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BOUNDARY BEHAVIOR AND ESTIMATES FOR SOLUTIONS OF EQUATIONS CONTAINING THE p-LAPLACIAN

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Abstract

We use "Hardy-type" inequalities to derive L^q estimates for solutions of equations containing the p-Laplacian with $p > 1$. We begin by deriving some inequalities using elementary ideas from an early article [B3] which has been largely overlooked. Then we derive L^q estimates of the boundary behavior of test functions of finite energy, and consequently of principal (positive) eigenfunctions of functionals containing the p-Laplacian. The estimates contain exponents known to be sharp when $p = 2$. These lead to estimates of the effect of boundary perturbation on the fundamental eigenvalue. Finally, we present global L^q estimates of solutions of the Cauchy problem for some initial-value problems containing the p-Laplacian.

I. Introduction

Our interest in this article is to derive potentially sharp L^q estimates for solutions of equations containing the p-Laplacian, in analogy with what is known for the usual Laplacian ($p = 2$), and to explore the consequences of those estimates.

The p-Laplacian has applications in several fields, including glaciology, non-Newtonian fluid flow, and flow through porous media. It has been intensively studied in the mathematical literature both because of these applications and because it is a model for understanding degenerate elliptic equations and non-convex functionals. We refer to the recent book [D3] for discussion and further references. Here we define the p-Laplacian in the weak sense, i.e., by considering the variational analysis of energy forms

$$R(\zeta) := \frac{\|\nabla\zeta(\mathbf{x})\|_{L^p}^p + \int V(\mathbf{x})|\zeta(\mathbf{x})|^p d^N x}{\|\zeta(\mathbf{x})\|_{L^p}^p} \tag{1.1}$$

with $\zeta(\mathbf{x}) \in C_c^\infty(\Omega)$, or by density $W_0^{1,p}$, where Ω is a connected open set in $\mathbb{R}^N$, and $V(\mathbf{x})$ is a given real-valued function. The nonlinear operator known as the p-Laplacian arises in the first variation of (1.1), which leads to the equation

$$-\Delta_p u + V(\mathbf{x})u^{p-1} = \lambda u^{p-1}, \tag{1.2}$$

where

$$\Delta_p\zeta := \nabla\cdot\left(|\nabla\zeta|^{p-2}\nabla\zeta\right). \tag{1.3}$$

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The behavior in the L^q sense of Dirichlet eigensolutions of elliptic linear operators ($p = 2$) near a boundary has been studied in [E2], [P1], [D2]. In particular it was shown in [D2] that sharp rates of decay can be derived from inequalities of "Hardy type",

$$c^2 \int_\Omega |\nabla\zeta|^2 \geq \int_\Omega \left|\frac{\zeta}{d(\mathbf{x})}\right|^2, \tag{1.4}$$

where $d(\mathbf{x})$ denotes the distance from $\mathbf{x}$ to the boundary of the domain Ω. (Actually, $d(\mathbf{x})$ may be any absolutely continuous function satisfying $|\nabla d| \leq 1$ on Ω).

We were inspired by the philosophy of these articles to seek analogous estimates for the p-Laplacian. L^q versions of (1.4) are known, with sharp constants, which would suffice for some of our purposes. We begin, however, by presenting a little–known but elementary way to derive inequalities of this type, building on an idea of Boggio [B3], which predates related inequalities by Barta [B1], Duffin [D4], Hardy [H1], and others. This is the content of section II.

In section III we derive some estimates of boundary decay of principal eigenfunctions of equations containing the p-Laplacian modeled on those of [D2] for elliptic second–order linear operators. The argument there is based on the spectral theorem, however, which is not available when $p \neq 2$, as the p-Laplacian is not even linear then. It was thus necessary to substantially replace many of the technical ideas of [D2], and in the course of this we were obliged to establish certain special algebraic inequalities (see Section IV). The constants involved in these inequalities determine the exponents appearing in the theorems, and we have striven to make them as sharp as possible. In Section V we use the estimates of Sections III and IV to estimate how the fundamental eigenvalue is affected by a boundary perturbation.

Finally, we turn our attention to the Cauchy problem for equations of the form

$$u^{p-2}u_t = \Delta_p u - V(\mathbf{x})u^{p-1},$$

and prove an L^q growth estimate for solutions.

In the interest of clarity we have restricted ourselves to Euclidean domains and p-Laplacians without weights, and we have not attempted to specify the widest class of potentials $V(\mathbf{x})$ for which our estimates remain valid. We anticipate few if any technical barriers in extending our results to manifolds or to $V(\mathbf{x})$ in function classes analogous to those treated in [S1].

Notation and terminology

A function or vector field is of class AC^1 if all components are differentiable by the Cartesian coordinates and the derivatives are absolutely continuous.

A *distance function* may be any absolutely continuous function $d(\mathbf{x})$ satisfying $|\nabla d| \leq 1$ a.e. on Ω. We invariably choose $d(\mathbf{x})$ as the distance from $\mathbf{x}$ to the boundary of Ω.

The *energy form* is the functional $R(\zeta)$ defined in (1.1).

The *Hardy constant* is the positive number defined in (3.3), which extends (1.4) to the case where $p \neq 2$.

The *index* p is a real number in $(1, \infty)$, and the *dual index* is $p' := p/(p-1)$.

The *inradius* of a domain Ω is the supremum of the radii of all balls included in Ω.

The p-Laplacian is the nonlinear operator defined in (1.3).

The *principal eigenvalue* which appears in (1.2) is

$$\lambda_1 = \inf_{\zeta \in W_0^{1,p}(\Omega), \zeta \not\equiv 0} \frac{\int_\Omega \left(|\nabla \zeta|^p + V(\mathbf{x})|\zeta|^p\right) d^N x}{\int_\Omega |\zeta|^p d^N x}. \tag{1.5}$$

Under the conditions of this article, the minimum is attained in the classical Sobolev space $W_0^{1,p}(\Omega)$; the minimizer is known as the *principal eigenfunction*.

A *regular domain* is a connected open set the boundary of which satisfies a uniform external ball condition (See [D1], p. 27). This condition is implied by the standard uniform external cone condition.

A *test function* is a smooth function of compact support in the domain Ω, and the set of these is denoted C_c^∞

II. Lower bounds to energy forms

In 1907, Boggio [B3] derived some lower bounds to the fundamental eigenvalue of the two-dimensional Laplacian by applying the divergence theorem to a well chosen expression containing two arbitrary differentiable functions. From the modern point of view, his result can be interpreted as a quadratic-form inequality for the Dirichlet energy form of a test function, which contains an arbitrary sufficiently smooth vector field, good choices of which lead to useful lower bounds (see below).

In this section we discuss extensions of Boggio's idea and connections with inequalities of Hardy and Rellich. To a certain extent the significance of the section is historical, as estimates we need for later sections can be found elsewhere in the literature. In addition to correcting the historical record, however, Boggio's idea is significant because it an elementary and efficient way to obtain useful inequalities of this type. (We have recently learned from E. Mitidieri, in response to a preprint version of this article, that he also has a preprint [M2] emphasizing the efficiency of deriving Hardy–type inequalities from the divergence theorem. Mitidieri's treatment is somewhat different from ours, and he was unaware of [B3].)

