Electronic Journal of Differential Equations, Vol. 2005(2005), No. 71, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

ON SYLVESTER OPERATOR EQUATIONS, COMPLETE TRAJECTORIES, REGULAR ADMISSIBILITY, AND STABILITY OF C_0-SEMIGROUPS

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Abstract. We show that the existence of a nontrivial bounded uniformly continuous (BUC) complete trajectory for a C_0-semigroup $T_A(t)$ generated by an operator A in a Banach space X is equivalent to the existence of a solution $\Pi = \delta_0$ to the homogenous operator equation $\Pi S|_{\mathcal{M}} = A\Pi$. Here $S|_{\mathcal{M}}$ generates the shift C_0-group $T_S(t)|_{\mathcal{M}}$ in a closed translation-invariant subspace $\mathcal{M}$ of $BUC(\mathbb{R}, X)$, and δ_0 is the point evaluation at the origin. If, in addition, $\mathcal{M}$ is operator-invariant and $0 \neq \Pi \in \mathcal{L}(\mathcal{M}, X)$ is any solution of $\Pi S|_{\mathcal{M}} = A\Pi$, then all functions $t \to \Pi T_S(t)|_{\mathcal{M}} f$, $f \in \mathcal{M}$, are complete trajectories for $T_A(t)$ in $\mathcal{M}$. We connect these results to the study of regular admissibility of Banach function spaces for $T_A(t)$; among the new results are perturbation theorems for regular admissibility and complete trajectories. Finally, we show how strong stability of a C_0-semigroup can be characterized by the nonexistence of nontrivial bounded complete trajectories for the sun-dual semigroup, and by the surjective solvability of an operator equation $\Pi S|_{\mathcal{M}} = A\Pi$.

1. Introduction

Consider the abstract Cauchy problem

$$\dot{x}(t) = Ax(t), \quad t \geq 0, \quad x(0) = x_0 \in X \tag{1.1}$$

where A generates a C_0-semigroup $T_A(t)$ in some Banach space X. It is well known that a unique mild solution $x(t) = T_A(t)x_0$, $t \geq 0$, of (1.1) always exists. However, sometimes there also exist so-called complete trajectories for $T_A(t)$. A complete trajectory for $T_A(t)$ is a continuous function $x : \mathbb{R} \to X$ such that $x(t) = T_A(t-s)x(s)$ for each $t, s \in \mathbb{R}$ for which $t \geq s$, and $x(0) = x_0$. Such a trajectory is nontrivial if it is not identically zero. Bounded nontrivial complete trajectories for $T_A(t)$ are important e.g. in the study of equations (1.1) on the whole real line [18, 19]; Vu has studied their existence and construction in [19]. His main

2000 *Mathematics Subject Classification.* 47D03.
Key words and phrases. Sylvester operator equation; regularly admissible space; complete nontrivial trajectory; C_0-semigroup; exponential stability; strong stability; exponential dichotomy.

Submitted April 17, 2005. Published June 30, 2005.

result asserts that if $T_A(t)$ is uniformly bounded and sun-reflexive, and its sun-dual semigroup $T_A^{\odot}(t)$ (see Subsection 1.1) is not strongly stable[1], then there exist nontrivial bounded complete trajectories provided one of the following conditions holds: $i\mathbb{R} \nsubseteq \sigma(A)$ or $\operatorname{ran}(T_A^{\odot}(t_0))$ is dense in $X^{\odot}$ for some $t_0 > 0$. Vu also shows in [19] that if the intersection of the approximate point spectrum of A and the imaginary axis is countable, then every bounded uniformly continuous complete trajectory for $T_A(t)$ is almost periodic provided X does not contain an isomorphic copy of c_0, the Banach space of sequences convergent to 0, or the trajectory itself is weakly compact.

A related problem for the inhomogenous abstract Cauchy problem

$$\dot{x}(t) = Ax(t) + f(t), \quad t \in \mathbb{R} \tag{1.2}$$

in X is the following [15, 21]. Let $\mathcal{M}$ be a closed translation-invariant operator-invariant (i.e. CTO, see Definition 1.1) subspace of $BUC(\mathbb{R}, X)$, the space of bounded uniformly continuous X-valued functions. We say that $\mathcal{M}$ is regularly admissible for $T_A(t)$ if for each $f \in \mathcal{M}$ there exists a unique mild solution $x \in \mathcal{M}$ of (1.2), i.e. for which

$$x(t) = T_A(t-s)x(s) + \int_s^t T_A(t-\tau)f(\tau)d\tau \quad \forall t \geq s, \quad t, s \in \mathbb{R} \tag{1.3}$$

Vu and Schüler [21] showed, among other things, that $\mathcal{M}$ is regularly admissible for $T_A(t)$ if and only if the operator equation $\Pi S|_{\mathcal{M}} = A\Pi + \delta_0$, where $S|_{\mathcal{M}} = \frac{d}{dx}|_{\mathcal{M}}$ and δ_0 is the point evaluation operator in $\mathcal{M}$ centered at the origin, has a unique solution $\Pi \in \mathcal{L}(\mathcal{M}, X)$ (see Section 2).

The main purpose of the present article is to interconnect the results in [19] and [21]. To avoid repetition we shall assume the reader to have access to these papers. Our main results are the following. We show that the existence of a nontrivial complete trajectory $x \in BUC(\mathbb{R}, X)$ for $T_A(t)$ is equivalent to the existence of a solution $\Pi = \delta_0$ to the homogenous operator equation $\Pi S|_{\mathcal{M}} = A\Pi$ for some closed translation-invariant subspace $\mathcal{M}$ of $BUC(\mathbb{R}, X)$. If, in addition, $\mathcal{M}$ is operator-invariant and $0 \neq \Pi \in \mathcal{L}(\mathcal{M}, X)$ is any solution of $\Pi S|_{\mathcal{M}} = A\Pi$, then all functions $t \to \Pi T_S(t)|_{\mathcal{M}} f$, $f \in \mathcal{M}$ are complete trajectories for $T_A(t)$ in $\mathcal{M}$. There are three remarkable features in these results. First of all, we do not need to assume e.g. the uniform boundedness of $T_A(t)$ or restrict $\sigma(A) \cap i\mathbb{R}$ in any explicit way to obtain nontrivial bounded complete trajectories. Secondly, the complete trajectories are known to be in $\mathcal{M}$ – hence we can conclude more than just boundedness of the trajectory. For example $\mathcal{M}$ could be the space $AP(\mathbb{R}, X)$ of X-valued almost periodic functions. Finally, these results also provide a way to construct bounded complete trajectories for $T_A(t)$ via the solution operators Π.

By combining our main results with those in [19, 21] we obtain several useful corollaries. For example, we immediately see that if $\mathcal{M}$ is regularly admissible for $T_A(t)$, then there cannot be complete nontrivial trajectories for $T_A(t)$ in $\mathcal{M}$. Since all CTO subspaces $\mathcal{M} \subset BUC(\mathbb{R}, X)$ are regularly admissible for an exponentially dichotomous semigroup $T_A(t)$ [21], exponentially dichotomous C_0-semigroups cannot have bounded uniformly continuous complete trajectories. Consequently the same is true for exponentially stable C_0-semigroups.

