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OSCILLATORY AND NON-OSCILLATORY SOLUTIONS OF DYNAMIC EQUATIONS WITH BOUNDED COEFFICIENTS

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Abstract. We analyze second-order half-linear dynamic equations on time scales. We prove oscillation and non-oscillation criteria which are sharp in the sense that the considered equations remain uncovered only for one setting of their coefficients. We continue in the research of the so-called conditional oscillation. The previously known conditionally oscillatory dynamic equations were allowed to contain only periodic coefficients. In the presented results, we deal with bounded coefficients (among others, our results cover coefficients which are periodic, almost periodic, having mean values, etc.). We point out that the results are new even for linear dynamic equations. We also provide corollaries to illustrate the novelty of our results.

1. Introduction

The dynamic equations on time scales are in the center of attention of many researchers almost from their introducing by Stefan Hilger in [23, 24]. The main benefit of the time scale calculus is the fact that one can work with an arbitrary non-empty closed subset $\mathbb{T}$ of $\mathbb{R}$. Especially, the differential calculus (when $\mathbb{T} = \mathbb{R}$) and the discrete calculus (when $\mathbb{T} = \mathbb{Z}$) are special cases of the time scale calculus. Nowadays, the time scale calculus is well advanced and widely studied, besides the pure theoretical reasons also for its applications in economy, natural sciences, etc. For the comprehensive overview, we refer to [5, 6]. We recall more details including the standard notation at the beginning of Section 2 of this paper.

Now, we describe the studied problem and give the corresponding literature overview of the results up to date. We are interested in the qualitative properties (namely, the oscillation behaviour) of the second-order half-linear dynamic equation

$$\big[r(t)\Phi(y^{\Delta})\big]^{\Delta} + \frac{s(t)}{t^{p-1}\sigma(t)}\Phi(y^{\sigma}) = 0, \quad \Phi(y) = |y|^{p-1}\operatorname{sgn} y, \quad p > 1, \tag{1.1}$$

with bounded, rd-continuous, positive coefficients r, s. Such equations are usually designated *half-linear equations* (it comes from [4]). The name follows from the fact that the solution spaces of these equations lack the additivity but they remain homogeneous. Of course, the linear equations form the special case of (1.1) given by

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$p = 2$. From this point of view, we can see half-linear equations as the gateway to non-linear equations. From another point of view, the function Φ is one dimensional p-Laplacian. Hence, the half-linear equations for $\mathbb{T} = \mathbb{R}$ are the scalar case of partial differential equations with p-Laplacian. Some partial differential equations can be reduced (under certain assumptions) to half-linear equations and, at the same time, some results obtained for half-linear equations can be extended to elliptic partial differential equations.

The Sturm theory is extendable verbatim for both of the half-linear equations and the time scale calculus (see, e.g., [1, 2]). Especially, if one solution of (1.1) oscillates, then all solutions oscillate as well. Therefore, we can classify the considered equations as oscillatory and non-oscillatory. Unfortunately, the need of working with differential and difference calculus (together with other closed subsets of $\mathbb{R}$) and the lack of additivity imply that many methods and tools are not available (or, at least, they have to be significantly improved) in the study of oscillatory and non-oscillatory solutions of (1.1).

The problem we are about to deal with is closely related to the so-called *conditional oscillation*. We say that the equation

$$\big[r(t)\Phi(y^{\Delta})\big]^{\Delta} + \gamma c(t)\Phi(y^{\sigma}) = 0, \tag{1.2}$$

where $\gamma \in \mathbb{R}$ and $r(t) > 0$, is *conditionally oscillatory* if there exists a positive constant Γ such that (1.2) is oscillatory for $\gamma > \Gamma$ and it is non-oscillatory for $\gamma < \Gamma$. This number Γ is usually called the critical oscillation constant. As one can easily see, the validity of many comparison theorems (especially, the Sturm's one) implies that conditionally oscillatory equations are ideal testing equations (see also Corollaries 4.5 and 4.6 below).

The approach to the conditional oscillation requires the calculation of the critical oscillation constant depending on coefficients of the treated equations. Hence, the coefficients have to be measurable in a certain sense. Typically, the periodic (almost periodic, etc.) coefficients are considered (see, e.g., [10, 13, 14, 17, 33, 37]). Our approach is different. We do not require the pure periodicity and not even any concrete generalization of periodicity. We only need the boundedness of coefficients. From certain inequalities which are valid for periodic coefficients, we deduce the oscillation behaviour of the given equation. This procedure leads to two theorems – one oscillation and one non-oscillation criterion (see Theorems 3.1 and 3.2 below). In addition, the nature of the criteria remains untouched for other types of equations (see Theorem 3.3 below).

To mention the literature foundations, we begin with the continuous case when $\mathbb{T} = \mathbb{R}$. The first result of this kind appeared in [27] in 1893 (about linear equations with constant coefficients). During the last decades, many results were published in this direction, we mention at least papers [15, 25, 28, 29, 30, 31].

The research of the discrete counterparts of the above mentioned results (i.e., for $\mathbb{T} = \mathbb{Z}$) was initiated by [34] in 1959. For the direction towards difference equations, we refer to [9, 17, 20, 44].

Finally, we recall our basic motivation which comes from [22, 45]. In [45], there is proved that the Euler-type dynamic equation

$$[r(t)y^{\Delta}]^{\Delta} + \frac{\gamma s(t)}{t\sigma(t)} y^{\sigma} = 0$$

with positive α-periodic coefficients is conditionally oscillatory with the critical oscillation constant

$$\Gamma = \frac{\alpha^2}{4}\Big(\int_t^{t+\alpha}\frac{\Delta\tau}{r(\tau)}\Big)^{-1}\Big(\int_t^{t+\alpha}s(\tau)\Delta\tau\Big)^{-1}, \quad t\in\mathbb{T}.$$

Then, in [22], the above result is extended to the Euler-type half-linear dynamic equation (again with positive α-periodic coefficients)

$$\big[r(t)\Phi(y^\Delta)\big]^\Delta + \frac{\gamma s(t)}{t^{(p-1)}\sigma(t)}\Phi(y^\sigma) = 0,$$

where $t^{(p)}$ is the generalized power function. We remark that this function is introduced in Section 4 together with a corollary of our results that covers equations with this power function in the second term (see Corollary 4.4 below). The value

$$\Gamma := \Big(\frac{\alpha(p-1)}{p}\Big)^p\Big(\int_t^{t+\alpha}r^{\frac{1}{1-p}}(\tau)\Delta\tau\Big)^{1-p}\Big(\int_t^{t+\alpha}s(\tau)\Delta\tau\Big)^{-1}, \quad t\in\mathbb{T},$$

is obtained as the resulting critical oscillation constant.

The paper is organized as follows. In the upcoming section, we mention the notations and we state the used lemmas. The results are formulated and proved in Section 3. Then, we present a lot of corollaries to illustrate the novelty of our results (Section 4) and we formulate several open problems (Section 5).

2. Preliminaries and auxiliary results

In this section, we recall the notion of time scales, including the standard notation, we prepare tools for our method, and we mention necessary lemmas. The time scale calculus itself is usually described as a unification of the continuous and discrete calculus. Nevertheless, time scale $\mathbb{T}$ is an arbitrary non-empty closed subset of real numbers, i.e., it covers much more cases than only $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}$.

Now, we mention the used notation (for details, see [5, 6]). A time scale interval $[a,b]\cap\mathbb{T}$ is denoted by $[a,b]_\mathbb{T}$. The forward jump operator, the backward jump operator, and the graininess is denoted by σ, ρ, and μ, respectively. For simplicity, we use the notation $f^\sigma := f\circ\sigma$ and $f^\rho := f\circ\rho$. The Δ-derivative of f is denoted by f^Δ and Δ-integral of f from a to b by $\int_a^b f(t)\Delta t$. Finally, $C_{\rm rd}(\mathbb{T})$ and $C^1_{\rm rd}(\mathbb{T})$ stand for the class of rd-continuous and rd-continuous Δ-differentiable functions defined on the time scale $\mathbb{T}$.

In this paper, we consider that the time scale $\mathbb{T}$ is α-periodic (i.e., there exists a constant $\alpha>0$ such that $t\in\mathbb{T}$ implies $t\pm\alpha\in\mathbb{T}$) which gives that $\mathbb{T}$ is infinite and $\sup\mathbb{T}=\infty$. We study the second-order half-linear dynamic equation

$$[r(t)\Phi(y^\Delta)]^\Delta + c(t)\Phi(y^\sigma) = 0, \quad \Phi(y)=|y|^{p-1}\operatorname{sgn} y, \quad p>1, \tag{2.1}$$

on the given time scale $\mathbb{T}$, where $c,r\in C_{\rm rd}(\mathbb{T})$ satisfy

$$0<\inf\{r(t):t\in\mathbb{T}\}\le\sup\{r(t):t\in\mathbb{T}\}<\infty, \tag{2.2}$$

$$0\le\inf\{c(t):t\in\mathbb{T}\}\le\sup\{c(t):t\in\mathbb{T}\}<\infty. \tag{2.3}$$

Naturally, any solution y of (2.1) satisfies $r\Phi(y^\Delta)\in C^1_{\rm rd}(\mathbb{T})$. Since we use integrals of $r^{1/(1-p)}(t)$, we also need $1/r\in C_{\rm rd}(\mathbb{T})$. Hence, the positivity of the infimum of r in (2.2) cannot be replaced by the condition $r(t)>0$ (see [35] for a more comprehensive discussion on this problem). Note that if we denote by q the number conjugated with the given number $p > 1$ (i.e., $p + q = pq$), then we immediately obtain the inverse function to $\Phi(y)$ in the form $\Phi^{-1}(y) = |y|^{q-1} \operatorname{sgn} y$.