Our generalization of Boggio's result to the situation of p–Laplacians is:

Theorem II.1. *Let Ω be a regular domain and $\zeta \in C_c^\infty(\Omega)$. Let $\mathbf{Q}$ be a vector field on Ω of class AC^1. Then*

$$\int_\Omega |\nabla \zeta|^p \geq \int_\Omega \left\{ \operatorname{div} \mathbf{Q} - (p-1)|\mathbf{Q}|^{p'} \right\} |\zeta|^p \, d^N x. \tag{2.1}$$

Remarks: Boggio's result corresponds to the case $p = 2$:

$$\int_\Omega |\nabla \zeta|^2 \geq \int_\Omega \zeta^2 \left(\operatorname{div} \mathbf{Q} - |\mathbf{Q}|^2 \right) d^2 x.$$

The basic estimates for inequalities of the Hardy type (see Section 5.3 of [D1]) result from choices for $\mathbf{Q}$ such as

$$\mathbf{Q} = -\mathrm{const.} \nabla Ln(x_1)$$

where x_1 is a Cartesian coordinate. We shall make similar choices below.

Proof:

$$0 = \int_\Omega \operatorname{div}(\mathbf{Q}|\zeta|^p) = \int_\Omega (\operatorname{div}\mathbf{Q})|\zeta|^p + p\int_\Omega |\zeta|^{p-2}\zeta\nabla\zeta\cdot\mathbf{Q}.$$

With $\mathbf{w} = |\zeta|^{p-2}\zeta\mathbf{Q}$, Young's inequality gives

$$|\nabla\zeta\cdot\mathbf{w}| \le \frac{1}{p}|\nabla\zeta|^p + \frac{1}{p'}|\mathbf{w}|^{p'} = \frac{1}{p}|\nabla\zeta|^p + \frac{1}{p'}|\zeta|^p|\mathbf{Q}|^{p'},$$

so

$$\int_\Omega |\nabla\zeta|^p \ge \int_\Omega \left(\operatorname{div}\mathbf{Q} - \frac{p}{p'}|\mathbf{Q}|^{p'}\right)|\zeta|^p.$$

◊

Our first application of this theorem is to derive a Hardy-type inequality with the known sharp constant [M1].

Corollary II.2. *Let* $\Omega \subset \mathbb{R}^N_+ = \{\mathbf{x}\in\mathbb{R}^N, x_1 > 0\}$, $\zeta \in C_c^\infty(\Omega)$, $p' = \frac{p}{p-1}$. *Then:*

$$\int_\Omega \frac{|\zeta|^p}{x_1^p} \le (p')^p \int_\Omega \left|\frac{\partial\zeta}{\partial x_1}\right|^p.$$

Proof: We use the one-dimensional version of Theorem II.1, with $\mathbf{Q} = (\frac{-\alpha}{x_1^{p-1}}, 0, \ldots, 0)$, finding $\operatorname{div}\mathbf{Q} = \frac{\alpha(p-1)}{x_1^p}$, and $|\mathbf{Q}|^{p'} = \frac{\alpha^{p'}}{x_1^p}$. Then for all $\alpha > 0$ we have:

$$\int_\Omega \left|\frac{\partial\zeta}{\partial x_1}\right|^p \ge (p-1)\int_\Omega (\alpha - \alpha^{p'})\left|\frac{\zeta}{x_1}\right|^p.$$

Now, $\alpha - \alpha^{p'}$ reaches its maximum for $p'\alpha^{p'-1} = 1$, which gives $\alpha = \frac{1}{(p')^{p-1}}$ and

$$\alpha - \alpha^{p'} = \frac{1}{(p')^{p-1}}\left[1 - \left[\frac{1}{(p')^{p-1}}\right]^{p'-1}\right] = \frac{1}{(p')^{p-1}}\left(1 - \frac{1}{p'}\right) = \frac{1}{(p')^{p-1}} \times \frac{1}{p}.$$

Hence $\int_\Omega |\nabla\zeta|^p \ge \frac{p-1}{p} \times \frac{1}{(p')^{p-1}}\int_\Omega \left|\frac{\zeta}{x_1}\right|^p$, and we obtain the desired result. ◊

Corollary II.3. *Let Ω be a regular domain in $\mathbb{R}^N$, and let $d(\mathbf{x})$ denote the distance from the boundary. Assume that the inradius of Ω is finite. Then, there exists $c_p < \infty$ such that, for any $\zeta \in W_0^{1,p}(\Omega)$ Hardy's inequality holds:*

$${c_p}^p \int_\Omega |\nabla\zeta|^p \ge \int_\Omega \left|\frac{\zeta}{d(\mathbf{x})}\right|^p. \tag{2.2}$$

Proof: Since the proof follows [D1], pp. 26-28, closely, we content ourselves with an outline, referring the reader to that source. If Ω is a region in $\mathbb{R}^N$, and $\mathbf{u}$ is a unit vector in $\mathbb{R}^N$, we define

$$d_{\mathbf{u}}(\mathbf{x}) = \min\{|t| : \mathbf{x} + t\mathbf{u} \notin \Omega\}$$

and an averaged distance to the boundary $m(\mathbf{x})$ by

$$\frac{1}{m(\mathbf{x})^p} = \int_{\|\mathbf{u}\|=1} \frac{dS(\mathbf{u})}{d_{\mathbf{u}}(\mathbf{x})^p},$$

where dS is the normalized surface measure on the unit sphere of $\mathbb{R}^N$. By averaging the estimate of Corollary II.2 over directions, with the origin always shifted to the edge of Ω, we obtain

$$\int_\Omega \frac{|\zeta|^p}{m(\mathbf{x})^p} \le (p')^p \int_\Omega |\nabla\zeta|^p$$

for all $\zeta \in C_c^\infty(\Omega)$. We now observe, as in [D1], that for regular domains with a finite inradius, one has the estimate

$$d(\mathbf{x}) \le m(\mathbf{x}) \le \gamma d(\mathbf{x})$$

for some constant γ computable from the inradius and the constants in the uniform sphere condition. Then we obtain Hardy's inequality (2.2), with the constant $c_p = \gamma p'$. By density the same inequality holds for $\zeta \in W_0^{1,p}(\Omega)$. $\diamondsuit$

Remark: Our further estimates are based on the minimal value of c_p such that (2.2) holds; in fact $c_p \ge p'$ as we have seen in the proof above.

We close the section with two corollaries which generalize the Rellich inequality for $p = 2$.

Corollary II.4. *Let Ω be any domain in $\mathbb{R}^N$, and $N > p$. Then for all $\zeta \in W_0^{1,p}(\Omega)$,*

$$\left(\frac{p}{N-p}\right)^{p-1} \int_\Omega |\nabla\zeta|^p d^N x \ge \int_\Omega \left|\frac{\zeta}{|\mathbf{x}|}\right|^p d^N x.$$

Proof sketch: We apply Theorem II.1 with the choice

$$\mathbf{Q}(\mathbf{x}) = \left(\frac{N-p}{p}\right)^{p-1} \frac{\mathbf{x}}{|\mathbf{x}|^p}.$$

Of course, this vector field is not AC^1 near the origin, so it must be regularized there, which accounts for the restriction that $N > p$. $\diamondsuit$

Corollary II.5. *Let Ω be a finite domain in $\mathbb{R}^N$, and $N > p > 2$. Then there exists a finite constant c_0 such that for all $\zeta \in W_0^{1,p}(\Omega)$.*

$$c_0^p \int_\Omega |\nabla\zeta|^p \, d^N x \ge \int_\Omega \frac{|\zeta|^p}{|\mathbf{x}|^2} d^N x$$

