EXISTENCE OF POSITIVE SOLUTIONS FOR FRACTIONAL LAPLACIAN SYSTEMS WITH CRITICAL GROWTH

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ABSTRACT. In this article, we show the existence of positive solution to the nonlocal system

$$(-\Delta)^s u + a(x)u = \frac{1}{2_s^*}H_u(u,v) \quad \text{in } \mathbb{R}^N,$$

$$(-\Delta)^s v + b(x)v = \frac{1}{2_s^*}H_v(u,v) \quad \text{in } \mathbb{R}^N,$$

$$u,v > 0 \quad \text{in } \mathbb{R}^N,$$

$$u,v \in \mathcal{D}^{s,2}(\mathbb{R}^N).$$

We also prove a global compactness result for the associated energy functional similar to that due to Struwe in [26]. The basic tools are some information from a limit system with $a(x) = b(x) = 0$, a variant of the Lion's principle of concentration and compactness for fractional systems, and Brouwer degree theory.

1. INTRODUCTION

In this article, we study the existence of positive solutions for the nonlocal elliptic system

$$(-\Delta)^s u + a(x)u = \frac{1}{2_s^*}H_u(u,v) \quad \text{in } \mathbb{R}^N,$$

$$(-\Delta)^s v + b(x)v = \frac{1}{2_s^*}H_v(u,v) \quad \text{in } \mathbb{R}^N, \qquad (1.1)$$

$$u,v > 0 \quad \text{in } \mathbb{R}^N,$$

$$u,v \in \mathcal{D}^{s,2}(\mathbb{R}^N),$$

with $s \in (0,1)$, $N > 2s$, H_u and H_v are the partial derivatives of the function H, where $H(u,v) \in \mathcal{C}^1(\mathbb{R}_+^2, \mathbb{R})$ is a homogeneous function satisfying suitable conditions that will be presented throughout later. The fractional Laplacian $(-\Delta)^s$, of a smooth function $u : \mathbb{R}^N \to \mathbb{R}$, is defined by

$$(-\Delta)^s u(x) := C(N,s)\,\mathrm{P.\,V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy,$$

where P.V. is a commonly used abbreviation for "in the Cauchy principal value sense" and $C(N, s) > 0$ denotes the normalization constant. The work space $\mathcal{D}^{s,2}(\mathbb{R}^N)$ is defined as the completion of $u \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ with respect to the Gagliardo semi-norm

$$[u] := \Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \Big)^{1/2}$$

According to [24, Propositions 3.4 and 3.6], we have that

$$\|u\|^2 = |(-\Delta)^{s/2}u|_{L^2}^2 = [u]^2,$$

by omitting the normalization $C(N, s)$. Notice that this space can be also characterized as

$$\mathcal{D}^{s,2}(\mathbb{R}^N) := \big\{ u \in L^{2_s^*}(\mathbb{R}^N); [u] < +\infty \big\},$$

where $2_s^* = 2N/(N - 2s)$ is the fractional critical Sobolev exponent. For an elementary introduction to the fractional Laplacian and fractional Sobolev spaces, we refer the interested readers to [22, 24] and references therein.

In recent years, the fractional Laplace operator has received attention, for both its applicability and for its purely mathematical properties. This operator can be seen as the infinitesimal generators of Lévy stable processes (see [4]) and arises in several areas such as physics, biology, anomalous diffusion, chemistry, and finance; see [4, 5, 18, 20]. For more details and applications, see [9, 17, 28, 29, 30] and the references therein.

In the case $s = 1$, $u = v$, and $H(u, u) = |u|^{2^*}$ with $2^* = 2N/(N - 2)$, system (1.1) reduces to the critical Schrödinger equation

$$\begin{aligned}
-\Delta u + a(x)u &= u^{\frac{N+2}{N-2}} \quad \text{in } \mathbb{R}^N, \\
u &\in \mathcal{D}^{1,2}(\mathbb{R}^N), \quad u \geq 0, \ N \geq 3,
\end{aligned} \tag{1.2}$$

which was studied by Benci and Cerami in the seminal paper [6]. In this article, we prove that (1.2) does not have a ground state solution and this fact generates some additional difficulties. To overcome these difficulties, the authors investigate the behavior of a Palais-Smale sequence estimate of the energy levels where the Palais-Smale condition fails. In that article, they proved that if $N \geq 3$ and $\|a\|_{N/2}$ is small enough, then the problem (1.2) has at least one positive solution. After this pioneering work, several other authors studied problems related to (1.2); see for example [2, 7, 8, 11, 13, 19, 21, 23] and references therein. Correia and Figueiredo [13] studied the following version of problem (1.2) for the fractional Laplacian,

$$\begin{aligned}
(-\Delta)^s u + a(x)u &= |u|^{2_s^* - 2}u \quad \text{in } \mathbb{R}^N, \\
u &> 0, \quad \text{in } \mathbb{R}^N, \\
u &\in \mathcal{D}^{s,2}(\mathbb{R}^N).
\end{aligned} \tag{1.3}$$