Next, we recall the notion of (non-) oscillatory equations. To do this, we define the *generalized zero* of a non-trivial solution y of (2.1) as a point $t \in \mathbb{T}$, where

$$r(t)y(t)y(\sigma(t)) \leq 0.$$

In the special case $y(t) = 0$, t is called the *common zero* of y. Further, a non-trivial solution y of (2.1) is said to be *oscillatory* on $\mathbb{T}$ if y has a generalized zero on $[\tau, \infty)_{\mathbb{T}}$ for every $\tau \in \mathbb{T}$, and it is said to be *non-oscillatory* otherwise. Finally, the time scale version of the half-linear Sturm type separation theorem (besides the references already given in Introduction, see also [36]) guarantees that if (2.1) has one (non-) oscillatory solution, then any non-trivial solution of (2.1) is (non-) oscillatory as well. Therefore, we can classify (2.1) as *oscillatory* or *non-oscilatory*.

The basis of our method is the transformation to the Riccati dynamic equation which can be introduced as follows. We consider a solution y of (2.1) such that $y(t)y^{\sigma}(t) \neq 0$, $t \in [t_1, t_2]_{\mathbb{T}}$, and we define

$$w(t) = \frac{r(t)\Phi(y^{\Delta}(t))}{\Phi(y(t))}. \tag{2.4}$$

The differentiation of (2.4) and its combination with (2.1) leads to the Riccati dynamic equation

$$w^{\Delta}(t) + c(t) + \mathcal{S}[w, r, \mu](t) = 0. \tag{2.5}$$

The term $\mathcal{S}[w, r, \mu]$ can be expressed as

$$\mathcal{S}[w, r, \mu] = \lim_{\lambda \to \mu} \frac{w}{\lambda} \left(1 - \frac{r}{\Phi[\Phi^{-1}(r) + \lambda \Phi^{-1}(w)]} \right)$$

and rewritten as

$$\mathcal{S}[w, r, \mu] = \begin{cases} \frac{p-1}{\Phi^{-1}(r)} |w|^q & \text{if applied to right-dense } t; \\ \frac{w}{\mu} \left(1 - \frac{r}{\Phi(\Phi^{-1}(r) + \mu \Phi^{-1}(w))} \right) & \text{if applied to right-scattered } t. \end{cases}$$

Taking into account the Lagrange mean value theorem on time scales (see, e.g., [5]), we finally come to $\mathcal{S}[w, r, \mu]$ in the form

$$\mathcal{S}[w, r, \mu](t) = \frac{(p-1)|w(t)|^q |\eta(t)|^{p-2}}{\Phi[\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(w(t))]}, \tag{2.6}$$

where $\eta(t)$ is between $\Phi^{-1}(r(t))$ and $\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(w(t))$.

The form of (2.5) with (2.6) is not sufficient enough for our method. Hence, we apply the transformation

$$z(t) = -t^{p-1} w(t) \tag{2.7}$$

which leads to the equation

$$\begin{aligned} z^{\Delta}(t) = c(t)(\sigma(t))^{p-1} &+ \frac{(p-1)(\sigma(t))^{p-1} |\eta(t)|^{p-2} |z(t)|^q}{t^p \Phi[\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(-z(t)/t^{p-1})]} \\ &+ \frac{(p-1)(\zeta(t))^{p-2} z(t)}{t^{p-1}}. \end{aligned} \tag{2.8}$$

Equation (2.8) is called the adapted generalized Riccati equation for the consistency with similar cases in the literature. In (2.8), $\zeta(t)$ is given by

$$\zeta(t) := \Big[\frac{(t^{p-1})^{\Delta}}{p-1}\Big]^{\frac{1}{p-2}} \tag{2.9}$$

and $\eta(t)$ is between $\Phi^{-1}(r(t))$ and $\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(-z(t)/t^{p-1})$ (see (2.6)). We emphasize that, rewriting (2.9) as $(t^{p-1})^{\Delta} = (p-1)(\zeta(t))^{p-2}$ and considering the Lagrange mean value theorem on time scales, it is seen that $\zeta(t)$ is well-defined and satisfies $t \le \zeta(t) \le \sigma(t)$.

We finish this section by lemmas that will be needed in the following section within the proofs of our results. The first lemma is well-known and describes the connection between the behaviour of solutions of (2.1) and the Riccati dynamic equation (2.5).

Lemma 2.1. *If* (2.1) *is non-oscillatory, then there exists a solution w of the associated generalized Riccati equation* (2.5) *such that $w(t) > 0$ for all large $t \in \mathbb{T}$. Moreover, w is decreasing for large t and satisfies*

$$\lim_{t\to\infty} w(t) = 0. \tag{2.10}$$

For a proof of the above lemma see [22, 36]. The second lemma is a consequence of Lemma 2.1 which is formulated for the adapted Riccati equation (2.8).

Lemma 2.2. *If* (2.1) *is non-oscillatory, then there exists a solution z of the associated adapted generalized Riccati equation* (2.8) *such that $z(t) < 0$ for all large $t \in \mathbb{T}$ and*

$$\lim_{t\to\infty} \frac{z(t)}{t^{p-1}} = 0. \tag{2.11}$$

The statement of the above lemma follows directly from Lemma 2.1, where (2.11) follows from (2.7) and (2.10). The last lemma deals with the adapted Riccati equation (2.8) as well and takes into account the time scale version of the Reid roundabout theorem.

Lemma 2.3. *If there exists a solution z of* (2.8) *which is negative for all large $t \in \mathbb{T}$, then* (2.1) *is non-oscillatory.*

Proof. The negative solution z of (2.8) gives the positive solution w of (2.5) for which we have the inequality $\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(w(t)) > 0$ for all large t. Using the well-known roundabout theorem (see, e.g., [36, p. 383]), we obtain that (2.1) is non-oscillatory. $\square$

3. Oscillation and non-oscillation

In this section, we formulate and prove our main results. At first, for reader's convenience, let us recall that we deal with the half-linear dynamic equation

$$\big[r(t)\Phi(y^{\Delta})\big]^{\Delta} + \frac{s(t)}{t^{p-1}\sigma(t)}\Phi(y^{\sigma}) = 0, \tag{3.1}$$

where $t \in \mathbb{T}$ is sufficiently large, functions r, s are rd-continuous, positive, and bounded and r satisfies $\inf\{r(t) : t \in [a,\infty)_{\mathbb{T}}\} > 0$ for some $a \in \mathbb{T}$. Now, we can formulate the our first theorem as follows.

Theorem 3.1. *Let $R, S > 0$ be given and let $q^p S R^{p-1} > 1$. If*

$$\frac{1}{\alpha}\int_t^{t+\alpha} r^{1-q}(\tau)\Delta\tau \geq R, \quad \text{and} \quad \frac{1}{\alpha}\int_t^{t+\alpha} s(\tau)\Delta\tau \geq S \tag{3.2}$$

for all large $t \in \mathbb{T}$, then (3.1) is oscillatory.

Proof. By contradiction, we suppose that (3.1) is non-oscillatory. Using Lemma 2.2, there exists a negative solution z of the associated adapted Riccati equation (see (2.8))

$$\begin{aligned} z^{\Delta}(t) = &\frac{s(t)(\sigma(t))^{p-2}}{t^{p-1}} + \frac{(p-1)(\sigma(t))^{p-1}|\eta(t)|^{p-2}|z(t)|^q}{t^p\,\Phi[\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(-z(t)/t^{p-1})]} \\ &+ \frac{(p-1)(\zeta(t))^{p-2}z(t)}{t^{p-1}} \end{aligned} \tag{3.3}$$

for all large t and $z(t)/t^{p-1} \to 0$ as $t \to \infty$. We recall that $\mu(t) \leq \alpha$, $t \leq \zeta(t) \leq \sigma(t) \leq t + \alpha$, and

$$\Phi^{-1}(r(t)) \leq \eta(t) \leq \Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(w(t)) = \Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(-z(t)/t^{p-1}).$$

Hence,

$$\lim_{t\to\infty} \frac{(\sigma(t))^{p-2}}{t^{p-2}} = 1, \quad \lim_{t\to\infty} \frac{\eta(t)}{\Phi^{-1}(r(t))} = 1, \quad \lim_{t\to\infty} \frac{\zeta(t)}{t} = \lim_{t\to\infty} \frac{\sigma(t)}{t} = 1. \tag{3.4}$$