Proof sketch: Here the choice is

$$\mathbf{Q}(\mathbf{x}) = \frac{\alpha\mathbf{x}}{|\mathbf{x}|^2},$$

which leads to a lower bound of the form

$$\frac{\alpha\,(N-2)}{|\mathbf{x}|^2} - \frac{(p-1)\alpha^{p'}}{|\mathbf{x}|^{p'}}.$$

For $p > 2$ and Ω finite, the constant α can be chosen sufficiently small so that the first term dominates the second throughout Ω. $\diamondsuit$

III. L^q boundary behavior for functions of finite energy

In this section we provide estimates of the boundary decay of test functions of finite energy and, consequently, the principal eigenfunction of equations of the form

$$-\Delta_p u + V(\mathbf{x})u^{p-1} = \lambda_1 u^{p-1}. \tag{3.1}$$

Recall that the energy form is defined by

$$R(\zeta) := \frac{\int_\Omega \left(|\nabla\zeta|^p + V(\mathbf{x})|\zeta|^p\right) d^N x}{\|\zeta\|_{L^p}^p}, \tag{3.2}$$

and that i.e., the principal eigenfunction is the positive function which minimizes this functional in $W_0^{1,p}$. Initially we consider $V \equiv 0$, after which we shall introduce a class of potentials V for which the minimizer exists and similar estimates pertain.

As in [D2], we base these estimates on the Hardy constant, i.e., given a distance function $d(\mathbf{x})$ as above, the minimal value of c_p such that for any $\zeta \in W_0^{1,p}(\Omega)$,

$$c_p{}^p \int_\Omega |\nabla\zeta|^p \geq \int_\Omega \left|\frac{\zeta}{d(\mathbf{x})}\right|^p. \tag{3.3}$$

As remarked above, we choose $d(\mathbf{x})$ as the distance from $\mathbf{x} \in \Omega$ to the boundary of Ω. The goal of this section is to replicate the boundary estimates of Section 3 of [D2] to the extent possible, replacing estimates based on the spectral theorem with integral inequalities as necessary.

The main theorems of this section are III.4 (for $V = 0$) and III.5.

The Hardy constant contains geometric information about the domain, and in some cases can be estimated exactly (e.g., [M1]; note that by convention, the constant c_p in this work is the reciprocal of ours and of [D2].). In Section II of this article, we established that any regular domain with a finite inradius has a finite Hardy constant. A higher value than the minimal $c_p \geq p'$ in (3.3) may arise depending on the geometry of Ω.

Here we assume that the value of c_p is known and explore the consequences for the eigenfunctions.

Our boundary estimates require an algebraic bound of the following form.

Basic Algebraic Bound

There are finite constants $\hat{m} \geq 1$ and $\hat{k} > 0$ such that for all $\mathbf{X} \in \mathbb{R}^N$, $\mathbf{Z} \in \mathbb{R}^N$:

$$\|\mathbf{X}+\mathbf{Z}\|_2^p \leq \hat{m}^p \|\mathbf{X}\|_2^p + \hat{k}\left(\|\mathbf{Z}\|_2^p + p\|\mathbf{Z}\|_2^{p-2}\mathbf{Z}\cdot\mathbf{X}\right) \tag{A}$$

In Section IV we shall identify constants $\hat{m}$ and $\hat{k}$ depending on p and N such that (A) is valid. For $p = 2$, they reduce to $\hat{m} = \hat{k} = 1$. More precisely, Section IV proves (A) with:

$$\begin{array}{llll}
p \geq 2, & N = 1: & \hat{m} = m = p-1 & \hat{k} = k = p^{2-p}(p-1)^{p-1} \\
p \geq 2, & N \geq 2: & \hat{m} = 2^{\frac{(p-2)}{2p}}(p-1) & \hat{k} = 2^{\frac{(p-2)}{2}} p^{2-p}(p-1)^{p-1} \\
1 < p \leq 2, & N = 1: & \hat{m} = m & \hat{k} = k = 1 \\
1 < p \leq 2, & N \geq 2: & \hat{m} = 2^{\frac{(2-p)}{2p}} m & \hat{k} = 1
\end{array}$$

Here, for $p < 2$, m is the constant defined in (4.6); by Lemma IV.2 we know that $m \geq 1$.

Lemma III.1. *With $\hat{m}$ and $\hat{k}$ such that (A) holds, for any $\varphi \geq 0$ which is piecewise C^1 and any $\zeta \in W_0^{1,p}(\Omega)$ such that $\Delta_p \zeta \in L^{p'}(\Omega)$,*

$$\int_\Omega |\nabla(\varphi\zeta)|^p \leq \hat{m}^p \int_\Omega |\zeta \nabla \varphi|^p + \hat{k} \int_\Omega \zeta \varphi^p (-\Delta_p \zeta).$$

Proof: Applying (A) with $\mathbf{X} = \zeta \nabla \varphi$ and $\mathbf{Z} = \varphi \nabla \zeta$, we get:

$$\int_\Omega |\zeta \nabla \varphi + \varphi \nabla \zeta|^p \leq \hat{m}^p \int_\Omega |\zeta \nabla \varphi|^p + \hat{k} \int_\Omega \left[|\varphi \nabla \zeta|^p + p \varphi^{p-1} \zeta |\nabla \zeta|^{p-2} \nabla \zeta \cdot \nabla \varphi \right].$$

Moreover,

$$\int_\Omega |\varphi \nabla \zeta|^p = \int_\Omega (\varphi^p |\nabla \zeta|^{p-2} \nabla \zeta) \cdot \nabla \zeta = -p \int_\Omega \varphi^{p-1} \zeta |\nabla \zeta|^{p-2} \nabla \zeta \cdot \nabla \varphi - \int_\Omega \zeta \varphi^p (\Delta_p \zeta),$$

and we obtain:

$$\int_\Omega |\nabla(\varphi\zeta)|^p \leq \hat{m}^p \int_\Omega |\zeta \nabla \varphi|^p + \hat{k} \int_\Omega \zeta \varphi^p (-\Delta_p \zeta)$$

as claimed. ♢

With $\hat{m}$ appearing in (A) and c_p in (3.3), we henceforth set

$$c = \hat{m} c_p$$

and we remark that $c \geq p$ in view of (A).

Lemma III.2. *Suppose that $c > p$ and that φ is a piecewise C^1 function such that $0 \leq \varphi \leq d(\mathbf{x})^{-1/c}$. Then for any $\zeta \in W_0^{1,p}(\Omega)$:*

$$\int_\Omega \varphi^{p^2} |\zeta|^p d^N x \leq (c_p)^{p^2/c} \left(\int_\Omega |\nabla \zeta|^p d^N x \right)^{p/c} \left(\int |\zeta|^p d^N x \right)^{1-p/c}.$$

Proof: Because $\varphi(\mathbf{x}) \leq d(\mathbf{x})^{-1/c}$,

$$\int_\Omega \varphi^{p^2} |\zeta|^p \leq \int_\Omega d^{-p^2/c} |\zeta|^{p^2 c^{-1} + p(c-p)c^{-1}},$$

which by Hölder's inequality is bounded by

$$\left(\int_\Omega \frac{|\zeta|^p}{d^p} \right)^{p/c} \left(\int_\Omega |\zeta|^p \right)^{1-p/c}.$$

With the Hardy inequality (3.3), we therefore obtain:

$$\int_\Omega \varphi^{p^2} \zeta^p \leq (c_p)^{p^2/c} \left(\int_\Omega |\nabla \zeta|^p \right)^{p/c} \left(\int_\Omega |\zeta|^p \right)^{1-p/c}.$$