They first proved a global compactness result for fractional Laplacian in $\mathbb{R}^N$, and then, by the compactness result above, and the Linking Theorem, they obtained the existence of high energy solutions for (1.3), provided that $a(x) \geq 0$ in $\mathbb{R}^N$ and

$$|a|_{L^{N/2s}} \leq S(2^{2s/N} - 1),$$

where S is the best constant for the Sobolev embedding $\mathcal{D}^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2^*_s}(\mathbb{R}^N)$; that is,

$$S := \inf_{u \in \mathcal{D}^{s,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \, dx}{(\int_{\mathbb{R}^N} |u|^{2^*_s} dx)^{2/2^*_s}}. \tag{1.4}$$

If $a(x) \equiv 0$, problem (1.3) reduces to

$$\begin{aligned} (-\Delta)^s u &= |u|^{2^*_s - 2} u \quad \text{in } \mathbb{R}^N, \\ u &> 0, \quad \text{in } \mathbb{R}^N, \\ u &\in \mathcal{D}^{s,2}(\mathbb{R}^N). \end{aligned} \tag{1.5}$$

It is known that problem (1.5) has the positive solution

$$\Phi_{\delta,b}(x) = c \Big(\frac{\delta}{\delta^2 + |x - b|^2} \Big)^{(N-2s)/2}, \quad x, b \in \mathbb{R}^N, \ \delta > 0, \tag{1.6}$$

and satisfies

$$\|\Phi_{\delta,b}\|^2 = S, \quad |\Phi_{\delta,b}|_{2^*_s} = 1. \tag{1.7}$$

Moreover, all positive solutions of (1.5) can be obtained by translation and scale changes, see [12].

Recently, Figueiredo and Silva [15] considered a variant of the Benci and Cerami's problem for the system of equations

$$\begin{aligned} -\Delta u + a(x) u &= \frac{1}{2^*} K_u(u, v) \quad \text{in } \mathbb{R}^N, \\ -\Delta v + b(x) v &= \frac{1}{2^*} K_v(u, v) \quad \text{in } \mathbb{R}^N, \\ u, v &> 0 \quad \text{in } \mathbb{R}^N, \\ u, v &\in \mathcal{D}^{1,2}(\mathbb{R}^N), \end{aligned} \tag{1.8}$$

where the nonlinearity $K(u, v) \in \mathcal{C}^1(\mathbb{R}^2_+, \mathbb{R})$ is a homogeneous function with certain assumptions (for more details see [14]). In that article, using the same techniques introduced by Benci and Cerami [6], they obtained the existence of high energy solutions for system (1.8), provided that $a(x), b(x) \geq 0$ in $\mathbb{R}^N$ and

$$s_0^2 |a|_{L^{N/2}} + t_0^2 |b|_{L^{N/2}} < S_K (2^{2/N} - 1),$$

where S_K denote the best constant of the embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \times \mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N) \times L^{2^*}(\mathbb{R}^N)$; that is,

$$S_K := \inf_{u,v \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} [|\nabla u|^2 + |\nabla v|^2] dx}{(\int_{\mathbb{R}^N} K(u, v) dx)^{2/2^*}},$$

with s_0, t_0 positives constant such that the pair $(s_0 \Psi_{\delta,y}, t_0 \Psi_{\delta,y})$ reaches S_K (see [14, Lemma 3]) and $\Psi_{\delta,y}$ are Talenti functions (see [1, 27]). Motivated by the works mentioned above, mainly by the ideas found in Benci and Cerami [6], Correia and Figueiredo [13] and Figueiredo and Silva [15], and that a bibliography review did not find any paper dealing with (1.1), we decided to investigate the this class of systems. This article concerns the existence of positive solution for system (1.1). However, we would like to point out that some estimates made in [6, 13, 15] are not immediate for our case because of the nonlocal character of fractional Laplacian. Some refined estimates were necessary, see Section 3 and Section 4.

