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MULTIPLE POSITIVE SOLUTIONS FOR NONHOMOGENEOUS SCHRÖDINGER-POISSON SYSTEMS WITH BERESTYCKI-LIONS TYPE CONDITIONS

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Abstract. In this article, we consider the multiplicity of solutions for nonhomogeneous Schrödinger-Poisson systems under the Berestycki-Lions type conditions. With the aid of Ekeland's variational principle, the mountain pass theorem and a Pohožaev type identity, we prove that the system has at least two positive solutions.

1. Introduction and statement of main results

In this article, we study the Schrödinger-Poisson system

$$\begin{gathered} -\Delta u + \lambda\phi u = g(u) + h(x) \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $g \in C(\mathbb{R},\mathbb{R})$ and $h \in L^2(\mathbb{R}^3)$. System (1.1) is also called Schrödinger-Maxwell equations, and arises in an interesting physical context. In fact, according to a classical model, the interaction of a charged particle with an electromagnetic field can be described by coupling the nonlinear Schrödinger's and Poisson's equations (we refer the reader to previous studies [4, 5, 21] and the references therein for more details on the physical aspects).

If $h(x) \equiv 0$, system (1.1) becomes the classical Schrödinger-Poisson system

$$\begin{gathered} -\Delta u + \lambda\phi u = g(u) \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{1.2}$$

which was introduced by Benci and Fortunato [4]. System (1.2) has been extensively studied under various hypotheses on the nonlinearity, see for example [1, 2, 3, 8, 9, 13, 19, 20, 22, 26, 27]. The case $g(u) = |u|^{p-2}u - u$, $p \in (2,6)$ has been studied in [3, 8, 9, 20]. D'Aprile and Mugnai [8] established the existence of a nontrivial radial solution for $p \in [4,6)$. On the other hand, the non-radial solution of system (1.2) was considered in [9] for $p \in (4,6)$. Ruiz [20] showed that system (1.2) has no solution when $p \in (2,3]$ and obtained a positive radial solution by using a constrained minimization method when $p \in (3,6)$. Azzollini and Pomponio [3]

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considered the ground state solutions of system (1.2) for $p \in (3,6)$. For the case that g is a general nonlinear term, we refer the reader to [1, 2, 22, 26]. Sun and Ma [22] dealt with the existence of a ground state solution for system (1.2) involving a 3-superlinear nonlinearity $g(u)$ satisfying the (AR) type condition, namely, there exists $\mu > 3$ such that $g(u)u \geq \mu G(u) > 0$ for all $u \in \mathbb{R}\backslash\{0\}$. After that, the authors in [26] improved the result of [22], who discussed the case that $g(u)$ is asymptotically 2-linear and obtained a ground state solution. Azzollini et al [2] assumed that $g \in C(\mathbb{R},\mathbb{R})$ and g satisfies the following conditions:

- (A1) $-\infty < \liminf_{t\to 0} g(t)/t \leq \limsup_{t\to 0} g(t)/t = -m < 0$,
- (A2) $-\infty < \limsup_{|t|\to\infty} g(t)/|t|^4 t \leq 0$,
- (A3) there exists $\zeta > 0$ such that $G(\zeta) := \int_0^\zeta g(s)ds > 0$.

They showed that system (1.2) admits a nontrivial radial solution for λ small enough. To guarantee the boundedness of Palais–Smale sequence in [2], the authors used a truncation argument in [16] and Struwe's monotonicity trick. By the way, (A1)–(A3) are known as the Berestycki-Lions conditions, introduced in [6]. There, the authors showed that (A1)–(A3) are "almost" necessary for the existence of nontrivial solutions to system (1.2) when $\lambda = 0$. In the sequel, Azzollini [1] assumed that g satisfies (A1)–(A3) and obtained a nontrivial non-radial solution for system (1.2) by using a concentration and compactness argument.

Next, we consider the nonhomogeneous case of system (1.1), that is $h(x) \not\equiv 0$. Salvatore [21] proved the existence of three radially symmetric solutions to system (1.1) with $g(u) = |u|^{p-2}u - u$, $p \in (4,6)$. Subsequently, Jiang et al [17] discussed system (1.1) for $p \in (2,6)$ and obtained two radial solutions when $|h|_2$ is small enough. Particularly, Zhang et al [28] studied the following nonhomogeneous Schrödinger-Poisson system

$$\begin{gathered} -\Delta u + Ku + \lambda\phi f(u) = g(u) + h(x) \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi = 2\lambda F(u) \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{1.3}$$

where $\lambda \geq 0$, K is a positive constant. They assumed the following conditions:

- (A4) $f \in C(\mathbb{R},\mathbb{R}_+)$, there exist $C > 0, \alpha \in (2,4)$ such that $f(t) \leq C(|t| + |t|^\alpha)$, $t \in \mathbb{R}$,
- (A5) $g \in C(\mathbb{R},\mathbb{R}_+)$, there exist $C > 0, p \in (2,6)$ such that $g(t) \leq C(|t| + |t|^{p-1})$, $t \in \mathbb{R}$,
- (A6) $\lim_{t\to 0^+} \frac{g(t)}{t} = 0$,
- (A7) $\lim_{t\to\infty} \frac{g(t)}{t} = l$, where $K < l \leq \infty$,
- (A8) $(x \cdot \nabla h) \in L^2(\mathbb{R}^3)$ is nonnegative, $h \in C^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ is a nonnegative radial function and there exists $M > 0$ such that $|h|_2 \leq M$.

Theorem 1.1 ([28]). *If* (A4)–(A8) *are satisfied, then there exists $\lambda_0 > 0$ such that system* (1.3) *has at least two positive radial solutions for $\lambda \in [0, \lambda_0)$.*

Since the condition (A7) implies that g is asymptotically linear or superlinear at infinity and g does not satisfy the (AR) condition, it is not easy to obtain a bounded Palais-Smale sequence. To overcome this difficulty, the authors in [28] used the method based on Struwe's monotonicity trick and cut-off function. For more results on the nonhomogeneous case, see [7, 10, 11, 18, 23, 24] and the references therein. However, in these papers, the hypotheses of nonlinear term are much stronger than the Berestycki-Lions type conditions.

