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FRACTIONAL POWER FUNCTION SPACES ASSOCIATED TO REGULAR STURM-LIOUVILLE PROBLEMS

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Abstract. Using spectral properties of the regular Sturm-Liouville problems, we construct a collection of abstract function spaces. Then we find the smallest index for which these spaces are mapped continuously in to the space of continuous functions. We also give some applications of these spaces for variational methods.

1. Introduction

Taking Sobolev spaces as models, we construct functional spaces, using Sturm-Liouville differential operators as a starting point in place of weak derivatives. The choice of these particular differential operators is due to their "good" spectral qualities. After giving some properties of this spaces, we will compare them with the space of continuous functions with the goal for obtaining an optimal index.

The principal arguments used here are the asymptotic behaviour of the eigenvalues and eigenfunctions associated to Sturm-Liouville problems, and the fact that the eigenvalues λ_n of regular Sturm-Liouville problems have the asymptotic behaviour $O(n^2)$, which is not necessarily the case for non-regular problems.

We conclude by presenting some applications of these spaces for using variational methods to solve boundary value problems.

2. Preliminaries

Definition 2.1. We call "regular Sturm-Liouville problem", a differential equation of the form

$$\frac{d}{dx}\big[p(x)\frac{d}{dx}y(x)\big] \pm q(x)y(x) + \lambda\rho(x)y(x) = 0 \tag{2.1}$$

associated with the boundary conditions

$$\begin{aligned} a_0y(a) + a_1y'(a) = 0 \\ b_0y(b) + b_1y'(b) = 0 \end{aligned} \tag{2.2}$$

where a, b, a_0, b_0, a_1, b_1 are finite real numbers, p is a C^1 strictly positive function over $[a, b]$, q is a continuous function over $[a, b]$, and ρ is a continuous strictly positive function on $[a, b]$.

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Theorem 2.2. *Consider the regular Sturm-Liouville problem*

$$\begin{gathered}\frac{d}{dx}[p(x)\frac{d}{dx}y(x)] \pm q(x)y(x) + \lambda\rho(x)y(x) = 0\\ a_0y(a) + a_1y'(a) = 0\\ b_0y(b) + b_1y'(b) = 0\end{gathered} \tag{2.3}$$

Then:

- (i) *Problem* (2.3) *admits a denumerable sequence* $\{\lambda_n\}_{n\in\mathbb{N}^*}$ *of real and simple eigenvalues, which can be ordered* $|\lambda_1| < |\lambda_2| < \dots < |\lambda_n| < \dots$
- (ii) *The eigenfunctions* $\{\phi_n\}_n$ *corresponding to the eigenvalues* $\{\lambda_n\}_n$, *are such that: for all* $i \neq j$, $\int_a^b \phi_i(x)\phi_j(x)\rho(x)dx = 0$, *we say that they are orthogonal in* $L^2_\rho((a,b))$ *(by* $L^2_\rho((a,b))$ *we mean* $L^2((a,b))$ *weighted by* $\rho(x)$*).*
- (iii) *The eigenfunctions* $\{\phi_n\}_n$ *form an orthogonal (orthonormal) basis of the Hilbert space* $L^2_\rho((a,b))$.

We will assume that $\{\phi_n\}_n$ to be orthonormal.

Liouville transformation. Consider the regular Sturm-Liouville operator

$$l = \frac{d}{dx}\big[p(x)\frac{dy}{dx}(x)\big] + q(x)$$

under the transformation T defined by

$$y \mapsto (Ty)(x) = |s'|^{1/2}y(s(x))$$

where s is a bijective differentiable function, the operator l becomes

$$\tilde{l} = \frac{d}{ds}\big[P(s)\frac{d}{ds}\big] + Q(s)$$

where

$$\begin{gathered}P(s) = p(x)s'(x)^2|_{x=x(s)}\\ Q(s) = s'(x)^{-1/2}\frac{d}{dx}\big[p(x)\frac{d}{dx}s'(x)^{1/2}\big] + q(x)|_{x=x(s)}\end{gathered}$$

and $x = x(s)$ is the inverse function of $s(x)$.