♢

Lemma III.3. *Let $\hat{m}$ and $\hat{k}$ be such that (A) holds, and let φ be any piecewise C^1 function such that $0 \le \varphi \le d(\mathbf{x})^{-1/c}$. Then for any $\zeta \in W_0^{1,p}(\Omega)$ such that $\Delta_p \zeta \in L^{p'}(\Omega)$:*

$$\int_\Omega |\nabla(\varphi\zeta)|^p d^N x \le \hat{m}^p \int_\Omega |\zeta \nabla \omega|^p d^N x + \hat{k}(c_p)^{p/c} \left(\int_\Omega |\nabla \zeta|^p d^N x \right)^{1/c} \\ \times \left(\int_\Omega |\zeta|^p d^N x \right)^{p^{-1}-c^{-1}} \left(\int_\Omega |-\Delta_p \zeta|^{p'} d^N x \right)^{1/p'} .$$

Proof: From Lemma III.1 we know that

$$\int_\Omega |\nabla(\varphi\zeta)|^p \le \hat{m}^p \int_\Omega |\zeta \nabla \varphi|^p + \hat{k} \int_\Omega \zeta \varphi^p(-\Delta_p \zeta).$$

Recall that $c \ge p$. If $c > p$, then by Hölder's inequality and Lemma III.2,

$$\begin{aligned} \left| \int_\Omega \zeta \varphi^p(-\Delta_p \zeta) \right| &\le \left(\int_\Omega \zeta^p \varphi^{p^2} \right)^{1/p} \left(\int_\Omega |-\Delta_p \zeta|^{p'} \right)^{1/p'} \\ &\le (c_p)^{p/c} \left(\int_\Omega |\nabla \zeta|^p \right)^{1/c} \left(\int_\Omega |\zeta|^p \right)^{1/p-1/c} \left(\int_\Omega |-\Delta_p \zeta|^{p'} \right)^{1/p'} , \end{aligned} \tag{3.4}$$

yielding the claim.

For $c = p$, since $\varphi^{p^2} \le d^{-p^2/c} = d^{-p}$ we have

$$\left| \int_\Omega (\zeta \varphi^p)(-\Delta_p \zeta) \right| \le \left(\int_\Omega |\zeta|^p \varphi^{p^2} \right)^{1/p} \left(\int_\Omega |-\Delta_p \zeta|^{p'} \right)^{1/p'} ,$$

which by Lemma III.2 is bounded by

$$c_p \left(\int_\Omega |\nabla \zeta|^p \right)^{1/p} \times \left(\int_\Omega |-\Delta_p \zeta|^{p'} \right)^{1/p'} .$$

Hence the same inequality holds in this case. ♢

Our next result, Theorem III.4, shows that integrals involving ζ on an ϵ-neighborhood of the boundary are bounded by expressions of the form $F \cdot \epsilon^s$, where F depends only on Ω, $\|\zeta\|_p$, $\|\nabla \zeta\|_p$, and $\|\Delta_p \zeta\|_{p'}$. When $p = 2$, and $\partial\Omega$ is smooth, our exponents s reduce to the sharp values as remarked in [D2].

We adopt some notation and other conventions of [D2]; in particular, for a given $\varepsilon > 0$, we define

$$\omega(\mathbf{x}) = (\max\{d(\mathbf{x}), \varepsilon\})^{-1/c} \tag{3.5}$$

and

$$\tau(\mathbf{x}) = \begin{cases} \varepsilon^{-1/c} & \text{if } 0 < d(\mathbf{x}) \le \varepsilon \\ c^{-1}\varepsilon^{-1-1/c}\left((1+c)\,\varepsilon - d(\mathbf{x})\right) & \text{if } \varepsilon < d(\mathbf{x}) \le (1+c)\,\varepsilon \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

(Recall that $c = \widehat{m} c_p$ with $\widehat{m}$ appearing in (A) and c_p in (3.3). We remark that both functions ω and τ satisfy the conditions of the functions φ appearing in Lemma III.1–Lemma III.3.)

Theorem III.4. *There are (identifiable) constants $K_{1,2}$ such that given any $\zeta \in W_0^{1,p}(\Omega)$ such that $\Delta_p\zeta \in L^{p'}(\Omega)$:*

(*i*)
$$\int_{\{d(\mathbf{x})<\epsilon\}\cap\Omega} \frac{|\zeta|^p}{d^p} d^N x \le$$
$$K_1 \epsilon^{p/c} \left(\int_\Omega |\nabla\zeta|^p d^N x\right)^{1/c} \left(\int_\Omega |\zeta|^p d^N x\right)^{p^{-1}-c^{-1}} \left(\int_\Omega (-\Delta_p\zeta)^{p'} d^N x\right)^{1/p'}$$
for all $\epsilon > 0$. Hence also,

(*ii*)
$$\int_{\{d(\mathbf{x})<\epsilon\}\cap\Omega} |\zeta|^p d^N x \le$$
$$K_1 \epsilon^{p+p/c} \left(\int_\Omega |\nabla\zeta|^p d^N x\right)^{1/c} \left(\int_\Omega |\zeta|^p d^N x\right)^{p^{-1}-c^{-1}} \left(\int_\Omega (-\Delta_p\zeta)^{p'} d^N x\right)^{1/p'}$$
for all $\epsilon > 0$. In addition,

(*iii*)
$$\int_{\{d(\mathbf{x})\le\varepsilon\}} |\nabla\zeta|^p d^N x \le K_2 F \varepsilon^{p/c},$$
where F depends only on Ω, $\|\zeta\|_p$, $\|\nabla\zeta\|_p$, and $\|\Delta_p\zeta\|_{p'}$ (and is implicitly specified by the last few lines of the proof). Recall that $c = \hat{m}c_p$.

Proof: We deduce from Lemmas III.2 and III.3 that
$$\int_\Omega \frac{|\omega\zeta|^p}{d^p} \le (\hat{m}c_p)^p \int_\Omega |\zeta\nabla\omega|^p + I, \tag{3.7}$$
where
$$I = \hat{k}(pc_p)^{p+p/c} \left(\int_\Omega |\nabla\zeta|^p\right)^{1/c} \left(\int_\Omega |\zeta|^p\right)^{p^{-1}-c^{-1}} \left(\int_\Omega |-\Delta_p\zeta|^{p'}\right)^{1/p'}.$$
Let $Y(\mathbf{x}) = \frac{\omega^p}{d^p} - c^p|\nabla\omega|^p$. For $d(\mathbf{x}) \ge \epsilon$, $|\nabla\omega| = \frac{1}{c}\frac{\omega}{d}$; hence $Y(\mathbf{x}) \ge 0$, and for $d(\mathbf{x}) < \epsilon$, $\nabla\omega(\mathbf{x}) = 0$, so $Y(\mathbf{x}) \ge \frac{1}{\epsilon^{p/c}d^p}$.

Rewriting (3.7) as
$$\int_\Omega |\zeta|^p Y \le I$$
we deduce that
$$\int_{\{d(\mathbf{x})<\epsilon\}\cap\Omega} \frac{|\zeta|^p}{d^p} \le \hat{k}(c_p)^{p+p/c}\epsilon^{p/c} I,$$
and hence we have part (i), from which (ii) is immediate.

