ON THE PEBBLING NUMBERS
OF GRAPHS

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DEDICATION

I wish to dedicate this thesis to my loving mother, Sharon Collison.

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I would like to thank some individuals that helped me accomplish the task of writing this thesis. First, I would like to thank Dr. Jian Shen for his guidance as my professor in graph theory and then later as my thesis advisor. He continually challenged me, but gave me support so I could be successful in each challenge. Also, I am grateful to Dr. Daniela Ferrero and Dr. Xingde Jia, my thesis committee members, for reading my work and giving me constructive feedback. Their comments have helped to fine-tune my efforts.

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ABSTRACT

ON THE PEBBLING NUMBERS OF GRAPHS

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Graph pebbling is an application which has evolved from the study of graph theory. The goal of pebbling in a graph is to use *pebbling steps* to move one pebble onto a designated *root* vertex. A pebbling step is produced by taking two pebbles from a vertex, moving one of them to an adjacent vertex, and throwing out the other pebble. The *pebbling number* of a graph G, denoted $f(G)$, is the smallest integer t such that for any distribution of t pebbles on the vertices of G, one pebble can be moved to any specified root vertex.

Within this thesis is an exploration into the origins of basic theorems and properties of the pebbling function. There will be displayed a relationship between a graph's pebbling number and such characteristics as diameter and number of vertices. Also, new improvements are made to existing upper bounds of this function for specific types of graphs. One such finding is for a complete graph K_n with r missing edges where $r \leq \frac{n}{2} - 1$ the pebbling number is equal to n which categorizes this type of graph as Class 0. Another result is for the Cartesian product of a clique K_2 and a graph G the pebbling number has the upper bound of $2f(G) + \frac{n}{2} - \frac{1}{2}$. Finally we use the idea of a spanning tree to prove that for any graph G with n vertices and diameter d, there exists an upper bound $f(G) \leq (2^d - 1)\lfloor \frac{n-1}{d} \rfloor + 2^{n-1-d\lfloor \frac{n-1}{d} \rfloor}$ which is an improvement of the known upper bound $f(G) \leq (2^d - 1)(n-1) + 1$.

CHAPTER 1

INTRODUCTION

Consider a graph G with edge set $E(G)$ and vertex set $V(G)$ from which one of the vertices is designated as the *root* r. Also, suppose there exist n pebbles distributed among the vertices of G. The goal of pebbling is to use *pebbling steps* to move one pebble onto the root vertex. A pebbling step is produced by taking two pebbles from a vertex, moving one of them to an adjacent vertex, and throwing out the other pebble. If $u, v \in V(G)$ where u and v are adjacent, then a pebbling step which moves in the direction from u to v is denoted $[u, v]$.

Given a distribution D of pebbles, if one can use pebbling steps to move a pebble to the root vertex, r, then D is said to be *r-solvable*. If D is *r-solvable* for all r, then D is said to be solvable. The *pebbling number* of a graph G, denoted $f(G)$, is the minimum number t such that for any distribution of t pebbles is solvable. In other words, $f(G)$ is the smallest integer t such that for any distribution of t pebbles on the vertices of G, one pebble can be moved to any specified root vertex [1].

For the purposes of this thesis, G will always refer to a simple connected graph and $f(G)$ will denote the pebbling number of G. The number of vertices in a graph G will be $n(G)$ or just n while the number of pebbles distributed among a graph G or among a set X of vertices will be $D(G)$ and $D(X)$ respectively. The diameter of a graph G may be called d.

To further understand to process of pebbling in a graph, the following figure will demonstrate how pebbling steps are conducted. The vertices are labeled as a, b, c, d, and the root. Above each vertex is a number indicating the number of pebbles on that vertex.

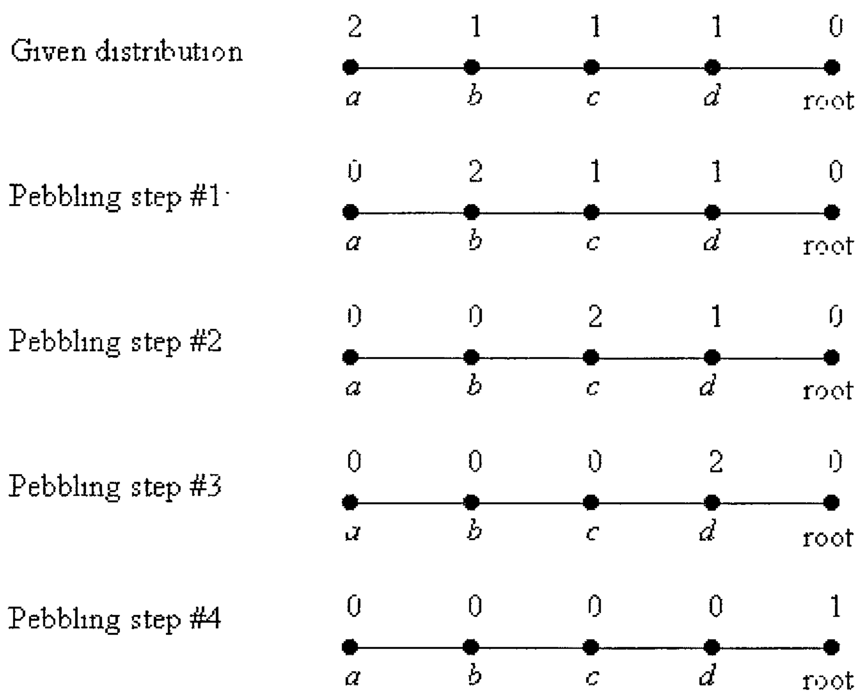

Figure 1: **Example of Pebbling Steps**

The given distribution has two pebbles on vertex a, one pebble on each of the remaining non-root vertices, and no pebble at the root. In the first pebbling step we pick up the two pebbles from a, throw one away, and move one to vertex b. After this step, the new distribution leaves no pebbles on a, two pebbles on b, and one pebble on each of c and d. For the second pebbling step, we take the two pebbles from b, throw one of them away, and move one to c. There are now two pebbles on c, one pebble on d, and still no pebble at the root. The third pebbling step is to pick up the two pebbles from c, throw one away, and move one to vertex d. After this step we have two pebbles at d and no pebbles distributed anywhere else on the graph. The fourth and final step is to pick up the two pebbles at d, throw one away, and move one to the root. After this series of pebbling steps we have successfully pebbled to the given root of the graph.

