int_{Q_{T^\lambda}}|v|^p\varphi_1\varphi_2dx\,dt\Big)^{1/p}
\end{aligned}
\tag{4.48}
$$

Consequently,

$$
\begin{aligned}
&\int_{Q_{T^\lambda}}|v|^p\varphi_1(t)\varphi_2(x)dx\,dt + CT^{\lambda(1-\gamma_1)}\int_{\mathbb{R}^N}u_0\varphi_2(x)dx\\
&\le \mathcal{A}\Big(\int_{Q_{T^\lambda}}|u|^q\varphi_1\varphi_2dx\,dt\Big)^{1/q},
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{Q_{T^\lambda}}|u|^q\varphi_1(t)\varphi_2(x)dx\,dt + CT^{\lambda(1-\gamma_2)}\int_{\mathbb{R}^N}v_0\varphi_2(x)dx\\
&\le \mathcal{B}\Big(\int_{Q_{T^\lambda}}|v|^p\varphi_1\varphi_2dx\,dt\Big)^{1/p},
\end{aligned}
$$

where
$$\mathcal{A} = \mathcal{A}(\gamma_1, q, q') + \mathcal{B}(q, q'), \quad \mathcal{B} = \mathcal{A}(\gamma_2, p, p') + \mathcal{B}(p, p').$$
Using inequalities (4.47) and (4.48) in the last two inequalities, we obtain
$$\begin{aligned}
&\int_{Q_{T^\lambda}} |v|^p \varphi_1(t)\varphi_2(x)\,dx\,dt + CT^{\lambda(1-\gamma_1)} \int_{\mathbb{R}^N} u_0\varphi_2(x)dx \\
&\leq \mathcal{A}\mathcal{B}^{1/q} \Big(\int_{Q_{T^\lambda}} |v|^p \varphi_1\varphi_2 dx\,dt \Big)^{\frac{1}{pq}}, \\
&\int_{Q_{T^\lambda}} |u|^q \varphi_1(t)\varphi_2(x)\,dx\,dt + CT^{\lambda(1-\gamma_2)} \int_{\mathbb{R}^N} v_0\varphi_2(x)dx \\
&\leq \mathcal{B}\mathcal{A}^{1/p} \Big(\int_{Q_{T^\lambda}} |u|^q \varphi_1\varphi_2 dx\,dt \Big)^{\frac{1}{pq}}.
\end{aligned}$$

