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EXISTENCE OF TIME-PERIODIC SOLUTIONS TO INCOMPRESSIBLE NAVIER-STOKES EQUATIONS IN THE WHOLE SPACE

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Abstract. In this article, we assume that the force field acting over a fluid is periodic on time and the velocity of the liquid is zero at spatial infinity. We prove the existence of time-periodic solutions to the system governing the motion of an incompressible fluid filling the whole space.

1. Introduction

It is well known that the incompressible Navier-Stokes equations moving in $\mathbb{R}^3$ with the action of time-periodic body force is an important research field. In 1959, Serrin [7] studied the existence of time-periodic solutions in bounded domains. Once we consider the case of unbounded domains, the Pioncare inequality is not valid anymore. Therefore, we need some different methods. So far, there exist many results in the literature in the case of unbounded domains. Maremonti [5] first showed the existence and uniqueness of time-periodic strong solutions, under the assumptions that the body force is the form of $\operatorname{curl}\Psi$ and the initial data are small enough. Later, Maremonti and Padula [6] showed the existence of time-periodic solutions, provided the body force has the property of symmetry. Recently, Galdi and Sohr [3] obtained the existence and uniqueness of the time-periodic solution on the condition that the force is sufficiently small and the force is the form of $\operatorname{div} F$. Salvi [8] also discussed the existence of time-periodic solutions with periodically moving boundaries, using the elliptic regularization.

In this paper, we will prove the existence of time-periodic solutions to incompressible Navier-Stokes equations. In [8], Salvi showed the similar result, using the different method. Moreover, in this paper, we will prove the similar result without the conditions on body force in [5, 3, 6].

This paper will be organized as follows. In section 2, we will introduce some notations. Employing the Galerkin method, in section 3, we show the existence and the uniqueness of the time-periodic solution to Navier-Stokes systems with the prescribed artificial terms. Finally, vanishing the effect of the artificial terms, we prove the main result we want to get.

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2. Notation

Assume that Ω is an open, connected domain. Denoted by $L^q(\Omega)$, $W^{m,q}(\Omega)$, $m \geq 0$, $1 \leq q \leq \infty$ the usual Lebesgue and Sobolev spaces. Let $\|\cdot\|_q$, $\|\cdot\|_{m,q}$ be the norms in $L^q(\Omega)$ and $W^{m,q}(\Omega)$, respectively. Denoted by $D(\Omega)$ the space of infinitely differentiable and divergence-free functions with compact support in Ω. For any ϕ, ψ, we define

$$(\Phi,\Psi) = \int_{\mathbb{R}^3} \Phi\Psi \,\mathrm{d}x$$

if the integral is finite. Denoted by $L^p((0,t);X)$ the set of functions Φ from $(0,t)$ into X such that $\int_0^t (|\Phi(\tau)|_X)^p \,\mathrm{d}\tau < \infty$, where X is a Banach space. Finally, denoted by $C((0,t);X)$ the continuous functions from (0,t) into X with norms $sup_{(0,t)}|\Phi|_X$.

We consider the following Navier-Stokes system in $(0,T)\times\mathbb{R}^3$:

$$\begin{gathered}
u_t(x,t) - \Delta u(x,t) + u(x,t)\cdot\nabla u(x,t) + \nabla p(x,t) = f(x,t),\\
\nabla\cdot u(x,t) = 0,\\
u(x,t) \to 0 \quad \text{as } |x|\to\infty,\\
u(x,0) = u_0(x),
\end{gathered} \tag{2.1}$$

which describes the motion of an incompressible viscous fluid, with viscosity $\upsilon = 1$, filling the whole space $\mathbb{R}^3$ and subject to a body force $f(x,t)$.

Our main result is as follows.

Theorem 2.1. *Let $u_0 \in H^{2,2}(\mathbb{R}^3)$ with $\nabla\cdot u_0 = 0$. Suppose that $f(x,t)$ belongs to $L^2((0,T);H^{2,2}(\mathbb{R}^3))$ and f is a time-periodic function with period T. Then the system* (2.1) *has at least one time-periodic solution u in $L^\infty(0,T;L^2(\mathbb{R}^3))$, with ∇u in $L^2(0,T;L^2(\mathbb{R}^3))$ in the sense of distributions.*

We will give the proof of Theorem 2.1 in section 4 by using a two-level approximation scheme based on the following system:

$$\begin{gathered}
u_t(x,t) - \Delta u(x,t) + u(x,t)\cdot\nabla u(x,t) + \varepsilon(-\Delta)^2 u(x,t) + \eta u(x,t) + \nabla p(x,t)\\
= f(x,t),\\
\nabla\cdot u(x,t) = 0,\\
u(x,t) \to 0 \quad \text{as } |x|\to\infty,\\
u(x,0) = u_0(x).
\end{gathered} \tag{2.2}$$

Where ε and η are positive numbers.