Within this thesis we will take a look at existing lower bounds and upper bounds for the pebbling number function and show how they were derived. Later we will improve upon the current upper bound for simple graphs. By creating a spanning tree and using its pebbling number as an upper bound, we prove that $f(\mathrm{G}) \leq (2^d - 1)\lfloor\frac{n-1}{d}\rfloor + 2^{n-1-d\lfloor\frac{n-1}{d}\rfloor}$. This strategy will require a proof of the upper bound for trees as well.

We will also take a look at Class 0 graphs which have the property $f(G) = n(G)$, and we will add a new type of graph to this category. Then, we will look at Graham's Conjecture for the Cartesian product of two graphs and attempt to prove the conjecture for a particular type of graph.

CHAPTER 2

EXISTING UPPER AND LOWER BOUNDS

We may begin by looking at upper and lower bounds for the pebbling function that have been previously proved by others. Note that we could receive a more specific result if we considered a particular type of graph, but here we are concerned with finding bounds that will apply for all simple graphs.

Fact 2.1 [1] *Let d= diam(G) and $n = n(G)$. Then*

$$\max\{n, 2^d\} \leq f(G) \leq (2^d - 1)(n - 1) + 1$$

.

Proof. In order to prove $f(G) > n - 1$, it suffices to prove that there is a distribution D_1 of $n - 1$ pebbles which is not solvable. Let $V = \{u_1, u_2, ..., u_n\}$. Define $D_1(u_i) = 1$ for all $1 \leq i \leq n - 1$ and $D_1(u_n) = 0$. Then D_1 is not solvable for the root vertex u_n.

In order to prove $f(G) > 2^d - 1$, it suffices to prove that there is a distribution D_2 of $2^d - 1$ pebbles which is not solvable. Since $diam(G) = d$, there are two vertices

u and v with distance d. Define $D_2(v) = 2^d - 1$ and $D_2(x) = 0$ for all $x \neq v$. Then D_2 is not solvable for the root vertex u. This holds since $\frac{D_2(v)}{2^{dist(u,v)}} = \frac{2^d-1}{2^d} < 1$.

Now, since $n \leq f(G)$ and $2^d \leq f(G)$, then $\max\{n, 2^d\} \leq f(G)$.

Finally, we will show $f(G) \leq (2^d - 1)(n-1) + 1$. Let there be a distribution of $(2^d - 1)(n-1) + 1$ pebbles on a graph G. If there exists a vertex with 2^d pebbles on it then we can pebble to the root, and we can show there must exist such a vertex. If we put $(2^d - 1)$ pebbles on each of the $(n-1)$ non-root vertices, then we have used all but one of the pebbles provided. This pebble must be placed on one of the $(n-1)$ vertices, call the vertex u, so that u now has 2^d pebbles. Consider the shortest path from u to the root r. Since $\text{diam}(G) = d$, it must be that $\text{dist}(u, r) \leq d$. Now, we want to make our pebbling moves toward the root only along this path; in fact we can make all of our pebbling moves with the use of only these 2^d pebbles. From u we can move 2^{d-1} pebbles to the vertex adjacent to u which is closer to the root r. From there we can move 2^{d-2} pebbles to the next vertex. Then 2^{d-3} to the next vertex and so on. If we make pebbling steps in this manner and if $\text{dist}(u, r) = d$, then we will end up with $2^{d-d} = 2^0 = 1$ pebble at the root. If $\text{dist}(u, r) < d$, then we could have more than one pebble at the root. In either case we know that we can pebble to r with this distribution of $(2^d - 1)(n-1) + 1$ pebbles. Therefore, $f(G) \leq (2^d - 1)(n-1) + 1$. ■

In Chapter 8 we will improve upon this upper bound. We can see that it is not likely that a graph must contain a vertex with 2^d pebbles in order to pebble to the root; we will show that it can be done with fewer pebbles and without the guarantee of a vertex containing 2^d pebbles.

CHAPTER 3

KNOWN GRAPHS AND THEIR PEBBLING NUMBERS

To fully understand the idea of the pebbling function and how pebbling numbers are assigned, we can look at some familiar graphs and their respective pebbling numbers.

First we may consider the graph of a path on $d+1$ vertices which would have a diameter of d.

Lemma 3.1 *Let P_{d+1} be a path on $d+1$ vertices. Then, $f(P_{d+1})=2^d$.*

Proof. Let P_{d+1} be a path on $d+1$ vertices $u_0, u_1, u_2, ..., u_d$.

Figure 2: **Path on $d+1$ Vertices**

For each $1 \leq i \leq d$. Let n_i be the number of pebbles on u_i. By definition, when a pebbling step is made, two pebbles are taken from a vertex, one is moved to an adjacent vertex, and one is discarded. Therefore in each pebbling step, a vertex can contribute at most half of its pebbles to an adjacent vertex. It follows that at most $n_1/2$ pebbles can be moved to the root. Now for the vertex v such that $\text{dist}(r, v) = 2$, it can move at most $n_2/2$ pebbles to the vertex adjacent to the root, and then from there at most $n_2/2^2$ can be moved to the root. In general, at most $n_i/2^i$ pebbles can be moved to the root from u_i.

To show $f(P_{d+1}) \leq 2^d$, we need to show that it is possible to move a pebble to any root for any distribution D of 2^d pebbles with $D(u_i) = n_i$

If the root is an endpoint, say u_0, then by the above argument, u_0 can receive $\frac{n_1}{2} + \frac{n_2}{2^2} + \frac{n_3}{2^3} + \ldots + \frac{n_d}{2^d} \geq \frac{n_1+n_2+ \quad +n_d}{2} = \frac{2^d}{2^d} = 1$ pebble.

If the root is a midpoint, say u_i with $1 \leq i \leq d-1$, then u_i is the endpoint of the left path of length i and is also the endpoint of the right path of length $d - i$. Since $2^d = 2^{d-1} + 2^{d-1} \geq 2^i + 2^{d-i}$, by the Pigeonhole Principle, either the left path P_{i+1} has 2^i pebbles or the right path P_{d-i+1} has 2^{d-i} pebbles. Thus u_i can receive a pebble from either the left or right path.

On the other hand, by Fact 2.1, $f(P_{d+1}) \geq 2^d$ since the diameter of P_{d+1} is d. Therefore, $f(P_{d+1}) = 2^d$. ■

Next we will consider the graph of a cycle on an even number of vertices, let us say C_{2k}.