In this article, we consider the following assumptions on $H = H(\ell, t)$:

(H0) H is 2_s^*-homogeneous, that is,
$$H(\theta\ell, \theta t) = \theta^{2_s^*} H(\ell, t), \quad \text{for each } \theta > 0, (\ell, t) \in \mathbb{R}_+^2;$$

(H1) there exists $c_1 > 0$ such that
$$|H_\ell(\ell, t)| + |H_t(\ell, t)| \leq c_1(\ell^{2_s^*-1} + t^{2_s^*-1}) \quad \text{for each } (\ell, t) \in \mathbb{R}_+^2;$$

(H2) $\nabla H(0, 1) = \nabla H(1, 0) = (0, 0)$;

(H3) $H(\ell, t) > 0$ for each $\ell, t > 0$;

(H4) $H_\ell(\ell, t) \geq 0$, $H_t(\ell, t) \geq 0$ for each $(\ell, t) \in \mathbb{R}_+^2$;

(H5) the 1-homogeneous function $\Psi(\ell^{2_s^*}, t^{2_s^*}) = H(\ell, t)$ is concave in $\mathbb{R}_+^2$.

On the functions $a, b : \mathbb{R}^N \to \mathbb{R}$, we assume the following conditions:

(H6) The functions a, b are positive in a set of positive measure;

(H7) $a, b \in L^q(\mathbb{R}^N)$ for all $q \in [p_1, p_2]$ with $1 < p_1 < N/2s < p_2$ and $p_2 < N/(4s - N)$ if $N < 4s$;

(H8) $\ell_0^2 |a|_{L^{N/2s}(\mathbb{R}^N)} + t_0^2 |b|_{L^{N/2s}(\mathbb{R}^N)} < S_H(2^{2s/N} - 1)$.

In assumption (H8), the expression S_H denotes the best constant of the immersion $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N) \times L^{2_s^*}(\mathbb{R}^N)$, namely

$$S_H := \inf_{u,v \in \mathcal{D}^{s,2}(\mathbb{R}^N) \backslash \{0\}} \frac{\int_{\mathbb{R}^N} [|(-\Delta)^{s/2} u|^2 + |(-\Delta)^{s/2} v|^2] dx}{\left(\int_{\mathbb{R}^N} H(u, v) dx\right)^{2/2_s^*}}.$$

Moreover, by [3, Lemma 2.3] there are ℓ_0 and t_0 positive such that S_H is attained by $(\ell_0 \Phi_{\delta,b}, t_0 \Phi_{\delta,b})$ and

$$M_H S_H = S, \tag{1.9}$$

where $M_H = \max_{\ell^2 + t^2 = 1} H(\ell, t)^{2/2_s^*} = H(\ell_0, t_0)^{2/2_s^*}$.

To state the main result this article, we consider the energy functional of calss $\mathcal{C}^1$, $\mathcal{J} : \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N) \to \mathbb{R}$, given by

$$\mathcal{J}(u, v) = \frac{1}{2} \|(u, v)\|^2 + \frac{1}{2} \int_{\mathbb{R}^N} (a(x)u^2 + b(x)v^2) dx - \frac{1}{2_s^*} \int_{\mathbb{R}^N} H(u, v) dx,$$

where $\|(u, v)\|^2 = \|u\|^2 + \|v\|^2$ is the norm in the space $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$ and

$$\mathcal{J}'(u, v)(\varphi, \psi) = \int_{\mathbb{R}^N} [(-\Delta)^{s/2} u (-\Delta)^{s/2} \varphi + (-\Delta)^{s/2} v (-\Delta)^{s/2} \psi] dx$$
$$+ \int_{\mathbb{R}^N} [a(x)u\varphi + b(x)v\psi] dx - \frac{1}{2_s^*} \int_{\mathbb{R}^N} [H_u(u, v)\varphi + H_v(u, v)\psi] dx$$

for all $(\varphi, \psi) \in \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$. We have the following existence result.

Theorem 1.1. *Assume that* (H0)–(H8) *hold. Then* (1.1) *has a positive solution* $(u_0, v_0) \in \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$ *with*

$$\frac{s}{N} S_H^{N/2s} < \mathcal{J}(u_0, v_0) < \frac{2s}{N} S_H^{N/2s}.$$

In some sense, the main result of this article expands the study made in [6, 13, 15], because we are considering a version of a paper for the fractional Laplacian. Moreover, we prove the version for fractional system in $\mathbb{R}^N$ of Struwe's Global Compactness result [26], which may be useful also in other context and has never appeared in the literature, to the best of our knowledge.

Before we finish this introduction, let us comment on some difficulties encountered in problem (1.1).

- The "double" lack of compactness due to the unboundedness of the domain, and the presence of the critical Sobolev exponent, which is related to the fact that embedding $D^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2^*_s}(\mathbb{R}^N)$ is not compact. Thus, the associated energy functional does not satisfy the Palais-Smale condition in general.
- The extension to nonlocal system involves some technical difficulties which are overcome with some refined estimates, as can be seen in Lemma 3.1, Theorem 3.2, and Section 4.