We are particularly interested in the case $P(s) \equiv 1$, which gives

$$p(x)s'(x)^2 = 1 \Rightarrow s(x) = \int \sqrt{1/p(x)}\,dx\,.$$

More general, the transformation

$$u = (p\rho)^{1/4}y, \quad t = \int_0^x \sqrt{\frac{\rho(\tau)}{p(\tau)}}\,d\tau, \quad c = \int_0^b \sqrt{\frac{\rho(\tau)}{p(\tau)}}\,d\tau$$

applied to

$$(py')' - qy + \lambda\rho y = 0 \quad \text{on } [0,b]$$

gives the simpler equation

$$u'' - ru + \lambda u = 0 \quad \text{on } [0,c],$$

where y is function of the variable x, u is function of the variable t,

$$r = (\frac{\varphi''}{\varphi}) + \frac{q}{\rho}, \quad \text{and} \quad \varphi = (p\rho)^{1/4}$$

The above transformation is often called Liouville transformation, it allows us to call "regular Sturm-Liouuville problem" every problem of the form

$$-y'' + ry = \lambda y$$

with boundary conditions. This problem is simpler than (2.3).

Asymptotic behaviour of eigenvalues and eigenfunctions. There are many methods to compute the asymptotic behaviour of the eigenvalues of a regular Sturm-Liouville, probably the most useful one is the Courant-Fisher method. We present here another method using Prüfer transformation [12].

Consider the regular Sturm-Liouville problem

$$\begin{gathered} -y'' + qy = \lambda y \\ y(0) = y(a) = 0 \,. \end{gathered}$$

The transformation

$$\tan\theta = \lambda^{1/2} \frac{y}{y'}$$

is called Prüfer transformation. When we differentiate both sides of the above equality, we obtain

$$\frac{\theta'}{\cos^2\theta} = \lambda^{1/2} \frac{(y')^2 - yy''}{(y')^2} = \lambda^{1/2}(1 + (\lambda - q)\frac{y}{(y')^2}) = \lambda^{1/2}(1 + (\lambda - q)\lambda^{-1}\tan^2\theta)$$

which gives

$$\begin{aligned} \theta' &= \cos^2\theta\big(\lambda^{1/2} + (\lambda - q)\lambda^{-\frac{1}{2}}\tan^2\theta\big) \\ &= \lambda^{1/2}\cos^2\theta + (\lambda - q)\lambda^{-\frac{1}{2}}\sin^2\theta \\ &= \lambda^{1/2} - q\lambda^{-\frac{1}{2}}\frac{1 - \cos 2\theta}{2} \\ &= \lambda^{1/2} - \frac{1}{2}q\lambda^{-\frac{1}{2}} + \frac{1}{2}q\lambda^{-\frac{1}{2}}\cos 2\theta \,. \end{aligned}$$

Integrating the last equation between 0 and a, we obtain

$$\theta(a) - \theta(0) = a\lambda^{1/2} - \frac{1}{2}\lambda^{-\frac{1}{2}}\int_0^a q(t)dt + \frac{1}{2}\lambda^{-\frac{1}{2}}\int_0^a q(t)\cos(2\theta(t))dt \,.$$

Using the boundary conditions, we have

$$\begin{gathered} y(0) = 0 \Rightarrow \tan\theta(0) = 0 \Rightarrow \theta(0) = 0 \\ y(a) = 0 \Rightarrow \tan\theta(a) = 0 \Rightarrow \theta(a) = (n+1)\pi, \quad n \in \mathbb{N} \,. \end{gathered}$$

Therefore,

$$(n+1)\pi = a\lambda_n^{1/2} - \frac{1}{2}\lambda_n^{-\frac{1}{2}}\int_0^a q(t)dt + \frac{1}{2}\lambda_n^{-\frac{1}{2}}\int_0^a q(t)\cos(2\theta(t))dt \,.$$