3. The Faedo-Galerkin Approximation

In this section, we will use the Faedo-Galerkin method to solve the system (2.2).

Theorem 3.1. *Let $u_0 \in H^{2,2}(\mathbb{R}^3)$ with $\nabla\cdot u_0 = 0$. Suppose that $f(x,t)$ belongs to $L^2((0,T);H^{2,2}(\mathbb{R}^3))$. Then* (2.2) *has unique solution u in $L^\infty(0,T;H^{2,2}(\mathbb{R}^3))$, with ∇u in $L^2(0,T;H^{2,2}(\mathbb{R}^3))$.*

Proof. It is easy to obtain the existence of Theorem 3.1 by applying the Galerkin method. Furthermore, from the equation, we have $u_t \in L^\infty(0,T;L^3(\mathbb{R}^3))$. Next, we prove the uniqueness as follows.

Suppose that u, v are two solutions to the system (2.2). Set $w = u - v$. By the direct calculation, we can get

$$\frac{d}{dt}\int_{\mathbb{R}^3} |w(x,t)|^2\,\mathrm{d}x + \varepsilon\int_{\mathbb{R}^3}(-\Delta)w(x,t)]^2\,\mathrm{d}x + \int_{\mathbb{R}^3}[\nabla w(x,t)]^2\,\mathrm{d}x + \eta\int_{\mathbb{R}^3}[w(x,t)]^2\,\mathrm{d}x \\ = -\int_{\mathbb{R}^3} w(x,t)\nabla u(x,t)w(x,t)\,\mathrm{d}x.$$

It is easy to prove u is in $L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^\infty(0,T;H^{2,2}(\mathbb{R}^3))$ by the energy method. Then, by the imbedding theorem $W^{1,r} \hookrightarrow L^\infty$, where $r > 3$, we have that u is in $L^\infty(0,T;L^\infty(\mathbb{R}^3))$. Furthermore, applying the interpolation theorem, $u \in L^\infty(0,T;L^3(\mathbb{R}^3))$. Therefore, we have $u \in C(0,T;L^3(\mathbb{R}^3))$ because u_t is in $L^\infty(0,T;L^3(\mathbb{R}^3))$.

Now, we decompose u as $u = u_1 + u_2$ such that $\|u_1\|_{L^\infty(0,T;L^3(\mathbb{R}^3))} \le \alpha$ and $\|u_2\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))} \le C_\alpha$, where $\alpha > 0$ is small enough. Concerning the term in the right hand side of above equality, we have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^3} w(x,s)\nabla u(x,s)w(x,s)\,\mathrm{d}x\Big| \\
&= \Big|\int_{\mathbb{R}^3} w(x,s)\nabla w(x,s)u(x,s)\,\mathrm{d}x\Big| \\
&\le \alpha\|\nabla w(x,t)\|_2\|w(x,t)\|_6 + C_\alpha\|w(x,t)\|_2\|\nabla w(x,t)\|_2 \\
&\le \alpha\|\nabla w(x,t)\|_2^2 + C\|w(x,t)\|_2^2.
\end{aligned}$$

Therefore, choosing $\alpha \le 1/2$ and using the Granwall's inequality, we can show that $w(x,t) = 0$. □

Theorem 3.2. *Under the assumptions in Theorem 3.1, the system* (2.2) *has unique time-periodic solution* $u \in L^\infty(0,T;H^{2,2}(\mathbb{R}^3))$.

Proof. The proof of uniqueness is similar to that of Theorem 3.1. Thus, we only need to show the existence.

Assume that $u_n(x,T)$ is the Galerkin approximation solutions of system (2.2). Then, we consider the map: $\Upsilon : u_n(x,0) \to u_n(x,T)$. It is easy to show that this map is continuous from the finite-dimensional space X_n to itself, where X_n are the finite dimensional spaces in the Galerkin approximation. More important, we also have the non-expansive property, i.e., if R is large enough which will be determined later and $\|u_n(x,0)\|_2 \le R$, then $\|u_n(x,T)\|_2 \le R$. For simplicity, we set $u_n(x,T) = u(x,t)$.