Let large $b \in \mathbb{T}$ be arbitrarily. We put

$$r^+ := \sup\{r(t) : t \in [b,\infty)_{\mathbb{T}}\}, \quad r^- := \inf\{r(t) : t \in [b,\infty)_{\mathbb{T}}\}, \tag{3.5}$$

$$s^+ := \sup\{s(t) : t \in [b,\infty)_{\mathbb{T}}\}, \quad s^- := \inf\{s(t) : t \in [b,\infty)_{\mathbb{T}}\}. \tag{3.6}$$

Recall that it holds (see also (2.2), (2.3))

$$0 < r^- \leq r^+ < \infty, \quad 0 \leq s^- \leq s^+ < \infty.$$

Let $\varepsilon > 0$ be sufficiently small. From (3.3) and (3.4), we obtain

$$\begin{aligned} &z^{\Delta}(t) \\ &\geq \frac{(1-\varepsilon)s(t)}{t} + \frac{(1-\varepsilon)(p-1)t^{p-1}(r^{q-1}(t))^{p-2}|z(t)|^q}{t^p\,r(t)} + \frac{(1+\varepsilon)(p-1)t^{p-2}z(t)}{t^{p-1}} \\ &\geq \frac{(1-\varepsilon)(p-1)}{t}\Big[\frac{s^-}{p-1} + \big(r^+\big)^{1-q}|z(t)|^q + \frac{1+\varepsilon}{1-\varepsilon}\,z(t)\Big] \\ &\geq \frac{(1-\varepsilon)(p-1)(-z(t))}{t}\Big[\Big(\frac{-z(t)}{r^+}\Big)^{q-1} - \frac{1+\varepsilon}{1-\varepsilon}\Big] \end{aligned}$$

for all large t. Since $\varepsilon > 0$ is arbitrary, the previous estimations prove that

$$z^{\Delta}(t) \geq 0 \quad \text{if } z(t) < -r^+ \text{ for large } t. \tag{3.7}$$

At the same time, we have (see (3.3) and (3.4))

$$\begin{aligned} |z^{\Delta}(t)| &\leq \frac{(1+\varepsilon)s(t)}{t} + \frac{(1+\varepsilon)(p-1)t^{p-1}(r^{q-1}(t))^{p-2}|z(t)|^q}{t^p\,r(t)} \\ &\quad + \frac{(1+\varepsilon)(p-1)t^{p-2}|z(t)|}{t^{p-1}} \\ &\leq \frac{(1+\varepsilon)(p-1)}{t}\Big[\frac{s^+}{p-1} + (r^-)^{1-q}|z(t)|^q + |z(t)|\Big] \end{aligned} \tag{3.8}$$

for large t. If we consider $z(t) \in (-2r^+, 0)$, then (3.8) implies

$$|z^\Delta(t)| \le \frac{(1+\varepsilon)(p-1)}{t}\Big[\frac{s^+}{p-1} + (r^-)^{1-q}(2r^+)^q + 2r^+\Big] = \frac{K}{t}, \tag{3.9}$$

where

$$K := (1+\varepsilon)\big[s^+ + (p-1)(r^-)^{1-q}(2r^+)^q + (p-1)2r^+\big]. \tag{3.10}$$

From (3.7) and (3.9), it is seen that there exists a negative number M such that $z(t) \in (M, 0)$ for all considered large t.

Now, we introduce the average function $z_{\rm ave}$ of the function z by the formula

$$z_{\rm ave}(t) := \frac{1}{\alpha}\int_t^{t+\alpha} z(\tau)\Delta\tau. \tag{3.11}$$

Evidently, $z_{\rm ave}(t) \in (M, 0)$ for all considered large t.

For large t (see (3.3), (3.4)), we have

$$\begin{aligned}
&z^\Delta_{\rm ave}(t)\\
&= \frac{1}{\alpha}\int_t^{t+\alpha} z^\Delta(\tau)\Delta\tau = \frac{1}{\alpha}\int_t^{t+\alpha} \frac{s(\tau)(\sigma(\tau))^{p-2}}{\tau^{p-1}}\,\Delta\tau\\
&\quad + \frac{1}{\alpha}\int_t^{t+\alpha} \frac{(p-1)(\sigma(\tau))^{p-1}|\eta(\tau)|^{p-2}|z(\tau)|^q}{\tau^p\,\Phi[\Phi^{-1}(r(\tau)) + \mu(\tau)\Phi^{-1}(-z(\tau)/\tau^{p-1})]}\,\Delta\tau\\
&\quad + \frac{1}{\alpha}\int_t^{t+\alpha} \frac{(p-1)(\zeta(\tau))^{p-2}z(\tau)}{\tau^{p-1}}\,\Delta\tau\\
&\ge \frac{1-\varepsilon}{\alpha}\int_t^{t+\alpha} \frac{s(\tau)}{\tau} + \frac{(p-1)r^{1-q}(\tau)|z(\tau)|^q}{\tau} + \frac{1+\varepsilon}{1-\varepsilon}\frac{(p-1)z(\tau)}{\tau}\Delta\tau.
\end{aligned} \tag{3.12}$$

Since

$$\begin{aligned}
&\Big|\int_t^{t+\alpha} \frac{s(\tau)}{\tau} + \frac{(p-1)r^{1-q}(\tau)|z(\tau)|^q}{\tau} + \frac{1+\varepsilon}{1-\varepsilon}\frac{(p-1)z(\tau)}{\tau}\Delta\tau\\
&- \frac{1}{t}\int_t^{t+\alpha} s(\tau) + (p-1)r^{1-q}(\tau)|z(\tau)|^q + \frac{1+\varepsilon}{1-\varepsilon}(p-1)z(\tau)\Delta\tau\Big|\\
&\le \int_t^{t+\alpha} \big(\frac{1}{t} - \frac{1}{\tau}\big)\Big(s^+ + (p-1)(r^-)^{1-q}|M|^q + \frac{1+\varepsilon}{1-\varepsilon}(p-1)|M|\Big)\Delta\tau\\
&\le \frac{\alpha^2}{t^2}\Big(s^+ + (p-1)(r^-)^{1-q}|M|^q + \frac{1+\varepsilon}{1-\varepsilon}(p-1)|M|\Big) = \frac{N(1-\varepsilon)}{t^2},
\end{aligned}$$

where

$$N := \frac{\alpha^2}{1-\varepsilon}\Big(s^+ + (p-1)(r^-)^{1-q}|M|^q + \frac{1+\varepsilon}{1-\varepsilon}(p-1)|M|\Big),$$

for large t, we obtain (see (3.12))

$$\begin{aligned}
z^\Delta_{\rm ave}(t) \ge \frac{1-\varepsilon}{\alpha t}\int_t^{t+\alpha} &\Big(s(\tau) + (p-1)r^{1-q}(\tau)|z(\tau)|^q\\
&+ \frac{1+\varepsilon}{1-\varepsilon}(p-1)z(\tau) - \frac{N}{t}\Big)\Delta\tau.
\end{aligned} \tag{3.13}$$

If we put

$$X(t) = \frac{q^{-p}}{t}\Big(\frac{1}{\alpha}\int_t^{t+\alpha} r^{1-q}(\tau)\,\Delta\tau\Big)^{-p/q}, \quad Y(t) = |z_{\rm ave}(t)|^q\,\frac{p}{qt\alpha}\int_t^{t+\alpha} r^{1-q}(\tau)\,\Delta\tau$$

for large t, then we have (see (3.13))

$$\begin{aligned} z^{\Delta}_{\rm ave}(t) \geq{}& \frac{1-\varepsilon}{\alpha t}\int_t^{t+\alpha} s(\tau)\Delta\tau - (1-\varepsilon)X(t) \\ &+ \frac{1-\varepsilon}{\alpha t}\int_t^{t+\alpha} (p-1)r^{1-q}(\tau)|z(\tau)|^q\Delta\tau - (1-\varepsilon)Y(t) \\ &+ \frac{1+\varepsilon}{\alpha t}\int_t^{t+\alpha} (p-1)z(\tau)\Delta\tau + (1-\varepsilon)X(t) + (1-\varepsilon)Y(t) \\ &- \frac{N(1-\varepsilon)}{t^2}. \end{aligned} \tag{3.14}$$

Considering this expression, for large t, we show the inequalities

$$\frac{L}{t} \leq \frac{1-\varepsilon}{\alpha t}\int_t^{t+\alpha} s(\tau)\Delta\tau - (1-\varepsilon)X(t), \tag{3.15}$$

$$\Big|\frac{1-\varepsilon}{\alpha t}\int_t^{t+\alpha} (p-1)r^{1-q}(\tau)|z(\tau)|^q\Delta\tau - (1-\varepsilon)Y(t)\Big| \leq \frac{L}{4t}, \tag{3.16}$$

$$-\frac{L}{4t} \leq \frac{1+\varepsilon}{\alpha t}\int_t^{t+\alpha} (p-1)z(\tau)\Delta\tau + (1-\varepsilon)X(t) + (1-\varepsilon)Y(t), \tag{3.17}$$

$$\frac{N(1-\varepsilon)}{t^2} \leq \frac{L}{4t}, \tag{3.18}$$

where $L := S(1-q^{-p}R^{1-p}S^{-1})/2 > 0$ and $\varepsilon > 0$ is arbitrarily small.