After inversion and using the fact that $\int_0^a q(t)dt < \infty$, and $\int_0^a q(t)\cos(2\theta(t))dt < \infty$, we obtain the asymptotic behaviour of the eigenvalues

$$\lambda_n = O(n^2).$$

This result will lead us to find the asymptotic behaviour of the associated eigenfunctions as follows: The solution of the equation $u'' - qu + \lambda u = 0$ which vanishes at 0 will satisfies the integral equation

$$u(t) = c \sin \sqrt{\lambda} t + \frac{1}{\lambda} \int_0^t q(\tau) u(\tau) \sin \sqrt{\lambda}(t - \tau) d\tau$$

where c is an arbitrary constant. The conditions $u(a) = 0$, and $\int_0^a u^2 dt = 1$, give

$$c = \sqrt{\frac{2}{a}} + O\big(\frac{1}{\sqrt{\lambda}}\big)$$

and then

$$u(t) - \sqrt{\frac{2}{a}} \sin \sqrt{\lambda} t = O\big(\frac{1}{\sqrt{\lambda}}\big) .$$

If λ_n is the n^{th} eigenvalue of the considered problem, the associated (normalized) eigenfunction is such that

$$\phi_n(t) = \sqrt{\frac{2}{a}} \sin \sqrt{\lambda_n} t + O\big(\frac{1}{\sqrt{\lambda_n}}\big) .$$

Since $\lambda_n = O(n^2)$, we get

$$\phi_n(t) = \sqrt{\frac{2}{a}} \sin \sqrt{\lambda_n} t + O\big(\frac{1}{n}\big) .$$

For more details, we refer the reader to [6], or [12].

3. Fractional power spaces associated to regular Sturm-Liouville problems

Let

$$ly := -y'' + ry = \lambda y$$

with boundary conditions be a regular Sturm-Liouville problem and let $\{\lambda_n\}$ and $\{\phi_n\}$ be as above. Consider a function $f \in L^2(a, b)$, so one can write $f = \sum a_n \phi_n$. Then for $s > 0$, we define

$$l^s f = \sum \lambda_n^s a_n \phi_n .$$

Without loss of generality, we assume that $\lambda_n > 1$.

Definition 3.1. Let

$$lu = \lambda u, \quad \text{on } \Omega = (a, b) \tag{3.1}$$

with boundary conditions be a regular Sturm-Liouville problem, that has $\{\lambda_n\}$ and $\{\phi_n\}$ as eigenvalues and eigenfunctions. For $s > 0$, we introduce the functional spaces associated to (3.1):

$$\begin{aligned} A^s &= \{u \in L^2(\Omega) : l^s u \in L^2(\Omega)\} \\ &= \{u = \sum a_n \phi_n : \sum |a_n|^2 \lambda_n^{2s} < \infty\} . \end{aligned}$$

These two sets are equal due to Parseval identity. We call the spaces A^s fractional power Sobolev spaces associated to (3.1).

The aim of this paper is to find for what exponents $s > 0$ the injection $A^s \hookrightarrow C([a, b])$ holds.

Properties of the spaces A^s. Most of the properties of the spaces A^s are deduced from those of L^2

(1) Let $u = \sum a_n\phi_n$, and $v = \sum b_n\phi_n$ be two elements of A^s. We define the scalar product in A^s by
$$(u,v)_{A^s} = (l^s u, l^s v)_{L^2} = \sum a_n b_n \lambda_n^{2s}\,.$$
and corresponding norm by
$$\|u\|_{A^s}^2 = (u,u)_{A^s} = (l^s u, l^s u)_{L^2} = \sum |a_n|^2 \lambda_n^{2s}\,.$$
Note that A^s becomes a Hilbert space, and l^s defines an isometry from A^s to $L^2(\Omega)$.

(2) We identify A^0 with L^2.

(3) We have continuous injections between the spaces A^s as follows: If $0 \leq s_1 \leq s_2$ then $A^{s_2} \hookrightarrow A^{s_1}$

(4) The space of test functions
$$\mathcal{D}(\Omega) = \{f \in \mathcal{C}^\infty(\Omega) : \operatorname{supp} f \text{ is a compact subset of } \Omega\}$$
is dense in A^s for every $s > 0$, where $\operatorname{supp} f = \overline{\{x \in \Omega;\ f(x) \neq 0\}}$.