This article is organized as follows. In Section 2, we study the limit system associated with (1.1). In Section 3, we give the complete descriptions for the Palais-Smale (PS) sequences for the functional $\mathcal{J}$. In Section 4, we prove some technical lemmas. In Section 5, we show the main result.

2. Limit problem

In this section, we give some results involving the limit problem that will be useful in our approach. We start with example of the function $H(u,v)$ that satisfies the conditions (H0)–(H5).

Let H be the function

$$H(u,v) := a|u|^{2^*_s} + \sum_{\alpha_i + \beta_i = 2^*_s} b_i |u|^{\alpha_i} |v|^{\beta_i} + c|v|^{2^*_s},$$

where $a, b_i, c \in \mathbb{R}$, $\alpha_i + \beta_i = 2^*_s$, $\alpha_i, \beta_i \geq 1$, $i \in \mathcal{I}$ with $\mathcal{I}$ a finite subset of $\mathbb{N}$. Then H satisfies conditions (H0)–(H5).

From the homogeneity condition (H0), have the so called Euller identity,

$$(u,v) \cdot \nabla H(u,v) = 2^*_s H(u,v). \tag{2.1}$$

Let us introduce the limit problem associated with (1.1),

$$
\begin{aligned}
(-\Delta)^s u &= \frac{1}{2^*_s} H_u(u,v) \quad \text{in } \mathbb{R}^N, \\
(-\Delta)^s v &= \frac{1}{2^*_s} H_v(u,v) \quad \text{in } \mathbb{R}^N, \\
u, v &> 0 \quad \text{in } \mathbb{R}^N, \\
u, v &\in \mathcal{D}^{s,2}(\mathbb{R}^N)
\end{aligned}
\tag{2.2}
$$

whose associated energy functional $\mathcal{J}_\infty : \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N) \to \mathbb{R}$ is

$$\mathcal{J}_\infty(u,v) = \frac{1}{2} \|(u,v)\|^2 - \frac{1}{2^*_s} \int_{\mathbb{R}^N} H(u,v) dx.$$

The next lemma states that the functional associated with the limit problem satisfies the Palais-Smale condition.

Lemma 2.1 ((PS)-condition for $\mathcal{J}_\infty$). *Let (u_n, v_n) be sequence $(PS)_c$ for $\mathcal{J}_\infty$. Then*

(a) *The sequence (u_n, v_n) is bounded in $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$;*

(b) *If $(u_n, v_n) \rightharpoonup (u,v)$ in $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$, then $\mathcal{J}'_\infty(u,v) = 0$;*

(c) *If $c \in (-\infty, \frac{s}{N} S_H^{N/2s})$, the $\mathcal{J}_\infty$ satisfies the $(PS)_c$ condition, i.e, up to a subsequence*

$$(u_n, v_n) \to (u,v) \quad \text{in } \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N).$$

Proof. (a) Since $\mathcal{J}_\infty(u_n, v_n) \to c$ and $\mathcal{J}'_\infty(u_n, v_n) \to 0$, and using (2.1), there exists $d > 0$ such that

$$d + \|(u_n, v_n)\| \geq \mathcal{J}_\infty(u_n, v_n) - \frac{1}{2^*_s} \mathcal{J}_\infty(u_n, v_n)(u_n, v_n) = \frac{s}{N} \|(u_n, v_n)\|^2 + o_n(1);$$

thus

$$\frac{s}{N} \|(u_n, v_n)\|^2 + o_n(1) \leq d + \|(u_n, v_n)\|.$$

which proves part (a).

(b) Since $(u_n, v_n) \rightharpoonup (u, v)$ in $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$, up to a subsequence, we have

$$(u_n, v_n) \to (u, v) \quad \text{in } L^q_{\text{loc}}(\mathbb{R}^N) \times L^q_{\text{loc}}(\mathbb{R}^N),$$

$$(u_n, v_n) \to (u, v) \quad \text{a.e. in } \mathbb{R}^N \times \mathbb{R}^N.$$

Using a denseness argument we obtain

$$\int_{\mathbb{R}^N} H_u(u_n, v_n)\varphi \, dx + \int_{\mathbb{R}^N} H_v(u_n, v_n)\psi \, dx \to \int_{\mathbb{R}^N} H_u(u, v)\varphi \, dx + \int_{\mathbb{R}^N} H_v(u, v)\psi \, dx$$

for all $\varphi, \psi \in \mathcal{D}^{s,2}(\mathbb{R}^N)$, which implies (b).