Inspired by the above works, especially by the results in [2, 17, 21, 28], we try to give the weakest conditions on g. Then, the purpose of this article is to obtain multiple positive solutions of system (1.1) under the Berestycki-Lions type conditions when $h(x) \not\equiv 0$ and λ is small enough. To state our results, we assume that $h \in L^2(\mathbb{R}^3)$ is a radial function, $h(x) \not\equiv 0$, satisfies (A1) and (A3), and the following two conditions are satisfied.

(A9) $\lim_{|t|\to\infty} \frac{g(t)}{|t|^4 t} = 0$,

(A10) $(x \cdot \nabla h) \in L^{\frac{6}{5}}(\mathbb{R}^3)$, where the gradient ∇h is in the weak sense.

Now we state our main results.

Theorem 1.2. *Suppose that $h \in L^2(\mathbb{R}^3)$ is a radial function and $h(x) \not\equiv 0$. Let* (A1), (A3), (A9), (A10) *hold. Then there exist $\lambda_0, \Lambda > 0$ such that* (1.1) *admits two nontrivial radial solutions for $\lambda \in (0, \lambda_0)$, $|h|_2 < \Lambda$.*

Moreover, if h satisfies an additional condition, we can prove the solution is positive.

Corollary 1.3. *If $h(x) \geq 0$ in $\mathbb{R}^3$ and the assumptions of Theorem 1.2 hold, then there exist $\tilde{\lambda_0}, \tilde{\Lambda} > 0$ such that system* (1.1) *admits two positive radial solutions for $\lambda \in (0, \tilde{\lambda_0})$, $|h|_2 < \tilde{\Lambda}$.*

Remark 1.4. (1) From the condition (A10), our h can change its sign and $(x \cdot \nabla h)$ belongs to a class of functions which are different from the ones that appear in (A8).

(2) Notice that, (A3) is much weaker than (A7) of Theorem 1.1. In fact, there exist many functions that satisfy (A3) but not (A7), for example, $g(t) = -t + \frac{4t^3}{1+t^4}$. It is not difficult to verify g satisfying (A1), (A3) and (A9), however, setting $K = 1$ in system (1.3) we have

$$\lim_{t\to\infty} \frac{g(t)+t}{t} = \lim_{t\to\infty} \frac{4t^2}{1+t^4} = 0 < K,$$

which implies that g does not satisfy (A7). Hence, Corollary 1.3 improves Theorem 1.1 and thus generalizes [17, Theorems 1.1, 1.2] and [21, Theorem 1.2].

It also should be pointed out that our methods are different from ones in [28], since our g does not satisfy $g \in C(\mathbb{R}, \mathbb{R}_+)$.

This article is organized as follows. In section 2, with the aid of Ekeland's variational principle that the first radial solution is a local minimizer u_1 with negative energy, Theorem 2.3. In section 3, we find the second radial solution u_2 with positive energy by Theorem 3.6 to complete the proof of Theorem 1.2, and finish the proof of Corollary 1.3.

Throughout this paper, we use the following notation:

- $L^p(\mathbb{R}^3)$ denotes the Lebesgue space with the usual norm $|u|_p = \big(\int_{\mathbb{R}^3} |u|^p \, dx\big)^{1/p}$.
- $H^1(\mathbb{R}^3)$ is the Hilbert space endowed with the norm $\|u\| = \big(\int_{\mathbb{R}^3} |\nabla u|^2 + u^2 \, dx\big)^{1/2}$.
- $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with the norm $\|u\|_D = \big(\int_{\mathbb{R}^3} |\nabla u|^2 \, dx\big)^{\frac{1}{2}}$.
- The best Sobolev constant of the embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ is defined by

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 \, dx}{(\int_{\mathbb{R}^3} |u|^6 \, dx)^{1/3}}.$$

- For each $p \in (2,6)$, there exists S_p such that $|u|_p \leq S_p\|u\|$ for all $u \in H^1(\mathbb{R}^3)$.
- $u^+ = \max\{u,0\}$, $u^- = \min\{u,0\}$; $(H^*, \|\cdot\|_*)$ denotes the dual space of $(H, \|\cdot\|)$.
- $S_r := \{u \in H^1(\mathbb{R}^3) : \|u\| = r\}$ and $B_r := \{u \in H^1(\mathbb{R}^3) : \|u\| < r\}$.
- C, C_i, a_i are various positive constants.
- $o_n(1)$ is a quantity tending to 0 as $n \to \infty$.

2. A weak solution with negative energy

We recall that, for every $u \in H^1(\mathbb{R}^3)$, the Lax-Milgram theorem implies that there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$-\Delta\phi_u = u^2. \tag{2.1}$$

Furthermore, we can write the integral expression for ϕ_u:

$$\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{u^2(y)}{|x-y|}dy.$$

For more properties of ϕ_u, we refer the reader to [2]. It follows from (2.1), the Hölder and Sobolev inequalities that

$$\|\phi_u\|_D^2 = -\int_{\mathbb{R}^3}\phi_u\Delta\phi_u\,dx = \int_{\mathbb{R}^3}\phi_u u^2\,dx \leq |\phi_u|_6|u|_{12/5}^2 \leq S^{-1/2}\|\phi_u\|_D|u|_{12/5}^2,$$

then there exists a constant $a_1 = S^{-1}S_{12/5}^4 > 0$ such that

$$\int_{\mathbb{R}^3}\phi_u u^2\,dx \leq S^{-1/2}\|\phi_u\|_D|u|_{12/5}^2 \leq S^{-1}|u|_{12/5}^4 \leq S^{-1}S_{12/5}^4\|u\|^4 = a_1\|u\|^4. \tag{2.2}$$

In this article, we consider system (1.1) in $H_r^1(\mathbb{R}^3)$. Define the functional I_λ : $H_r^1(\mathbb{R}^3) \to \mathbb{R}$ by

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,dx + \frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_u u^2\,dx - \int_{\mathbb{R}^3}G(u)\,dx - \int_{\mathbb{R}^3}h(x)u\,dx.$$

It is standard to prove that I_λ is of class C^1 whose derivative is given by

$$\langle I'_\lambda(u), v\rangle = \int_{\mathbb{R}^3}\nabla u\cdot\nabla v\,dx + \lambda\int_{\mathbb{R}^3}\phi_u uv\,dx - \int_{\mathbb{R}^3}g(u)v\,dx - \int_{\mathbb{R}^3}h(x)v\,dx,$$

for all $v \in H_r^1(\mathbb{R}^3)$. Then, if u is a critical point of I_λ, the couple (u,ϕ_u) is a solution of (1.1). For simplicity, in many cases we just say that $u \in H_r^1(\mathbb{R}^3)$, instead of $(u,\phi_u) \in H_r^1(\mathbb{R}^3)\times D^{1,2}(\mathbb{R}^3)$, is a weak solution of system (1.1).