Now, applying Young's inequality, we obtain
$$\begin{aligned}
&(pq-1) \int_0^{T^\lambda} \int_{\mathbb{R}^N} |v|^p \varphi_2(x)\varphi_1(t)\,dx\,dt + CpqT^{\lambda(1-\gamma_1)} \int_{\mathbb{R}^N} u_0(x)\varphi_2(x)dx \\
&\leq (pq-1)\big(\mathcal{A}\mathcal{B}^{1/q}\big)^{\frac{pq}{pq-1}}, \\
&(pq-1) \int_0^{T^\lambda} \int_{\mathbb{R}^N} |u|^q \varphi_2(x)\varphi_1(t)\,dx\,dt + CpqT^{\lambda(1-\gamma_2)} \int_{\mathbb{R}^N} v_0(x)\varphi_2(x)dx \\
&\leq (pq-1)\big(\mathcal{B}\,\mathcal{A}^{1/p}\big)^{\frac{pq}{pq-1}}.
\end{aligned}$$
At this stage, using the change of variables, $x = T^2 y$, $t = T^\lambda \tau$, with $\lambda > 0$ to be chosen later, we obtain
$$\begin{aligned}
&\int_0^{T^\lambda} \int_{\mathbb{R}^N} |v|^p \varphi_2(x)\varphi_1(t)dx\,dt + T^{\lambda(1-\gamma_1)} \int_{\mathbb{R}^N} u_0\varphi_2(x)dx \\
&\leq C\Big(T^{-\lambda\gamma_1 + (\lambda+2N)\frac{1}{q'}} + T^{-4+(\lambda+2N)\frac{1}{q'}}\Big)\Big(\int_{Q_{T^\lambda}} |u|^q \varphi_1(t)\varphi_2(x)dx\,dt\Big)^{1/q} \\
&\leq CT^{-\lambda\gamma_1+(\lambda+2N)\frac{1}{q'}} \Big(\int_{Q_{T^\lambda}} |u|^q \varphi_1\varphi_2 dx\,dt\Big)^{1/q}.
\end{aligned}$$
Analogously, we have
$$\begin{aligned}
&\int_0^{T^\lambda} \int_{\mathbb{R}^N} |u|^q \varphi_2(x)\varphi_1(t)dx\,dt + CT^{\lambda(1-\gamma_2)} \int_{\mathbb{R}^N} v_0\varphi_2(x)dx \\
&\leq C\Big(T^{-\lambda\gamma_2 + (\lambda+2N)\frac{1}{p'}} + T^{-4+(\lambda+2N)\frac{1}{p'}}\Big)\Big(\int_{Q_{T^\lambda}} |v|^p \varphi_1\varphi_2 dx\,dt\Big)^{1/p} \\
&= CT^{-\lambda\gamma_2+(\lambda+2N)\frac{1}{p'}} \Big(\int_{Q_{T^\lambda}} |v|^p \varphi_1\varphi_2 dx\,dt\Big)^{1/p}.
\end{aligned}$$
Choosing $\gamma_1\lambda = 4$, we have
$$\begin{aligned}
&\int_0^{T^\lambda} \int_{\mathbb{R}^N} |v|^p \varphi_2(x)\varphi_1(t)dx\,dt + T^{\lambda(1-\gamma_1)} \int_{\mathbb{R}^N} u_0\varphi_2(x)dx \\
&\leq C\Big(T^{-\frac{4}{q\gamma_1}\gamma_2 + (\lambda+2N)\frac{1}{p'}\frac{1}{q} - 4 + (\frac{4}{\gamma_1}+2N)\frac{1}{q'}} + T^{-4\frac{1}{q} + (\lambda+2N)\frac{1}{p'q} - 4 + (\frac{4}{\gamma_1}+2N)\frac{1}{q'}}\Big)
\end{aligned}$$

$$\times\Big(\int_{Q_{T^\lambda}}|v|^p\varphi_1\varphi_2dx\,dt\Big)^{\frac{1}{pq}},$$

and

$$\begin{aligned}
&\int_0^{T^\lambda}\int_{\mathbb{R}^N}|u|^q\varphi_2(x)\varphi_1(t)dx\,dt+CT^{\lambda(1-\gamma_2)}\int_{\mathbb{R}^N}v_0\varphi_2(x)dx\\
&\le C\Big(T^{-\lambda\gamma_2+(\lambda+2N)\frac{1}{p'}}+T^{-4+(\lambda+2N)\frac{1}{p'}}\Big)\Big(\int_{Q_{T^\lambda}}|v|^p\varphi_1\varphi_2dx\,dt\Big)^{1/p}\\
&\le C\Big(T^{-\lambda\gamma_2+(\lambda+2N)\frac{1}{p'}}+T^{-4+(\lambda+2N)\frac{1}{p'}}\Big)T^{-4\frac{1}{p}+(\frac{4}{\gamma_1}+2N)\frac{1}{pq'}}\\
&\quad\times\Big(\int_{Q_{T^\lambda}}|u|^q\varphi_1\varphi_2dx\,dt\Big)^{\frac{1}{pq}}\\
&=C\Big(T^{-\frac{4}{\gamma_1}\gamma_2+(\frac{4}{\gamma_1}+2N)\frac{1}{p'}-4\frac{1}{p}+(\frac{4}{\gamma_1}+2N)\frac{1}{pq'}}+T^{-4+(\frac{4}{\gamma_1}+2N)\frac{1}{p'}-4\frac{1}{p}+(\frac{4}{\gamma_1}+2N)\frac{1}{pq'}}\Big)\\
&\quad\times\Big(\int_{Q_{T^\lambda}}|u|^q\varphi_1(t)\varphi_2dx\,dt\Big)^{\frac{1}{pq}}.
\end{aligned}$$