(5) We define the space A^∞ as $A^\infty = \bigcap_{s\in\mathbb{N}} A^s$ equipped with the family of semi-norms $\{\|u\|_{A^s}\}_{s\in\mathbb{N}}$ it is a metrisable space with the metric
$$d(u,v) = \sum_{j=1}^{\infty} 2^{-j}\frac{\|u-v\|_{A^j}}{1+\|u-v\|_{A^j}}\,.$$

(6) For negative exponents $s < 0$, we define
$$\begin{aligned} A^s &= \{u \in \mathcal{E}'(\Omega) : l^s u \in L^2(\Omega)\} \\ &= \{u = \sum a_n \tilde{\phi}_n : \sum |a_n|^2 \lambda_n^{2s} < \infty\}\,, \end{aligned}$$
where $\mathcal{E}'(\Omega)$ is the space of the distribution with compact support; it is the topological dual of the space $\mathcal{C}^\infty(\Omega)$). Its elements are defined as follows: T is in $\mathcal{E}'(\Omega)$ if there exist $c > 0$, $m \in \mathbb{N}$ and K compact subset of Ω such that
$$|\langle T, f\rangle| \leq c \sum_{\alpha\leq m} \sup_{x\in k} |\frac{d^\alpha f}{dx^\alpha}| \quad \forall f \in \mathcal{C}^\infty(\Omega)\,.$$
For the justification of this statement, see for example [13].

Remark 3.2. To make sure that the spaces A^s are well defined, we assume that $\lambda_n > 1$. If (3.1) admits a finite number of negative eigenvalues, we consider the operator $(l + (1-\lambda_*))$ instead of l, where λ_* is the smallest eigenvalue of l.

If (3.1) admits an infinite number of negative and a finite number of positive eigenvalues, we consider the operator $((1+\lambda_*)Id - l)$ in stead of l, where λ_* is the largest positive eigenvalue of l.

In this paper, we will not consider the case when (3.1) admits other distribution of eigenvalues, which is the case of some singular periodic problems.

Theorem 3.3. *Let A^s be as above, then $A^s \hookrightarrow C(\bar{\Omega})$ whenever $s > 1/4$.*

Proof. Let $u \in \mathcal{D}(\Omega)$, then $u(x) = \sum_{n\in\mathbb{N}^*} a_n\phi_n(x)$, where
$$a_n = a_n(u) = \int_a^b u(x)\phi_n(x)dx = (u(x), \phi_n(x))_{L^2}\,.$$

Using integration by parts, we obtain

$$a_n(lu) = (lu, \phi_n)_{L^2} = (u, l\phi_n)_{L^2} = (u, \lambda_n \phi_n)_{L^2} = \lambda_n (u, \phi_n)_{L^2},$$

so that $a_n(lu) = \lambda_n a_n(u)$. Then we iterate this procedure to obtain

$$a_n(l^p u) = \lambda_n^p a_n(u)\,.$$

Using Hölder inequality, in the other side we have

$$\begin{aligned}
|a_n(l^p u)| &= \Big|\int_a^b l^p u \phi_n dx\Big| \\
&\le \Big(\int_a^b |l^p u|^2 dx\Big)^{1/2} \Big(\int_a^b |\phi_n|^2 dx\Big)^{1/2} \\
&\le \Big(\int_a^b |l^p u|^2 dx\Big)^{1/2} < \infty\,.
\end{aligned}$$

Therefore, $a_n(l^p u) = O(1)$ and $a_n(l^p u) = O(n^{2p}) a_n(u)$ imply $a_n(u) = O(n^{-2p})$ for every $p \in \mathbb{N}$. In other words, if $u \in \mathcal{D}(\Omega)$ then $\{a_n(u)\}_n$ is a rapidly decreasing sequence. As consequence of this statement, the series $\sum_{n\in\mathbb{N}^*} a_n \phi_n(x)$ converges uniformly to $u \in \mathcal{D}(\Omega)$ and in $L^2(\Omega)$. Since $u(x) = \sum a_n \phi_n(x)$,

$$|u(x)| \le \sum |a_n \phi_n(x)| = \sum \Big|a_n \lambda_n^s \frac{\phi_n(x)}{\lambda_n^s}\Big|\,.$$