Theorem 3.2 [1] *For $k \geq 1$, $f(C_{2k}) = 2^k$.*

Proof. Let G be the cycle C_{2k} which has $diam(G) = k$. By Fact 2.1 we know that $2^k \leq f(C_{2k})$, so we must show that $2^k = f(C_{2k})$. Let r be the root of the graph G and label the rest of the vertices $1, 2, \ldots, k-1, k, k+1, \ldots, 2k-1$ starting at the root and moving clockwise so that k is the vertex of G with $\text{dist}(k, r) = k$. Now, partition the vertices such that $\{1, 2, \ldots, k-1\} = X$ and $\{k+1, k+2, \ldots, 2k-1\} = Y$. Let

$D(X) = x$ and $D(Y) = y$, and without loss of generality let us assume that $x \geq y$.

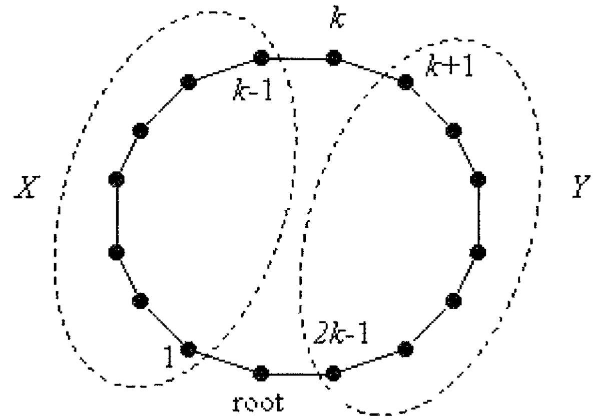

Figure 3: **Partition of Vertices of C_{2k}**

Case 1: Assume that vertex k contains all 2^k pebbles. By Lemma 3.1 we can pebble to r along the path of $V(X)$ or the path of $V(Y)$ since $dist(k, r) = k$.

Case 2: Assume that k contains less than 2^k pebbles. The number of pebbles on the vertex k can be expressed as $2^k - x - y$. Since $D(X) \geq D(Y)$, we will choose to move pebbles along the path which utilizes the vertices of X.

Subcase 2.1: Suppose $x \geq 2^{k-1}$. Since the distance from any of the vertices in X to the root r is less than $k-1$, we can pebble to r given this distribution.

Subcase 2.2: Suppose $x \leq 2^{k-1} - 1$. Since k contains $2^k - x - y$ pebbles, we know we can move $\lfloor \frac{2^k - x - y}{2} \rfloor$ of them to the vertex $k - 1 \in X$. If we move these $\lfloor \frac{2^k - x - y}{2} \rfloor$ pebbles to X, we would now have $\lfloor \frac{2^k - x - y}{2} \rfloor + x$ pebbles within X. Note that

$$\begin{aligned}
\lfloor \tfrac{2^k - x - y}{2} \rfloor + x &\geq \lfloor \tfrac{2^k - x - y}{2} \rfloor + \tfrac{2x}{2} \\
&\geq \lfloor \tfrac{2^k - x - y + 2x}{2} \rfloor \\
&= \lfloor \tfrac{2^k + x - y}{2} \rfloor \geq \lfloor \tfrac{2^k}{2} \rfloor \\
&= \tfrac{2^k}{2} = 2^{k-1}.
\end{aligned}$$

Thus there is now some distribution of 2^{k-1} pebbles in X, and since every vertex of X is within a distance of $k-1$ from the root r, we can pebble to r. Since $2^k \leq f(C_{2k})$ and $f(C_{2k}) \geq 2^k$, we have proved that $f(C_{2k})=2^k$ for $k \geq 1$. ■

CHAPTER 4

PEBBLING NUMBERS OF GRAPHS AND SUBGRAPHS

Another method for creating an upper bound for the pebbling number of a graph may be to look at a subgraph which has the same vertex set. A subgraph G' of G is called a spanning subgraph if $V(G') = V(G)$.

Lemma 4.1 *Let G be a graph and let G' be a connected spanning subgraph of G. Then $f(G) \leq f(G')$.*

Figure 4: **Example of a Graph G and a Spanning Graph G'**

Proof. Let G have any distribution of $f(G')$ pebbles. Then we can pebble to the root of G using only the edges of G'. Thus $f(G) \leq f(G')$. ■

CHAPTER 5

CLASS 0 GRAPHS

Graphs may be categorized into two groups according to their pebbling number. Recall from Fact 2.1 that $f(G) \geq n(G)$. Now, graphs that are Class 0 satisfy the condition that $f(G) = n(G)$ where $n(G)$ is the number of vertices of the graph. Class 1 graphs are all other graphs for which $f(G) > n(G)$ [1].

One example of a Class 0 graph is K_n, the complete graph on n vertices. We can easily show that $f(K_n) = n$ since the root will always be adjacent to every other vertex in the graph, and we may assume that the root has no pebble. If any vertex contains at least two pebbles it would satisfy the requirement for pebbling, and we can prove the existence of such a vertex using the Pigeonhole Principle. Let us instead consider the case of a complete graph minus a particular number of edges.

Theorem 5.1 *Let G be a complete graph K_n with r missing edges where $r \leq \frac{n}{2} - 1$. Then $f(G) = n$.*

Proof. By Fact 2.1, it suffices to prove that $f(G) \leq n$. This will be done by first showing that G has a Class 0 subgraph.

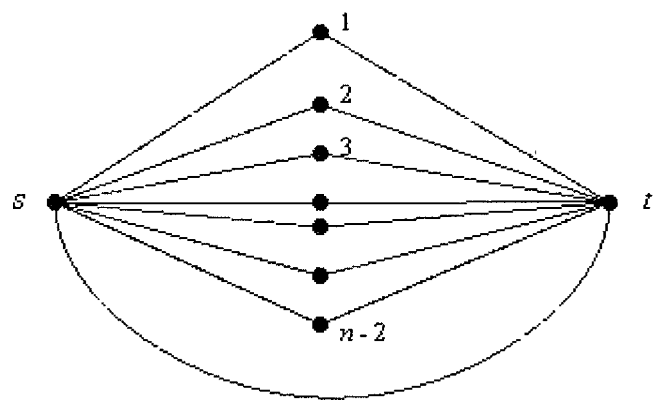

Figure 5: **Subgraph of G**

We will demonstrate the existence of the spanning subgraph shown in Figure 5 by proving that G contains two vertices s and t with degree $n-1$. This forces all other remaining vertices to have edges connecting to both s and t.

Now, a complete graph K_n has $n(n-1)/2$ edges. Thus, G has at least $\frac{n(n-1)}{2} - (\frac{n}{2} - 1) = \frac{n(n-2)}{2} + 1$ edges. If a graph has at least $\frac{n(n-2)}{2} + 1$ edges then the sum of the degrees of the vertices must be at least $2(\frac{n(n-2)}{2} + 1) = n(n-2) + 2$.