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[1]A C_0-semigroup $T(t)$ in a Banach space Z is strongly stable if $\lim_{t\to\infty} T(t)z = 0$ for every $z \in Z$

In Section 4 we shall show that the existence of nontrivial bounded complete trajectories for $T_A(t)$ is a fragile property; arbitrarily small bounded additive perturbations to the generator A may destroy it. On the other hand, we shall show that the *nonexistence* of such trajectories may be a stable property even under certain unbounded additive perturbations to A. We also show that regular admissibility of $\mathcal{M}$ for $T_A(t)$ may sustain some unbounded additive perturbations to A. Hence we have another situation in which the nonexistence of bounded complete trajectories in $\mathcal{M}$ is not affected by perturbations to A.

We conclude this article with some new characterizations for strong stability of a C_0-semigroup $T_A(t)$. We shall show that if $T_A(t)$ is uniformly bounded and $\sigma_A(A) \cap i\mathbb{R}$ is countable, then $T_A(t)$ is *not* strongly stable if and only if the sun-dual semigroup $T_A^{\odot}(t)$ has a nontrivial bounded complete trajectory. We also show that strong stability of $T_A(t)$ is equivalent to the existence of a surjective solution to the operator equation $\Pi S|_{\mathcal{M}} = A\Pi$ for a closed translation-invariant subspace $\mathcal{M} \subset C_0(\mathbb{R}_+, X)$.

1.1. **Preliminaries.** As in the above, let X be a Banach space and consider a C_0-semigroup $T_A(t)$ in X generated by A. The spectrum, point spectrum, approximate point spectrum and resolvent set of A are denoted by $\sigma(A)$, $\sigma_P(A)$, $\sigma_A(A)$ and $\rho(A)$ respectively. A^* denotes the adjoint operator of A and for every $\lambda \in \rho(A)$ we denote by $R(\lambda, A)$ the resolvent operator of A. A linear operator $\Delta_A : \mathcal{D}(\Delta_A) \subset X \to X$ is called A-bounded if $\mathcal{D}(A) \subset \mathcal{D}(\Delta_A)$ and for some nonnegative constants a, b we have

$$\|\Delta_A x\| \le a\|x\| + b\|Ax\| \quad \forall x \in \mathcal{D}(A) \tag{1.4}$$

If the Banach space X is not reflexive, then the adjoint semigroup $T_A^*(t)$ is not necessarily strongly continuous. However, the subspace

$$X^{\odot} = \{\phi \in X^* \mid T_A^*(t)\phi \text{ is strongly continuous}\} \tag{1.5}$$

is closed in X^* and invariant for $T_A^*(t)$. Additionally, $X^{\odot} = \overline{\mathcal{D}(A^*)}$ and the restriction $T_A^*(t)|_{X^{\odot}}$ defines a strongly continuous semigroup in $X^{\odot}$, the so-called sun-dual semigroup $T_A^{\odot}(t)$ [9, 19].

We denote the Banach space (with sup-norm) of bounded uniformly continuous functions $t \to X$ by $BUC(\mathbb{R}, X)$. The shift operators $T_S(t)$, $t \in \mathbb{R}$, are defined for each $f \in BUC(\mathbb{R}, X)$ as $T_S(t)f = f(\cdot + t)$. It is clear that $T_S(t)$ constitutes a strongly continuous group in $BUC(\mathbb{R}, X)$. Its infinitesimal generator is the differential operator $S = \frac{d}{dx}$ with a suitable domain of definition. Clearly the restrictions $T_S(t)|_{\mathcal{M}}$ of the shift group to closed (in the sup-norm) translation-invariant subspaces $\mathcal{M} \subset BUC(\mathbb{R}, X)$ are also strongly continuous. The infinitesimal generator of such a restriction $T_S(t)|_{\mathcal{M}}$ is denoted by $S|_{\mathcal{M}}$. Of special interest are the so-called CTO (closed translation-invariant operator-invariant) subspaces $\mathcal{M}$ of $BUC(\mathbb{R}, X)$:

Definition 1.1. *A* sup-*norm closed translation-invariant function space* $\mathcal{M} \subset BUC(\mathbb{R}, X)$ *is operator-invariant if for each* $C \in \mathcal{L}(\mathcal{M}, X)$ *and every* $f \in \mathcal{M}$ *the function* $t \to CT_S(t)f$ *is in* $\mathcal{M}$.

Several interesting function spaces are CTO. For example: Continuous p-periodic X-valued functions, almost periodic functions $\mathbb{R} \to X$ and functions in $BUC(\mathbb{R}, X)$ whose Carleman spectrum is contained in a given closed subset Λ of $i\mathbb{R}$, the imaginary axis. Recall that almost periodic functions are those which can be uniformly approximated by trigonometric polynomials [2], and that the Carleman spectrum

$sp(f)$ of a function $f \in BUC(\mathbb{R}, X)$ is defined as the set of singularities of its Carleman transform

$$\widetilde{f}(\lambda) = \begin{cases} \int_0^\infty e^{-\lambda t} f(t)dt, & \Re(\lambda) > 0 \\ -\int_{-\infty}^0 e^{-\lambda t} f(t)dt, & \Re(\lambda) < 0 \end{cases} \tag{1.6}$$

on $i\mathbb{R}$. The reader is referred to [2, 11, 21] for more details.

In this article we shall use the well known fact that for every closed translation-invariant subspace $\mathcal{M} \subset BUC(\mathbb{R}, X)$ there exists a sequence $(\mathcal{M}_n)_{n\in\mathbb{N}} \subset \mathcal{M}$ of closed translation-invariant subspaces with the following properties [16, 21]:

(1) $\mathcal{M}_n \subset \mathcal{M}_{n+1}$ for every $n \in \mathbb{N}$.
(2) $S_n = S|_{\mathcal{M}_n}$ is a bounded operator for every $n \in \mathbb{N}$.
(3) $\sigma(S_n) \subset \sigma(S|_{\mathcal{M}})$ for every $n \in \mathbb{N}$.
(4) $\cup_{n\in\mathbb{N}}\mathcal{M}_n$ is dense in $\mathcal{M}$.

2. Mild and Strong Solutions of $\Pi S|_{\mathcal{M}} = A\Pi + \Delta$

Let $\mathcal{M} \subset BUC(\mathbb{R}, X)$ be a closed translation-invariant function space and let $\Delta \in \mathcal{L}(\mathcal{M}, X)$. As before, we assume that A generates the C_0-semigroup $T_A(t)$ in X. In this section we shall study the operator equation

$$\Pi S|_{\mathcal{M}} = A\Pi + \Delta \tag{2.1}$$

which will play a prominent role throughout this article. Equation (2.1) is a special instance of general linear Sylvester type operator equations. Such equations have a long history: For classical finite-dimensional results the reader is referred to [10] and to the excellent survey article [8]. The treatment of Bhatia and Rosenthal [8] actually also covers the case of bounded linear operators in infinite-dimensional spaces. Many of these results can be generalized for unbounded operators which may or may not generate C_0-semigroups. Such results can be found e.g. in [3, 14, 20, 21].

Vu and Schüler [21] concentrated on the unique solvability of (2.1) for each Δ. They showed that it is equivalent to the regular admissibility of $\mathcal{M}$ for $T_A(t)$. It turns out, however, that also nonunique solutions of (2.1) have importance. We shall see in the next section that the existence of a nontrivial solution $\Pi = \delta_0$ to the homogenous equation $\Pi S|_{\mathcal{M}} = A\Pi$ — which implies nonuniqueness of solutions of (2.1) — is equivalent to the existence of nontrivial bounded uniformly continuous complete trajectories for $T_A(t)$. In order to establish this result we consider two types of solutions for (2.1):

Definition 2.1. An operator $\Pi \in \mathcal{L}(\mathcal{M}, X)$ is called a strong solution of (2.1) if $\Pi(\mathcal{D}(S|_{\mathcal{M}})) \subset \mathcal{D}(A)$ and $\Pi S|_{\mathcal{M}} f = A\Pi f + \Delta f$ for every $f \in \mathcal{D}(S|_{\mathcal{M}})$.