For part (iii), we first note that
$$\int_{\{d(\mathbf{x})<\varepsilon\}} |\nabla\zeta|^p d^N x \le \varepsilon^{p/c} \int_\Omega |\nabla(\tau\zeta)|^p d^N x, \tag{3.8}$$

and then apply Lemma III.1 to conclude that

$$\int_\Omega |\nabla(\tau\zeta)|^p d^N x \leq \widehat{m}^p \int_{\{d(\mathbf{x})<(1+c)\varepsilon\}} |\zeta\nabla\tau|^p d^N x + \widehat{k} c_p^{p/c} \left(\int_\Omega |\nabla\zeta|^p d^N x\right)^{1/c} \times \left(\int_\Omega |\zeta|^p d^N x\right)^{1/p-1/c} \left(\int_\Omega |-\Delta_p\zeta|^{p\prime} d^N x\right)^{1/p\prime}.$$

Now,

$$\int_{\{d(\mathbf{x})<(1+c)\varepsilon\}} |\zeta\nabla\tau|^p d^N x \leq \left(\frac{1}{c\varepsilon^{1+1/c}}\right)^p \int_{\{d(\mathbf{x})<(1+c)\varepsilon\}} |\zeta|^p d^N x,$$

which is bounded by quantities independent of ϵ according to part (ii). Together with (3.8), this yields (iii). ♢

Next we obtain a similar estimate for (3.1) for nonzero $V(\mathbf{x})$, for which the coefficient of ϵ^s is given in terms of $\|\zeta\|_p, R(\zeta)$, and $\|-\Delta_p\zeta + V(\mathbf{x})|\zeta|^{p-2}\zeta\|_{p'}$.

We shall assume that $V(\mathbf{x}) = V_1(\mathbf{x}) + V_2(\mathbf{x})$, where $V_1(\mathbf{x}) \geq 0$ and there exist finite constants A, B, α, β, with $\alpha < 1$, such that $|V_2|$ satisfies

$$(i) \qquad \int_\Omega |V_2|^{p'} |\zeta|^p d^N x \leq A \int_\Omega |\nabla\zeta|^p d^N x + B \int_\Omega |\zeta|^p d^N x$$

and

$$\text{(ii)} \qquad \int_\Omega |V_2||\zeta|^p d^N x \leq \alpha \int_\Omega |\nabla\zeta|^p d^N x + \beta \int_\Omega |\zeta|^p d^N x \tag{3.9}$$

for all $\zeta \in C_c^\infty(\Omega)$.

We remark that using the results of Section II, (3.9) will hold, for example, provided that $|V_2|^{p'} < \frac{C_1}{d^p}$ + bounded function $\Leftrightarrow |V_2| < C_2 d^{-(p-1)}$ + bounded function for some constants $C_{1,2}$, since this implies that $|V_2| < \frac{1}{c_p^p}\frac{1}{d^p}$ + bounded function.

Theorem III.5. *Given Hardy's inequality* (3.3) *with* $c = \hat{m}c_p > p$, *assume that* V *satisfies (3.9) and that* $\zeta \in W_0^{1,p}$ *with* $-\Delta_p\zeta + V|\zeta|^{p-2}\zeta \in W_0^{1,p} \cap L^{p'}(\Omega)$. *Then there are quantities* $F_{1,2}$ *depending only on* Ω, $\|\zeta\|_p$, $R(\zeta)$, *and* $\|-\Delta_p\zeta + V(\mathbf{x})|\zeta|^{p-2}\zeta\|_{p'}$ *such that*

$$(i) \qquad \int_{\{d(\mathbf{x})<\epsilon\}\cap\Omega} \frac{|\zeta|^p}{d^p} d^N x \leq F_1 \epsilon^{p/\hat{m}c_p}$$

for all $\epsilon > 0$. *Hence also,*

$$(ii) \qquad \int_{\{d(\mathbf{x})<\epsilon\}\cap\Omega} |\zeta|^p d^N x \leq F_1 \epsilon^{p+p/\hat{m}c_p}$$

for all $\epsilon > 0$. *In addition,*

$$(iii) \qquad \int_{\{d(\mathbf{x})\leq\varepsilon\}} |\nabla\zeta|^p d^N x \leq K_2 F_2 \varepsilon^{p/c}.$$

Proof: We proceed as in the proof of Theorem III.4 until the stage where we call on Lemma III.3. Instead of dominating $\int \zeta\omega^p(-\Delta_p\zeta)$ as in (3.4), we bound it above by

$$\begin{aligned}&\int \zeta\omega^p(-\Delta_p\zeta + V_1|\zeta|^{p-2}\zeta)\\ \leq &\left(\int |\zeta|^p\omega^{p^c}\right)^{1/p}\left(\|-\Delta_p\zeta + V|\zeta|^{p-2}\zeta\|_{p'} + \|V_2|\zeta|^{p-2}\zeta\|_{p'}\right).\end{aligned}$$

The claim requires that we control the final term, which to the p' power is

$$\begin{aligned}\int |V_2|^{p'}|\zeta|^p \leq & A\int |\nabla\zeta|^p + B\int |\zeta|^p\\ \leq & A\left(\int |\nabla\zeta|^p + V|\zeta|^p + |V_2\zeta^p|\right) + B\|\zeta\|_p^p\\ \leq &(AR(\zeta)+B)\|\zeta\|_p^p + A\int |V_2\zeta^p|,\end{aligned}$$

so it remains to control $\int |V_2\zeta^p|$. This we do using part (ii) of (3.7) as follows.

$$\int |V_2\zeta^p| \leq \alpha\left(\int |\nabla\zeta|^p + V|\zeta|^p + |V_2\zeta^p|\right) + \beta\int |\zeta|^p,$$

so

$$\int |V_2\zeta^p| \leq \frac{1}{1-\alpha}\,(\alpha R(\zeta)+\beta)\|\zeta\|_p^p.$$

$\diamondsuit$

IV. Some inequalities

In this section we establish a family of elementary but refined algebraic inequalities, needed to apply the estimates of Section III to the p-Laplacian for various values of p.

First we establish some algebraic inequalities for a binomial in a scalar real variable x, taken to the power p. Then we use them to derive vectorial inequalities which imply the basic algebraic bound (A) of Section III.

Lemma IV.1. *For $p \geq 2$ and $x \in \mathbb{R}$,*

$$|x-1|^p \leq (p-1)^p + p^{2-p}(p-1)^{p-1}\left(|x|^p - p|x|^{p-2}x\right). \tag{4.1}$$

Remark: Essentially we dominate the left side by a constant plus two terms from its expansion for large $|x|$. The inequality is sharp in the sense that the constant $(p-1)^p$ on the right is minimal.

Proof: Because of the absolute values, we need to consider separately three cases, $1 < x$, $0 \leq x \leq 1$, and $x < 0$.

Case 1. For $0 < x < 1$, we let

$$f_2(x) = (1-x)^{-p}[(p-1)^p + p^{p-2}(p-1)^{p-1}(x^p - px^{p-1})],$$

and calculate the derivative

$$f_2'(x) = p^{3-p}(p-1)^p(1-x)^{-p-1}[p^{p-2} - x^{p-2}] > 0,$$

so the minimal value of f_2 on this interval is $f_2(0) = (p-1)^p \geq 1$.

Case 2. For $1 < x$, we claim that

$$f_1(x) = (x-1)^{-p}[(p-1)^p + p^{p-2}(p-1)^{p-1}(x^p - px^{p-1})]$$

achieves its unique minimum for $x = p$. This is because a calculation reveals that

$$f_1'(x) = p^{3-p}(p-1)^p(x-1)^{-p-1}[x^{p-2} - p^{p-2}],$$

which is zero uniquely for $x = p$ and otherwise has the same sign as $x - p$.