Suppose that the degree of all n vertices is less than $n-1$. Then each of the n vertices may have degree at most $n-2$. If we take the sum of the degrees of all the vertices at this point, we will only have $n(n-2)$. The sum must be increased by two. Now, since we cannot have any loops within our graph, we will need to add an edge from one of the vertices to some other vertex. This will increase the degree of each of those two vertices to $n-1$. Thus we have shown that there must exist at least two vertices with degree $n-1$. Thus G contains a spanning subgraph G' shown in Figure 5.

By Fact 2.1 and Lemma 4.1, $n \leq f(G) \leq f(G')$; thus it suffices to show that $f(G') \leq n$, we will first consider the case where the root of the graph G' is one of the vertices with degree $n-1$ (see s or t in figure 5). In this case the root has distance one to all other vertices, so if there are two pebbles on any adjacent vertex we can pebble to the root. Given a distribution of n pebbles that are to be placed on $n-1$ vertices, we can easily see by the Pigeonhole Principle that at least one of the vertices must contain at least two pebbles. Thus, for this case we can pebble to the root with n pebbles.

Next, consider the case where the root is one the vertices with degree two. Let the degree two vertices be labeled $1, 2, ..., n-2$. Without loss of generality, let $n-2$ be the root. Let D be any distribution of n pebbles on the vertices of G'. Let $D(i) = n_i$ for $1 \leq i \leq n-3$, $D(s) = n_s$, $D(t) = n_t$, and $D(n-2) = 0$. Since

$$\begin{aligned}(n_s + \textstyle\sum_{i=1}^{n-3} \frac{n_i - 1}{2}) + (n_t + \sum_{i=1}^{n-3} \frac{n_i-1}{2}) &= n_s + n_t + \textstyle\sum_{i=1}^{n-3} n_i - 1 \\ &= |D| - n + 3 \\ &= 3\end{aligned}$$

By the Pigeonhole Principle either $n_s + \sum_{i=1}^{n-3} \frac{n_i-1}{2} \geq 2$ or $n_t + \sum_{i=1}^{n-3} \frac{n_i-1}{2} \geq 2$. Without loss of generality, let us assume that $n_s + \sum_{i=1}^{n-3} \frac{n_i-1}{2} \geq 2$. Now, since each vertex i with $1 \leq i \leq n-3$ can move $\lfloor \frac{n_i}{2} \rfloor$ pebbles to the vertex s, then s will eventually have $n_s + \sum_{i=1}^{n-3} \frac{n_i}{2} \geq n_s + \sum_{i=1}^{n-3} \frac{n_i-1}{2} \geq 2$ pebbles. Therefore one pebble can be moved to the root vertex $n-2$ from s.

For any distribution of n pebbles on the vertices of G' we can pebble to the root, thus $f(G') = n$.

We are given the initial lower bound of $n \leq f(G)$ from Fact 2.1. Then we have shown that G' is a spanning subgraph of G, and proved that $f(G') \leq n$. Now using Lemma 4.1 we know that $f(G) \leq f(G')$. Therefore, $f(G) = n$ where G is a complete graph K_n with r missing edges where $r \leq \frac{n}{2} - 1$. ■

CHAPTER 6

CARTESIAN PRODUCTS OF GRAPHS

We will define the Cartesian product $G_1 \Box G_2$ of two graphs to be a graph with a vertex set $\{(v_1, v_2) | v_1 \in V(G_1) \text{ and } v_2 \in V(G_2)\}$ and an edge set $\{((v_1, v_2), (w_1, w_2)) | (v_1 = w_1 \text{ and } (v_2, w_2) \in E(G_2)) \text{ or } (v_2 = w_2 \text{ and } (v_1, w_1) \in E(G_1))\}$ [1].

Conjecture 6.1 (Graham) *For all G_1 and G_2, we have that $f(G_1 \Box G_2) \leq f(G_1)f(G_2)$.*

Graham's conjecture has been proven true for distinct situations. One of these situations is a clique times a graph which has a property known as the 2-pebbling property. A graph possesses this property if two pebbles can be moved to a root r given any distribution of $2f(G) - q + 1$ pebbles where q is the number of vertices containing a pebble [2]. We are interested in discovering if Graham's conjecture holds for a clique times any given graph without consideration of the 2-pebbling property. As of now, we have been unable to prove that Graham's conjecture holds for this case, but we have made some progress in that direction.

Let us look at the graph of $K_2 \Box G$ which can be thought of as two copies of the graph G with edges connecting their corresponding vertices.

Theorem 6.2 *Let K_2 be a clique and G a graph, then $f(K_2 \square G) \leq 2f(G) + \frac{n}{2} - \frac{1}{2}$.*

Proof. Let $f(G) = f$ for convenience, and let X_1 be one copy of the graph while X_2 is the other copy. We will refer to the number of pebbles distributed within X_1 as x_1 and the number in X_2 as x_2 such that $x_1 + x_2 \geq 2f + \frac{n}{2} - \frac{1}{2}$. Without loss of generality assume that the root is located within X_2.

Now, if $x_2 \geq f$, then we can pebble to the root just within X_2. Also, if $x_1 \geq 2f$ then we could move two pebbles to the vertex of X_1 that is adjacent to r and then we could pebble to the root. Thus, we will assume that $x_2 \leq f - 1$ and that $x_1 \leq 2f - 1$.

We may assume that there are not enough pebbles within X_2 to pebble to the root. Our remedy will be to move some pebbles from X_1 to X_2. We can partition X_1 into vertices containing an even number of pebbles and vertices containing an odd number of pebbles.

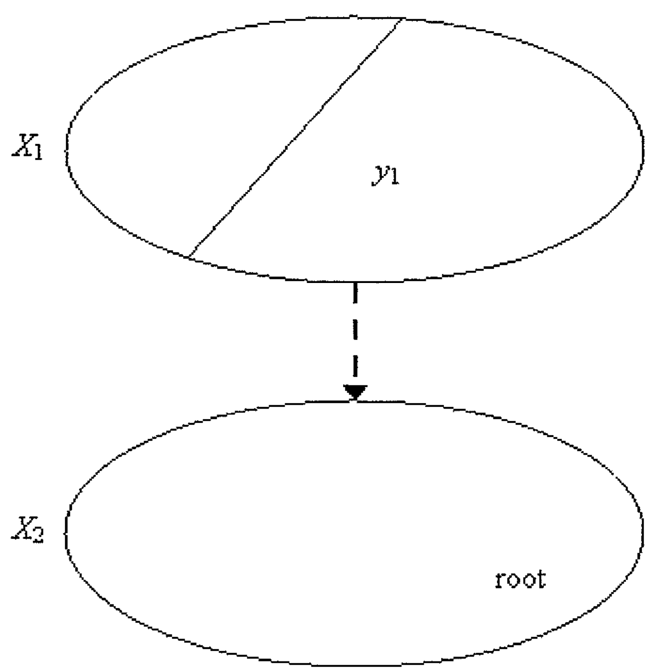

Figure 6: **Partition of X_1 and X_2**

Let y_1 be the number of vertices containing an odd number of pebbles. Then $y_1 \leq n$ and we can move $\frac{x_1 - y_1}{2}$ pebbles into X_2 by making pebbling steps from a vertex in X_1 to its neighbor in X_2.