Lemma 2.1. *Suppose that $h \in L^2(\mathbb{R}^3)$ is a radial function and $h(x) \not\equiv 0$. Let* (A1) *and* (A9) *hold, then there exist $r, \alpha, \Lambda > 0$ such that $I_\lambda|_{S_r} \geq \alpha$ holds for $\lambda > 0$, $|h|_2 < \Lambda$.*

Proof. By (A1) and (A9), there exist $L, C > 0$ such that

$$G(t) \leq -Lt^2 + C|t|^6 \quad \forall t \in \mathbb{R}. \tag{2.3}$$

For each $\lambda > 0$, by the Hölder and Sobolev inequalities,

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,dx + L\int_{\mathbb{R}^3}u^2\,dx - C\int_{\mathbb{R}^3}|u|^6\,dx - |h|_2|u|_2 \\ &\geq \min\big\{\frac{1}{2}, L\big\}\|u\|^2 - CS^{-3}\|u\|^6 - |h|_2\|u\| \\ &= \|u\|\Big(\min\big\{\frac{1}{2}, L\big\}\|u\| - CS^{-3}\|u\|^5 - |h|_2\Big). \end{aligned} \tag{2.4}$$

Setting $f(t)=\min\{\frac12,L\}t-CS^{-3}t^5$ for $t\geq 0$, there exists

$$r=\Big(\frac{\min\{\frac12,L\}}{5CS^{-3}}\Big)^{1/4}>0$$

such that

$$\max_{t\geq 0}f(t)=f(r)=\frac{4(\min\{\frac12,L\})^{5/4}}{5(5CS^{-3})^{1/4}}=\Lambda.$$

Hence from (2.4) we deduce that if $|h|_2<\Lambda$, there exists $\alpha=\Lambda-|h|_2>0$ satisfying $I_\lambda|_{S_r}\geq\alpha$ for all $\lambda>0$. This completes the proof. $\square$

As in [6], we set

$$\begin{gathered}
g_1(t)=\begin{cases}(g(t)+mt)^+, & t\geq 0,\\ (g(t)+mt)^-, & t\leq 0,\end{cases}\\
g_2(t)=g_1(t)-g(t)-mt \quad \text{for } t\in\mathbb{R}.
\end{gathered}\tag{2.5}$$

Clearly, g_1 and g_2 satisfy

$$\lim_{t\to 0}\frac{g_1(t)}{t}=\lim_{|t|\to\infty}\frac{g_1(t)}{|t|^4t}=0,\tag{2.6}$$

$$g_2(t)t\geq 0 \quad \text{for all } t\in\mathbb{R}.\tag{2.7}$$

Lemma 2.2. *Suppose that* (A1) *and* (A9) *hold. Let $\{u_n\}\subset H^1_r(\mathbb{R}^3)$ be a bounded Palais-Smale sequence of I_λ, then $\{u_n\}$ has a convergent subsequence in $H^1_r(\mathbb{R}^3)$.*

Proof. Since $\{u_n\}$ is bounded, up to a subsequence, there exists $u\in H^1_r(\mathbb{R}^3)$ such that

$$\begin{gathered}
u_n\rightharpoonup u \quad \text{in } H^1_r(\mathbb{R}^3),\\
u_n\to u \quad \text{in } L^s(\mathbb{R}^3),\quad s\in(2,6),\\
u_n\to u \quad \text{a.e. in } \mathbb{R}^3.
\end{gathered}$$

We now show that $u_n\to u$ in $H^1_r(\mathbb{R}^3)$. We recall that $\{u_n\}$ is a bounded Palais-Smale sequence for I_λ, namely, $\{I_\lambda(u_n)\}$ is bounded and $I'_\lambda(u_n)\to 0$. Combining this with (2.5), we have

$$\begin{aligned}
&\langle I'_\lambda(u_n)-I'_\lambda(u),u_n-u\rangle\\
&=\int_{\mathbb{R}^3}|\nabla(u_n-u)|^2+m(u_n-u)^2\,dx+\lambda\int_{\mathbb{R}^3}(\phi_{u_n}u_n-\phi_u u)(u_n-u)\,dx\\
&\quad -\int_{\mathbb{R}^3}(g_1(u_n)-g_1(u))(u_n-u)\,dx+\int_{\mathbb{R}^3}(g_2(u_n)-g_2(u))(u_n-u)\,dx\\
&\geq\min\{1,m\}\|u_n-u\|^2+\lambda\int_{\mathbb{R}^3}(\phi_{u_n}u_n-\phi_u u)(u_n-u)\,dx\\
&\quad -\int_{\mathbb{R}^3}(g_1(u_n)-g_1(u))(u_n-u)\,dx\\
&\quad +\int_{\mathbb{R}^3}g_2(u_n)u_n-g_2(u_n)u-g_2(u)(u_n-u)\,dx.
\end{aligned}\tag{2.8}$$

It is clear that

$$\langle I'_\lambda(u_n)-I'_\lambda(u),u_n-u\rangle\to 0 \quad \text{as } n\to\infty.\tag{2.9}$$

When $n \to \infty$, by the Hölder and Sobolev inequalities, we easily obtain

$$\int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_u u)(u_n - u)\,dx \le (|\phi_{u_n}|_6 |u_n|_{12/5} + |\phi_u|_6 |u|_{12/5}) |u_n - u|_{12/5} \to 0. \tag{2.10}$$

Combining Strauss's lemma with (2.6) (see [6, Theorem A.I]), we have

$$\int_{\mathbb{R}^3} (g_1(u_n) - g_1(u))(u_n - u)\,dx \to 0 \quad \text{as } n \to \infty. \tag{2.11}$$

From (2.7) and Fatou's lemma, one deduces that

$$\liminf_{n\to\infty} \int_{\mathbb{R}^3} g_2(u_n) u_n \,dx \ge \int_{\mathbb{R}^3} g_2(u) u\,dx.$$

Furthermore, one obtains

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} g_2(u_n)u_n - g_2(u_n)u - g_2(u)(u_n - u)\,dx \ge 0 \quad \text{as } n \to \infty. \tag{2.12}$$