Then by Hölder inequality,

$$|u(x)| \le \Big(\sum |a_n^2 \lambda_n^{2s}|\Big)^{1/2} \Big(\sum \Big|\frac{\phi_n^2(x)}{\lambda_n^{2s}}\Big|\Big)^{1/2}.$$

Since the ϕ_n's are uniformly bounded [12], we have

$$|u(x)| \le \|u\|_{A^s} \Big(\sum \Big|\frac{d}{\lambda_n^{2s}}\Big|\Big)^{1/2},$$

where d is a real constant. Since $\lambda_n = O(n^2)$, we obtain

$$\frac{d}{\lambda_n^{2s}} \sim \frac{d}{n^{4s}}$$

In conclusion if $s > \frac{1}{4}$, then $|u(x)| \le c\|u\|_{A^s}$, where c is a constant independent of u, and

$$\|u\|_{C(\bar\Omega)} \le c\,\|u\|_{A^s}\,. \tag{3.2}$$

Now consider $f \in A^s$, by the denseness of $\mathcal{D}(\Omega)$ in A^s, there exists a sequence $\{\varphi_n\} \subset \mathcal{D}(\Omega)$ such that

$$\varphi_n \underset{A^s}{\longrightarrow} f\,. \tag{3.3}$$

Then $\{\varphi_n\}_n$ is a Cauchy sequence in A^s, the inequality (3.2) implies that the sequence $\{\varphi_n\}_n$ is also a Cauchy one in $C(\bar\Omega)$ and then

$$\varphi_n \underset{C(\bar\Omega)}{\longrightarrow} \varphi \in C(\bar\Omega)\,. \tag{3.4}$$

Then (3.3) and (3.4) give the conclusion $f = \varphi$ $a.e$ in Ω. □

Now we proof the optimality of the index 1/4, in the sense that if $s_0 < 1/4$ then continuity of $A^{s_0} \hookrightarrow C(\bar{\Omega})$ may not hold. For this end let us consider the equation

$$\begin{gathered} -u'' = \lambda u \\ u(0) = u(\pi) = 0 \end{gathered}$$

which has $\lambda_n = n^2$ as eigenvalues and $\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$ as corresponding eigenfunctions. Let the associated spaces be

$$A^s = \big\{u \in L^2((0,\pi)) : u = \sum_{n\geq 1} a_n\sqrt{\frac{2}{\pi}}\sin(nx),\ \sum_{n\geq 1} a_n^2 n^{4s} < \infty\big\}$$

and consider the function

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < \frac{\pi}{4} \\ 1 & \text{if } \frac{\pi}{4} \leq x \leq \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} < x \leq \pi\,. \end{cases}$$

Since $f(x) \in L^2((0,\pi))$, we have $f(x) = \sum_{n\geq 1} a_n\sqrt{\frac{2}{\pi}}\sin(nx)$, with

$$a_n = \sqrt{\frac{2}{\pi}}\int_0^\pi f(x)\sin(nx)dx = \sqrt{\frac{2}{\pi}}\int_{\pi/4}^{\pi/2}\sin(nx)dx = \sqrt{\frac{2}{\pi}}\frac{\cos(n\pi/4) - \cos(n\pi/2)}{n}$$

thus $|a_n| \leq \sqrt{\frac{2}{\pi}}\,\frac{2}{n}$ and $a_n^2 \leq 8/(\pi n^2)$. Then

$$\sum_{n\geq 1} a_n^2 n^{4s} \leq \frac{8}{\pi}\sum_{n\geq 1}\frac{1}{n^{2-4s}}\,.$$

Since the series in the right hand side converges for $2 - 4s > 1$ i.e, $s < 1/4$, we obtain

$$\|f\|_{A^s} = \sum_{n\geq 1} a_n^2 n^{4s} < \infty \quad \forall s < \frac{1}{4}$$

in conclusion $f \in A^s$ for $s < 1/4$ and $f(x)$ is not continuous nor equal a.e. to a continuous function.

Remark 3.4. For the limiting case $s = \frac{1}{4}$ we do not have a definitive answer yet.

4. Applications

In this section we give some applications of the functional spaces A^s introduced above.