Definition 2.2. An operator $\Pi \in \mathcal{L}(\mathcal{M}, X)$ is called a mild solution of (2.1) if

$$\Pi T_S(t)|_{\mathcal{M}} f = T_A(t)\Pi f + \int_0^t T_A(t-s)\Delta T_S(s)|_{\mathcal{M}} f ds \tag{2.2}$$

for every $f \in \mathcal{M}$ and every $t \geq 0$.

The main result of this section shows that mild and strong solutions of (2.1) coincide. Hence we may refer to them as just solutions of (2.1).

Theorem 2.3. *An operator* $\Pi \in \mathcal{L}(\mathcal{M}, X)$ *is a mild solution of* (2.1) *if and only if it is a strong solution of* (2.1).

Proof. Assume first that $\Pi \in \mathcal{L}(\mathcal{M}, X)$ is a strong solution of the (2.1). Let $f \in \mathcal{D}(S|_{\mathcal{M}})$ be arbitrary. Then since $\Pi(\mathcal{D}(S|_{\mathcal{M}})) \subset \mathcal{D}(A)$, we have for every $t \geq 0$ that

$$\begin{aligned} \Pi T_S(t)|_{\mathcal{M}} f - T_A(t)\Pi f &= \big|_{\tau=0}^{t} T_A(t-\tau)\Pi T_S(\tau)|_{\mathcal{M}} f d\tau && (2.3)\\ &= \int_0^t \frac{d}{d\tau} T_A(t-\tau)\Pi T_S(\tau)|_{\mathcal{M}} f d\tau && (2.4)\\ &= \int_0^t T_A(t-\tau)[\Pi S|_{\mathcal{M}} - A\Pi] T_S(\tau)|_{\mathcal{M}} f d\tau && (2.5)\\ &= \int_0^t T_A(t-\tau)\Delta T_S(\tau)|_{\mathcal{M}} f d\tau && (2.6) \end{aligned}$$

because $T_S(\tau)|_{\mathcal{M}} f \in \mathcal{D}(S|_{\mathcal{M}})$ for every $\tau \geq 0$. Since $\mathcal{D}(S|_{\mathcal{M}})$ is dense in $\mathcal{M}$, we must have that

$$\Pi T_S(t)|_{\mathcal{M}} f = T_A(t)\Pi f + \int_0^t T_A(t-\tau)\Delta T_S(\tau)|_{\mathcal{M}} f d\tau \quad \forall f \in \mathcal{M} \quad \forall t \geq 0 \tag{2.7}$$

In other words Π is a mild solution of (2.1).

Assume then that $\Pi \in \mathcal{L}(\mathcal{M}, X)$ is a mild solution of (2.1). We first show that $\Pi(\mathcal{D}(S|_{\mathcal{M}})) \subset \mathcal{D}(A)$. Let $f \in \mathcal{D}(S|_{\mathcal{M}})$. Then for every $h > 0$

$$\begin{aligned} \frac{T_A(h)\Pi f - \Pi f}{h} &= \frac{T_A(h)\Pi f - \Pi T_S(h)|_{\mathcal{M}} f}{h} + \frac{\Pi T_S(h)|_{\mathcal{M}} f - \Pi f}{h} && (2.8)\\ &= -\frac{\int_0^h T_A(h-\tau)\Delta T_S(\tau)|_{\mathcal{M}} f d\tau}{h} + \frac{\Pi T_S(h)|_{\mathcal{M}} f - \Pi f}{h} && (2.9) \end{aligned}$$

which by the boundedness of Π shows that $\Pi f \in \mathcal{D}(A)$; also observe that the function $t \to \Delta T_S(t)|_{\mathcal{M}} f$ is continuously differentiable so that the convolution in (2.9) is differentiable. Moreover, we see that $A\Pi f = -\Delta f + \Pi S|_{\mathcal{M}} f$ for each $f \in \mathcal{D}(S|_{\mathcal{M}})$. Consequently Π is a strong solution of (2.1). □

Remark 2.4. As mentioned in the introductory section, the special case $\Delta = \delta_0 \in \mathcal{L}(\mathcal{M}, X)$ has turned out to be particularly important in the qualitative theory of differential equations. Theorem 2.3 immediately reveals why this is so. Clearly $f(t) = \delta_0 T_S(t)|_{\mathcal{M}} f$ for every $f \in \mathcal{M}$ and $t \in \mathbb{R}$ and hence if $\Pi \in \mathcal{L}(\mathcal{M}, X)$ is a solution of the operator equation $\Pi S|_{\mathcal{M}} = A\Pi + \delta_0$, then (2.2) reads

$$\Pi T_S(t)|_{\mathcal{M}} f = T_A(t)\Pi f + \int_0^t T_A(t-s) f(s) ds, \quad t \geq 0 \tag{2.10}$$

so that for $x(0) = \Pi f$ the right hand side of (2.10) is the mild solution of the inhomogenous differential equation $\dot{x}(t) = Ax(t) + f(t)$, $t \geq 0$. If in addition, $\mathcal{M}$ is a CTO subspace of $BUC(\mathbb{R}, X)$, then this mild solution $t \to \Pi T_S(t)|_{\mathcal{M}} f$ is in $\mathcal{M}$ for every $f \in \mathcal{M}$. Consequently we may deduce e.g. the existence of periodic mild solutions from solvability of the operator equation $\Pi S|_{\mathcal{M}} = A\Pi + \delta_0$ in a suitable space $\mathcal{M}$. We shall not pursue this discussion any further; the interested reader is referred to [12, 21] for a related discussion.

The operator equation (2.1) has also been studied as an operator equation $\tau_{A,S|_{\mathcal{M}}}\Pi = \Delta$ in the literature [3]. Here $\tau_{A,S|_{\mathcal{M}}}$ is an (unbounded) operator on

$\mathcal{L}(\mathcal{M}, X)$ defined as follows.

$$\mathcal{D}(\tau_{A,S|_\mathcal{M}}) = \big\{X \in \mathcal{L}(\mathcal{M}, X) : X(\mathcal{D}(S|_\mathcal{M})) \subset \mathcal{D}(A),\ \exists Y \in \mathcal{L}(\mathcal{M}, X) : \\ Yu = XS|_\mathcal{M}u - AXu\ \forall u \in \mathcal{D}(S|_\mathcal{M})\big\} \tag{2.11a}$$

$$\tau_{A,S|_\mathcal{M}} X = Y \tag{2.11b}$$

It can be shown that $\tau_{A,S|_\mathcal{M}}$ is a closed operator on $\mathcal{L}(\mathcal{M}, X)$ [3]. The following result is then evident.

Proposition 2.5. *Equation* (2.1) *has a unique solution for every* $\Delta \in \mathcal{L}(\mathcal{M}, X)$ *if and only if* $0 \in \rho(\tau_{A,S|_\mathcal{M}})$. *The homogenous equation* $\Pi S|_\mathcal{M} = A\Pi$ *has a nontrivial solution if and only if* $0 \in \sigma_P(\tau_{A,S|_\mathcal{M}})$.