After moving these pebbles from X_1 into X_2, X_2 has $x_2 + \frac{x_1 - y_1}{2}$ pebbles. Since $x_1 + x_2 \geq 2f + \frac{n}{2} - \frac{1}{2}$, $x_1 \leq 2f - 1$, and $y_1 \leq n$, we have

$$\begin{aligned} x_2 + \tfrac{x_1 - y_1}{2} &= (x_1 + x_2) - \tfrac{x_1}{2} - \tfrac{y_1}{2} \\ &\geq (2f + \tfrac{n}{2} - \tfrac{1}{2}) - \tfrac{2f-1}{2} - \tfrac{n}{2} \\ &\geq f. \end{aligned}$$

So, X_2 has at least f pebbles and thus we can pebble to the root within X_2. We have proved that we can always pebble to the root if $x_1 + x_2 \geq 2f + \frac{n}{2} - \frac{1}{2}$. So $f(K_2 \square G) \leq 2f + \frac{n}{2} - \frac{1}{2}$. ■

CHAPTER 7

PEBBLING NUMBERS OF TREES

In this section we want to create an upper bound for the pebbling number of trees. We will first create some lemmas that will help give us leverage as we define our upper bound. Our ultimate goal is to use this upper bound and apply it to non-tree graphs as well. This idea will be explored in depth in the next section.

For a rooted tree T where the root is given, define $f(T, r)$ to be the minimum number t for every distribution of t pebbles such that one can always move one pebble to the root r. Note that $f(T) = \max\{f(T, r)\}$ for $r \in V(T)$.

Lemma 7.1 *Suppose T is a path with endpoints u and v and suppose the root r is a mid-vertex with $dist(r, u) = l_1$ and $dist(r, v) = l_2$. Then*

$$f(T, r) = f(P_{l_1+1}) + f(P_{l_2+1}) - 1 = 2^{l_1} + 2^{l_2} - 1.$$

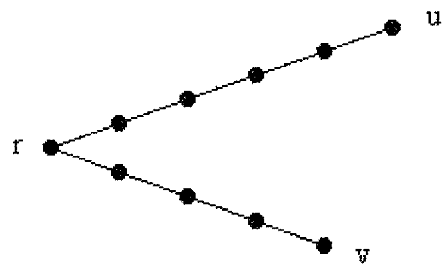

Figure 7: **Tree T with Two Leaves u and v**

Proof. Let there be any distribution of $f(P_{l_1+1}) + f(P_{l_2+1}) - 1$ pebbles on the vertices of T. By the Pigeonhole Principle, either the path with endpoints u and r has at least $f(P_{l_1+1})$ or the path with endpoints v and r has at least $f(P_{l_2+1})$ pebbles. If the former is true, one can pebble the root within the path with endpoints with endpoints u and r since $\text{dist}(u, r) = l_1$. If the latter is true, one can pebble to the root within the path with endpoints v and r since $\text{dist}(v, r) = l_2$. Thus $f(T, r) \leq f(P_{l_1+1}) + f(P_{l_2+1}) - 1$.

To show $f(T, r) > f(P_{l_1+1}) + f(P_{l_2+1}) - 2$, we can position $f(P_{l_1+1}) - 1$ pebbles at u and $f(P_{l_2+1}) - 1$ pebbles at v. Then no pebbles can be sent to the root from either side.

Therefore $f(T, r) = f(P_{l_1+1}) + f(P_{l_2+1}) - 1$. The second part of Lemma 7.1 follows from Lemma 3.1. ■

Lemma 7.2: *Let T be the tree defined in Lemma 7.1. Let S be a path with endpoints, m and n. Suppose r is the root of S with $dist(r, m) = l_1 + 1$ and $dist(r, n) = l_2 - 1$. If $l_1 \geq l_2$, then $f(T, r) < f(S, r)$.*

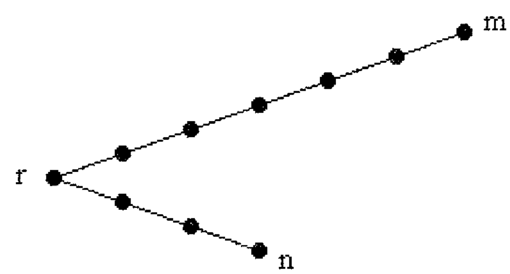

Figure 8: **Tree S with Two Leaves m and n**

Proof. By Lemma 7.1, we have $f(T,r) = 2^{l_1}+2^{l_2}-1$ and $f(S,r) = 2^{l_1+1}+2^{l_2-1}-1$. Now we will show that $2^{l_1} + 2^{l_2} - 1 < 2^{l_1+1} + 2^{l_2-1} - 1$.

Since we have assumed that $l_1 \geq l_2$, we can say

$$\frac{2^{l_2}}{2} < 2^{l_1}$$
$$2^{l_2} - \frac{2^{l_2}}{2} < 2(2^{l_1}) - 2^{l_1}$$
$$2^{l_2} - 2^{l_2-1} < 2^{l_1+1} - 2^{l_1}$$
$$2^{l_1} + 2^{l_2} < 2^{l_1+1} + 2^{l_2-1}$$
$$2^{l_1} + 2^{l_2} - 1 < 2^{l_1+1} + 2^{l_2-1} - 1$$

Therefore, $f(T,r) < f(S,r)$. ∎

This lemma is important to proving our proposed upper bound of the pebbling number of trees. Since we are searching for an upper bound of the pebbling function, we want to consider the type of graph that would have the largest pebbling number. We have shown that we can always construct a spanning tree with the same diameter but a larger pebbling number by increasing one path and simultaneously decreasing another. This new graph will produce our upper bound

If we are given a tree with n vertices and diameter d, we will use this idea to construct the next tree.