Example 1. For a finite interval (α,β) in $\mathbb{R}$, consider the problem

$$\begin{gathered} Tu := u^{(4)} = f \quad \text{on } (\alpha,\beta) \\ u''(\alpha) = u''(\beta) = 0 \\ u'''(\alpha) = u'''(\beta) = 0 \end{gathered} \tag{4.1}$$

with an appropriate f. We want to solve this equation using the next well known theorem in a space A^s.

Theorem 4.1 (Lax Milgram). *Let H be a Hilbert space and H' its dual. Let $a(u,v)$ be a continuous coercive bilinear form aver $H \times H$, then for each $f \in H'$ there exists a unique $u \in H$ such that*

$$a(u,v) = \langle f, v\rangle \quad \forall v \in H\,,$$

where $\langle \cdot,\cdot\rangle$ denotes the duality bracket between H and H'. In addition, if the bilinear form a is symmetric then the solution u is characterized by

$$\frac{1}{2}a(u,u) - \langle f,u\rangle = \min_{v\in H}\{\frac{1}{2}a(v,v) - \langle f,v\rangle\}$$

To solve problem (4.1) we consider the corresponding bilinear form

$$a(u,v) = \int_\alpha^\beta u''v''dx\,.$$

We remark that this bilinear form is not coercive in the Sobolev space $H^2((\alpha,\beta))$. To see that consider the affine function $u = cx + d$ so we have

$$a(u,u) = \int_\alpha^\beta (u'')^2 dx = 0\,,$$

but

$$\|u\|_{H^2}^2 = \int_\alpha^\beta u^2 dx + \int_\alpha^\beta (u')^2 dx + \int_\alpha^\beta (u'')^2 dx \neq 0\,.$$

So that one can not apply the Lax Milgram theorem to prove the existence of solutions in $H^2((\alpha,\beta))$. On the other hand, if we consider the same bilinear form in the space A^1 associated to the problem

$$\begin{gathered} lu := -u'' = \lambda u\\ u(\alpha) = u(\beta) = 0\,, \end{gathered}$$

we have

$$a(u,u) = \int_\alpha^\beta (u'')^2 dx = \|u\|_{A^1}^2$$

where u'' is regarded in the sense

$$u = \sum a_n\phi_n, \quad u'' = \sum \lambda_n a_n \phi_n\,.$$

Then the coercivity of a holds and leads to the existence of solutions in A^1.

Example 2. For an interval (a,b), consider the semi-linear problem

$$lu = g(u) + h \quad \text{on } (a,b) \tag{4.2}$$

associated to boundary value conditions, where l is a Sturm-Liouville operator. In this example we present a method based on the Ky Fan-Von-Neumann theorem for finding solutions in a convenient fractional space associated with the Sturm-Liouville problem $lu = \lambda u$. Before this we recall some basic definitions.

Definition 4.2. Let X be a Banach space, and $J : X \to \mathbb{R}$ be an application. We say that J is lower semi-continuous (l.s.c), if for every $\alpha \in \mathbb{R}$, the set $[J \leq \alpha] := \{x \in X : J(x) \leq \alpha\}$ is closed. We say that J is upper semi-continuous (u.s.c) if $(-J)$ is lower semi-continuous.

Let $A,\ B$ be two sets, and let $L : A \times B \to \mathbb{R}$ be an application, a point $(x^*, y^*) \in A \times B$ is said to be a saddle point if for all $x \in A$ and all $y \in B$, $L(x^*, y) \leq L(x^*, y^*) \leq L(x, y^*)$.

Theorem 4.3 (Ky Fan-Von-Neumann [18]). *Let X and Y be two reflexive Banach spaces; and let $H_1 \subset X$ and $H_2 \subset Y$ be convex closed subsets. Suppose that $L : H_1 \times H_2 \to \mathbb{R}$ is convex-concave i.e., for all $x \in H_1$, $L(x, .)$ is concave (u.s.c) on H_2, and for all $y \in H_2$, $L(., y)$ is convex (l.s.c) on H_1. Moreover if H_1 (or H_2) is unbounded we suppose that there exists y_0 (or x_0) such that $\lim_{\|x\|\to+\infty} L(x, y_0) = +\infty$ (or $\lim_{\|y\|\to+\infty} L(x_0, y) = -\infty$), then L will posses a saddle point.*

If the function L is concave and $L(x, .)$, $L(., y)$ are G-differentiable, then we have an equivalence between the following two assertions

(i) $(x^*, y^*) \in H_1 \times H_2$ is a saddle point of L in $H_1 \times H_2$.