The inequality (3.18) is evidently valid for all $t \geq 4N(1-\varepsilon)/L$. Now, we prove (3.15). We have (see (3.2))

$$\begin{aligned} &\frac{1-\varepsilon}{\alpha t}\int_t^{t+\alpha} s(\tau)\Delta\tau - (1-\varepsilon)X(t) \\ &= \frac{1-\varepsilon}{t}\Big[\frac{1}{\alpha}\int_t^{t+\alpha} s(\tau)\Delta\tau - q^{-p}\Big(\frac{1}{\alpha}\int_t^{t+\alpha} r^{1-q}(\tau)\Delta\tau\Big)^{-p/q}\Big] \\ &\geq \frac{1-\varepsilon}{t}[S - q^{-p}R^{-p/q}] \\ &= \frac{(1-\varepsilon)S}{t}[1 - q^{-p}R^{1-p}S^{-1}] = \frac{(1-\varepsilon)2L}{t}. \end{aligned} \tag{3.19}$$

To obtain (3.15), it suffices to choose $\varepsilon \leq 1/2$ in (3.19).

Since $z(t) \in (M, 0)$, from (3.3) and (3.4) it follows that

$$\begin{aligned} |z^{\Delta}(t)| \leq{}& \frac{s(t)(\sigma(t))^{p-2}}{t^{p-1}} + \frac{(p-1)(\sigma(t))^{p-1}|\eta(t)|^{p-2}|z(t)|^q}{t^p\,\Phi[\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(-z(t)/t^{p-1})]} \\ &- \frac{(p-1)(\zeta(t))^{p-2}z(t)}{t^{p-1}} \\ \leq{}& (1+\varepsilon)\Big(\frac{s(t)}{t} + \frac{(p-1)r^{1-q}(t)|z(t)|^q}{t} + \frac{(p-1)|z(t)|}{t}\Big) \\ ={}& \frac{(1+\varepsilon)\big(s^+ + (p-1)\left(r^-\right)^{1-q}|M|^q + (p-1)|M|\big)}{t}. \end{aligned} \tag{3.20}$$

We denote

$$P := 2\left(s^+ + (p-1)\left(r^-\right)^{1-q}|M|^q + (p-1)|M|\right).$$

Of course, for large t and $\varepsilon < 1$, (3.20) implies

$$|z(t+i) - z(t+j)| \le \frac{\alpha P}{t}, \quad i, j \in [0, \alpha],\ t+i, t+j \in \mathbb{T}.$$

Especially (see directly (3.11)), we know that

$$|z(\tau) - z_{\mathrm{ave}}(t)| \le \frac{\alpha P}{t} \tag{3.21}$$

for all considered large t and $\tau \in [t, t+\alpha]_{\mathbb{T}}$.

Further, since the function $y = |x|^q$ is continuously differentiable on $[M, 0]$, there exists $A > 0$ for which

$$\big||y|^q - |z|^q\big| \le A|y - z|, \quad y, z \in [M, 0]. \tag{3.22}$$

Hence, we have (see (3.21) and (3.22))

$$\begin{aligned}
&\Big|\frac{1-\varepsilon}{\alpha t}\int_t^{t+\alpha} (p-1) r^{1-q}(\tau)|z(\tau)|^q \Delta\tau - (1-\varepsilon)Y(t)\Big| \\
&= \frac{1-\varepsilon}{t}\Big|\frac{1}{\alpha}\int_t^{t+\alpha} (p-1) r^{1-q}(\tau)|z(\tau)|^q \Delta\tau - |z_{\mathrm{ave}}(t)|^q \frac{p}{q\alpha}\int_t^{t+\alpha} r^{1-q}(\tau)\,\Delta\tau\Big| \\
&\le \frac{1-\varepsilon}{t}\cdot\frac{p-1}{\alpha}\int_t^{t+\alpha} r^{1-q}(\tau)\big||z(\tau)|^q - |z_{\mathrm{ave}}(t)|^q\big|\Delta\tau \\
&\le \frac{1-\varepsilon}{t}\cdot\frac{(p-1)\left(r^-\right)^{1-q}}{\alpha}\int_t^{t+\alpha} \big||z(\tau)|^q - |z_{\mathrm{ave}}(t)|^q\big|\Delta\tau \\
&\le \frac{1-\varepsilon}{t}\cdot\frac{(p-1)\left(r^-\right)^{1-q}}{\alpha}\int_t^{t+\alpha} A\,|z(\tau) - z_{\mathrm{ave}}(t)|\,\Delta\tau \\
&\le \frac{1-\varepsilon}{t}\cdot(p-1)\left(r^-\right)^{1-q}\cdot A\frac{\alpha P}{t}.
\end{aligned}$$

For large t, the previous computation gives (3.16).

It remains to prove (3.17). It holds

$$\begin{aligned}
&\frac{1+\varepsilon}{\alpha t}\int_t^{t+\alpha} (p-1) z(\tau)\Delta\tau + (1-\varepsilon)X(t) + (1-\varepsilon)Y(t) \\
&= \frac{1}{t}\Big[(1+\varepsilon)(p-1)z_{\mathrm{ave}}(t) + (1-\varepsilon)q^{-p}\Big(\frac{1}{\alpha}\int_t^{t+\alpha} r^{1-q}(\tau)\,\Delta\tau\Big)^{-p/q} \\
&\quad + (1-\varepsilon)|z_{\mathrm{ave}}(t)|^q \frac{p}{q\alpha}\int_t^{t+\alpha} r^{1-q}(\tau)\,\Delta\tau\Big].
\end{aligned} \tag{3.23}$$

We recall the Young inequality which says that

$$\frac{U^p}{p} + \frac{V^q}{q} \ge UV \tag{3.24}$$

holds for any numbers $U, V \ge 0$. If one puts $U = (ptX(t))^{1/p}$ and $V = (qtY(t))^{1/q}$, then

$$\begin{aligned}
UV &= (ptX(t))^{1/p}(qtY(t))^{1/q} \\
&= \frac{p^{1/p}}{q}\Big(\frac{1}{\alpha}\int_t^{t+\alpha} r^{1-q}(\tau)\,\Delta\tau\Big)^{-1/q}|z_{\mathrm{ave}}(t)|\Big(\frac{p}{\alpha}\int_t^{t+\alpha} r^{1-q}(\tau)\,\Delta\tau\Big)^{1/q},
\end{aligned}$$

i.e.,

$$UV = \frac{p^{\frac{1}{p}+\frac{1}{q}}}{q} |z_{\rm ave}(t)| = (p-1)|z_{\rm ave}(t)| \,,$$

and

$$\begin{aligned} \frac{U^p}{p} &= tX(t) = q^{-p}\Big(\frac{1}{\alpha}\int_t^{t+\alpha} r^{1-q}(\tau)\,\Delta\tau\Big)^{-p/q}, \\ \frac{V^q}{q} &= tY(t) = |z_{\rm ave}(t)|^q \frac{p}{q\alpha}\int_t^{t+\alpha} r^{1-q}(\tau)\,\Delta\tau. \end{aligned}$$

For these numbers U, V, inequality (3.24) gives

$$\begin{aligned} &q^{-p}\Big(\frac{1}{\alpha}\int_t^{t+\alpha} r^{1-q}(\tau)\,\Delta\tau\Big)^{-p/q} \\ &+ |z_{\rm ave}(t)|^q \frac{p}{q\alpha}\int_t^{t+\alpha} r^{1-q}(\tau)\,\Delta\tau - (p-1)|z_{\rm ave}(t)| \geq 0. \end{aligned} \tag{3.25}$$

Since $\varepsilon > 0$ can be arbitrarily small, from (3.23) and (3.25), we obtain (3.17).

Altogether, (3.14) together with (3.15), (3.16), (3.17), and (3.18) guarantee

$$z_{\rm ave}^{\Delta}(t) \geq \frac{L}{t} - \frac{L}{4t} - \frac{L}{4t} - \frac{L}{4t} = \frac{L}{4t} \tag{3.26}$$

for all large t. From (3.26) it follows

$$\lim_{t\to\infty} z_{\rm ave}(t) = \infty.$$

In particular, $z_{\rm ave}$ is positive at least in one point which is a contradiction. The proof is complete. □

Theorem 3.2. *Let $R, S > 0$ be given and let $q^p S R^{p-1} < 1$. If*

$$\frac{1}{\alpha}\int_t^{t+\alpha} r^{1-q}(\tau)\Delta\tau \leq R, \quad \frac{1}{\alpha}\int_t^{t+\alpha} s(\tau)\Delta\tau \leq S \tag{3.27}$$

for all large $t \in \mathbb{T}$, then (3.1) *is non-oscillatory.*

Proof. Henceforth, we consider a sufficiently small number $\varepsilon > 0$. Let $t_0 \in \mathbb{T}$ be sufficiently large. In particular, we assume that, for all $t \geq t_0$, it holds (see (3.4))

$$1-\varepsilon \leq \big(\frac{\sigma(t)}{t}\big)^{p-2} \leq 1+\varepsilon, \quad 1-\varepsilon \leq \big(\frac{\zeta(t)}{t}\big)^{p-2} \leq 1+\varepsilon, \tag{3.28}$$

$$\sqrt{1-\varepsilon} \leq \big(\frac{\sigma(t)}{t}\big)^{p-1} \leq \sqrt{1+\varepsilon}, \tag{3.29}$$

$$\sqrt{1-\varepsilon} \leq \Big(\frac{\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(-z_0/t^{p-1})}{\Phi^{-1}(r(t))}\Big)^{p-2} \leq \sqrt{1+\varepsilon}, \tag{3.30}$$

$$\Phi\left(\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(-z_0/t^{p-1})\right) \leq r(t)(1+\varepsilon), \tag{3.31}$$

where z_0 in (3.30), (3.31) is an arbitrary number from interval $(-2r^+, 0)$. As in the proof of Theorem 3.1 (see (3.5), (3.6)), we introduce r^+, r^-, s^+, s^-.