Proposition 2.5 is particularly useful if $T_A(t)$ is a holomorphic semigroup or if $S|_\mathcal{M}$ is bounded. By the results of Arendt, Räbiger and Sourour [3], in both cases $\sigma(\tau_{A,S|_\mathcal{M}}) = \sigma(A) + \sigma(S|_\mathcal{M})$. We shall, however, use Proposition 2.5 in a different context in Section 4: We make use of the well known fact that bounded invertibility of a closed operator is preserved under small (but possibly unbounded) additive perturbations.

3. Complete Trajectories, Regular Admissibility and $\Pi S|_\mathcal{M} = A\Pi$

The main results of this article are Theorem 3.1 and Theorem 3.3 below. They connect the existence of nontrivial bounded uniformly continuous complete trajectories for $T_A(t)$ to the nonunique solvability of the homogenous operator equation $\Pi S|_\mathcal{M} = A\Pi$. Consequently they provide the link between the articles [19] and [21] mentioned in the introductory section.

Theorem 3.1. *Let A generate a C_0-semigroup $T_A(t)$ in X. Then the following are equivalent.*

(1) *There exists a nontrivial bounded uniformly continuous complete trajectory $x(t)$ for $T_A(t)$.*
(2) *There exists a nontrivial closed translation-invariant subspace $\mathcal{M}$ of $BUC(\mathbb{R}, X)$ in which δ_0 solves the operator equation $\Pi S|_\mathcal{M} = A\Pi$.*
(3) *There exists a nontrivial closed translation-invariant subspace $\mathcal{M}$ of $BUC(\mathbb{R}, X)$ for which every $x \in \mathcal{M}$ is a bounded uniformly continuous complete trajectory for $T_A(t)$.*

Proof. Since by Theorem 2.3 mild and strong solutions of the operator equation (2.1) coincide, we may restrict our attention to mild solutions. We show $1 \implies 2 \implies 3 \implies 1$.

$1 \implies 2$: Assume that $x \in BUC(\mathbb{R}, X)$ is a nontrivial bounded complete trajectory for $T_A(t)$. Let $\mathcal{M} = \overline{\text{span}}\{x(\cdot + t) \mid t \in \mathbb{R}\}$ where closure is taken in the sup-norm. Then $\mathcal{M} \neq 0$ is a closed translation invariant subspace of $BUC(\mathbb{R}, X)$ and clearly $\delta_0 \in \mathcal{L}(\mathcal{M}, X)$. Moreover $x(t) = \delta_0 T_S(t)|_\mathcal{M} x$ for each $t \in \mathbb{R}$. Furthermore, for any $\tau \geq 0$ and $s \in \mathbb{R}$ we have

$$x(\tau + s) = \delta_0 T_S(\tau)|_\mathcal{M} T_S(s)|_\mathcal{M} x = T_A(\tau)x(s) = T_A(\tau)\delta_0 T_S(s)|_\mathcal{M} x \tag{3.1}$$

since x is a complete trajectory for $T_A(t)$. This shows that $\delta_0 T_S(\tau)|_\mathcal{M} x(\cdot + s) = T_A(\tau)\delta_0 x(\cdot + s)$ for each $\tau \geq 0$ and $s \in \mathbb{R}$ because $T_S(s)|_\mathcal{M} x = x(\cdot + s)$. In other words δ_0 is a mild solution of the operator equation $\Pi S|_\mathcal{M} = A\Pi$ in the set $\{x(\cdot + s) \mid s \in \mathbb{R}\}$. Upon extensions by linearity and continuity we immediately have that for $\mathcal{M}$ as in the above, the equation $\Pi S|_{\mathcal{M}} = A\Pi$ has a nontrivial mild solution $\Pi = \delta_0$.

$2 \implies 3$: Assume that the homogenous equation $\Pi S|_{\mathcal{M}} = A\Pi$ has a mild solution $\delta_0 \in \mathcal{L}(\mathcal{M}, X)$. Let $f \in \mathcal{M}$. Then $f(t) = \delta_0 T_S(t)|_{\mathcal{M}} f$ for every $t \in \mathbb{R}$. Furthermore for every $t, s \in \mathbb{R}$ such that $t \geq s$ we have

$$\begin{aligned} T_A(t-s)f(s) &= T_A(t-s)\delta_0 T_S(s)|_{\mathcal{M}} f \\ &= \delta_0 T_S(t-s)|_{\mathcal{M}} T_S(s)|_{\mathcal{M}} f \\ &= \delta_0 T_S(t)|_{\mathcal{M}} f = f(t) \end{aligned}$$

This shows that every $f \in \mathcal{M}$ is a complete nontrivial trajectory for $T_A(t)$.

$3 \implies 1$: This is trivial.

□

We state the following corollary to emphasize that in parts 2 and 3 of Theorem 3.1 the closed translation invariant spaces are equal.

Corollary 3.2. *Let $T_A(t)$ be a C_0-semigroup in X generated by A, and let $\mathcal{M} \subset BUC(\mathbb{R}, X)$ be a closed and translation-invariant subspace. Then every $x \in \mathcal{M}$ is a complete trajectory for $T_A(t)$ if and only if δ_0 is a solution of the operator equation $\Pi S|_{\mathcal{M}} = A\Pi$.*

Proof. Assume that every $x \in \mathcal{M}$ is a bounded complete trajectory for $T_A(t)$. Then for any $\tau \geq 0$ and $s \in \mathbb{R}$ we have $x(\tau + s) = \delta_0 T_S(\tau)|_{\mathcal{M}} T_S(s)|_{\mathcal{M}} x = T_A(\tau)x(s) = T_A(\tau)\delta_0 T_S(s)|_{\mathcal{M}} x$ for each $x \in \mathcal{M}$, because every $x \in \mathcal{M}$ is a complete trajectory for $T_A(t)$. Consequently δ_0 is a mild solution of the operator equation $\Pi S|_{\mathcal{M}} = A\Pi$ in the set $\{x(\cdot + s) \mid s \in \mathbb{R}\}$ for each $x \in \mathcal{M}$. Since $\mathcal{M}$ is translation-invariant, we have $\mathcal{M} = \cup_{x \in \mathcal{M}} \{x(\cdot + s) \mid s \in \mathbb{R}\}$. This shows that δ_0 is a mild solution of the operator equation $\Pi S|_{\mathcal{M}} = A\Pi$. The converse claim is contained in the proof of Theorem 3.1. □

In the above results we assumed that $\mathcal{M}$ is a closed and translation-invariant subspace of $BUC(\mathbb{R}, X)$. If $\mathcal{M}$ is in addition CTO, then also other nontrivial solutions of the homogenous operator equation $\Pi S|_{\mathcal{M}} = A\Pi$ yield nontrivial bounded complete trajectories for $T_A(t)$:

Theorem 3.3. *Let $T_A(t)$ be a C_0-semigroup in X generated by A. Then the following assertions are equivalent for a given CTO space $0 \neq \mathcal{M} \subset BUC(\mathbb{R}, X)$.*