(ii) For all $(x, y) \in H_1 \times H_2$,

$$\langle \partial_1 L(x^*, y), x - x^* \rangle \geq 0$$
$$\langle \partial_2 L(x, y^*), x - x^* \rangle \leq 0\,.$$

This equivalence gives a characterization of the saddle points.

Let $\{\lambda_k\}_k$ ($\lambda_k \geq 1$) and $\{\varphi_k\}_k$ be the eigenvalues and the eigenfunctions of the problem $lu = \lambda u$ associated with the same boundary conditions as those associated with (4.2).

In (4.2) $g(u)$ is a non linear function , and h is in $L^2((a, b))$. We will assume that $g : \mathbb{R} \to \mathbb{R}$ and there exist $k \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}^+$ such that for all $s, t \in \mathbb{R}$, with $s \neq t$

$$\lambda_k < \alpha \leq \frac{g(s) - g(t)}{s - t} \leq \beta < \lambda_{k+1}\,. \tag{4.3}$$

Under these conditions (4.2) admits a solution u in the space

$$A^{1/2} = \{u \in L^2(a, b) : u = \sum a_n\varphi_n,\ \sum a_n^2\lambda_n < \infty\}\,.$$

To prove the existence of such a solution we put

$$J(u) = \frac{1}{2}(l^{1/2}u, l^{1/2}u) - \int_a^b G(u(x))dx - \int_a^b h(x)u(x)dx$$

where $G(s) = \int_a^s g(t)dt$. The symbol $(\cdot, \cdot)$ will denote the inner product in $L^2(a, b)$ and $(\cdot, \cdot)_{A^{1/2}}$ the inner product in $A^{1/2}$ and $\langle \cdot, \cdot \rangle$ will denote a duality bracket For every $v \in A^{1/2}$, we have

$$\begin{aligned}\langle J'(u), v\rangle &= (l^{1/2}u, l^{1/2}v) - \int_a^b g(u(x))v(x)dx - \int_a^b h(x)v(x)dx\\ &= (u, v)_{A^{1/2}} + (g(u), v) - (h, v)\,.\end{aligned}$$

we define the spaces

$$H_1 = \oplus_{n\leq k}\mathbb{R}\varphi_n \quad\text{and}\quad H_2 = \oplus_{n\geq k+1}\mathbb{R}\varphi_n$$

where $\mathbb{R}\varphi_n = \{c\varphi_n; c \in \mathbb{R}\}$. One can remark that $A^{1/2} = H_1 \oplus^\perp H_2$ (direct and orthogonal sum). Let L be the mapping defined on $H_1 \times H_2$ by

$$L(v_1, v_2) = J(v_1 + v_2)\,.$$

We will show that L posses a saddle point, which is the wanted solution. Hypothesis (4.3) gives

$$0 < \alpha \leq \frac{g(v_1 + v_2) - g(w_1 + v_2)}{v_1 - w_1};$$

thus

$$\alpha(v_1 - w_1)^2 \leq [g(v_1 + v_2) - g(w_1 + v_2)](v_1 - w_1)\,.$$

After integration, we obtain

$$\alpha \|v_1 - w_1\|_{L^2}^2 \le ([g(v_1+v_2) - g(w_1+v_2)], (v_1 - w_1)) . \tag{4.4}$$