Using (2.9)–(2.12) in (2.8), we conclude that $u_n \to u$ in $H_r^1(\mathbb{R}^3)$ as $n \to \infty$. This completes the proof. □

Theorem 2.3. *Suppose that $h \in L^2(\mathbb{R}^3)$ is a radial function and $h(x) \not\equiv 0$. Let* (A1) *and* (A9) *hold, then* (1.1) *has a nontrivial solution $u_1 \in H_r^1(\mathbb{R}^3)$ with $I_\lambda(u_1) < 0$ for $\lambda > 0$, $|h|_2 < \Lambda$, where Λ is given by Lemma 2.1.*

Proof. We choose a $\varphi \in H_r^1(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} h(x)\varphi(x)\,dx > 0$. Moreover, by (A1) and (A9), for any $\delta > 0$, there exists $C_\delta > 0$ such that

$$|G(t)| \le C_\delta |t|^2 + \delta |t|^6 \quad \text{for all } t \in \mathbb{R}. \tag{2.13}$$

Hence, for $t > 0$ small enough, we have

$$I_\lambda(t\varphi) \le \int_{\mathbb{R}^3} \frac{t^2}{2}|\nabla\varphi|^2 + \frac{\lambda t^4}{4}\phi_\varphi \varphi^2\,dx + \int_{\mathbb{R}^3} C_\delta t^2|\varphi|^2 + \delta t^6 |\varphi|^6\,dx - \int_{\mathbb{R}^3} th\varphi\,dx < 0.$$

Thus, we obtain $c_0 = \inf_{u\in \bar{B}_r} I_\lambda(u) < 0$, where r is given by Lemma 2.1. Furthermore, by Ekeland's variational principle [12], there is a minimizing sequence $\{u_n\} \subset \bar{B}_r$ of c_0 such that

$$c_0 \le I_\lambda(u_n) \le c_0 + \frac{1}{n},$$
$$I_\lambda(\varphi) \ge I_\lambda(u_n) - \frac{1}{n}\|\varphi - u_n\| \quad \text{for all } \varphi \in \bar{B}_r.$$

It is standard to show that $\{u_n\}$ is a bounded $(PS)_{c_0}$ sequence of I_λ. By Lemma 2.2 we prove that $\{u_n\}$ possesses a convergent subsequence. Thus, we conclude that there exists $u_1 \in H_r^1(\mathbb{R}^3)$ such that $I_\lambda(u_1) = c_0 < 0$ and $I'_\lambda(u_1) = 0$. So we completed the proof. □

3. A weak solution with positive energy

Following [2], we introduce a cut-off function $\chi \in C^\infty(\mathbb{R}_+, [0,1])$ satisfying

$$\chi(t) = 1, \quad t \in [0,1],$$
$$0 \le \chi(t) \le 1, \quad t \in (1,2),$$
$$\chi(t) = 0, \quad t \ge 2,$$
$$|\chi'|_\infty \le 2$$

and the truncated functional $I_{\lambda,T}:H_r^1(\mathbb{R}^3)\to\mathbb{R}$ is defined as

$$
\begin{aligned}
&I_{\lambda,T}(u)\\
&=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,dx+\frac{\lambda}{4}h_T(u)\int_{\mathbb{R}^3}\phi_u u^2\,dx-\int_{\mathbb{R}^3}G(u)\,dx-\int_{\mathbb{R}^3}h(x)u\,dx\\
&=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2+mu^2\,dx+\frac{\lambda}{4}h_T(u)\int_{\mathbb{R}^3}\phi_u u^2\,dx-\int_{\mathbb{R}^3}G_1(u)-G_2(u)+h(x)u\,dx,
\end{aligned}
$$

where $T>0$, $h_T(u)=\chi(T^{-2}\|u\|^2)$. It is standard to prove that $I_{\lambda,T}$ is of class C^1 whose derivative is given by

$$
\begin{aligned}
&\langle I'_{\lambda,T}(u),v\rangle\\
&=\int_{\mathbb{R}^3}\nabla u\cdot\nabla v+muv\,dx+\lambda h_T(u)\int_{\mathbb{R}^3}\phi_u uv\,dx\\
&\quad+\frac{a_{\lambda,T}(u)}{2}\int_{\mathbb{R}^3}\nabla u\cdot\nabla v+uv\,dx-\int_{\mathbb{R}^3}g_1(u)v-g_2(u)v\,dx-\int_{\mathbb{R}^3}hv\,dx,
\end{aligned}
\tag{3.1}
$$

for all $v\in H_r^1(\mathbb{R}^3)$, where

$$
a_{\lambda,T}(u)=\lambda T^{-2}\chi'(T^{-2}\|u\|^2)\int_{\mathbb{R}^3}\phi_u u^2\,dx. \tag{3.2}
$$

For T sufficiently large and λ sufficiently small, we can find a critical point u such that $\|u\|\le T$ and prove that u is a critical point of I_λ.

We shall use the following Pohožaev type identity. The proof can be done similarly to that in [2] and details are omitted here.

Lemma 3.1. *Suppose that $h\in L^2(\mathbb{R}^3)$ is a radial function and $h(x)\not\equiv 0$. Let* (A1), (A9), (A10) *hold and $u\in H_r^1(\mathbb{R}^3)$ is a weak solution of* (1.1)*, then the following Pohožaev type identity holds*

$$
P_\lambda(u):=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,dx+\frac{5\lambda}{4}\int_{\mathbb{R}^3}\phi_u u^2\,dx-\int_{\mathbb{R}^3}3G(u)+3hu+(x\cdot\nabla h)u\,dx=0.
$$

Lemma 3.2. *For $8\lambda\widetilde{T}<\min\{1,m\}$, every bounded Palais-Smale sequence of $I_{\lambda,T}$ admits a convergent subsequence in $H_r^1(\mathbb{R}^3)$, where $\widetilde{T}=a_1T^2$, a_1 is given by* (2.2).