1. *There exists a nonzero operator $\Pi \in \mathcal{L}(\mathcal{M}, X)$ such that for every $f \in \mathcal{M}$, the function $t \to \Pi T_S(t)|_{\mathcal{M}} f$ is a complete trajectory for $T_A(t)$ in $\mathcal{M}$.*
2. *The homogenous operator equation $\Pi S|_{\mathcal{M}} = A\Pi$ has a nontrivial solution $\Pi \in \mathcal{L}(\mathcal{M}, X)$.*
3. *There exists an operator $\Delta \in \mathcal{L}(\mathcal{M}, X)$ such that the operator equation $\Pi S|_{\mathcal{M}} = A\Pi + \Delta$ has at least two distinct solutions.*
4. *The operator $\tau_{A,S|_{\mathcal{M}}}$ defined in (2.11) has 0 as its eigenvalue.*

Proof. We show $1 \iff 2 \iff 3$ and $2 \iff 4$:

$1 \iff 2$: First assume that for every $f \in \mathcal{M}$ the functions $t \to x_f(t) = \Pi T_S(t)|_{\mathcal{M}} f$ are complete trajectories for $T_A(t)$ in $\mathcal{M}$. Hence for each $f \in \mathcal{M}$ and $\tau \geq 0$

and $s \in \mathbb{R}$ we have

$$\begin{aligned} x_f(\tau + s) &= \Pi T_S(\tau + s)|_{\mathcal{M}} f \\ &= \Pi T_S(\tau)|_{\mathcal{M}} T_S(s)|_{\mathcal{M}} f \\ &= T_A(\tau) x_f(s) \\ &= T_A(\tau) \Pi T_S(s)|_{\mathcal{M}} f \end{aligned}$$

This shows that $\Pi T_S(\tau)|_{\mathcal{M}} f(\cdot + s) = T_A(\tau)\Pi f(\cdot + s)$ for each $f \in \mathcal{M}$, $\tau \geq 0$ and $s \in \mathbb{R}$. As we let $s = 0$ we see that Π satisfies the operator equation $\Pi S|_{\mathcal{M}} = A\Pi$.

Conversely assume that the operator equation $\Pi S|_{\mathcal{M}} = A\Pi$ has a nonzero mild solution $\Pi \in \mathcal{L}(\mathcal{M}, X)$. Let $f \in \mathcal{M}$ and define the function $x_f : \mathbb{R} \to X$ such that $x(t) = \Pi T_S(t)|_{\mathcal{M}} f$ for each $t \in \mathbb{R}$. Since $\mathcal{M}$ is CTO, $x_f \in \mathcal{M}$. Furthermore for every $t, s \in \mathbb{R}$ such that $t \geq s$ we have

$$\begin{aligned} T_A(t-s) x_f(s) &= T_A(t-s)\Pi T_S(s)|_{\mathcal{M}} f \\ &= \Pi T_S(t-s)|_{\mathcal{M}} T_S(s)|_{\mathcal{M}} f \\ &= \Pi T_S(t)|_{\mathcal{M}} f = x_f(t) \end{aligned}$$

because $T_S(t)|_{\mathcal{M}} f = f(\cdot + t) \in \mathcal{M}$ for each $t \in \mathbb{R}$. This shows that for every $f \in \mathcal{M}$ the function x_f is a complete nontrivial trajectory for $T_A(t)$ in $\mathcal{M}$.

$2 \Longleftrightarrow 3$: This is trivial.

$2 \Longleftrightarrow 4$: This is contained in Proposition 2.5.

□

Remark 3.4. Vu [19] studied bounded uniformly continuous and almost periodic complete nontrivial trajectories for $T_A(t)$. Theorem 3.1 and Theorem 3.3 provide more flexibility. For example, in Theorem 3.3 one may look for p-periodic continuous complete trajectories or complete trajectories $x \in BUC(\mathbb{R}, X)$ such that the Carleman spectrum $sp(x)$ of x is contained in some closed set $\Lambda \subset i\mathbb{R}$.

Remark 3.5. Theorem 3.3 also provides a way to construct nontrivial complete trajectories in $\mathcal{M} \subset BUC(\mathbb{R}, X)$ for $T_A(t)$ via nontrivial solutions of the homogenous operator equation $\Pi S|_{\mathcal{M}} = A\Pi$.

The following result is of fundamental importance, since it provides a simple necessary condition for the existence of a nontrivial bounded complete trajectory for $T_A(t)$, and since this condition allows us to combine our results with the regular admissibility theory of Vu and Schüler [21]. Because of its importance we choose to give two separate proofs for this result.

Theorem 3.6. *Let $\mathcal{M}$ be a nontrivial closed translation-invariant subspace of $BUC(\mathbb{R}, X)$ and assume that A generates a C_0-semigroup $T_A(t)$ in X. If $\sigma(S|_{\mathcal{M}}) \cap \sigma(A) = \emptyset$, then there are no nontrivial complete trajectories for $T_A(t)$ in $\mathcal{M}$.*

Proof 1. Assume, conversely, that there exists a nontrivial complete trajectory x for $T_A(t)$ in $\mathcal{M}$. Then by Proposition 3.5 in [19] $sp(x) = \sigma(S_x)$ where S_x is the restriction of $S|_{\mathcal{M}}$ to the space $\overline{\operatorname{span}}\{x(\cdot + t) \mid t \in \mathbb{R}\}$. Consequently $sp(x) \subset \sigma(S|_{\mathcal{M}})$, and $sp(x) \cap \sigma(A) = \emptyset$. But by Proposition 3.7 in [19] $sp(x) \subset \sigma_A(A)$ which implies $sp(x) = \emptyset$. According to Wiener's Tauberian Theorem [19] this is possible only if x is identically zero — a contradiction. □

Proof 2. Assume again, conversely, that there exists a nontrivial complete trajectory x for $T_A(t)$ in $\mathcal{M}$. By Theorem 3.1 there exists a nontrivial closed translation-invariant subspace $\mathcal{N} \subset \mathcal{M}$ in which the operator equation $\Pi S|_{\mathcal{N}} = A\Pi$ has a nontrivial solution. Then by a result stated in Subsection 1.1 there exists another nontrivial closed translation-invariant subspace $\mathcal{N}_0 \subset \mathcal{N}$ in which the restriction $S|_{\mathcal{N}_0}$ is a nonzero bounded operator. Moreover the operator equation $\Pi S|_{\mathcal{N}_0} = A\Pi$ also has a nontrivial solution. But this is impossible since $\sigma(S|_{\mathcal{N}_0}) \cap \sigma(A) \subset \sigma(S|_{\mathcal{M}}) \cap \sigma(A) = \emptyset$ and the boundedness of $S|_{\mathcal{N}_0}$ imply that the only solution of $\Pi S|_{\mathcal{N}_0} = A\Pi$ is the zero operator (see Section 2 in [21]). □

Throughout the following corollaries A generates a C_0-semigroup $T_A(t)$ in X.