On other hand, for every $z \in H_1$ we have

$$(lz, z) = (l^{1/2}z, l^{1/2}z) \le \lambda_k \|z\|_{L^2}^2 \tag{4.5}$$

because z implies $z = \sum_{n=0}^{k} a_n \varphi_n$ which implies $lz = \sum_{n=0}^{k} a_n \lambda_n \varphi_n$ Then

$$(lz, z) = \Big(\sum_{n=0}^{k} a_n \lambda_n \varphi_n, \sum_{n=0}^{k} a_n \varphi_n\Big) = \sum_{n=0}^{k} a_n^2 \lambda_n$$

by the orthogonality of the φ_n's. Then

$$(lz, z) \le \lambda_k \sum_{n=0}^{k} a_n^2$$

because $\lambda_n \le \lambda_k$ for all $n \le k$. Then $(lz, z) \le \lambda_k \|z\|_{L^2}^2$. Using (4.4) and (4.5)

$$\begin{aligned}
&\langle \partial_1 L(v_1, v_2) - \partial_1 L(w_1, v_2), v_1 - w_1\rangle \\
&= (lv_1 - g(v_1+v_2) - h - lw_1 + g(w_1+v_2) + h, v_1 - w_1) \\
&= (lv_1 - g(v_1+v_2) - h - lw_1 + g(w_1+v_2) + h, v_1 - w_1) \\
&= (l(v_1 - w_1) - (g(v_1+v_2) - g(w_1+v_2)), v_1 - w_1) \\
&\le \lambda_k \|v_1 - w_1\|_{L^2}^2 - \alpha \|v_1 - w_1\|_{L^2}^2 .;
\end{aligned}$$

so that

$$\langle \partial_1 L(v_1, v_2) - \partial_1 L(w_1, v_2), v_1 - w_1\rangle \le -(\alpha - \lambda_k)\|v_1 - w_1\|_{L^2}^2$$

this shows that $-L(., v_2)$ is a strictly convex and coercive function (on L^2), in other words $-L(., v_2)$ is strictly concave. Since $\|v_1\|_{L^2} \le \|v_1 t\|_{A^{1/2}}$, we obtain

$$\lim_{\|v_1\|_{L^2} \to +\infty} L(v_1, v_2) = -\infty \Rightarrow \lim_{\|v_1\|_{L^2} \to +\infty} L(v_1, v_2) = -\infty$$

By a similar reasoning, and using the second inequality in (4.3) we show that $L(v_1, .)$ is strictly convex and coercive.

Since L being continuous, using the Ky Fan-Von-Neumann theorem, we conclude that L admits a saddle point $(u_1^*, u_2^*) \in H_1 \times H_2$. Using the characterization of the saddle point

$$\langle \partial_1 L(u_1^*, u_2), u_1 - u_1^*\rangle \ge 0 \quad \forall (u_1, u_2) \in H_1 \times H_2 \tag{4.6}$$

and the fact that H_1 is a vector space, we have for every $u_1 \in H_1$, $(u_1 + u_1^*)$ and $(-u_1 + u_1^*)$ are in H_1, so by substituting u_1 by $(u_1 + u_1^*)$ then by $(-u_1 + u_1^*)$, in the expression (4.6) we obtain

$$\langle \partial_1 L(u_1^*, u_2), u_1\rangle \ge 0 \quad \forall (u_1, u_2) \in H_1 \times H_2 .$$

In particular,

$$\langle \partial_1 L(u_1^*, u_2^*), u_1\rangle = 0 \quad \forall u_1 \in H_1$$

and, in the same way,

$$\langle \partial_1 L(u_1^*, u_2^*), u_2\rangle = 0 \quad \forall u_2 \in H_2 .$$

Therefore,

$$\langle J'(u_1^* + u_2^*), u_2\rangle = \langle \partial_1 L(u_1^*, u_2^*), u_2\rangle = 0 .$$

Finally

$$\langle J'(u_1^* + u_2^*), u\rangle = 0$$

for $u \in A^{1/2}$ with $u = u_1 + u_2$ and

$$u^* = u_1^* + u_2^* \in A^{1/2},$$

which is solution of (4.2) in the weak sense $\langle J'(u^*), v\rangle = 0$.

Conclusion. In this work, we constructed functional spaces related to regular Sturm-Liouville problems, but we can do it for singular spaces and particularly those giving orthogonal polynomials and other special functions (with some modifications). Following the same procedure, we can replace Sturm-Liouville operators by differential operator including partial differential operators having similar spectral properties.

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