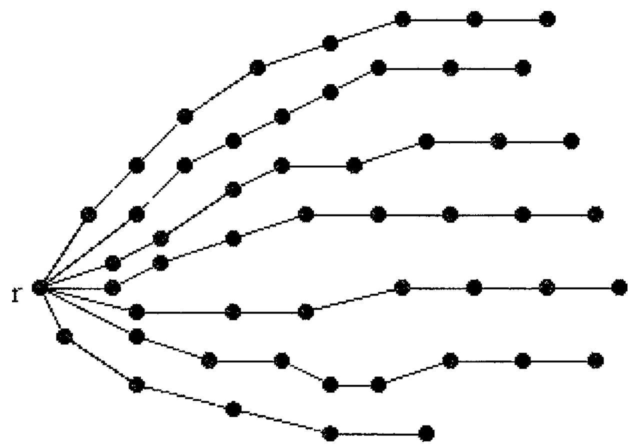

Figure 9: **Tree T on n Vertices**

Lemma 7.3: *Let T be a tree with n vertices such that there are only paths extending from r as in Figure 9. Suppose the distance between r to any other vertices is at most d, then*

$$f(T,r) \leq (2^d-1)\lfloor\frac{n-1}{d}\rfloor + 2^{n-1-d\lfloor\frac{n-1}{d}\rfloor}$$

.

Proof. Let T have t paths $P_1, P_2, ..., P_t$ from the root with lengths $l_1, l_2, ..., l_t$ respectively. Then $l_1 + l_2 + ... + l_t = n-1$ and $1 \leq l_i \leq d$ for all $1 \leq i \leq t$. For any distribution of $f(P_{l_1+1}) + f(P_{l_2+1}) + ... + f(P_{l_t+1}) - t + 1$ pebbles, by the Pigeonhole Principle there exists an i, $1 \leq i \leq t$, such that P_i has at least $f(P_{l_i+1})$ pebbles. Then the root r can receive a pebble within P_i. Thus

$$f(T,r) \leq f(P_{l_1+1}) + f(P_{l_2+1}) + ... + f(P_{l_t+1}) - t + 1$$

$$= 2^{l_1} + 2^{l_2} + ... + 2^{l_t} - t + 1.$$

To find the maximal value of $2^{l_1} + 2^{l_2} + ... + 2^{l_t} - t + 1$ subject to the constraint

$l_1 + l_2 + ... + l_t = n - 1$, by Lemma 7.2 one may assume that all paths $P_1, P_2, ..., P_t$, but at most one, have length d. We may have another path extending from the root r that has length $n - 1 - d\lfloor\frac{n-1}{d}\rfloor$. So there are $\lfloor\frac{n-1}{d}\rfloor$ paths of length d and one path of length $n - 1 - d\lfloor\frac{n-1}{d}\rfloor$ if $n - 1 - d\lfloor\frac{n-1}{d}\rfloor \geq 1$. Thus,

$$\begin{aligned} f(T, r) &\leq 2^{l_1} + 2^{l_2} + ... + 2^{l_t} - t + 1 \\ &= \textstyle\sum_{i=1}^{t}(2^{l_i} - 1) + 1 \\ &\leq [(2^d - 1)\lfloor\tfrac{n-1}{d}\rfloor] + (2^{n-1-d\lfloor\frac{n-1}{d}\rfloor} - 1) + 1 \\ &= (2^d - 1)\lfloor\tfrac{n-1}{d}\rfloor + 2^{n-1-d\lfloor\frac{n-1}{d}\rfloor} \end{aligned}$$

Therefore,

$$f(T, r) \leq (2^d - 1)\lfloor\tfrac{n-1}{d}\rfloor + 2^{n-1-d\lfloor\frac{n-1}{d}\rfloor}. \quad \blacksquare$$

Now, let us consider the case where a tree does not consist of only paths extending from the root. There may exist a portion of the graph that branches into two paths such that the parent vertex is not the root r.

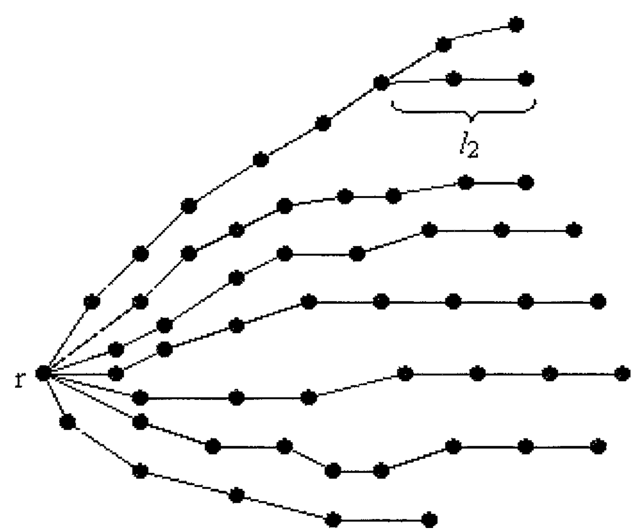

Figure 10: **Tree G with a Branch**

Lemma 7.4: *Let G be a tree, as in Figure 10, containing a branch such that the parent vertex is not r. Suppose the distance between r and any other vertex is at most d. Then,*

$$f(G,r) \leq (2^d-1)\lfloor\frac{n-1}{d}\rfloor + 2^{n-1-d\lfloor\frac{n-1}{d}\rfloor}.$$

Proof. Let G' be the graph of $G-l_2$. Similar to the Lemma 7.1, we see that the pebbling number of the entire graph G would be $f(G,r) \leq f(G',r)+2^{l_2}-1$. G' now has $n-l_2$ vertices and the pebbling number can be described as $f(G',r) \leq (2^d-1)\lfloor\frac{n-l_2-1}{d}\rfloor + 2^{n-l_2-1-d\lfloor\frac{n-l_2-1}{d}\rfloor}$.

Thus we can say that

$$f(G,r) \leq (2^d-1)\lfloor\frac{n-l_2-1}{d}\rfloor + 2^{n-l_2-1-d\lfloor\frac{n-l_2-1}{d}\rfloor} + 2^{l_2}-1.$$

We want to show that this is less than or equal to $(2^d-1)\lfloor\frac{n-1}{d}\rfloor+2^{n-1-d\lfloor\frac{n-1}{d}\rfloor}$ which is the upper bound for a tree with no branch. This is equivalent to showing,

$$2^{l_2}-1 \leq (2^d-1)(\lfloor\frac{n-1}{d}\rfloor-\lfloor\frac{n-l_2-1}{d}\rfloor)+2^{n-1-d\lfloor\frac{n-1}{d}\rfloor}-2^{n-l_2-1-d\lfloor\frac{n-l_2-1}{d}\rfloor}. \qquad (1)$$

First, suppose $d=l_2$. Then $\lfloor\frac{n-1}{d}\rfloor-\lfloor\frac{n-l_2-1}{d}\rfloor=1$ and $n-1-d\lfloor\frac{n-1}{d}\rfloor = n-l_2-1-d\lfloor\frac{n-l_2-1}{d}\rfloor$ gives

$$(2^d-1) = (2^d-1)(\lfloor\frac{n-1}{d}\rfloor-\lfloor\frac{n-l_2-1}{d}\rfloor)+2^{n-1-d\lfloor\frac{n-1}{d}\rfloor}-2^{n-l_2-1-d\lfloor\frac{n-l_2-1}{d}\rfloor}.$$

Thus, for this case $f(G,r) \leq (2^d-1)\lfloor\frac{n-1}{d}\rfloor+2^{n-1-d\lfloor\frac{n-1}{d}\rfloor}$.