This property can be showed as follows: We multiply the equation (2.2) by $2u(x,t)$. Since $f(x,t) \in L^2((0,T);L^2(\mathbb{R}^3))$, by interpolating inequality, Young inequality, Sobolev inequality, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3} |u(x,t)|^2\,\mathrm{d}x + 2\varepsilon\int_{\mathbb{R}^3}[(-\Delta)^{\frac{\sigma}{2}}u(x,t)]^2\,\mathrm{d}x \\
&+ 2\int_{\mathbb{R}^3}[\nabla u(x,t)]^2\,\mathrm{d}x + 2\eta\int_{\mathbb{R}^3}[u(x,t)]^2\,\mathrm{d}x \\
&= 2\int_{\mathbb{R}^3} f(x,t)u(x,t)\,\mathrm{d}x \\
&\le 2\|f(x,t)\|_2\|u(x,t)\|_2 \\
&\le \eta\|u(x,t)\|_2^2 + C_\eta\|f(x,t)\|_2^2.
\end{aligned}$$

Therefore, we have

$$\frac{d}{dt}\int_{\mathbb{R}^3} |u(x,t)|^2 \,\mathrm{d}x + \eta\|u(x,t)\|_2^2 \le C\|f(x,t)\|_2^2.$$

From this inequality, we obtain

$$\frac{d}{dt}(e^{\eta t}\|u(x,t)\|_2^2) \le e^{\eta T}\int_0^T \|f(x,t)\|_2^2 \,\mathrm{d}t.$$

Integrating in (0,T) with respect to t, we get

$$e^{\eta T}\|u(x,t)\|_2^2 - \|u_0(x)\|_2^2 \le Te^{\eta T}\int_0^T \|f(x,t)\|_2^2 \,\mathrm{d}t.$$

Therefore,

$$e^{\eta T}\|u(x,t)\|_2^2 \le \|u_0(x)\|_2^2 + C.$$

Since $\|u_0(x)\|_2^2 \le R^2$, we have

$$e^{\eta T}\|u(x,t)\|_2^2 \le R^2 + C.$$

Let $R^2 + C \le e^{\eta T}R^2$. Then, we have $\|u(x,t)\|_2^2 \le R^2$. This completes the proof of the property.

Then, by the Brouwer fixed point theorem, the map Υ has a fixed point, that is to say that there exists a function $u(x,t)$ such that $u(x,0) = u(x,T)$. Therefore, for any fixed n, we obtained a time-periodic solution u_n to the Galerkin approximation equations of system (2.2). At last, applying the Galerkin method again, we finish the proof of Theorem 3.2. □

4. The Limit as the Artificial Term Vanishes

In section 3, we have proved the existence and uniqueness of the time-periodic solution to system (2.2) for any given positive numbers ε and η. Unfortunately, these solutions may depend on the coefficients ε and η. The aim of this section is to prove the similar result for the original Navier-Stokes system (2.1). For this purpose, we need to take the limit of solution $u_{\varepsilon,\eta}$ to the system (2.2) as $\varepsilon \to 0$ and $\eta \to 0$ in system (2.2). We begin with the following a priori estimates of L^2 norms which are independent of ε and η.

Lemma 4.1. *If $u_{\varepsilon,\eta}$ is the solution to system* (2.2)*, then we have the following inequalities:*

$$\sup_{(0,T)} \|u_{\varepsilon,\eta}(x,t)\|_2^2 \le C_1 \quad \text{and} \quad \int_0^T \|\nabla u_{\varepsilon,\eta}(x,t)\|_2^2 \,\mathrm{d}t \le C_2,$$

where C_1 and C_2 are positive constants, independent of ε and η.

The proof of the lemma above follows when multiplying (2.2) by 2u(x,t) and then doing direct calculations.

Lemma 4.2. *Assume that $u_{\varepsilon,\eta}$ is the sequence of functions satisfying the inequalities in Lemma 4.1. Then, extracting subsequence if necessary, we have the following results:*

$$\begin{gathered}
u_{\varepsilon,\eta} \to u_\eta \quad \text{weakly in } L^2(0,T;L^2(\mathbb{R}^3)),\\
u_{\varepsilon,\eta} \to u_\eta \quad \text{weak* in } L^\infty(0,T;L^2(\mathbb{R}^3)),\\
\nabla u_{\varepsilon,\eta} \to \nabla u_\eta \quad \text{weakly in} \quad L^2(0,T;L^2(\mathbb{R}^3)).
\end{gathered}$$

Lemma 4.3. *Assume that u_η is the sequence of functions obtaining in Lemma 4.2. Then, extracting subsequence if necessary, we have the following results:*

$$\begin{aligned} u_\eta \to u \quad &\textit{weakly in } L^2(0,T;L^2(\mathbb{R}^3)),\\ u_\eta \to u \quad &\textit{weak* in } L^\infty(0,T;L^2(\mathbb{R}^3)),\\ \nabla u_\eta \to \nabla u \quad &\textit{weakly in } L^2(0,T;L^2(\mathbb{R}^3)). \end{aligned}$$

The two lemmas above are standard theorems in functional analysis. Now, we prove our main theorem.