Therefore, using the ε-Young inequality, we obtain

$$\int_{\mathbb{R}^N}u_0(x)\varphi_2(x)dx\le CT^{\delta_1}, \tag{4.49}$$

$$\int_{\mathbb{R}^N}v_0(x)\varphi_2(x)dx\le CT^{\delta_2}, \tag{4.50}$$

where

$$\begin{aligned}
\delta_1=\max\Big\{&(-\frac{4}{q\gamma_1}\gamma_2+(\frac{4}{\gamma_1}+2N)\frac{1}{p'}\frac{1}{q}-4+(\frac{4}{\gamma_1}+2N)\frac{1}{q'})\frac{pq}{pq-1}+\frac{4}{\gamma_1}(\gamma_1-1),\\
&(-4\frac{1}{q}+(\frac{4}{\gamma_1}+2N)\frac{1}{p'q}-4+(\frac{4}{\gamma_1}+2N)\frac{1}{q'})\frac{pq}{pq-1}+\frac{4}{\gamma_1}(\gamma_1-1)\Big\},
\end{aligned}$$

and

$$\begin{aligned}
\delta_2=\max\Big\{&(-\frac{4}{\gamma_1}\gamma_2+(\frac{4}{\gamma_1}+2N)\frac{1}{p'}-4\frac{1}{p}+(\frac{4}{\gamma_1}+2N)\frac{1}{pq'})\frac{pq}{pq-1}+\frac{4}{\gamma_1}(\gamma_2-1),\\
&(-4+(\frac{4}{\gamma_1}+2N)\frac{1}{p'}-4\frac{1}{p}+(\frac{4}{\gamma_1}+2N)\frac{1}{pq'})\frac{pq}{pq-1}+\frac{4}{\gamma_1}(\gamma_2-1)\Big\}.
\end{aligned}$$

The condition (3.2) leads to either $\delta_1<0$ or $\delta_2<0$. Then as $T\to\infty$, the right-hand side of (4.49) (resp. (4.50)) tends to zero while the left-hand side tends to $\int_{\mathbb{R}^N}u_0(x)dx>0$ (resp. $\int_{\mathbb{R}^N}v_0(x)dx>0$); a contradiction.

We repeat the same argument for $\gamma_2\lambda=4$ to conclude the proof of Theorem 3.6.

Case $p=1,q>1$ (the case $p>1,q=1$ is treated similarly). We still use the weak formulation of the solution and argue by contradiction. Let us set

$$\mathcal{I}=\int_0^T\int_\Omega v\varphi\,dx\,dt,\quad \mathcal{J}=\Big(\int_0^T\int_\Omega u^q\varphi\,dx\,dt\Big)^{1/q}.$$

Then, applying Holder's inequality as above, we obtain

$$\mathcal{I}+\int_0^T\int_\Omega u_0D^{\gamma_1}_{t|T}\varphi\,dx\,dt\le\mathcal{J}(\mathcal{A}+\mathcal{B}), \tag{4.51}$$

where

$$\mathcal{A}=\Big(\int_0^T\int_\Omega \varphi^{-\frac{q'}{q}}|\Delta\varphi|^{q'}\,dx\,dt\Big)^{1/q'},\quad \mathcal{B}=\Big(\int_0^T\int_\Omega \varphi^{-\frac{q'}{q}}|D^{\gamma_1}_{t|T}\varphi|^{q'}\,dx\,dt\Big)^{1/q'};$$

and

$$\begin{aligned}\mathcal{J}^q+\int_0^T\int_\Omega v_0\,D^{\gamma_1}_{t|T}\varphi\,dx\,dt&\le \lambda\int_0^T\int_\Omega v\varphi\,dx\,dt+\int_0^T\int_\Omega vD^{\gamma_2}_{t|T}\varphi\,dx\,dt\\ &\le(\lambda+\varepsilon)\mathcal{I},\end{aligned}\tag{4.52}$$