Corollary 3.7. *If a given CTO space $\mathcal{M} \subset BUC(\mathbb{R}, X)$ is regularly admissible for $T_A(t)$, then there cannot be complete nontrivial trajectories for $T_A(t)$ in $\mathcal{M}$.*

Proof. By Corollary 3.2 in [21] we have $\sigma(S|_{\mathcal{M}}) \cap \sigma(A) = \emptyset$. By Theorem 3.6 there cannot be complete nontrivial trajectories in $\mathcal{M}$. □

Corollary 3.8. *Let $\mathcal{M}$ be a CTO subspace of $BUC(\mathbb{R}, X)$ and suppose that $\sigma(T_A(1)) \cap \sigma(T_S(1)|_{\mathcal{M}}) = \emptyset$. Then there cannot be complete nontrivial trajectories for $T_A(t)$ in $\mathcal{M}$.*

Proof. By Corollary 2.4 and Theorem 3.1 in [21] $\mathcal{M}$ is regularly admissible for $T_A(t)$. By Corollary 3.7 there cannot be complete nontrivial trajectories for $T_A(t)$ in $\mathcal{M}$. □

Corollary 3.9. *Assume that there are no complete trajectories for $T_A(t)$ in a CTO subspace $\mathcal{M}$ of $BUC(\mathbb{R}, X)$ and that the operator equation $\Pi S|_{\mathcal{M}} = A\Pi + \delta_0$ has a solution $\Pi \in \mathcal{L}(\mathcal{M}, X)$. Then $\mathcal{M}$ is regularly admissible.*

Proof. By Theorem 3.3, Π must be the unique solution of the operator equation $\Pi S|_{\mathcal{M}} = A\Pi + \delta_0$. The result follows by Theorem 3.1 in [21]. □

Recall that $T_A(t)$ is exponentially dichotomous if there exists a bounded projection operator P on X and positive constants M, ω such that

1. $PT_A(t) = T_A(t)P$ for all $t \geq 0$.
2. $\|T_A(t)x_0\| \leq Me^{-\omega t}\|x_0\|$ for all $x_0 \in \operatorname{ran}(P)$ and all $t \geq 0$.
3. The restriction $T_A(t)|_{\ker(P)}$ extends to a C_0-group and $\|T_A(-t)|_{\ker(P)}x_0\| \leq Me^{-\omega t}\|x_0\|$ for all $x_0 \in \ker(P)$ and all $t \geq 0$.

Clearly if $T_A(t)$ is exponentially stable, then it is also exponentially dichotomous. Vu ([19], Example 2.7) showed that there are no complete bounded trajectories for the diffusion semigroup on $C_0(\mathbb{R})$. The following result implies that the same is in fact true for all exponentially stable semigroups.

Corollary 3.10. *Let $T_A(t)$ be exponentially dichotomous. Then there cannot exist nontrivial bounded uniformly continuous complete trajectories for $T_A(t)$.*

Proof. By Theorem 4.1 in [21] the space $BUC(\mathbb{R}, X)$ is regularly admissible for $T_A(t)$. The result follows by Corollary 3.7. □

The last corollary of Theorem 3.3 provides a sufficient condition for the almost periodicity of a nontrivial complete trajectory for $T_A(t)$.

Corollary 3.11. *Let $\sigma_A(A) \cap i\mathbb{R}$ be countable and assume that the space X does not contain a subspace which is isomorphic to c_0 (the Banach space of numerical sequences which converge to zero). Let $\mathcal{M}$ be a CTO subspace of $BUC(\mathbb{R}, X)$. If the operator equation $\Pi S|_{\mathcal{M}} = A\Pi$ has a nontrivial solution $\Pi \in \mathcal{L}(\mathcal{M}, X)$, then $x_f(t) = \Pi T_S(t) f$ is an almost periodic complete trajectory for $T_A(t)$ for each $f \in \mathcal{M}$.*

Proof. By Theorem 3.10 in [19] all bounded uniformly continuous bounded trajectories are almost periodic. By Theorem 3.3, the function $t \to \Pi T_S(t) f$ is a complete trajectory in $\mathcal{M} \subset BUC(\mathbb{R}, X)$ for every $f \in \mathcal{M}$. □

4. Some Perturbation Results

Consider again a closed translation-invariant subspace $\mathcal{M}$ of $BUC(\mathbb{R}, X)$. Clearly for every $f \in \mathcal{M}$ the trajectory $T_S(t)|_{\mathcal{M}} f$ of the left shift group is bounded and complete, and it is in $\mathcal{M}$. However, for every $\epsilon > 0$ the semigroup $T_{S-\epsilon I}(t)$ generated by $S - \epsilon I$ in $\mathcal{M}$ is exponentially stable. By Corollary 3.10 there are no nontrivial bounded complete trajectories for $T_{S-\epsilon I}(t)$ in $\mathcal{M}$, and hence the existence of nontrivial bounded complete trajectories for a semigroup is a fragile property; arbitrarily small bounded additive perturbations to the generator may destroy it. On the other hand, in this section we shall provide conditions under which the *nonexistence* of nontrivial bounded complete trajectories is not destroyed by small unbounded (but possibly structured) additive perturbations to the generator A.

Proposition 4.1. *Let A generate a C_0-semigroup $T_A(t)$ in X. Let $\mathcal{M}$ be a closed translation-invariant subspace of $BUC(\mathbb{R}, X)$ and let $\sigma(A) \cap \sigma(S|_{\mathcal{M}}) = \emptyset$. Let $\Delta_A : \mathcal{D}(\Delta_A) \subset X \to X$ be a linear A-bounded operator such that*

1. *$A + \Delta_A$ with domain $\mathcal{D}(A)$ generates a C_0-semigroup $T_{A+\Delta_A}(t)$ in X.*
2. *The A-boundedness constants a, b in (1.4) satisfy*

$$\sup_{i\omega \in \sigma(S|_{\mathcal{M}})} a\|R(i\omega, A)\| + b\|AR(i\omega, A)\| < 1 \tag{4.1}$$

Then there are no nontrivial complete trajectories in $\mathcal{M}$ for $T_A(t)$ and the same holds for the perturbed C_0-semigroup $T_{A+\Delta_A}(t)$.

Proof. By Theorem IV.3.17 in [13], $\sigma(S|_{\mathcal{M}}) \subset \rho(A + \Delta_A)$. The result then follows by Theorem 3.6. □

It is well known that if A generates an analytic or contractive C_0-semigroup, then so does $A + \Delta_A$ under rather mild additional conditions for the A-bounded perturbation Δ_A [9].

We next prove that regular admissibility of $\mathcal{M}$ for $T_A(t)$ is also preserved under certain additive perturbations to A. According to Corollary 3.7 we then have another situation in which the nonexistence of bounded complete trajectories in $\mathcal{M}$ is not affected by such perturbations. In order to establish this result, we need some notation. Let $\mathcal{M}$ be a CTO subspace of $BUC(\mathbb{R}, X)$. Let $\Delta_A : \mathcal{D}(\Delta_A) \subset X \to X$ be a closed linear operator such that $\mathcal{D}(A) \subset \mathcal{D}(\Delta_A)$ and such that $A - \Delta_A$ (with domain $\mathcal{D}(A)$) generates a C_0-semigroup in X. Define another linear operator $\underline{\Delta_A} : \mathcal{D}(\underline{\Delta_A}) \subset \mathcal{L}(\mathcal{M}, X) \to \mathcal{L}(\mathcal{M}, X)$ such that

$$\mathcal{D}(\underline{\Delta_A}) = \{X \in \mathcal{L}(\mathcal{M}, X) \mid A_\Delta X \in \mathcal{L}(\mathcal{M}, X)\} \tag{4.2a}$$

$$\underline{\Delta_A} X = A_\Delta X \quad \forall X \in \mathcal{D}(\underline{\Delta_A}) \tag{4.2b}$$

Proposition 4.2. *In the above notation assume that $\mathcal{M}$ is regularly admissible for $T_A(t)$. Let*

$$M = \sup\{\|\Pi\| \mid \Pi S|_{\mathcal{M}} = A\Pi + \Delta, \|\Delta\| = 1\} \tag{4.3}$$