Let z be the solution of the associated adapted Riccati equation (see (2.8))

$$\begin{aligned} z^{\Delta}(t) = &\frac{s(t)(\sigma(t))^{p-2}}{t^{p-1}} + \frac{(p-1)(\sigma(t))^{p-1}|\eta(t)|^{p-2}|z(t)|^q}{t^p\,\Phi[\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(-z(t)/t^{p-1})]} \\ &+ \frac{(p-1)(\zeta(t))^{p-2}z(t)}{t^{p-1}} \end{aligned} \tag{3.32}$$

satisfying

$$z(t_0) = -\Big(\frac{q}{\alpha}\int_{t_0}^{t_0+\alpha}\frac{\Delta\tau}{r^{q-1}(\tau)}\Big)^{1-p} > -r^+. \tag{3.33}$$

To prove the theorem, it suffices to show that this solution is negative for all $t \in [t_0,\infty)_{\mathbb{T}}$. As in the proof of Theorem 3.1 (see (3.8)), for $t \geq t_0$, we have (see (3.28), (3.29), and (3.30))

$$\begin{aligned} |z^\Delta(t)| &\leq \frac{(1+\varepsilon)s(t)}{t} + \frac{(1+\varepsilon)(p-1)(r^{q-1}(t))^{p-2}|z(t)|^q}{tr(t)} + \frac{(1+\varepsilon)(p-1)|z(t)|}{t} \\ &\leq \frac{(1+\varepsilon)(p-1)}{t}\Big[\frac{s^+}{p-1} + (r^-)^{1-q}|z(t)|^q + |z(t)|\Big] \\ &\leq \frac{(1+\varepsilon)(p-1)}{t}\Big[\frac{s^+}{p-1} + (r^-)^{1-q}(2r^+)^q + 2r^+\Big] \end{aligned}$$

if $z(t) \in (-2r^+,0)$. This means that if $z(t) \in (-2r^+,0)$ for some $t \geq t_0$, then

$$\big|z^\Delta(t)\big| \leq \frac{K}{t} \tag{3.34}$$

for a given positive number K (see (3.10)). Especially, we know that

$$z(t) \in (-2r^+,0), \quad t \in [t_0, t_0+\alpha]_{\mathbb{T}}. \tag{3.35}$$

Indeed, it follows from (3.33).

If

$$-2r^+ < z(t) < -r^+\Big(\frac{(1+\varepsilon)^2}{1-\varepsilon}\Big)^{\frac{1}{q-1}},$$

then from (3.32), using (3.28), (3.29), (3.30), and (3.31), we obtain

$$\begin{aligned} &z^\Delta(t) \\ &\geq \frac{(1-\varepsilon)s(t)}{t} + \frac{(1-\varepsilon)(p-1)(r^{q-1}(t))^{p-2}|z(t)|^q}{tr(t)(1+\varepsilon)} + \frac{(1+\varepsilon)(p-1)z(t)}{t} \\ &\geq \frac{(1-\varepsilon)(p-1)}{t}\Big[\frac{(r^+)^{1-q}}{1+\varepsilon}|z(t)|^q + \frac{1+\varepsilon}{1-\varepsilon}z(t)\Big] \\ &\geq \frac{(1-\varepsilon)(p-1)(-z(t))}{t}\Big[\Big(\frac{-z(t)}{r^+}\Big)^{q-1}\frac{1}{1+\varepsilon} - \frac{1+\varepsilon}{1-\varepsilon}\Big] > 0. \end{aligned} \tag{3.36}$$

We add that we can consider $\varepsilon > 0$ such that $z^\Delta(t) > 0$ whenever $z(t) \leq -3r^+/2$. The previous fact together with (3.34) imply that any negative solution z satisfies $z(t) \in (-2r^+,0)$ for all large t if $z(t_0) > -r^+$. We prove that the solution z of (3.32) satisfying (3.33) is negative.

From (3.34), we have

$$|z(t_0) - z(\tau)| < \frac{K\alpha}{t_0}, \quad \tau \in [t_0, t_0+\alpha]_{\mathbb{T}}. \tag{3.37}$$

Similarly as in the proof of Theorem 3.1, we define

$$z_{\rm ave}(t_0) := \frac{1}{\alpha}\int_{t_0}^{t_0+\alpha} z(\tau)\Delta\tau. \tag{3.38}$$

We know that (see (3.35), (3.37), and (3.38))

$$z_{\rm ave}(t_0) \in (-2r^+,0) \quad \text{and} \quad |z_{\rm ave}(t_0) - z(t_0)| < \frac{K\alpha}{t_0}. \tag{3.39}$$

We have (see (3.28), (3.29), and (3.30) with (3.32))

$$
\begin{aligned}
&z^{\Delta}_{\rm ave}(t_0)\\
&= \frac{1}{\alpha}\int_{t_0}^{t_0+\alpha} z^{\Delta}(\tau)\Delta\tau = \frac{1}{\alpha}\int_{t_0}^{t_0+\alpha} \frac{s(\tau)(\sigma(\tau))^{p-2}}{\tau^{p-1}}\,\Delta\tau\\
&\quad + \frac{1}{\alpha}\int_{t_0}^{t_0+\alpha} \frac{(p-1)(\sigma(\tau))^{p-1}|\eta(\tau)|^{p-2}|z(\tau)|^q}{\tau^p\Phi[\Phi^{-1}(r(\tau))+\mu(\tau)\Phi^{-1}(-z(\tau)/\tau^{p-1})]}\,\Delta\tau\\
&\quad + \frac{1}{\alpha}\int_{t_0}^{t_0+\alpha} \frac{(p-1)(\zeta(\tau))^{p-2}z(\tau)}{\tau^{p-1}}\,\Delta\tau\\
&\le \frac{1+\varepsilon}{\alpha}\int_{t_0}^{t_0+\alpha} \frac{s(\tau)}{\tau} + \frac{(p-1)r^{1-q}(\tau)|z(\tau)|^q}{\tau} + \frac{1-\varepsilon}{1+\varepsilon}\frac{(p-1)z(\tau)}{\tau}\Delta\tau.
\end{aligned}
\tag{3.40}
$$

In addition, it holds (see (3.35))

$$
\begin{aligned}
&\Big|\int_{t_0}^{t_0+\alpha} \frac{s(\tau)}{\tau} + \frac{(p-1)r^{1-q}(\tau)|z(\tau)|^q}{\tau} + \frac{1-\varepsilon}{1+\varepsilon}\frac{(p-1)z(\tau)}{\tau}\Delta\tau\\
&- \frac{1}{t_0}\int_{t_0}^{t_0+\alpha} s(\tau) + (p-1)r^{1-q}(\tau)|z(\tau)|^q + \frac{1-\varepsilon}{1+\varepsilon}(p-1)z(\tau)\Delta\tau\Big|\\
&\le \int_{t_0}^{t_0+\alpha}\Big(\frac{1}{t_0}-\frac{1}{\tau}\Big)\Big(s^+ + (p-1)(r^-)^{1-q}(2r^+)^q + \frac{1-\varepsilon}{1+\varepsilon}(p-1)2r^+\Big)\Delta\tau\\
&\le \frac{\alpha^2}{t_0^2}\Big(s^+ + (p-1)(r^-)^{1-q}(2r^+)^q + \frac{1-\varepsilon}{1+\varepsilon}(p-1)2r^+\Big)\\
&= \frac{N(1+\varepsilon)}{t_0^2}
\end{aligned}
$$

for a positive constant N. Hence, from (3.40), we obtain

$$
\begin{aligned}
z^{\Delta}_{\rm ave}(t_0) \le \frac{1+\varepsilon}{\alpha t_0}\int_{t_0}^{t_0+\alpha}\Big(&s(\tau) + (p-1)r^{1-q}(\tau)|z(\tau)|^q\\
&+ \frac{1-\varepsilon}{1+\varepsilon}(p-1)z(\tau) + \frac{N}{t_0}\Big)\Delta\tau.
\end{aligned}
\tag{3.41}
$$