Proof. Let $\{u_n\}$ be a bounded Palais-Smale sequence of $I_{\lambda,T}$. Repeating the proof of Lemma 2.2, we easily obtain

$$
\begin{aligned}
o_n(1)&=\langle I'_{\lambda,T}(u_n)-I'_{\lambda,T}(u),u_n-u\rangle\\
&\ge\min\{1,m\}\langle u_n,u_n-u\rangle-\max\{1,m\}\langle u,u_n-u\rangle\\
&\quad+\lambda h_T(u_n)\int_{\mathbb{R}^3}\phi_{u_n}u_n(u_n-u)\,dx-\lambda h_T(u)\int_{\mathbb{R}^3}\phi_u u(u_n-u)\,dx\\
&\quad+\frac{a_{\lambda,T}(u_n)}{2}\langle u_n,u_n-u\rangle-\frac{a_{\lambda,T}(u)}{2}\langle u,u_n-u\rangle\\
&\quad-\int_{\mathbb{R}^3}(g_1(u_n)-g_1(u))(u_n-u)\,dx+\int_{\mathbb{R}^3}(g_2(u_n)-g_2(u))(u_n-u)\,dx\\
&=\Big(\min\{1,m\}+\frac{a_{\lambda,T}(u_n)}{2}\Big)\langle u_n,u_n-u\rangle,
\end{aligned}
$$

this shows that $\big(\min\{1,m\} + \frac{a_{\lambda,T}(u_n)}{2}\big)\langle u_n, u_n - u\rangle \to 0$ as $n \to \infty$. By (2.2) and (3.2), we have

$$|a_{\lambda,T}(u_n)| \le \lambda T^{-2}|\chi'(T^{-2}\|u_n\|^2)|\Big|\int_{\mathbb{R}^3} \phi_{u_n} u_n^2\,dx\Big| < 8\lambda\widetilde{T}. \tag{3.3}$$

For $8\lambda\widetilde{T} < \min\{1,m\}$, we conclude that

$$\min\{1,m\} + \frac{a_{\lambda,T}(u_n)}{2} \ge \min\{1,m\} - 4\lambda\widetilde{T} > 0.$$

Then $\langle u_n, u_n - u\rangle \to 0$. Combining with $u_n \rightharpoonup u$ in $H_r^1(\mathbb{R}^3)$, this implies that $u_n \to u$ in $H_r^1(\mathbb{R}^3)$. The proof is complete. □

Next, we prove that the functional $I_{\lambda,T}$ possesses a mountain pass geometry.

Lemma 3.3. *Suppose that $h \in L^2(\mathbb{R}^3)$ is a radial function and $h(x) \not\equiv 0$. Let* (A1), (A3), (A9) *hold, then there exist $r, \alpha, \Lambda > 0$ such that for $\lambda > 0$ and $|h|_2 < \Lambda$, we have*

(i) *$I_{\lambda,T}|_{S_r} \ge \alpha$,*
(ii) *there exists a function $\omega \in H_r^1(\mathbb{R}^3)\backslash\{0\}$ such that $\|\omega\| > r$ and $I_{\lambda,T}(\omega) < 0$.*

Proof. (i) Repeating the proof of Lemma 2.1, we prove the statement holds.
(ii) For any given $R > 1$, we define

$$\omega_R(x) = \begin{cases} \zeta, & |x| \le R,\\ \zeta(R+1-|x|), & R < |x| \le R+1,\\ 0, & |x| > R+1. \end{cases}$$

Through a direct calculation, we conclude that

$$\begin{gathered} \int_{\mathbb{R}^3} |\nabla\omega_R|^2\,dx = \zeta^2\,\mathrm{meas}\{B_{R+1} - B_R\},\\ \int_{\mathbb{R}^3} G(\omega_R)\,dx \ge G(\zeta)\,\mathrm{meas}\{B_R\} - \mathrm{meas}\{B_{R+1} - B_R\}(\max_{s\in[0,\zeta]}|G(s)|),\\ \int_{\mathbb{R}^3} |h(x)\omega_R|\,dx \le \Lambda\zeta(\mathrm{meas}\{B_{R+1}\})^{1/2}, \end{gathered}$$

where meas$\{\cdot\}$ denotes Lebesgue measure. Then there exist some $C_i > 0$ ($i = 1,2,3,4,5$) such that

$$\int_{\mathbb{R}^3} |\nabla\omega_R|^2\,dx \le C_1R^2, \tag{3.4}$$

$$\int_{\mathbb{R}^3} G(\omega_R)\,dx \ge C_2R^3 - C_3R^2, \tag{3.5}$$

$$\int_{\mathbb{R}^3} |h(x)\omega_R|\,dx \le C_4(R+1)^{3/2}. \tag{3.6}$$

Defining $\omega_{R,\theta} : \omega_R(\frac{x}{\theta})$ for $\theta > 0$ and combining with (3.4)–(3.6), we obtain

$$
\begin{aligned}
I_{\lambda,T}(\omega_{R,\theta}) &= \frac{\theta}{2}\int_{\mathbb{R}^3} |\nabla\omega_R|^2\,dx + \frac{\lambda}{4}h_T(\omega_{R,\theta})\int_{\mathbb{R}^3} \phi_{\omega_{R,\theta}}\omega_{R,\theta}^2\,dx \\
&\quad - \theta^3\int_{\mathbb{R}^3} G(\omega_R)\,dx - \int_{\mathbb{R}^3} h\omega_{R,\theta}\,dx \\
&\le \frac{\theta}{2}C_1R^2 - \theta^3(C_2R^3 - C_3R^2) + \theta^{3/2}C_4(R+1)^{3/2} \\
&\quad + \frac{\lambda}{4}\chi\Big(\frac{\theta\int_{\mathbb{R}^3}|\nabla\omega_R|^2\,dx + \theta^3\int_{\mathbb{R}^3}\omega_R^2\,dx}{T^2}\Big)\int_{\mathbb{R}^3}\phi_{\omega_{R,\theta}}\omega_{R,\theta}^2\,dx.
\end{aligned}
\tag{3.7}
$$

Therefore, we can choose $R > 1$ and $\theta > 0$ sufficiently large such that $\|\omega_{R,\theta}\| > \max\{r, \sqrt{2}T\}$ and $I_{\lambda,T}(\omega_{R,\theta}) < 0$. Namely, (ii) holds. This completes the proof. $\square$

Set $\theta > 0$ and $\bar{\omega_R} = \omega_R(\cdot/\theta)$. Define

$$
\gamma(t) = \begin{cases} 0, & t = 0, \\ \bar{\omega_R}(\cdot/t), & 0 < t \le 1. \end{cases}
$$

It is easy to see that γ is a continuous path from 0 to $\bar{\omega_R}$. Then, by Lemma 3.3, we define the mountain pass level

$$
c = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I_{\lambda,T}(\gamma(t)) > 0,
$$

where the set of paths $\Gamma := \big\{\gamma \in C([0,1], H_r^1(\mathbb{R}^3)) : \gamma(0) = 0 \text{ and } I_{\lambda,T}(\gamma(1)) < 0\big\}$.