If $\underline{\Delta_A}$ is $\tau_{A,S|_{\mathcal{M}}}$-bounded with the boundedness constants a, b in (1.4) *satisfying $aM + b < 1$, then $\mathcal{M}$ is regularly admissible for $T_{A-\Delta_A}(t)$.*

Proof. First observe that by Theorem 3.1 in [21] regular admissibility of $\mathcal{M}$ for $T_A(t)$ is equivalent to the unique solvability of the operator equation $\Pi S|_{\mathcal{M}} = A\Pi + \Delta$ for every $\Delta \in \mathcal{L}(\mathcal{M}, X)$. Consequently by Proposition 2.5 we have $0 \in \rho(\tau_{A,S|_{\mathcal{M}}})$ and $\Pi S|_{\mathcal{M}} = A\Pi + \Delta$ if and only if $\Pi = \tau^{-1}_{A,S|_{\mathcal{M}}}\Delta$. Hence

$$\|\tau^{-1}_{A,S|_{\mathcal{M}}}\| = \sup_{\|\Delta\|=1} \|\tau^{-1}_{A,S|_{\mathcal{M}}}\Delta\| = \sup_{\|\Delta\|=1} \{\|\Pi\| \mid \Pi S|_{\mathcal{M}} = A\Pi + \Delta\} = M \tag{4.4}$$

By our assumptions $\underline{\Delta_A}$ is $\tau_{A,S|_{\mathcal{M}}}$-bounded, with the boundedness constants a, b in (1.4) satisfying $a\|\tau^{-1}_{A,S|_{\mathcal{M}}}\| + b < 1$. Theorem IV.1.16 in [13] then implies that the operator $\tau_{A,S|_{\mathcal{M}}} + \underline{\Delta_A}$ with domain $\mathcal{D}(\tau_{A,S|_{\mathcal{M}}})$ is also boundedly invertible. But for each $X \in \mathcal{D}(\tau_{A,S|_{\mathcal{M}}})$ and $u \in \mathcal{D}(S|_{\mathcal{M}})$ we have

$$\begin{aligned}
[\tau_{A,S|_{\mathcal{M}}} + \underline{\Delta_A}]Xu &= XS|_{\mathcal{M}}u - AXu + \underline{\Delta_A}Xu \\
&= XS|_{\mathcal{M}}u - AXu + \Delta_A Xu \\
&= XS|_{\mathcal{M}}u - (A - \Delta_A)Xu
\end{aligned}$$

which shows that for every $\Delta \in \mathcal{L}(\mathcal{M}, X)$ the operator equation $XS|_{\mathcal{M}} - (A - \Delta_A)X = \Delta$ has a unique solution $X = \Pi_\Delta \in \mathcal{L}(\mathcal{M}, X)$. By Theorem 3.1 in [21] this implies regular admissibility of $\mathcal{M}$ for $T_{A-\Delta_A}(t)$. □

Remark 4.3. For bounded additive perturbations $\Delta_A \in \mathcal{L}(X)$ to A the content of Proposition 4.2 may be formulated in a much simpler way: There exists $\epsilon > 0$ such that whenever $\|\Delta_A\| < \epsilon$, the space $\mathcal{M}$ is regularly admissible for $T_{A+\Delta_A}(t)$.

Remark 4.4. It follows from Theorem 5.1 in [21] that regular admissibility of a space $\mathcal{M}$ is not destroyed by certain sufficiently continuous and small nonlinear perturbations to A. Theorem 4.2 is, however, not entirely contained in this result of Vu and Schüler, because we allow for a degree of unboundedness in the additive perturbation operator Δ_A. Furthermore, their proof relies on a fixed point argument, and consequently it is rather different from ours.

5. On Strong Stability of C_0-semigroups

Exponential stability of a C_0-semigroup can be completely characterized in many equivalent ways: There are the well-known conditions of the Datko Theorem [2], and a condition of Vu and Schüler [21] according to which exponential stability of a C_0-semigroup $T_A(t)$ is equivalent to the uniform boundedness of $T_A(t)$ and the unique solvability of the operator equation $\Pi S = A\Pi + \delta_0$. On the other hand, it has turned out that strong stability of a C_0-semigroup is considerably more difficult to characterize. Since the pioneering work of Arendt, Batty, Lyubich and Vu [1, 17] this question has received much attention in the literature; the reader is referred to [2, 4, 7, 5, 6] and the references therein. It is obvious that a strongly stable C_0-semigroup $T_A(t)$ is uniformly bounded and that $\sigma_P(A^*) \cap i\mathbb{R} = \emptyset$. On the other hand, the ABLV Theorem states that if $T_A(t)$ is uniformly bounded, $\sigma_P(A^*) \cap i\mathbb{R} = \emptyset$ *and* $\sigma(A) \cap i\mathbb{R}$ is countable, then $T_A(t)$ is strongly stable.

We next present new characterizations for strong stability of a C_0-semigroup $T_A(t)$ in terms of nontrivial bounded complete trajectories for the sun-dual semigroup $T_A^{\odot}(t)$ and nontrivial solvability of an operator equation $\Pi S|_{\mathcal{M}} = A\Pi$.

Theorem 5.1. *Assume that $\sigma_A(A) \cap i\mathbb{R}$ is countable and that $T_A(t)$ is a uniformly bounded C_0-semigroup in X generated by A. Then there exists a nontrivial bounded complete trajectory for the sun-dual semigroup $T_A^{\odot}(t)$ if and only if $T_A(t)$ is not strongly stable.*

Proof. Assume first that $T_A(t)$ is not strongly stable. Then $\sigma(A) \cap i\mathbb{R} = \sigma_A(A) \cap i\mathbb{R} \neq i\mathbb{R}$, which by Theorem 2.3 in [19] immediately shows that there exists a nontrivial bounded complete trajectory for the sun-dual semigroup $T_A^{\odot}(t)$.

For the converse, suppose that there exists a nontrivial bounded complete trajectory f for the sun-dual semigroup $T_A^{\odot}(t)$. Since $T_A(t)$ is uniformly bounded, the sun-dual semigroup $T_A^{\odot}(t)$ is uniformly bounded, and hence $f \in BUC(\mathbb{R}, X^{\odot})$. Then $sp(f) \subset \sigma(A^{\odot}) \cap i\mathbb{R} \subset \sigma(A) \cap i\mathbb{R}$ by Proposition 3.7 in [19] and Proposition IV.2.18 in [9]. This shows that $sp(f)$ is a closed countable subset of the imaginary axis, and so it must contain an isolated point. Consider the closed translation-invariant subspace $\mathcal{M}_f = \overline{\operatorname{span}}\{f(\cdot+t) \mid t \in \mathbb{R}\}$ of $BUC(\mathbb{R}, X^{\odot})$ and the restriction $T_S(t)|_{\mathcal{M}_f}$ of the translation group $T_S(t)$ to $\mathcal{M}_f$. By Theorem 3.1 and Corollary 3.2 every $g \in \mathcal{M}_f$ is a complete trajectory for $T_A^{\odot}(t)$. Furthermore, the generator S_f of this restriction $T_S(t)|_{\mathcal{M}_f}$ has an isolated point $i\lambda \in i\mathbb{R}$ in its spectrum because $\sigma(S_f) = sp(f)$ by Proposition 3.5 in [19]. It then follows from Gelfand's Theorem (cf. [2] Corollary 4.4.9) that $i\lambda$ must be an eigenvalue of S_f. Hence there exists a nonzero $g \in \mathcal{M}_f$ such that $T_S(t)|_{\mathcal{M}_f} g = e^{i\lambda t} g$ for each $t \in \mathbb{R}$. Now the function $t \to \delta_0 T_S(t)|_{\mathcal{M}_f} g = g(t) = g(0)e^{i\lambda t}$ is a (nontrivial) complete trajectory for $T_A^{\odot}(t)$ in $\mathcal{M}_f$. It is easy to see that this implies $i\lambda \in \sigma_P(A^{\odot}) \cap i\mathbb{R} = \sigma_P(A^*) \cap i\mathbb{R}$. Consequently $T_A(t)$ cannot be strongly stable. □