Case 3. For convenience, for the case when $x < 0$, we replace x by $-x$. Thus we need to show that for $x > 0$,

$$(1+x)^p \leq (p-1)^p + p^{2-p}(p-1)^{p-1}\left(x^p + px^{p-1}\right), \tag{4.2}$$

or in other words that

$$f_3(x) := \frac{(p-1)^p + p^{2-p}(p-1)^{p-1}(x^p + px^{p-1})}{(1+x)^p} \geq 1. \tag{4.3}$$

Again we differentiate, finding

$$f_3'(x) = p^{3-p}(p-1)^p(1+x)^{-p-1}(x^{p-2} - p^{p-2}),$$

which reveals that f_3' vanishes uniquely at p and elsewhere has the same sign as $x - p$. Hence $f_3(x) \geq f_3(p) = (2p-1)\left(\frac{p-1}{p+1}\right)^{p-1}$.

It remains to show that $f_3(p) \geq 1$, or equivalently that $f_4(y) \geq 1$ for $y \geq 2$ where

$$f_4(y) = (2y-1)\left(\frac{y-1}{y+1}\right)^{y-1}.$$

We note that $f_4(2) = 1$. We prove now that $f_4' > 0$:

$$f_4'(y) = f_4(y)B(y),$$

where $B(y) = \frac{2}{2y-1} + \frac{2}{y+1} + Ln\left(\frac{y-1}{y+1}\right)$. Hence we wish to prove that $B(y) > 0$, which is true for $y = 2$. Now,

$$B'(y) = 4\frac{N(y)}{D(y},$$

with $D(y) = (2y-1)^2(y+1)^2(y-1) > 0$ and $N(y) = -y^3 + 3y^2 - 3y + 2$. Since $N'(y) = -3(y-1)^2 < 0$, $N \leq 0$ and thus $B'(y) < 0$, i.e., B is a decreasing function. As y tends to ∞, $B(y) \to 0$. Hence $B > 0$ and $f_4' > 0$ for $y > 2$. Therefore $f_4(y) \geq 1$ for all $y \geq 2$. $\diamondsuit$

Lemma IV.2. *For $p \leq 2$ and $x \in \mathbb{R}$,*

$$|x-1|^p \leq m_p^p + \Big(|x|^p - p|x|^{p-2}x\Big)\,, \tag{4.5}$$

where m_p^p is defined by

$$m_p^p = \max_{0\leq x\leq 1}((p-x)x^{p-1} + (1-x)^p). \tag{4.6}$$

Remarks: In comparison with Lemma IV.1, for $p \geq 2$, the second constant on the right has been simplified to 1, while the first one has a different form. Both sharp inequalities trivialize to the same identity for $(x-1)^2$ when p becomes 2.

Observe that $m_2^2 = \max(1) = 1$, and that if $h_p\,(x) := (p-x)x^{p-1} + (1-x)^p$, then $m_p^p \geq \max(h(0), h(1)) = \max(1, p-1)$.

Proof: We need to show $|x-1|^p \leq m_p^p + (|x|^p - p|x|^{p-2}x)$ for $x \in \mathbb{R}$. As before, we consider three cases.

Case 1. $0 \leq x \leq 1$. The desired bound holds by the definition of m_p^p.

Case 2, $x \geq 1$. Let

$$\phi = (x-1)^p, \qquad \psi = m_p^p + x^p - px^{p-1}.$$

We see that $\phi(1) = 0 < \psi(1)$ and define

$$r := \frac{\psi'}{\phi'} = \frac{(x-(p-1))x^{p-2}}{(x-1)^{p-1}}.$$

It is easy to see that $\lim_{x\downarrow 1} r(x) = +\infty$ and $\lim_{x\to\infty} r(x) = 1$, and to calculate that $r'(x) = (\text{positive}) \times (p-2) < 0$ on this interval. Thus $r > 1$, which implies the bound in this case.

Case 3. $x < 0$. As before, it is convenient to redefine $x \leftrightarrow -x$ and compare the functions

$$\phi = (1+x)^p \text{ and } \psi = m^p + x^p + px^{p-1}$$

for $x > 0$. We define $r = \psi'/\phi'$, and calculate as for case 2 that $r' = \text{positive} \times (p-2) < 0$. By examining the limits $\lim_{x\downarrow 0} r(x) = +\infty$ and $\lim_{x\to\infty} r(x) = 1$, we conclude that $r(x) > 1$ on this interval, implying the desired bound. ♢

We now proceed to deduce vectorial inequalities from the scalar inequalities of Lemma IV.1 and Lemma IV.2.

Lemma IV.3. *For $p > q > 1$, the following inequalities hold $\forall\ \mathbf{Y} \in \mathbb{R}^N$,*

$$\|\mathbf{Y}\|_p \underbrace{\leq}_{(1)} \|\mathbf{Y}\|_q \underbrace{\leq}_{(2)} N^{(p-q)/pq}\|\mathbf{Y}\|_p.$$

where $\|\mathbf{Y}\|_p = \left\{ \sum\limits_{i=1}^{N} |y_i|^p \right\}^{1/p}$.

Proof: (1) By a homothety, it is sufficient to consider the case

$$\sum_{i=1}^{N} |y_i|^q \geq 1 \quad \text{with} \quad |y_i| \leq 1, \forall\, i = 1, \dots, N$$
$$\sum_{i=1}^{N} |y_i|^p \leq \sum_{i=1}^{N} |y_i|^q$$

so that

$$\left(\sum_{i=1}^{N} |y_i|^p\right)^q \leq \left(\sum_{i=1}^{N} |y_i|^q\right)^q \leq \left(\sum_{i=1}^{N} |y_i|^q\right)^p \Rightarrow \|\mathbf{Y}\|_p \leq \|\mathbf{Y}\|_q$$

(2) Letting $x_i = |y_i|^q$, by convexity we have

$$\left(\frac{x_1 + \cdots + x_N}{N}\right)^{p/q} \leq \frac{1}{N}\left(x_1^{p/q} + \cdots + x_N^{p/q}\right)$$
$$\frac{1}{N^{p/q}} (|y_1|^q + \cdots + |y_N|^q)^{p/q} \leq \frac{1}{N}(|y_1|^p + \cdots + |y_N|^p)$$
$$\|\mathbf{Y}\|_q \leq (N^{p/q-1})^{1/p} \|\mathbf{Y}\|_p = N^{(p-q)/pq} \|\mathbf{Y}\|_p.$$

♢

Remarks: The constant 1 in (1) is optimal: take $y_2 = \cdots = y_N = 0$. The constant $N^{(p-q)/pq}$ in (2) is likewise optimal: take $y_1 = y_2 = \cdots = y_N = 1$; in that case, (2) becomes $N^{1/q} \leq N^{(p-q)/pq} N^{1/p}$.