thanks to the ε-Young inequality and where we have chosen φ as a the first eigenfunction of the spectral problem

$$-\Delta\varphi=\lambda\varphi, x\in B_T(0),\quad \varphi_{|\partial\Omega}=0\,,$$

where ($\Omega=B_T(0)\subset\mathbb{R}^N$ is the ball centered in zero and of radius T and $\partial\Omega$ is the boundary of Ω). Adding equation (4.52) to $(\lambda+\varepsilon)$ times equation (4.51), we obtain

$$\mathcal{J}^q+(\lambda+\varepsilon)\int_0^T\int_\Omega u_0D^{\gamma_1}_{t|T}\varphi\,dx\,dt+\int_0^T\int_\Omega v_0\,D^{\gamma_2}_{t|T}\varphi\,dx\,dt\le(\lambda+\varepsilon)\mathcal{J}(\mathcal{A}+\mathcal{B}),$$

whereupon,

$$\mathcal{J}^{q-1}\le\mathcal{A}+\mathcal{B}.$$

Replacing $\varphi(x)$ by $\varphi(\frac{x}{T})$ and passing to the new variables $y=T^{-1}x$ and $\tau=T^{-1}t$, and then letting T go to infinity, we obtain a contradiction whenever $q<1+\frac{2}{N}$. □

Proof of Theorem 3.7. Let $u_0,v_0\in C_0(\mathbb{R}^N)$ be nonnegative and (u,v) be the corresponding solution of (1.1)-(1.2). We proceed by contradiction. Assume that (u,v) exists globally in time, that is (u,v) exists in $(0,t_*(u_0,v_0))$, for all $t_*(u_0,v_0)>0$. Let $T\in(0,t_*(u_0,v_0))$ be arbitrarily fixed.

Taking χ as test-function and setting

$$\begin{gathered}X(t):=\int_{\mathbb{R}^N}u(t,x)\chi(x)dx,\quad Y(t):=\int_{\mathbb{R}^N}v(t,x)\chi(x)dx\\ Z(t)=\int_\Omega\chi(u(t,x)+v(t,x))dx,\quad Z_0=\int_{\mathbb{R}^N}(u_0+v_0)\chi(x)\,dx.\end{gathered}$$

It follows from (1.1)-(1.2) that

$$\begin{gathered}{}^CD^\gamma_{0|t}\int_{\mathbb{R}^N}u(t,x)\chi(x)dx-\int_{\mathbb{R}^N}u\Delta\chi(x)dx=\int_{\mathbb{R}^N}|v(t,x)|^p\chi(x)dx,\ t\in(0,T),\\ {}^CD^\gamma_{0|t}\int_{\mathbb{R}^N}v(t,x)\chi(x)dx-\int_{\mathbb{R}^N}v\Delta\chi(x)dx=\int_{\mathbb{R}^N}|u(t,x)|^q\chi(x)dx,\ t\in(0,T),\end{gathered}\tag{4.53}$$

supplemented with the initial conditions

$$X(0)=\int_{\mathbb{R}^N}u_0(x)\chi(x)dx,\quad Y(0)=\int_{\mathbb{R}^N}v_0(x)\chi(x)dx.\tag{4.54}$$

From (4.53)-(4.54), we have

$$\begin{aligned}&D^\gamma_{0|t}([Z-Z_0])(t)-\int_{\mathbb{R}^N}(u(t,x)+v(t,x))\Delta\chi(x)dx\\ &=\int_{\mathbb{R}^N}(|v(t,x)|^p+|u(t,x)|^q)\chi(x)dx,\quad t\in(0,T).\end{aligned}\tag{4.55}$$

We observe that

$$\int_{\mathbb{R}^N} v(x,t)\chi(x)\,dx = \int_{\mathbb{R}^N} v(x,t)\chi^{\frac{1}{p}}(x)\chi^{1-\frac{1}{p}}(x)\,dx.$$