In the following theorem we shall characterize strongly stable semigroups by the solvability of an operator equation $\Pi S|_{\mathcal{M}} = A\Pi$. However, in contrast to the previous sections, here $\mathcal{M}$ is a closed translation-invariant subspace of $C_0(\mathbb{R}_+, X) = \{f \in BUC([0,\infty), X) \mid \lim_{t\to\infty} f(t) = 0\}$, and $S|_{\mathcal{M}}$ generates the strongly continuous left shift semigroup in $\mathcal{M}$.

Theorem 5.2. *Let $X \neq \{0\}$ and let $T_A(t)$ be a C_0-semigroup in X generated by A. Then $T_A(t)$ is strongly stable if and only if there exists a nontrivial closed translation-invariant subspace $\mathcal{M} \subset C_0(\mathbb{R}_+, X)$ such that the operator equation $\Pi S|_{\mathcal{M}} = A\Pi$ has a surjective solution $\Pi \in \mathcal{L}(\mathcal{M}, X)$*[2].

Proof. Let $T_A(t)$ be strongly stable and let $\mathcal{M} = \overline{\operatorname{span}}\{T_A(\cdot)x \mid x \in X\}$ where closure is taken in the sup-norm. Then $0 \neq \mathcal{M} \subset C_0(\mathbb{R}_+, X)$. Let $\Pi = \delta_0$, the point evaluation operator in $\mathcal{M}$ centered at the origin. Then $\delta_0 \in \mathcal{L}(\mathcal{M}, X)$ and δ_0 is surjective; for any $x \in X$ we have $x = \delta_0 T_A(t)x$. Moreover, for every trajectory $f_x(t) = T_A(t)x$ we have $\delta_0 T_S(t)|_{\mathcal{M}} f_x = f_x(t) = T_A(t)x = T_A(t)\delta_0 f_x$. Extension by continuity and linearity shows that $\delta_0 T_S(t)|_{\mathcal{M}} = T_A(t)\delta_0$ throughout $\mathcal{M}$ for each

[2]In analogy to Section 2, by a surjective solution of $\Pi S|_{\mathcal{M}} = A\Pi$ we mean a bounded linear surjective operator Π such that $\Pi(\mathcal{D}(S|_{\mathcal{M}})) \subset \mathcal{D}(A)$ and $\Pi S|_{\mathcal{M}} f = A\Pi f$ for each $f \in \mathcal{D}(S|_{\mathcal{M}})$.

$t \geq 0$. Let $f \in \mathcal{D}(S|_{\mathcal{M}})$. Then

$$\begin{aligned}\frac{T_A(h)\delta_0 f - \delta_0 f}{h} &= \frac{T_A(h)\delta_0 f - \delta_0 T_S(h)|_{\mathcal{M}} f}{h} + \frac{\delta_0 T_S(h)|_{\mathcal{M}} f - \delta_0 f}{h} \\ &= \frac{\delta_0 T_S(h)|_{\mathcal{M}} f - \delta_0 f}{h} \quad \forall h > 0\end{aligned} \tag{5.1}$$

which by the boundedness of δ_0 shows that $\delta_0 f \in \mathcal{D}(A)$ and that $A\delta_0 f = \delta_0 S|_{\mathcal{M}} f$ for each $f \in \mathcal{D}(S|_{\mathcal{M}})$. Consequently δ_0 is a surjective solution of $\Pi S|_{\mathcal{M}} = A\Pi$.

Conversely, assume that there exists a nontrivial closed translation-invariant subspace $\mathcal{M} \subset C_0(\mathbb{R}_+, X)$ such that the operator equation $\Pi S|_{\mathcal{M}} = A\Pi$ has a surjective solution $\Pi \in \mathcal{L}(\mathcal{M}, X)$. Then since $\Pi(\mathcal{D}(S|_{\mathcal{M}})) \subset \mathcal{D}(A)$, we have for every $t \geq 0$ and $f \in \mathcal{D}(S|_{\mathcal{M}})$ that

$$\begin{aligned}\Pi T_S(t)|_{\mathcal{M}} f - T_A(t)\Pi f &= \Big|_{\tau=0}^{t} T_A(t-\tau)\Pi T_S(\tau)|_{\mathcal{M}} f d\tau \\ &= \int_0^t \frac{d}{d\tau} T_A(t-\tau)\Pi T_S(\tau)|_{\mathcal{M}} f d\tau \\ &= \int_0^t T_A(t-\tau)[\Pi S|_{\mathcal{M}} - A\Pi] T_S(\tau)|_{\mathcal{M}} f d\tau = 0\end{aligned}$$

and by continuity $\Pi T_S(t)|_{\mathcal{M}} f - T_A(t)\Pi f = 0$ for each $f \in \mathcal{M}$ and $t \geq 0$. Let $x \in X$ be arbitrary. Then by the surjectivity of Π there exists $f \in \mathcal{M}$ such that $x = \Pi f$. Moreover,

$$\lim_{t\to\infty} T_A(t)x = \lim_{t\to\infty} T_A(t)\Pi f = \lim_{t\to\infty} \Pi T_S(t)|_{\mathcal{M}} f = 0 \tag{5.2}$$

since $T_S(t)|_{\mathcal{M}}$ is strongly stable and $\Pi \in \mathcal{L}(\mathcal{M}, X)$. Consequently $T_A(t)$ is strongly stable. $\square$

In a very similar way we obtain the following corollary.

Corollary 5.3. *Let $X \neq \{0\}$ and let $T_A(t)$ be a C_0-semigroup in X generated by A. Then $T_A(t)$ is strongly stable if and only if $T_A(t)$ is uniformly bounded and there exists a nontrivial closed translation-invariant subspace $\mathcal{M} \subset C_0(\mathbb{R}_+, X)$ such that the operator equation $\Pi S|_{\mathcal{M}} = A\Pi$ has a solution $\Pi \in \mathcal{L}(\mathcal{M}, X)$ such that* $\operatorname{ran}(\Pi)$ *is dense in X.*

Remark 5.4. Theorem 5.2 and Corollary 5.3 are related to, but independent of, a result of Batty [4]. He showed that if $T_S(t)$ is a C_0-semigroup in some Banach space Y with generator S, if $T_A(t)$ is a uniformly bounded C_0-semigroup in X with generator A, if $\sigma(S) \cap i\mathbb{R}$ is countable and $\sigma_P(A^*) \cap i\mathbb{R} = \emptyset$, and if $\Pi T_S(t) = T_A(t)\Pi$ for some $\Pi \in \mathcal{L}(Y, X)$ with a dense range, then $T_A(t)$ is strongly stable. In the above, we had to assume that $T_S(t)$ is the translation semigroup in some $\mathcal{M} \subset C_0(\mathbb{R}_+, X)$. However, we also obtained complete characterizations for strong stability.

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