Lemma IV.4. *Suppose that for* $m \geq 1$ *and* $k > 0$ *it has been established that*

$$\forall y, z \in \mathbb{R} : |y - z|^p \leq m^p |z|^p + k|y|^p - kp|y|^{p-2} yz. \tag{4.7}$$

Then the following inequalities hold for any $\mathbf{Y}$ *and* $\mathbf{Z} \in \mathbb{R}^n$*:*

(i) For $p \geq 2$, $\|\mathbf{Y} - \mathbf{Z}\|_2^p \leq 2^{(p/2)-1} \left\{ m^p \|\mathbf{Z}\|_2^p + k\|\mathbf{Y}\|_2^p - kp\|\mathbf{Y}\|_2^{p-2} \mathbf{Y} \cdot \mathbf{Z} \right\}$

(ii) For $1 < p \leq 2$, $\|\mathbf{Y} - \mathbf{Z}\|_2^p \leq 2^{1-(p/2)} m^p \|\mathbf{Z}\|_2^p + k\|\mathbf{Y}\|_2^p - kp\|\mathbf{Y}\|_2^{p-2} \mathbf{Y} \cdot \mathbf{Z}$*.*

Proof: Since the formulae (i) and (ii) are not changed by rotation or if we replace $\mathbf{X}$ and $\mathbf{Y}$ by any homothetic vectors, it is sufficient to consider the case where

$$\mathbf{Y} = (1, 0, \dots, 0) \text{ and } \mathbf{Z} = (z_1, z_2, 0, \dots, 0).$$

(i) Observe that from Lemma IV.3 we have

$$|z_1|^p + |z_2|^p \leq \{|z_1|^2 + |z_2|^2\}^{p/2}. \tag{4.8}$$

We get

$$\begin{aligned}
\|\mathbf{Y}-\mathbf{Z}\|_2^p &= \left\{(z_1-1)^2+z_2^2\right\}^{p/2} = 2^{p/2}\left\{\frac{(z_1-1)^2+z_2^2}{2}\right\}^{p/2} \\
&\le 2^{(p/2)-1}\left\{(z_1-1)^p+z_2^p\right\} \qquad \text{by convexity} \\
&\le 2^{(p/2)-1}\left\{m^p|z_1|^p+k-kpz_1+m^p|z_2|^p\right\} \qquad \text{from (4.7)} \\
&\le 2^{(p/2)-1}\left\{m^p(|z_1|^2+|z_2|^2)^{p/2}+k-kpz_1\right\} \qquad \text{from (4.8)} \\
&\le 2^{(p/2)-1}\left\{m^p\|\mathbf{Z}\|_2^p+k\|\mathbf{Y}\|_2^p-kp\|\mathbf{Y}\|_2^{p-2}\mathbf{Y}\cdot\mathbf{Z}\right\}.
\end{aligned}$$

(ii) Since $2>p$, from Lemma IV.3 we find

$$\begin{aligned}
\|\mathbf{Y}-\mathbf{Z}\|_2^p &= \left\{(z_1-1)^2+z_2^2\right\}^{p/2} \le |z_1-1|^p+|z_2|^p \\
&\le m^p(|z_1|^p+|z_2|^p)+k-kpz_1 \qquad \text{from (4.7)} \\
&\le m^p2^{1-(p/2)}(z_1^2+z_2^2)^{p/2}+k-kpz_1.
\end{aligned}$$

From the second relation of Lemma IV.3, we obtain here:

$$(|z_1|^p+|z_2|^p)^{1/p} \le 2^{(1/p)-(1/2)}(z_1^2+z_2^2)^{1/2},$$

and hence

$$\|\mathbf{Y}-\mathbf{Z}\|_2^p \le m^p2^{1-(p/2)}\|\mathbf{Z}\|_2^p+k\|\mathbf{Y}\|_2^p-kp\|\mathbf{Y}\|^{p-2}\mathbf{Y}\cdot\mathbf{Z}.$$

$\diamondsuit$

By combining the lemmas of this section, we obtain the estimates needed for Section III.

Proposition IV.5. *For any* $\mathbf{X}$ *and* $\mathbf{Z}\in\mathbb{R}^n$,

(i) For $p\ge 2$:

$$\|\mathbf{X}+\mathbf{Z}\|_2^p \le 2^{(p/2)-1}\left\{(p-1)^p\|\mathbf{X}\|_2^p+p^{2-p}(p-1)^{p-1}\left(\|\mathbf{Z}\|_2^p+p\|\mathbf{Z}\|_2^{p-2}\mathbf{Z}\cdot\mathbf{X}\right)\right\}.$$

(ii) For $1<p\le 2$:

$$\|\mathbf{X}+\mathbf{Z}\|_2^p \le 2^{1-(p/2)}m_p^p\|\mathbf{X}\|_2^p+\|\mathbf{Z}\|_2^p+p\|\mathbf{Z}\|_2^{p-2}\mathbf{Z}\cdot\mathbf{X},$$

where m_p^p is defined in (4.6).

V. Perturbation of the boundary

In this section we use the results stated in Section III to estimate how the first eigenvalue of the p-Laplacian, or the p-Laplacian plus a potential, depends on the domain. Again we follow ideas of [E2] and [D2]. More precisely, we wish to compare the fundamental eigenvalues for Ω and for the retracted domain $\Omega_\varepsilon=\{\mathbf{x}\in\Omega/d(\mathbf{x})>\varepsilon\}$. We shall find it convenient to define $\Gamma_\varepsilon=\{\mathbf{x}\in\Omega/d(\mathbf{x})<\varepsilon\}$ and $S_\varepsilon=\Omega_\varepsilon\cap\Gamma_{2\varepsilon}$.

We denote by $\lambda_1(\Omega)$ the first eigenvalue of the Dirichlet p-Laplacian on Ω. By the variational principle, we have

$$\lambda_1(\Omega)\le\lambda_1(\Omega_\varepsilon).$$

Our main result in this section is the following

Theorem V.1. *There exists a positive constant k depending only on p, N, and Ω, such that for ε sufficiently small,*

$$\lambda_1(\Omega_\varepsilon) \leq \lambda_1(\Omega) + k\varepsilon^{\frac{p}{\tilde{m}c_p}}.$$

Proof: We introduce $\mu : \Omega \longrightarrow [0; +\infty)$ defined by

$$\mu(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in \Gamma_\varepsilon, \\ \varepsilon^{-1}(d(\mathbf{x}) - \varepsilon) & \text{if } \mathbf{x} \in S_\varepsilon, \\ 1 & \text{if } \mathbf{x} \in \Omega_{2\varepsilon}. \end{cases}$$

Let ϕ_1 be the first eigenfunction of the Dirichlet p-Laplacian on Ω such that $\|\phi_1\|_{L^p} = 1$. We have

$$\begin{aligned} \int_\Omega (|\nabla(\mu\phi_1)|^p - |\nabla\phi_1|^p) &= \int_{\Gamma_{2\varepsilon}} (|\nabla(\mu\phi_1)|^p - |\nabla\phi_1|^p) \\ &\leq \int_{S_\varepsilon} (|\nabla(\mu\phi_1)|^p - |\nabla\phi_1|^p) \\ &\leq \int_{S_\varepsilon} \left[\left(|\nabla\phi_1| + |\frac{\phi_1}{\varepsilon}| \right)^p - |\nabla\phi_1|^p \right] \\ &\leq p \int_{S_\varepsilon} |\frac{\phi_1}{\varepsilon}| \left(|\nabla\phi_1| + |\frac{\phi_1}{\varepsilon}| \right)^{p-1} \\ &\leq K \int_{S_\varepsilon} |\frac{\phi_1}{\varepsilon}|^p + K \left(\int_{S_\varepsilon} |\frac{\phi_1}{\varepsilon}|^p \right)^{\frac{1}{p}} \left(\int_{S_\varepsilon} |\nabla\phi_1|^p \right)^{\frac{1}{p'}}. \end{aligned}$$