Lemma 3.4. *Suppose that $h \in L^2(\mathbb{R}^3)$ is a radial function and $h(x) \not\equiv 0$. Let* (A1), (A3), (A9), (A10) *hold. Then there exists $\{u_n\}$ satisfying $I_{\lambda,T}(u_n) \to c$, $P_{\lambda,T}(u_n) \to 0$, and $I'_{\lambda,T}(u_n) \to 0$ in $[H_r^1(\mathbb{R}^3)]^*$, where*

$$
\begin{aligned}
P_{\lambda,T}(u_n) &= \Big(\frac{1}{2} + \frac{a_{\lambda,T}(u_n)}{4}\Big)\int_{\mathbb{R}^3}|\nabla u_n|^2\,dx + \frac{3a_{\lambda,T}(u_n)}{4}\int_{\mathbb{R}^3} u_n^2\,dx \\
&\quad + \frac{5\lambda}{4}h_T(u_n)\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\,dx \\
&\quad - \int_{\mathbb{R}^3} 3G(u_n)\,dx + 3hu_n + (x\cdot\nabla h)u_n\,dx.
\end{aligned}
\tag{3.8}
$$

Proof. Following Jeanjean [15], we define the map $\Phi : \mathbb{R} \times H_r^1(\mathbb{R}^3) \to H_r^1(\mathbb{R}^3)$ for $\sigma \in \mathbb{R}$ and $v \in H_r^1(\mathbb{R}^3)$ by $\Phi(\sigma, v)(x) = v(e^{-\sigma}x)$. For every $\sigma \in \mathbb{R}$ and $v \in H_r^1(\mathbb{R}^3)$, the functional $I_{\lambda,T} \circ \Phi$ is computed as

$$
\begin{aligned}
I_{\lambda,T}(\Phi(\sigma, v)) &= \frac{e^\sigma}{2}\int_{\mathbb{R}^3}|\nabla v|^2\,dx + \frac{\lambda e^{5\sigma}}{4}h_T(v(e^{-\sigma}x))\int_{\mathbb{R}^3}\phi_v v^2\,dx \\
&\quad - e^{3\sigma}\int_{\mathbb{R}^3} G(v) + h(e^\sigma x)v\,dx.
\end{aligned}
$$

In view of (A1), (A3), and (A9), $I_{\lambda,T} \circ \Phi$ is continuously Fréchet-differential on $\mathbb{R} \times H_r^1(\mathbb{R}^3)$. We also define

$$
\bar{c} = \inf_{\bar{\gamma}\in\bar{\Gamma}}\sup_{t\in[0,1]} (I_{\lambda,T}\circ\Phi)(\bar{\gamma}(t)),
$$

where the class $\bar\Gamma = \{\bar\gamma \in C([0,1], \mathbb{R}\times H_r^1(\mathbb{R}^3)) : \bar\gamma(0) = (0,0),\ (I_{\lambda,T}\circ\Phi)(\bar\gamma(1)) < 0\}$. Since $\Gamma = \{\Phi\circ\bar\gamma : \bar\gamma \in \bar\Gamma\}$, we verify that $c = \bar c$. Let $\bar\gamma = (0,\gamma)$, for every $\varepsilon \in (0, \frac{c}{2})$, there exists $\gamma \in \Gamma$ such that

$$\sup(I_{\lambda,T}\circ\Phi)(0,\gamma) \le c+\varepsilon.$$

Then, by [25, Theorem 2.8], there exists $(\sigma, v) \in \mathbb{R}\times H_r^1(\mathbb{R}^3)$ such that

(a) $c - 2\varepsilon \le (I_{\lambda,T}\circ\Phi)(\sigma,v) \le c+2\varepsilon$,
(b) $\operatorname{dist}\{(\sigma,v),(0,\gamma)\} \le 2\sqrt{\varepsilon}$, where $\operatorname{dist}\{(\sigma,v),(\tau,\vartheta)\} = (|\sigma-\tau|^2+\|v-\vartheta\|^2)^{1/2}$,
(c) $\|(I_{\lambda,T}\circ\Phi)'(\sigma,v)\| \le 2\sqrt{\varepsilon}$.

Then there exists a sequence $\{(\sigma_n, v_n)\} \subset \mathbb{R}\times H_r^1(\mathbb{R}^3)$ such that, as $n\to\infty$, we have

$$\sigma_n \to 0,\quad (I_{\lambda,T}\circ\Phi)(\sigma_n,v_n) \to c,\quad (I_{\lambda,T}\circ\Phi)'(\sigma_n,v_n)\to 0 \quad \text{in } [H_r^1(\mathbb{R}^3)]^*.$$

It is easy to prove that, for every $(h,\iota) \in \mathbb{R}\times H_r^1(\mathbb{R}^3)$,

$$(I_{\lambda,T}\circ\Phi)'(\sigma_n,v_n)[h,\iota] = I'_{\lambda,T}(\Phi(\sigma_n,v_n))[\Phi(\sigma_n,\iota)] + P_{\lambda,T}(\Phi(\sigma_n,v_n))h. \tag{3.9}$$

Then, taking $u_n = \Phi(\sigma_n, v_n)$, we obtain $I_{\lambda,T}(u_n)\to c$. Further, set $(h,\iota) = (1,0)$ and $(h,\iota) = (0,\Phi(-\sigma_n,\psi))$ in (3.9) in order, we conclude that $P_{\lambda,T}(u_n)\to 0$ and $\langle I'_{\lambda,T}(u_n),\psi\rangle \to 0$. As a consequence, we have

$$I_{\lambda,T}(u_n)\to c,\quad P_{\lambda,T}(u_n)\to 0,\quad I'_{\lambda,T}(u_n)\to 0 \quad \text{in } [H_r^1(\mathbb{R}^3)]^*. \tag{3.10}$$