If we put

$$
\begin{gathered}
X(t_0) = q^{-p}\Big(\frac{1}{\alpha}\int_{t_0}^{t_0+\alpha} r^{1-q}(\sigma)\,\Delta\sigma\Big)^{-p/q},\\
Y(t_0) = \frac{p}{q^{p+1}}\Big(\frac{1}{\alpha}\int_{t_0}^{t_0+\alpha} r^{1-q}(\sigma)\,\Delta\sigma\Big)^{1-p},
\end{gathered}
\tag{3.42}
$$

then we have (see (3.41))

$$
\begin{aligned}
z^{\Delta}_{\rm ave}(t_0) \le \frac{1+\varepsilon}{\alpha t_0}&\int_{t_0}^{t_0+\alpha} s(\tau) - X(t_0)\Delta\tau\\
&+ \frac{1+\varepsilon}{\alpha t_0}\int_{t_0}^{t_0+\alpha} (p-1)r^{1-q}(\tau)|z(\tau)|^q - Y(t_0)\Delta\tau\\
&+ \frac{1-\varepsilon}{\alpha t_0}\int_{t_0}^{t_0+\alpha} (p-1)z(\tau)\Delta\tau + \frac{1+\varepsilon}{t_0}X(t_0) + \frac{1+\varepsilon}{t_0}Y(t_0)\\
&+ \frac{N(1+\varepsilon)}{t_0^2}.
\end{aligned}
\tag{3.43}
$$

Now we prove that $z^{\Delta}_{\rm ave}(t_0) < 0$. To prove this inequality, it suffices to show (see (3.43)) that

$$\frac{1+\varepsilon}{\alpha}\int_{t_0}^{t_0+\alpha} s(\tau) - X(t_0)\Delta\tau \le L, \tag{3.44}$$

$$\Big|\frac{1+\varepsilon}{\alpha}\int_{t_0}^{t_0+\alpha}(p-1)r^{1-q}(\tau)|z(\tau)|^q - Y(t_0)\Delta\tau\Big| \le \frac{-L}{4}, \tag{3.45}$$

$$\frac{1-\varepsilon}{\alpha}\int_{t_0}^{t_0+\alpha}(p-1)z(\tau)\Delta\tau + (1+\varepsilon)\,X(t_0) + (1+\varepsilon)\,Y(t_0) \le \frac{-L}{4}, \tag{3.46}$$

$$\frac{N(1+\varepsilon)}{t_0} \le \frac{-L}{4}, \tag{3.47}$$

where $L := S(1 - q^{-p}R^{1-p}S^{-1}) < 0$.

To obtain (3.47), it suffices to choose sufficiently large t_0. Further, we show that (3.44) is valid. We have (see (3.27), (3.42))

$$\begin{aligned}
&\frac{1+\varepsilon}{\alpha}\int_{t_0}^{t_0+\alpha} s(\tau) - X(t_0)\Delta\tau \\
&= \frac{1+\varepsilon}{\alpha}\int_{t_0}^{t_0+\alpha} s(\tau) - q^{-p}\Big(\frac{1}{\alpha}\int_{t_0}^{t_0+\alpha} r^{1-q}(\sigma)\,\Delta\sigma\Big)^{-p/q}\Delta\tau \\
&\le (1+\varepsilon)\big(S - q^{-p}R^{-p/q}\big) \\
&= (1+\varepsilon)\,S\left(1 - q^{-p}R^{1-p}S^{-1}\right) = (1+\varepsilon)L \le L,
\end{aligned}$$

i.e., we obtain (3.44).

Now we prove (3.45). Let $B > 0$ be such that

$$\big||y|^q - |z|^q\big| \le B|y-z|, \quad y, z \in [-2r^+, 0]. \tag{3.48}$$

Using (3.33), (3.35), (3.37), (3.42), and (3.48), we have

$$\begin{aligned}
&\Big|\frac{1+\varepsilon}{\alpha}\int_{t_0}^{t_0+\alpha}(p-1)r^{1-q}(\tau)|z(\tau)|^q - Y(t_0)\Delta\tau\Big| \\
&= \Big|\frac{1+\varepsilon}{\alpha}\int_{t_0}^{t_0+\alpha}(p-1)r^{1-q}(\tau)|z(\tau)|^q\Delta\tau \\
&\quad - \frac{p(1+\varepsilon)}{q^{p+1}}\Big(\frac{1}{\alpha}\int_{t_0}^{t_0+\alpha} r^{1-q}(\tau)\,\Delta\tau\Big)^{1-p}\Big| \\
&= \Big|\frac{1+\varepsilon}{\alpha}\int_{t_0}^{t_0+\alpha}(p-1)r^{1-q}(\tau)|z(\tau)|^q\Delta\tau \\
&\quad - \frac{p(1+\varepsilon)}{q}|z(t_0)|^q\Big(\frac{1}{\alpha}\int_{t_0}^{t_0+\alpha} r^{1-q}(\tau)\,\Delta\tau\Big)\Big| \\
&\le \frac{(1+\varepsilon)(p-1)}{\alpha}\int_{t_0}^{t_0+\alpha} r^{1-q}(\tau)\big||z(\tau)|^q - |z(t_0)|^q\big|\Delta\tau \\
&\le \frac{(1+\varepsilon)(p-1)\left(r^-\right)^{1-q}}{\alpha}\int_{t_0}^{t_0+\alpha}\big||z(\tau)|^q - |z(t_0)|^q\big|\Delta\tau \\
&\le \frac{(1+\varepsilon)(p-1)\left(r^-\right)^{1-q}}{\alpha}\int_{t_0}^{t_0+\alpha} B\,|z(\tau) - z(t_0)|\,\Delta\tau
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(1+\varepsilon)(p-1)\left(r^{-}\right)^{1-q}}{\alpha} \int_{t_0}^{t_0+\alpha} B\frac{K\alpha}{t_0}\Delta\tau \\
&\leq \frac{BK\alpha(1+\varepsilon)(p-1)\left(r^{-}\right)^{1-q}}{t_0}.
\end{aligned} \tag{3.49}
$$

For sufficiently large t_0, inequality (3.45) follows from (3.49) immediately.

It remains to prove (3.46). We have (see (3.38), (3.42))

$$
\begin{aligned}
&\frac{1-\varepsilon}{\alpha} \int_{t_0}^{t_0+\alpha} (p-1)z(\tau)\Delta\tau + (1+\varepsilon)\, X(t_0) + (1+\varepsilon)\, Y(t_0) \\
&= (1-\varepsilon)(p-1)z_{\rm ave}(t_0) + (1+\varepsilon)q^{-p}\Big(\frac{1}{\alpha}\int_{t_0}^{t_0+\alpha} r^{1-q}(\tau)\,\Delta\tau\Big)^{-p/q} \\
&\quad + (1+\varepsilon)\frac{p}{q^{p+1}}\Big(\frac{1}{\alpha}\int_{t_0}^{t_0+\alpha} r^{1-q}(\tau)\,\Delta\tau\Big)^{1-p}.
\end{aligned} \tag{3.50}
$$

Let us assume that

$$
z_{\rm ave}(t_0) = z(t_0) = -\Big(\frac{q}{\alpha}\int_{t_0}^{t_0+\alpha} r^{1-q}(\tau)\,\Delta\tau\Big)^{1-p}.
$$

Then, (3.50) gives

$$
\begin{aligned}
&\lim_{\varepsilon\to 0^+} \frac{1-\varepsilon}{\alpha} \int_{t_0}^{t_0+\alpha} (p-1)z(\tau)\Delta\tau + (1+\varepsilon)\, X(t_0) + (1+\varepsilon)\, Y(t_0) \\
&= -(p-1)\Big(\frac{q}{\alpha}\int_{t_0}^{t_0+\alpha} r^{1-q}(\tau)\,\Delta\tau\Big)^{1-p} + q^{-p}\Big(\frac{1}{\alpha}\int_{t_0}^{t_0+\alpha} r^{1-q}(\tau)\,\Delta\tau\Big)^{-p/q} \\
&\quad + \frac{p}{q^{p+1}}\Big(\frac{1}{\alpha}\int_{t_0}^{t_0+\alpha} r^{1-q}(\tau)\,\Delta\tau\Big)^{1-p} = 0,
\end{aligned} \tag{3.51}
$$

where we use the fact that $(p-1)q^{1-p} = q^{-p} + p/q^{p+1}$ (consider $p = 1+p/q$). Since the considered terms continuously depend on ε, using (3.39), we can see that (3.51) implies (3.46) for large t_0.

Finally, using (3.44), (3.45), (3.46), and (3.47) in (3.43), we have

$$
z^{\Delta}_{\rm ave}(t_0) \leq \frac{1}{t_0}\Big(L - \frac{L}{4} - \frac{L}{4} - \frac{L}{4}\Big) = \frac{L}{4t_0} < 0. \tag{3.52}
$$

Of course, (3.52) means (see (3.38))

$$
z^{\Delta}_{\rm ave}(t_0) = \frac{1}{\alpha}\int_{t_0}^{t_0+\alpha} z^{\Delta}(\tau)\Delta\tau = \frac{z(t_0+\alpha) - z(t_0)}{\alpha} < 0,
$$

i.e., $z(t_0+\alpha) < z(t_0)$. From the above processes (we can replace t_0 by t), we obtain the following implication. If $z(t) = z(t_0)$ for some $t \in (t_0,\infty)_{\mathbb{T}}$, then $z(t+\alpha) < z(t)$ and $z(\tau) < 0$ for all $\tau \in [t, t+\alpha]_{\mathbb{T}}$.