From Theorem III.4, we deduce that

$$\int_\Omega (|\nabla(\mu\phi_1)|^p - |\nabla\phi_1|^p) \leq K'\varepsilon^{\frac{p}{\tilde{m}c_p}} + K''\varepsilon^{\frac{p}{\tilde{m}c_p}\left(\frac{1}{p} + \frac{1}{p'}\right)} \leq K\varepsilon^{\frac{p}{\tilde{m}c_p}}.$$

Hence

$$\int_\Omega |\nabla(\mu\phi_1)|^p \leq \lambda_1(\Omega) + K\varepsilon^{\frac{p}{\tilde{m}c_p}}.$$

From the variational principle we conclude that

$$\int_\Omega |\nabla(\mu\phi_1)|^p \geq \lambda_1(\Omega_\varepsilon) \int_\Omega |\mu\phi_1|^p.$$

Now,

$$\begin{aligned} \int_\Omega |\phi_1|^p &= \int_\Omega |\mu\phi_1 + (1-\mu)\phi_1|^p \\ &\leq \int_\Omega |\mu\phi_1|^p + \int_\Omega (1-\mu)^p |\phi_1|^p \\ &\leq \int_{\Gamma_{2\varepsilon}} |\phi_1|^p + \int_\Omega |\mu\phi_1|^p \\ &\leq K\varepsilon^{\frac{p}{\tilde{m}c_p} + p} + \int_\Omega |\mu\phi_1|^p. \end{aligned}$$

Thus

$$\int_\Omega |\nabla(\mu\phi_1)|^p \geq \lambda_1(\Omega_\varepsilon)\left[1 - K\varepsilon^{\frac{p}{\tilde{m}c_p}+p}\right],$$

and hence for ε sufficiently small

$$\begin{aligned}\lambda_1(\Omega_\varepsilon) \leq& \frac{\lambda_1(\Omega) + K\varepsilon^{\frac{p}{\tilde{m}c_p}}}{1 - K\varepsilon^{p+\frac{p}{\tilde{m}c_p}}}\\ \leq& \lambda_1(\Omega) + K(1 + 2\lambda_1(\Omega))\varepsilon^{\frac{p}{\tilde{m}c_p}}\\ \leq& \lambda_1(\Omega) + k\varepsilon^{\frac{p}{\tilde{m}c_p}}.\end{aligned}$$

$\diamondsuit$

Estimates of this type apply, with the same power of ε under conditions as in Section III, to the p-Laplacian with a potential.

VI. $L^s(\Omega)$ estimates for solutions of $|u|^{p-2}u_t = \Delta_p u - V(\mathbf{x})|u|^{p-2}u$

In this section we turn our attention to the Cauchy problem for evolution equations of the form

$$|u|^{p-2}u_t = \Delta_p u - V(\mathbf{x})|u|^{p-2}u. \tag{6.1}$$

The reason for the factor $|u|^{p-2}$ on the left side is that it guarantees that the equation is homogeneous (see the definition (1.3) of the p-Laplacian).

In this section, we assume that $V(\mathbf{x}) = V_1(\mathbf{x}) + V_2(\mathbf{x})$, where $V_1(\mathbf{x}) \geq 0$ and $|V_2|$ satisfies a bound of the form

$$\int_\Omega |V_2|\,|\zeta|^p \; d^N x \leq \alpha \int_\Omega |\nabla\zeta|^p d^N x + \beta \int_\Omega |\zeta|^p \; d^N x, \tag{6.2}$$

with $\alpha < \infty$. We recall that in Section II we provided some criteria for this bound; for instance, by Corollary II.4, if $N > p$, then the negative part of $V(\mathbf{x})$ may be bounded in magnitude by a sufficiently small constant, proportional to α, times a sum of terms with local divergences of the form $\frac{1}{|\mathbf{x}-\mathbf{x_0}|^p}$.

Belyi and Semenov [B2] and Liskevich [L1] have shown that for certain linear differential operators the growth in time t of $\|u(t,x)\|_{L^p(\Omega)}$ can be estimated when the negative part of V is relatively form bounded. In this section we show that similar estimates are valid for solutions of (6.1). We consider only classical solutions of (6.1) on regular domains, with vanishing Dirichlet boundary conditions, and content ourselves with two theorems, which sufficiently well illustrate the idea.

Theorem VI.1. *Assume that u is a classical solution of equation (6.1), u belongs to $W_0^{1,p}(\Omega) \cap L^s(\Omega)$, $s \geq p$, and $-\Delta_p u \in L^\infty(\Omega)$. Assume moreover that the potential $V(\mathbf{x})$ satisfies (6.2) with $\alpha \leq (s+1-p)\left(\frac{p}{s}\right)^p$. Let $f_{s,u}(t) := \|u(t;\mathbf{x})\|_{L^s(\Omega)}$. Then*

$$f_{s,u}(t) \leq f_{s,u}(0)\exp(\beta t).$$

Proof: We write $r = s - p$ and multiply (6.1) by $|u|^r u$ and integrate. We find

$$\begin{aligned}\frac{1}{p+r}\frac{d}{dt}\int_\Omega |u|^{p+r} &= \int \left\{|u|^r u \nabla\cdot(|\nabla u|^{p-2}\nabla u) - V|u|^{p+r}\right\} \\ &\le -\int\left\{\nabla(|u|^r u)\cdot|\nabla u|^{p-2}\nabla u\right\} + \int |V_2||u|^{p+r} \\ &= -(r+1)\int\{|u|^r|\nabla u|^p\} + \int |V_2||u|^{p+r} \\ &\le -(r+1)\int |u|^r|\nabla u|^p + \alpha\int\left|\nabla\left(u^{(p+r)/p}\right)\right|^p + \beta\int |u|^{(p+r)} \\ &= \left(\alpha\left(\frac{p+r}{p}\right)^p - (r+1)\right)\int |u|^r|\nabla u|^p + \beta\int|u|^{(p+r)}\end{aligned}.$$

The assumption on α makes the first term in the final line ≤ 0, so we drop it, obtaining

$$\frac{d}{dt}\|u\|_s^s \le \beta s\|u\|_s^s,$$

which implies the claim. ♢

Theorem VI.2. *Assume that u is a positive solution of a differential equation for which the differential inequality*

$$|u|^{p-2}u_t \le \Delta_p u - V(\mathbf{x})|u|^{p-2}u. \tag{6.3}$$

holds, that $u \in W_0^{1,p}(\Omega)\cap L^s(\Omega)$, $s \ge p$, *and* $-\Delta_p u \in L^\infty(\Omega)$. *Assume moreover that the potential* $V(\mathbf{x})$ *satisfies (6.2) with* $\alpha \le (s+1-p)\left(\frac{p}{s}\right)^p$. *Let*

$$f_{s,u}(t) := \|u(t;\mathbf{x})\|_{L^s(\Omega)}.$$

Then

$$f_{s,u}(t) \le f_{s,u}(0)\exp(\beta t).$$

Proof: Exactly as for Theorem VI.1; positivity matters because the proof requires the inequality to be multiplied by a power of u. ♢

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In Corollary II.4, the formula

$$\Big(\frac{p}{N-p}\Big)^{p-1}\int_\Omega |\nabla\zeta|^p d^N x \geq \int_\Omega \Big|\frac{\zeta}{|\mathbf{x}|}\Big|^p d^N x.$$

should be replaced by

$$\Big(\frac{p}{N-p}\Big)^{p}\int_\Omega |\nabla\zeta|^p d^N x \geq \int_\Omega \Big|\frac{\zeta}{|\mathbf{x}|}\Big|^p d^N x.$$

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