Proof of Theorem 2.1. Given a test function $\Psi \in D(\mathbb{R}^3))$, multiplying (2.2) by Ψ, we obtain

$$\begin{aligned} &(\frac{d}{dt}u_{\varepsilon,\eta}(x,t),\Psi) + \int_{\mathbb{R}^3} u_{\varepsilon,\eta}(x,t)\nabla u_{\varepsilon,\eta}(x,t)\Psi\,\mathrm{d}x\\ &+ (\nabla u_{\varepsilon,\eta}(x,t),\nabla\Psi) + \varepsilon((-\Delta)u_{\varepsilon,\eta}(x,t),(-\Delta)\Psi) + \eta(u(x,t),\Psi)\\ &= (f,\Psi). \end{aligned}$$

Integrating by parts, we have

$$\begin{aligned} &-(u_{\varepsilon,\eta}(x,t),\frac{d}{dt}\Psi) + \int_{\mathbb{R}^3} u_{\varepsilon,\eta}(x,t)\nabla u_{\varepsilon,\eta}(x,t)\Psi\,\mathrm{d}x\\ &-(u_{\varepsilon,\eta}(x,t),\Delta\Psi) + \varepsilon(u_{\varepsilon,\eta}(x,t),(-\Delta)^2\Psi) + \eta(u_{\varepsilon,\eta}(x,t),\Psi)\\ &= (f,\Psi). \end{aligned}$$

Since $(u_{\varepsilon,\eta}(x,t),(-\Delta)^2\Psi) \le \sqrt{C_1}\|(-\Delta)^2\Psi\|_2$, it follows that $\varepsilon(u_{\varepsilon,\eta}(x,t),(-\Delta)^2\Psi)$ approaches 0, as $\varepsilon \to 0$.

We claim that $\int_{\mathbb{R}^3} u_{\varepsilon,\eta}(x,t)\nabla u_{\varepsilon,\eta}(x,t)\Psi\,\mathrm{d}x \to \int_{\mathbb{R}^3} u_\eta\nabla u_\eta\Psi\,\mathrm{d}x$, as ε approaches 0. Indeed, let $B_R = \{x \in \mathbb{R}^3 : |x| \le R\}$. Because the sequence $u_{\varepsilon,\eta}$ is bounded in $L^2(0,T;H^{2,2}(B_R))$, and by a compactness argument (see [4, Chap.1, Thm 5.1]) there is a subsequence satisfying the above property in B_R. Letting $R \to \infty$, then we get our claim. Therefore, let $\varepsilon \to 0$ in the above equation, we have

$$-(u_\eta(x,t),\frac{d}{dt}\Psi) + \int_{\mathbb{R}^3} u_\eta(x,t)\nabla u_\eta(x,t)\Psi\,\mathrm{d}x - (u_\eta(x,t),\Delta\Psi) + \eta(u_\eta,\Psi) = (f,\Psi).$$

On the other hand, because $u_{\varepsilon,\eta}(x,T) \to u_\eta(x,T)$ weakly in $L^2(\mathbb{R}^3)$, $u_{\varepsilon,\eta}(x,0) \to u_\eta(x,0)$ weakly in $L^2(\mathbb{R}^3)$, and $u_{\varepsilon,\eta}(x,T) = u_{\varepsilon,\eta}(x,0)$, we can get $u_\eta(x,T) = u_\eta(x,0)$ in the sense of distributions.

Finally, as in the proof of the case of $\varepsilon \to 0$, letting $\eta \to 0$, we obtain

$$-(u(x,t),\frac{d}{dt}\Psi) + \int_{\mathbb{R}^3} u(x,t)\nabla u(x,t)\Psi\,\mathrm{d}x - (u(x,t),\Delta\Psi) = (f,\Psi).$$

Moreover, we also have $u(x,T) = u(x,0)$ in the sense of distributions. This completes the proof of Theorem 2.1. □

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