Next, suppose $d>l_2$. Let $r=(n-1)\bmod d$ with $0\leq r\leq d-1$.

Case 1: $l_2 \leq r$

Then $\lfloor\frac{n-1}{d}\rfloor$=$\lfloor\frac{n-l_2-1}{d}\rfloor$, and inequality (1) becomes

$$2^{l_2}-1 \leq (2^d-1)0+2^{n-1-d\lfloor\frac{n-1}{d}\rfloor}-2^{n-l_2-1-d\lfloor\frac{n-1}{d}\rfloor}$$

which may be manipulated into

$$2^{l_2} + 2^{n-l_2-1-d\lfloor\frac{n-1}{d}\rfloor} - 1 \leq 2^{n-1-d\lfloor\frac{n-1}{d}\rfloor}.$$

This inequality is of the form $2^a + 2^b - 1 \leq 2^{a+b}$ where $a \geq 1$ and $b \geq 1$. We will show that this new inequality is true. Without loss of generality we may assume $a \leq b$.

$$\begin{aligned}2^a + 2^b - 1 &< 2^a + 2^b \\ &\leq 2^b + 2^b \\ &= 2^{b+1} \\ &\leq 2^{a+b}.\end{aligned}$$

Thus $2^a+2^b-1 \leq 2^{a+b}$ for $a \geq 1$ and $b \geq 1$. Therefore, $f(G,r) \leq (2^d-1)\lfloor\frac{n-1}{d}\rfloor + 2^{n-1-d\lfloor\frac{n-1}{d}\rfloor}$ for the case where $l_2 \leq r$.

Case 2: $l_2 \geq r+1$

Then $\lfloor\frac{n-1}{d}\rfloor = \lfloor\frac{n-l_2-1}{d}\rfloor + 1$, and inequality (1) becomes

$$2^{l_2} - 1 \leq (2^d-1)1 + 2^{n-1-d\lfloor\frac{n-1}{d}\rfloor} - 2^{n-l_2-1-d(\lfloor\frac{n-1}{d}\rfloor-1)}$$

$$2^{l_2} - 1 \leq 2^d - 1 + 2^{n-1-d\lfloor\frac{n-1}{d}\rfloor} - 2^{n-l_2-1-d\lfloor\frac{n-1}{d}\rfloor+d}$$

which may be manipulated into

$$2^{l_2} + 2^{n-l_2-1-d\lfloor\frac{n-1}{d}\rfloor+d} \leq 2^d + 2^{n-1-d\lfloor\frac{n-1}{d}\rfloor}.$$

Note that $d \geq l_2$ and $d \geq n-l_2-1-d\lfloor\frac{n-l_2-1}{d}\rfloor = n-l_2-1-d(\lfloor\frac{n-1}{d}\rfloor-1) = n-l_2-1-d\lfloor\frac{n-1}{d}\rfloor+d$. The above inequality is of the form $2^a+2^b \leq 2^c+2^{a+b-c}$ where $a \geq 1$, $b \geq 1$, and $c \geq \max\{a,b\}$. We will show that this new inequality is true. Without loss of generality we may assume $a \leq b$.

Subcase 2.1: Suppose $c = \max\{a,b\}$. Then $a+b-c = min\{a,b\}$ and thus

$$2^a + 2^b = 2^c + 2^{a+b-c}$$

For this subcase, $2^a + 2^b \leq 2^c + 2^{a+b-c}$.

Subcase 2.2: Suppose $c > \max\{a, b\}$.

$$\begin{aligned} 2^a + 2^b &\leq 2^{\max\{a,b\}} + 2^{\max\{a,b\}} \\ &= 2^{\max\{a,b\}+1} \\ &\leq 2^c \\ &\leq 2^c + 2^{a+b-c} \end{aligned}$$

For this subcase, $2^a + 2^b \leq 2^c + 2^{a+b-c}$.

We have shown inequality (1) to be true for all cases. This is equivalent to proving that

$f(\mathrm{G,r}) \leq (2^d - 1)\lfloor\frac{n-l_2-1}{d}\rfloor + 2^{n-l_2-1-d\lfloor\frac{n-l_2-1}{d}\rfloor} + 2^l_2 - 1 \leq (2^d - 1)\lfloor\frac{n-1}{d}\rfloor + 2^{n-1-d\lfloor\frac{n-1}{d}\rfloor}$.

Therefore, $f(\mathrm{G,r}) \leq (2^d - 1)\lfloor\frac{n-1}{d}\rfloor + 2^{n-1-d\lfloor\frac{n-1}{d}\rfloor}$ for a tree with a branch as in Figure 10. ■

We now know that a graph with this branch will have a pebbling number that is smaller than or equal to the one for a graph with no such branch. This is an important conclusion because it allows us to ignore the case where a tree has such a branch for the purpose of creating our upper bound for the pebbling function.

Theorem 7.5 *For any tree T with distance between the root to any other vertex at most d,*

$$f(T, r) \leq (2^d - 1)\lfloor\frac{n-1}{d}\rfloor + 2^{n-1-d\lfloor\frac{n-1}{d}\rfloor}.$$

Proof. Using induction, the preceding lemmas allow us to make this generalization about $f(T, r)$ for trees. ■

CHAPTER 8

UPPER BOUND FOR THE PEBBLING FUNCTION

Suppose T is a rooted tree with the distance from the root r to any other vertex is at most d. We have shown that the upper bound for the pebbling function $f(T, r)$ is $f(T, r) \leq (2^d - 1)\lfloor \frac{n-1}{d} \rfloor + 2^{n-1-d\lfloor \frac{n-1}{d} \rfloor}$.

Now, we want to point out that this upper bound could be applied to any non-tree graph as well.

Theorem 8.1 *For any graph G with n vertices and diameter d,* $f(\mathrm{G}) \leq (2^d - 1)\lfloor \frac{n-1}{d} \rfloor + 2^{n-1-d\lfloor \frac{n-1}{d} \rfloor}$.