Thus, we complete the proof. □

Lemma 3.5. *Under the assumptions of Lemma 3.4, let $\{u_n\}$ be given by (3.10). Then there exist $T_0 > 1$ and $\lambda_0 > 0$ satisfying $17\lambda_0 T_0^2 \widetilde{T_0} < \min\{1,m\}$ such that $\|u_n\| \le T_0$ for $\lambda \in (0,\lambda_0)$.*

Proof. Motivated by [28], we argue by contradiction. Suppose for every $T > 1$, there exists $\lambda_T > 0$ satisfying $17\lambda_T T^2 \widetilde{T} < \min\{1,m\}$ such that

$$\limsup_{n\to\infty} \|u_{n,\lambda_T}\| > T. \tag{3.11}$$

For simplicity, we denote u_{n,λ_T}, λ_T by u_n, λ respectively. By (3.8) and (3.10), $\{u_n\}$ satisfies the identity

$$\begin{aligned}
&\Big(\frac12 + \frac{a_{\lambda,T}(u_n)}{4}\Big)\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx + \frac{3a_{\lambda,T}(u_n)}{4}\int_{\mathbb{R}^3} u_n^2\,dx + \frac{5\lambda}{4}h_T(u_n)\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\,dx\\
&= \int_{\mathbb{R}^3} 3G_1(u_n) - 3G_2(u_n) - \frac{3m}{2}u_n^2 + 3hu_n\,dx + \int_{\mathbb{R}^3}(x\cdot\nabla h)u_n\,dx + o_n(1).
\end{aligned} \tag{3.12}$$

Actually, since $I_{\lambda,T}(u_n)\to c$,

$$\begin{aligned}
&\frac12\int_{\mathbb{R}^3} |\nabla u_n|^2 + mu_n^2\,dx + \frac{\lambda}{4}h_T(u_n)\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\,dx\\
&= \int_{\mathbb{R}^3} G_1(u_n) - G_2(u_n)\,dx + \int_{\mathbb{R}^3} hu_n\,dx + c + o_n(1).
\end{aligned} \tag{3.13}$$

By (A10), there exists a function $\xi(x) \in L^{\frac65}(\mathbb{R}^3)$ such that $|\nabla h(x)||x| \le \xi(x)$ for any $x\in\mathbb{R}^3$. Then from (A10) and (3.12)–(3.13) it follows that

$$\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx$$

$$
\begin{aligned}
&\le 3c+\frac{3|a_{\lambda,T}(u_n)|}{4}\|u_n\|^2+\frac{\lambda}{2}h_T(u_n)\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\,dx-\int_{\mathbb{R}^3}(x\cdot\nabla h)u_n\,dx+o_n(1)\\
&\le 3c+\frac{3|a_{\lambda,T}(u_n)|}{4}\|u_n\|^2+\frac{\lambda}{2}h_T(u_n)\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\,dx+|\xi|_{\frac{6}{5}}|u_n|_6+o_n(1)\\
&\le 3c+\frac{3|a_{\lambda,T}(u_n)|}{4}\|u_n\|^2+\frac{\lambda}{2}h_T(u_n)\int_{\mathbb{R}^3}\phi_{u_n}u_n^2+C\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2\,dx\Big)^{1/2}+o_n(1),
\end{aligned}
$$

we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3}|\nabla u_n|^2\,dx-C\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2\,dx\Big)^{1/2}\\
&\le 3c+\frac{3|a_{\lambda,T}(u_n)|}{4}\|u_n\|^2+\frac{\lambda}{2}h_T(u_n)\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\,dx+o_n(1).
\end{aligned}
\tag{3.14}
$$

Next, we turn to the estimate of right-hand side of (3.14). By the definition of χ, we obtain

$$
\lambda h_T(u_n)\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\,dx<4\lambda T^2\widetilde{T}. \tag{3.15}
$$

By Lemma 3.3 and (3.7), there exists $a_2>0$ such that

$$
\begin{aligned}
c&\le \max_\theta I_{\lambda,T}(\omega_{R,\theta}(x))\\
&\le \max_\theta\Big\{\frac{\theta}{2}C_1R^2-\theta^3(C_2R^3-C_3R^2)+\theta^{3/2}C_4(R+1)^{3/2}\Big\}\\
&\quad+\max_\theta\frac{\lambda}{4}\chi\Big(\frac{\|\omega_{R,\theta}(x)\|^2}{T^2}\Big)\int_{\mathbb{R}^3}\phi_{\omega_{R,\theta}}\omega_{R,\theta}^2\,dx\\
&=a_2+A_\lambda(T).
\end{aligned}
\tag{3.16}
$$

If $\|\omega_{R,\theta}(x)\|^2\ge 2T^2$, then $\chi\Big(\frac{\|\omega_R(\frac{x}{\theta})\|^2}{T^2}\Big)=0$. Thus, by (3.15), $A_\lambda(T)\le\lambda T^2\widetilde{T}$. By (3.3), one has

$$
\frac{3|a_{\lambda,T}(u_n)|}{4}\|u_n\|^2<12\lambda T^2\widetilde{T}. \tag{3.17}
$$

Then, by (3.14)–(3.17) we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^3}|\nabla u_n|^2\,dx-C\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2\,dx\Big)^{1/2}&\le 3(a_2+\lambda T^2\widetilde{T})+12\lambda T^2\widetilde{T}+2\lambda T^2\widetilde{T}+o_n(1)\\
&=3a_2+17\lambda T^2\widetilde{T}+o_n(1),
\end{aligned}
$$

which yields

$$
\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2\,dx\Big)^{1/2}\le\frac{C}{2}+\sqrt{\frac{C^2}{4}+3a_2+17\lambda T^2\widetilde{T}+o_n(1)}. \tag{3.18}
$$

Meanwhile, since $I'_{\lambda,T}(u_n)\to 0$, by (2.6) and (3.1),

$$
\begin{aligned}
&\Big(\min\{1,m\}+\frac{a_{\lambda,T}(u_n)}{2}\Big)\|u_n\|^2+\lambda h_T(u_n)\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\,dx+\int_{\mathbb{R}^3}g_2(u_n)u_n\,dx\\
&\le\int_{\mathbb{R}^3}g_1(u_n)u_n\,dx+\int_{\mathbb{R}^3}hu_n\,dx+o_n(1)\\
&\le\varepsilon|u_n|_2^2+C|u_n|_6^6+|h|_2\|u_n\|+o_n(1).
\end{aligned}
\tag{3.19}
$$