Since the function χ satisfies $\int_{\mathbb{R}^N} \chi(x)\,dx = 1$, then it yields by Hölder's inequality that

$$\int_{\mathbb{R}^N} v(x,t)\chi(x)\,dx \le \Big(\int_{\mathbb{R}^N} |v(x,t)|^p\chi(x)\,dx\Big)^{1/p}.$$

So

$$\int_{\mathbb{R}^N} |v(x,t)|^p\chi(x)\,dx \ge \Big(\int_{\mathbb{R}^N} v(x,t)\chi(x)\,dx\Big)^p = Y^p(t). \tag{4.56}$$

Similarly, we obtain

$$\int_{\mathbb{R}^N} |u(x,t)|^q\chi(x)\,dx \ge \Big(\int_{\mathbb{R}^N} u(x,t)\chi(x)\,dx\Big)^q = X^q(t). \tag{4.57}$$

Using estimates (4.56), (4.57) in (4.55) and the fact that the funtion χ satisfies $\Delta\chi \ge -\chi$, it yields

$$D^{\gamma}_{0|t}([Z - Z_0]) + Z(t) \ge Y^p(t) + X^q(t), \quad t \in (0,T). \tag{4.58}$$

By adding $Z(t)$ to the two members of (4.58), we obtain

$$\begin{aligned} D^{\gamma}_{0|t}([Z(t) - Z_0]) + 2Z(t) &\ge Y^p(t) + X^q(t) + X(t) + Y(t) \\ &\ge Y^p(t) + X^q(t) + X(t). \end{aligned}$$

We assume that $q \ge p$, by using the fact that $X^q(t) + X(t) \ge X^p(t)$ and

$$(a+b)^r \le 2^{r-1}(a^r + b^r), \quad a, b > 0,\ r \ge 1,$$

we obtain

$$D^{\gamma}_{0|t}([Z(t) - Z_0]) + 2Z(t) \ge 2^{1-p}Z(t)^p. \tag{4.59}$$

We put $F(y) = 2^{1-p}y^p - 2y$, the function F is convex on $(0,\infty)$ (since $F \in C^2(0,+\infty)$, $F'' \ge 0$).

Writing $\partial_t(k * [Z - Z_0])(t)$ instead of $D^{\gamma}_{0|t}([Z(t) - Z_0])$ with $k(t) = \frac{t^{-\gamma}}{\Gamma(1-\gamma)}$ in (4.59), we obtain

$$\partial_t(k * [Z - Z_0])(t) \ge F(Z(t)), \quad t \in (0,T). \tag{4.60}$$

It is clear that $F(y) > 0$ and $F'(y) > 0$ for all $y > 2^{\frac{p}{p-1}} := \alpha_1$.

Suppose now that $Z_0 > \alpha_1$. We claim that (4.60) implies that $Z(t) > \alpha_1$ for all $t \in (0,T)$. In fact, for $Z(0) = Z_0 > \alpha_1$, we have by continuity of Z, there exists $\delta \in (0,T]$ such that $Z(t) > \alpha_1$ for all $t \in (0,\delta)$. This implies that $F(Z(t)) > 0$ for all $t \in (0,\delta)$.

By the comparison principle, it follows that $Z(t) \ge Z_0$ for all $t \in (0,\delta)$. Setting

$$\delta_1 := \sup\{s \in (0,T) : Z(t) \ge Z_0 \ \ t \in (0,s)\},$$

then $\delta_1 > 0$. We want to show that $\delta_1 = T$.