To complete the proof, it suffices to show the existence of $\delta > 0$ (depending only on r, s, and α) such that if $z(t) \in (-\delta - z(t_0), -z(t_0))$ for some $t \in (t_0,\infty)_{\mathbb{T}}$, then $z(t+\alpha) < z(t)$ and $z(\tau) < 0$ for all $\tau \in [t, t+\alpha]_{\mathbb{T}}$. The initial value $z(t_0)$ is not used in the derivatives of (3.44), (3.45), and (3.47). Evidently, (3.46) is valid if we replace (3.33) by a sufficiently small perturbation. Hence, such a number δ exists (see (3.52) and the process above). Its existence implies that $z(t)$ is negative for all $t \in [t_0,\infty)_{\mathbb{T}}$ (consider again (3.35), (3.36)).

Altogether, we have proved that the solution z of the initial value problem given by (3.32) and (3.33) satisfies the inequality $z(t) < 0$ for every $t \in [t_0, \infty)_{\mathbb{T}}$ (where t_0 is sufficiently large). Then, the non-oscillation of (3.1) comes directly from Lemma 2.3. □

Combining the previous two results, we obtain the following main result of this paper about a simple type of equations.

Theorem 3.3. *Let us consider $A, B > 0$ and the equation*

$$\big[a(t)\Phi(y^\Delta)\big]^\Delta + \frac{b(t)}{t^p}\Phi(y^\sigma) = 0, \tag{3.53}$$

where $t \in \mathbb{T}$ is sufficiently large, functions a, b are rd-continuous, positive, and bounded, and function a satisfies $\liminf_{t\to\infty} a(t) > 0$.

(i) *If $q^p B A^{p-1} > 1$ and*

$$\frac{1}{\alpha}\int_t^{t+\alpha} a^{1-q}(\tau)\Delta\tau \geq A, \quad \frac{1}{\alpha}\int_t^{t+\alpha} b(\tau)\Delta\tau \geq B \tag{3.54}$$

for all large t, then (3.53) is oscillatory.

(ii) *If $q^p B A^{p-1} < 1$ and*

$$\frac{1}{\alpha}\int_t^{t+\alpha} a^{1-q}(\tau)\Delta\tau \leq A, \quad \frac{1}{\alpha}\int_t^{t+\alpha} b(\tau)\Delta\tau \leq B \tag{3.55}$$

for all large t, then (3.53) is non-oscillatory.

Proof. The theorem follows from Theorems 3.1 and 3.2, where we put

$$r(t) = a(t) \quad \text{and} \quad s(t) = \frac{\sigma(t)b(t)}{t}$$

for all considered t. For any number $\vartheta > 0$, using the limit

$$\lim_{t\to\infty}\frac{\sigma(t)}{t} = 1,$$

we have

$$\frac{1}{\alpha}\int_t^{t+\alpha} s(\tau)\Delta\tau = \frac{1}{\alpha}\int_t^{t+\alpha}\frac{\sigma(\tau)b(\tau)}{\tau}\Delta\tau \geq B - \vartheta$$

for all large t in case (i) and

$$\frac{1}{\alpha}\int_t^{t+\alpha} s(\tau)\Delta\tau = \frac{1}{\alpha}\int_t^{t+\alpha}\frac{\sigma(\tau)b(\tau)}{\tau}\Delta\tau \leq B + \vartheta$$

for all large t in case (ii). Thus, it suffices to consider $R = A$ and replace S by $S \pm \vartheta$ for a sufficiently small number $\vartheta > 0$ so that $q^p(S-\vartheta)R^{p-1} = q^p(B-\vartheta)A^{p-1} > 1$ in (i) and $q^p(S+\vartheta)R^{p-1} = q^p(B+\vartheta)A^{p-1} < 1$ in (ii). □

4. Corollaries

In this section, we emphasize some equations which are covered by results of Section 3. Especially, we formulate special cases which bring new results. The first direct corollary of Theorem 3.3 is its formulation for linear equations (we point out that Theorem 3.3 is new even for the case $p = 2$).

Corollary 4.1. *Let us consider $A, B > 0$ and the equation*

$$\big[a(t)y^{\Delta}\big]^{\Delta} + \frac{b(t)}{t^2} y^{\sigma} = 0, \tag{4.1}$$

where $t \in \mathbb{T}$ is sufficiently large, functions a, b are rd-continuous, positive, and bounded, and function a satisfies $\liminf_{t\to\infty} a(t) > 0$.

(i) *If $4AB > 1$ and*

$$\frac{1}{\alpha}\int_t^{t+\alpha} \frac{1}{a(\tau)}\Delta\tau \ge A, \quad \frac{1}{\alpha}\int_t^{t+\alpha} b(\tau)\Delta\tau \ge B$$

for all large t, then (4.1) is oscillatory.

(ii) *If $4AB < 1$ and*

$$\frac{1}{\alpha}\int_t^{t+\alpha} \frac{1}{a(\tau)}\Delta\tau \le A, \quad \frac{1}{\alpha}\int_t^{t+\alpha} b(\tau)\Delta\tau \le B$$

for all large t, then (4.1) is non-oscillatory.

Further, we can formulate a direct new corollary also for equations with asymptotically periodic coefficients. Recall that a function is said to be asymptotically periodic if it can be written as a sum of two functions, say f_1 and f_2, such that the function f_1 is periodic and the function f_2 vanishes at infinity, i.e., $\lim_{t\to\infty} f_2(t) = 0$.

Corollary 4.2. *Let us consider the equation*

$$\big[F(t)\Phi(y^{\Delta})\big]^{\Delta} + \frac{G(t)}{t^p}\Phi(y^{\sigma}) = 0, \tag{4.2}$$

where $t \in \mathbb{T}$ is sufficiently large, functions F, G are rd-continuous, positive, and asymptotically periodic. Let $F \equiv f_1 + f_2$ and $G \equiv g_1 + g_2$, where the functions f_1 and g_1 are α-periodic and the functions f_2 and g_2 vanish at infinity. Let us denote

$$A := \frac{1}{\alpha}\int_t^{t+\alpha} f_1^{1-q}(\tau)\Delta\tau, \quad B := \frac{1}{\alpha}\int_t^{t+\alpha} g_1(\tau)\Delta\tau$$

for an arbitrarily given $t \in \mathbb{T}$. Equation (4.2) is oscillatory if $q^p B A^{p-1} > 1$; and it is non-oscillatory if $q^p B A^{p-1} < 1$.

We note that, concerning the studied problem, the most general case analyzed in the literature (see [22]) deals with periodic coefficients. Hence, Corollary 4.2 is new in the linear case as well and we can formulate the following new result (which also follows from Corollary 4.1).

Corollary 4.3. *Let us consider the equation*

$$\big[F(t)y^{\Delta}\big]^{\Delta} + \frac{G(t)}{t^2} y^{\sigma} = 0, \tag{4.3}$$

where $t \in \mathbb{T}$ is sufficiently large, functions F, G are rd-continuous, positive, and asymptotically periodic. Let $F \equiv f_1 + f_2$ and $G \equiv g_1 + g_2$, where the functions f_1 and g_1 are α-periodic and the functions f_2 and g_2 vanish at infinity. Let us denote

$$A := \frac{1}{\alpha}\int_t^{t+\alpha} \frac{\Delta\tau}{f_1(\tau)}, \quad B := \frac{1}{\alpha}\int_t^{t+\alpha} g_1(\tau)\Delta\tau\,.$$

Equation (4.3) is oscillatory if $4AB > 1$; and it is non-oscillatory if $4AB < 1$.

Equations treated in this paper are often considered with the so-called generalized power function in the literature. Especially, for the difference equations ($\mathbb{T} = \mathbb{Z}$), it is well-described in [26, Definition 2.3] as the falling factorial power. To introduce the notion of the generalized power function on time scales (see, e.g., [22]), we recall the definition of the n-th composition of operator ρ (see, e.g., [6]) which reads as

$$\rho^0(t) := t, \quad \rho^1(t) := \rho(t), \quad \rho^2(t) := \rho(\rho(t)), ; \dots, \ \rho^n(t) = \rho(\rho^{n-1}(t)).$$

We remark that, in the case of a time scale bounded from below, i.e., $-\infty < a = \min \mathbb{T}$, we put $\rho^n(a) = a, n \in \mathbb{N} \cup \{0\}$.

Then, the generalized power function with natural or zero exponent is given by

$$t^{(n)} = \begin{cases} t\rho(t) \cdots \rho^{n-1}(t), & n \in \mathbb{N} \setminus \{1\}; \\ t, & n = 1; \\ 1, & n = 0. \end{cases}$$

For the considered real exponent $p > 1$, we use the floor function $\lfloor p \rfloor$ which gives the greatest integer less than or equal to p. The generalized power function is given by

$$t^{(p)} = t^{(\lfloor p \rfloor)} \Big\{ \Big(\rho^{\lfloor p-1 \rfloor}(t)\Big)^{1-p+\lfloor p \rfloor} \Big(\rho^{\lfloor p \rfloor}(t)\Big)^{p-\lfloor p \rfloor} \Big\}^{p-\lfloor p \rfloor}.$$

Of course, the generalized power function recalled above is continuous, increasing in p for large $t \in \mathbb{T}$, and it is asymptotically equivalent to the standard power function t^p, i.e.,

$$\lim_{t\to\infty} \frac{t^{(p)}}{t^p} = 1, \tag{4.4}$$

which is proved in [22, Lemma 2.4].