Thus, by (2.7) and (3.18)–(3.19) we obtain

$$
\begin{aligned}
&\Big(\min\{1,m\}+\frac{a_{\lambda,T}(u_n)}{2}-\varepsilon\Big)\|u_n\|^2-|h|_2\|u_n\| \\
&\le C|u_n|_6^6+o_n(1) \\
&\le CS^{-3}\|u_n\|_D^6+o_n(1) \\
&\le \frac{C}{S^3}\Big(\frac{C}{2}+\sqrt{\frac{C^2}{4}+3a_2+17\lambda T^2\widetilde{T}+o_n(1)}\Big)^6+o_n(1).
\end{aligned}
$$

Since $17\lambda T^2\widetilde{T}<\min\{1,m\}$, by (3.3) we have

$$
\min\{1,m\}+\frac{a_{\lambda,T}(u_n)}{2}\ge \min\{1,m\}-4\lambda\widetilde{T}>\frac{\min\{1,m\}}{2},
$$

we easily get

$$
\begin{aligned}
&\Big(\frac{\min\{1,m\}}{2}-\varepsilon\Big)\|u_n\|^2-|h|_2\|u_n\| \\
&\le \frac{C}{S^3}\Big(\frac{C}{2}+\sqrt{\frac{C^2}{4}+3a_2+1+o_n(1)}\Big)^6+o_n(1).
\end{aligned}
\tag{3.20}
$$

By (3.11), (3.20) is impossible for $T>1$ large enough. This completes the proof. □

Theorem 3.6. *Suppose that $h\in L^2(\mathbb{R}^3)$ is a radial function and $h(x)\not\equiv 0$. Let* (A1), (A3), (A9), (A10) *hold. Then there exist $\lambda_0,\Lambda>0$ such that* (1.1) *has a nontrivial solution $u_2\in H_r^1(\mathbb{R}^3)$ with $I_\lambda(u_2)>0$ for $\lambda\in(0,\lambda_0)$, $|h|_2<\Lambda$.*

Proof. Combining Lemmas 3.1–3.5 and the mountain pass theorem, we can find a critical point u_2 for $I_{\lambda,T}$ at c when λ and $|h|_2$ are sufficiently small. By Lemma 3.5, $\{u_n\}$ is a $(PS)_c$ sequence of $I_{\lambda,T}$ and satisfies $\|u_n\|\le T$, which implies that u_2 is a critical point for I_λ at c. Then we prove that there exist $\lambda_0,\Lambda>0$, such that (1.1) has a nontrivial radial solution u_2 with $I_\lambda(u_2)=c>0$ for $\lambda\in(0,\lambda_0)$, $|h|_2<\Lambda$. □

Proof of Theorem 1.2. It follows from Theorems 2.3 and 3.6, □

Proof of Corollary 1.3. For to this end, we construct a new system

$$
\begin{gathered}
-\Delta u+\lambda\phi u=\tilde{g}(u)+h(x)\quad \text{in } \mathbb{R}^3,\\
-\Delta\phi=u^2\quad \text{in } \mathbb{R}^3,
\end{gathered}
\tag{3.21}
$$

where $\tilde{g}:\mathbb{R}\to\mathbb{R}$ is defined by

$$
\tilde{g}(t)=\begin{cases}-mt, & t\le 0,\\ g(t), & t\ge 0,\end{cases}
$$

and define the energy functional $J_\lambda: H_r^1(\mathbb{R}^3)\mapsto\mathbb{R}$, corresponding to system (3.21), as

$$
J_\lambda(u)=\frac12\int_{\mathbb{R}^3}|\nabla u|^2\,dx+\frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_u u^2\,dx-\int_{\mathbb{R}^3}\tilde{G}(u)\,dx-\int_{\mathbb{R}^3}h(x)u\,dx,
$$

where $\tilde{G}(t)=\int_0^t\tilde{g}(s)ds$. It is standard to prove that J_λ is a well defined C^1-functional. Then, under the assumptions of Theorem 1.2, there exist $\tilde{\lambda_0},\tilde{\Lambda}>0$ such that system (3.21) has two nontrivial radial solutions $\tilde{u}_1,\tilde{u}_2$ for $\lambda\in(0,\tilde{\lambda_0})$,

$|h|_2 < \tilde{\Lambda}$, which satisfy that $J_\lambda(\tilde{u}_1) < 0 < J_\lambda(\tilde{u}_2)$. Further, letting $\tilde{u}_1^-$ be a test function, one has

$$\langle J'_\lambda(\tilde{u}_1), \tilde{u}_1^- \rangle = \int_{\mathbb{R}^3} |\nabla \tilde{u}_1^-|^2 \, dx + \int_{\mathbb{R}^3} \phi_{\tilde{u}_1^-} (\tilde{u}_1^-)^2 \, dx + \int_{\mathbb{R}^3} m |\tilde{u}_1^-|^2 \, dx - \int_{\mathbb{R}^3} h \tilde{u}_1^- \, dx,$$

which implies that $\tilde{u}_1^- = 0$ from $h(x) \geq 0$ in $\mathbb{R}^3$, so $\tilde{u}_1(x) \geq 0$ in $\mathbb{R}^3$. Namely, $\tilde{u}_1$ is also the nonnegative radial solution of (1.1) from the definition of $\tilde{g}$. By (A1) and (A9), there exist some $\bar{L} > 0$ such that

$$g(t) \geq -\bar{L}(|t| + |t|^5) \quad \text{for all } t \in \mathbb{R}. \tag{3.22}$$

It is clear that $\tilde{u}_1$ solves the equation

$$-\Delta u + \lambda \phi u + \bar{L}(1 + u^4) u = g(u) + \bar{L}(u + u^5) + h(x).$$

From the regular estimates of elliptic equations, we may deduce that $\tilde{u}_1 \in L^\infty_{\rm loc}(\mathbb{R}^3)$ and $\phi_{\tilde{u}_1} \in L^\infty_{\rm loc}(\mathbb{R}^3)$. Therefore, there exists $C(\Omega) > 0$ such that $-\Delta \tilde{u}_1 + C(\Omega)\tilde{u}_1 \geq 0$ in any bounded domain Ω. Applying the strong maximum principle [14, Theorem 8.9], we derive that $\tilde{u}_1(x) > 0$ in $\mathbb{R}^3$. Similarly, it can be proved that $\tilde{u}_2(x) > 0$ in $\mathbb{R}^3$. The proof is complete. □

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