Indeed, if $\delta_1 < T$, then by setting $s = t - \delta_1$ for $t \in (\delta_1, T)$ and $\tilde{Z}(s) = Z(s+\delta_1)$, $s \in (0, T-\delta_1)$, it follows from positivity of $Z - Z_0$ on $(0,\delta_1)$ and k being non-increasing that

$$\partial_s(k * [\tilde{Z} - Z_0])(s) \ge \partial_t(k * [Z - Z_0])(s+\delta_1), \quad s \in (0, T-\delta_1). \tag{4.61}$$

From (4.60) and (4.61) we deduce that

$$\partial_s(k * [\tilde{Z} - Z_0])(s) \ge F(\tilde{Z}(s)), \quad s \in (0, T-\delta_1).$$

This time-shifting property can be already found in [34]. So we may repeat the argument from above to see that there exists $\tilde{\delta} \in (0, T - \delta_1]$ such that $\tilde{Z}(s) \geq Z_0$ for all $s \in (0, \tilde{\delta})$. This leads to a contradiction with the definition of δ_1.

Hence, the assumption $\delta_1 < T$ was not true. This proves the claim. Knowing that $Z(t) \geq Z_0 > \alpha_1$ for all $t \in (0, T)$ it follows from (4.60) that

$$ {}^C D^{\gamma}_{0|t} Z(t) = \partial_t (k * [Z - Z_0])(t) \geq F(Z(t)) > 0, \quad \text{for all } t \in (0, T). \tag{4.62} $$

Therefore the function $Z(t)$ satisfying (4.62) is an upper solution of the problem

$$ {}^C D^{\gamma}_{0|t} y = F(y) = 2^{1-p} y^p - 2y, \quad y(0) = Z_0, \tag{4.63} $$

we have by comparison principle $Z(t) \geq y(t)$ (see [15, Theorem 2.3],[16, Theorem 4.10.]).

On the other hand, since $F(0) \geq 0$, $F(y) > 0$ and $F'(y) > 0$, for all $y \geq Z_0 > 2^{\frac{p}{p-1}}$. It then follows from Lemma 3.8, that $v(t) = w(\frac{t^{\gamma}}{\Gamma(\gamma+1)})$ is a lower solution for (4.63) (which means

$$ {}^C D^{\gamma}_{0|t} v \leq F(v) = 2^{1-p} v^p - 2v, \quad v(0) = Z_0 \leq Z_0), $$

where $w(t)$ solves the ordinary differential equation

$$ \frac{dw}{dt} = F(w) = 2^{1-p} w^p - 2w, \quad w(0) = Z_0. \tag{4.64} $$

By the comparison principle (see [15, Theorem 2.3],[16, Theorem 4.10.]), we obtain $y(t) \geq v(t)$. So, by solving the Cauchy problem (4.64), which is equivalent to

$$ \frac{d}{dt}(e^{2t} w) = 2^{1-p} e^{2(1-p)t} (e^{2t} w)^p, \ w(0) = Z(0), $$

in which the explicit blow-up solution is

$$ w(t) = (\frac{e^{2(1-p)t} - 1}{2^p} + Z_0^{1-p})^{\frac{1}{1-p}} e^{-2t} $$

which blows up in finite time $t_{**} = \frac{\ln(1 - 2^p Z_0^{1-p})}{2(1-p)}$. By the comparison principle (see [15, Theorem 2.3],[16, Theorem 4.10.]), we conclude that

$$ Z(t) \geq y(t) \geq v(t) = w\big(\frac{t^{\gamma}}{\Gamma(\gamma+1)}\big) = \Big(\frac{e^{2(1-p)\frac{t^{\gamma}}{\Gamma(\gamma+1)}} - 1}{2^p} + Z_0^{1-p}\Big)^{\frac{1}{1-p}} e^{-2\frac{t^{\gamma}}{\Gamma(\gamma+1)}} $$

which in turn leads to $Z(t)$ blows-up in finite time at $\bar{t}_{**} \leq [\frac{\ln(1-2^p Z_0^{1-p})}{2(1-p)} \Gamma(\gamma+1)]^{1/\gamma}$. Thus the same holds for the solution (u, v) of (1.1)-(1.2), which in turn leads to a contradiction. □

Remark 4.2. Similar results were obtained in [40, Theorem 3.5] using another method, while the authors did not address the estimation of the time blow up.

Acknowledgements. This project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, Saudi Arabia, under grant no. RG-39-130-38. The authors, therefore, acknowledge with thanks DSR technical and financial support.

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