Therefore, we can reformulate the results to the form common in the literature. For example, Theorem 3.3 implies the following corollary (it suffices to use (4.4)).

Corollary 4.4. *Let us consider $A, B > 0$ and the equation*

$$\big[a(t)\Phi(y^\Delta)\big]^\Delta + \frac{b(t)}{t^{(p)}}\Phi(y^\sigma) = 0, \tag{4.5}$$

where $t \in \mathbb{T}$ is sufficiently large, functions a, b are rd-continuous, positive, and bounded, and function a satisfies $\liminf_{t\to\infty} a(t) > 0$.

- (i) *If $q^p B A^{p-1} > 1$ and (3.54) is valid for all large t, then (4.5) is oscillatory.*
- (ii) *If $q^p B A^{p-1} < 1$ and (3.55) is valid for all large t, then (4.5) is non-oscillatory.*

Next, we can combine our results with comparison theorems to test the non-oscillation of equations which are not covered by the presented results directly. We demonstrate this fact by the combination of the Sturm–Picone comparison theorem (see [36, Theorem 3]) with Theorem 3.3 to decide whether certain equations with the second coefficients changing their sign are non-oscillatory.

Corollary 4.5. *Let us consider (3.53) with rd-continuous and bounded coefficients a, b satisfying $\liminf_{t\to\infty} a(t) > 0$. If $q^p B A^{p-1} < 1$, where*

$$\frac{1}{\alpha}\int_t^{t+\alpha} a^{1-q}(\tau)\Delta\tau \le A, \quad \frac{1}{\alpha}\int_t^{t+\alpha} \max\{0, b(\tau)\}\,\Delta\tau \le B$$

for all large $t \in \mathbb{T}$, then (3.53) is non-oscillatory.

The combination of the well-known Sturm type comparison theorems (see, e.g., [1, 2]) with Theorem 3.3 leads to the following Kneser-type (non-) oscillation criteria.

Corollary 4.6. *Let us consider the equation*

$$\big[\Phi(y^\Delta)\big]^\Delta + f(t)\Phi(y^\sigma) = 0, \tag{4.6}$$

where f is a rd-continuous function.

(i) *If there exist $B > 0$ and a rd-continuous, positive, and bounded function b such that*

$$\liminf_{t\to\infty} \frac{t^p f(t)}{b(t)} > B \geq \frac{\alpha}{q^p}\Big[\int_t^{t+\alpha} b(\tau)\Delta\tau\Big]^{-1}$$

for all large $t \in \mathbb{T}$, then (4.6) is oscillatory.

(ii) *If there exist $B > 0$ and a rd-continuous, positive, and bounded function b such that*

$$\limsup_{t\to\infty} \frac{t^p f(t)}{b(t)} < B \leq \frac{\alpha}{q^p}\Big[\int_t^{t+\alpha} b(\tau)\Delta\tau\Big]^{-1}$$

for all large $t \in \mathbb{T}$, then (4.6) is non-oscillatory.

To formulate the last corollary, we recall the notion of mean values. We say that a function $f : \mathbb{T} \to \mathbb{R}$ has the *mean value*

$$\overline{f} := \lim_{n\to\infty} \frac{1}{n\alpha} \int_t^{t+\alpha n} f(\tau)\,\Delta\tau$$

if the limit exists uniformly with respect to all $t \in \mathbb{T}$. Using this concept, we obtain the last direct consequence of Theorem 3.3 which reads as follows.

Corollary 4.7. *Let us consider the equation*

$$\big[c^{1-p}(t)\Phi(y^\Delta)\big]^\Delta + \frac{d(t)}{t^p}\Phi(y^\sigma) = 0, \tag{4.7}$$

where functions c, d have mean values $\overline{c}, \overline{d}$, functions c^{1-p}, d are rd-continuous, positive, and bounded, and function c satisfies $\liminf_{t\to\infty} c^{1-p}(t) > 0$.

(i) *If $q^p\overline{d} > \overline{c}^{1-p}$, then (4.7) is oscillatory.*

(ii) *If $q^p\overline{d} < \overline{c}^{1-p}$, then (4.7) is non-oscillatory.*

We add that Corollary 4.7 in the case $\mathbb{T} = \mathbb{R}$ is the main result of [15] for a non-negative and bounded coefficient d.

5. Open problems

At the end of our paper, we describe several possible applications of the presented results as well as open problems which seem to be solvable with the help of the results of this paper. We mention that none of the open problems formulated in this section is solved for linear dynamic equations. Hence, similarly to the results presented in the previous sections, the anticipated results for half-linear equations would imply new results for linear equations as their special cases. A big advantage of this approach is also the fact that new results obtained in this way are applicable for both ordinary (linear and half-linear) and partial differential equations on time scales (for some basics, see, e.g., [3]). For a concrete application of the theory of the conditional oscillation in partial differential equations, we refer at least to [15].

Now, we present a list of open problems and possible topics for future research.

(A) The first open problem is to get rid of the requirement on periodicity of the used time scale. Of course, the time scale can be redefined and supplemented to a periodic one in some cases. We point out that the methods applied in this paper use the requirement on periodicity substantially.

(B) The second direction originates from the continuous case, namely from [7, 8, 12, 32]. In those papers, the perturbations of half-linear differential equations which preserve the conditional oscillation are identified. The form of such perturbed equations is

$$\Big[\Big(r_0(t)+\sum_{i=1}^{n}\frac{r_i(t)}{\mathrm{Log}_{(i)}^2(t)}\Big)^{1-p}\Phi(y')\Big]'+\Big(s_0(t)+\sum_{i=1}^{n}\frac{s_i(t)}{\mathrm{Log}_{(i)}^2(t)}\Big)\frac{\Phi(y)}{t^p}=0, \tag{5.1}$$

where

$$\mathrm{Log}_{(k)}\,t=\prod_{i=1}^{k}\log_{(i)}t,\quad \log_{(1)}t=\log t,\quad \log_{(i+1)}t=\log\Big(\log_{(i)}t\Big),\quad i,k\in\mathbb{N}.$$

In addition, using the oscillatory properties of (5.1) with $n=m+1$, it is possible to obtain the behaviour of (5.1) with $n=m$ in the critical case, which is not solvable directly (see also part (C) in this list). To find a dynamic equation similar to (5.1) which is conditionally oscillatory remains an open problem.

(C) In Theorem 3.3, it is seen that the critical case remains unsolved. It may be also considered as the case $\gamma=\Gamma$ in the sense of (1.2). Such a situation is typical also in the continuous and discrete cases (for differential and difference equations). As far as we know, the critical case is solved only for differential equations ($\mathbb{T}=\mathbb{R}$) with (sums of) periodic coefficients (see, e.g., [18, 19]). We should emphasize that solving the critical case of more general equations than those with periodic coefficients is the most likely impossible in general. This conjecture is based on the methods described in [43] which lead to constructions of almost periodic functions such that the resulting equation with almost periodic coefficients obtained from these constructions can be oscillatory or non-oscillatory in the critical case. Note that the methods of such constructions are used in the discrete case as well (see [16, 42]). Therefore, it is not possible to describe the oscillation behaviour of such equations in general (see also [21, Section 5] and [44, Remark 19]).

(D) Once the results are available for half-linear equations, the natural questions involve partial differential equations on one side and non-linear equations on the other side. Regarding the non-linear equations, the possible continuation and application of the results presented in this paper include, e.g., the equations

$$\big[r(t)\Phi_a(y^\Delta)\big]^\Delta+c(t)\Phi_b(y^\sigma)=0, \tag{5.2}$$

$$\big[r(t)(y^\Delta)\big]^\Delta+c(t)g(y^\sigma)=0, \tag{5.3}$$

$$\big[r(t)\Phi(y^\Delta)\big]^\Delta+c(t)g(y^\sigma)=0, \tag{5.4}$$

where Φ_a and Φ_b (in (5.2)) stand for p-Laplacian with $p=a$ and $p=b$, respectively. The function g (in (5.3) and (5.4)) satisfies the sign condition $yg(y)>0$ whenever $y\neq 0$ (see, e.g., [38, 39, 40, 41] for results in the continuous case).

(E) Another direction of research originates from the need to study equations whose potential is asymptotically different from $s(t)/t^p$ and to find the form of the first coefficient which preserves the conditional oscillation. Inspired by the continuous case, we conjecture that such an equation is the equation

$$\big[t^{\alpha} r(t)\Phi(y^{\Delta})\big]^{\Delta} + \frac{s(t)}{t^{p-\alpha}}\Phi(y^{\sigma}) = 0,$$

where $\alpha \in \mathbb{R} \setminus \{p-1\}$. Some partial results are also known in the discrete case. For this point, the main references are [11, 20].

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