Proof. For any given root in G, we can perform an algorithm to find a spanning tree rooted at r. One such appropriate algorithm would be the Breadth-First Search. Within this process we would start at the root vertex r and pick up all adjacent edges. After this step we have reached all vertices that have distance one to the root. From the first distance one vertex we will pick up any edges that are adjacent to it if it will not create a cycle. We will repeat this process of picking up edges for all distance one vertices. Then from the first distance two vertex we will pick up adjacent edges providing it does not create a cycle. This process will continue until all vertices of the graph are reached. When this is completed, the original vertex set, along with the edges we picked up, will be our spanning tree, denoted T.

Since G has diameter d, the distance between r and any vertex in T is at most d. By Lemma 4.1, $f(G) \leq f(T)$ and by Theorem 7.5, $f(T) \leq (2^d-1)\lfloor\frac{n-1}{d}\rfloor+2^{n-1-d\lfloor\frac{n-1}{d}\rfloor}$.

Therefore, we have $f(\mathrm{G}) \leq (2^d-1)\lfloor\frac{n-1}{d}\rfloor+2^{n-1-d\lfloor\frac{n-1}{d}\rfloor}$. ■

Now, we will prove that this new upper bound for the pebbling function is an improvement over the former upper bound of $(2^d-1)(n-1)+1$. To do this we must show that

$$(2^d-1)\lfloor\frac{n-1}{d}\rfloor+2^{n-1-d\lfloor\frac{n-1}{d}\rfloor} \leq (2^d-1)(n-1)+1. \tag{2}$$

It is important to note that since d represents the diameter of a graph and n represents the number of vertices of that graph, $1 \leq d \leq n-1$.

Case 1: $d=1$

For this case we have

$$\begin{aligned}(2^d-1)\lfloor\tfrac{n-1}{d}\rfloor+2^{n-1-d\lfloor\frac{n-1}{d}\rfloor} &= (2-1)\lfloor n-1\rfloor+2^{n-1-\lfloor n-1\rfloor} \\ &=(2^1-1)(n-1)+1 \\ &=(2^d-1)(n-1)+1.\end{aligned}$$

Thus inequality (2) is true for the case $d=1$.

Case 2: $d \geq 2$

First, let us look at $2^{n-1-d\lfloor\frac{n-1}{d}\rfloor}$. We can express $n-1$ as

$$n-1=dx+r$$

meaning that $n-1$ is equal to some multiple of d plus a remainder r.

Thus,

$$\begin{aligned}
2^{n-1-d\lfloor\frac{n-1}{d}\rfloor} &= 2^{(dx+r)-d\lfloor\frac{dx+r}{d}\rfloor} \\
&= 2^{(dx+r)-d\lfloor\frac{dx}{d}+\frac{r}{d}\rfloor} \\
&= 2^{(dx+r)-d\lfloor x+\frac{r}{d}\rfloor} \\
&= 2^{(dx+r)-d\lfloor x\rfloor-d\lfloor\frac{r}{d}\rfloor} \\
&= 2^{(dx+r)-dx-d\lfloor\frac{r}{d}\rfloor} \\
&= 2^{r-d\lfloor\frac{r}{d}\rfloor}.
\end{aligned}$$

Now, since $r < d$,

$$\begin{aligned}
2^{r-d\lfloor\frac{r}{d}\rfloor} &= 2^{r-d(0)} \\
&= 2^{r}.
\end{aligned}$$

And since $r \leq d-1$,

$$2^{r} \leq 2^{d-1}.$$

This leads us to the conclusion that

$$2^{n-1-d\lfloor\frac{n-1}{d}\rfloor} \leq 2^{d-1}.$$

We will use this fact to show that inequality (2) is true.

$$\begin{aligned}
(2^d-1)\lfloor\tfrac{n-1}{d}\rfloor + 2^{n-1-d\lfloor\frac{n-1}{d}\rfloor} &\leq (2^d-1)\lfloor\tfrac{n-1}{d}\rfloor + 2^{d-1} \\
&\leq (2^d-1)\lfloor\tfrac{n-1}{d}\rfloor + 2^{d}-1 \\
&\leq (2^d-1)(\lfloor\tfrac{n-1}{d}\rfloor + 1).
\end{aligned}$$

Now to finish proving the inequality $(2^d-1)(\lfloor\frac{n-1}{d}\rfloor+1) \leq (2^d-1)(n-1)+1$ we will prove the equivalent statement that $(2^d-1)(\lfloor\frac{n-1}{d}\rfloor+1)-(2^d-1)(n-1) \leq 1$.

$$\begin{aligned}
(2^d-1)(\lfloor\tfrac{n-1}{d}\rfloor+1)-(2^d-1)(n-1) &= (2^d-1)[(\lfloor\tfrac{n-1}{d}\rfloor+1)-(n-1)] \\
&= (2^d-1)[\lfloor\tfrac{n-1}{d}\rfloor+1-n+1]
\end{aligned}$$

$$= (2^d - 1)[\lfloor \tfrac{n-1}{d} \rfloor - n + 2].$$

Since we are considering the case where $d \geq 2$, we know that $\frac{n-1}{d} \leq \frac{n-1}{2} \leq n - 2$.

So,

$$\begin{aligned}(2^d - 1)[\lfloor \tfrac{n-1}{d} \rfloor - n + 2] &\leq (2^d - 1)[(n-2) - n + 2] \\ &= (2^d - 1)(0) \\ &\leq 1.\end{aligned}$$

Thus we have shown that $(2^d - 1)(\lfloor \frac{n-1}{d} \rfloor + 1) - (2^d - 1)(n-1) \leq 1$. This inequality is equivalent to $(2^d - 1)(\lfloor \frac{n-1}{d} \rfloor + 1) \leq (2^d - 1)(n-1) + 1$.

Since $(2^d - 1)\lfloor \frac{n-1}{d} \rfloor + 2^{n-1-d\lfloor \frac{n-1}{d} \rfloor} \leq (2^d - 1)(\lfloor \frac{n-1}{d} \rfloor + 1) \leq (2^d - 1)(n-1) + 1$, we have proved that $(2^d - 1)\lfloor \frac{n-1}{d} \rfloor + 2^{n-1-d\lfloor \frac{n-1}{d} \rfloor} \leq (2^d - 1)(n-1) + 1$. The two upper bounds are the same when $d = 1$; however, our bound is stronger when $d \geq 2$. ■

We have successfully improved the upper bound for the pebbling number of any simple graph G using the strategy of constructing a spanning tree.

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VITA

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