(c) Consider the sequence $(w_n, z_n) = (u_n - u, v_n - v)$ and note that

$$o_n(1) = \mathcal{J}'_\infty(u_n, v_n)(u_n, v_n) = \|(u_n, v_n)\|^2 - \frac{1}{2^*_s} \int_{\mathbb{R}^N} [H_u(u_n, v_n)u_n + H_v(u_n, v_n)v_n] \, dx$$

or

$$o_n(1) = \|(w_n, z_n)\|^2 + \|(u, v)\|^2 - \frac{1}{2^*_s} \int_{\mathbb{R}^N} H_u(w_n + u, z_n + v)(w_n + u)dx$$

$$- \frac{1}{2^*_s} \int_{\mathbb{R}^N} H_v(w_n + u, z_n + v)(z_n + v) \, dx.$$

From [3, Lemma 7.2], we have

$$\|(w_n, z_n)\|^2 + \|(u, v)\|^2 - \frac{1}{2^*_s} \int_{\mathbb{R}^N} H_u(w_n, z_n)w_n \, dx - \frac{1}{2^*_s} \int_{\mathbb{R}^N} H_v(w_n, z_n)z_n \, dx$$

$$- \frac{1}{2^*_s} \int_{\mathbb{R}^N} H_u(u, v)u \, dx - \frac{1}{2^*_s} \int_{\mathbb{R}^N} H_v(u, v)v \, dx = o_n(1)$$

Now using (b) and (2.1) we have

$$\|(w_n, z_n)\|^2 - \int_{\mathbb{R}^N} H(w_n, z_n)dx = o_n(1).$$

Up to a subsequence, we conclude that there exists $L \geq 0$ such that

$$\lim_{n \to +\infty} \|(w_n, z_n)\|^2 = \lim_{n \to +\infty} \int_{\mathbb{R}^N} H(w_n, z_n)dx = L.$$

Suppose, by contradiction, that $L > 0$. Using the inequality

$$S_H \Big(\int_{\mathbb{R}^N} H(w_n, z_n)dx \Big)^{2/2^*_s} \leq \|(w_n, z_n)\|^2$$

we obtain

$$L \geq S_H L^{2/2^*_s} \Longrightarrow L \geq S_H^{N/2s}.$$

Since $\mathcal{J}_\infty(u, v) = \frac{s}{N} \|(u, v)\|^2 \geq 0$ and

$$c = \frac{s}{N} \|(w_n, z_n)\|^2 + \mathcal{J}_\infty(u, v) + o_n(1),$$

it follows that

$$c = \frac{s}{N}\|(w_n, z_n)\|^2 + \mathcal{J}_\infty(u, v) + o_n(1) \geq \frac{s}{N}\|(w_n, z_n)\|^2 + o_n(1) \geq \frac{s}{N}L \geq \frac{s}{N}S_H^{N/2s},$$

which is a contradiction. Therefore, $L = 0$ and so $\|u_n - u\|^2 \to 0$ and $\|v_n - v\|^2 \to 0$. $\qquad\square$

3. A COMPACTNESS RESULT

We start this section by establishing the following technical lemma for $\mathcal{J}_\infty$ which will be useful for proving our compactness theorem.

Lemma 3.1. *Let (u_n, v_n) be a $(PS)_c$ sequence for the functional $\mathcal{J}_\infty$ with $(u_n, v_n) \rightharpoonup (0,0)$ and $(u_n, v_n) \nrightarrow (0,0)$. Then, there are sequences $(R_n) \subset \mathbb{R}^+$, $(x_n) \subset \mathbb{R}^N$ and $(u_0, v_0) \in D^{s,2}(\mathbb{R}^N) \times D^{s,2}(\mathbb{R}^N)$ nontrivial solution of $(\mathcal{S}_\infty)$ and a sequence (τ_n, ζ_n) which is $(PS)_{\tilde{c}}$ for the $\mathcal{J}_\infty$ such that, up to a subsequence of (u_n, v_n), we have*

$$\tau_n(x) = u_n(x) - R_n^{\frac{N-2s}{2}} u_0(R_n(x - x_n)) + o_n(1),$$

$$\zeta_n(x) = v_n(x) - R_n^{\frac{N-2s}{2}} v_0(R_n(x - x_n)) + o_n(1).$$

Proof. Let $(u_n, v_n) \subset \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$ be a $(PS)_c$ sequence for the functional $\mathcal{J}_\infty$, i.e.,

$$\mathcal{J}_\infty(u_n, v_n) \to c \quad \text{and} \quad \mathcal{J}_\infty'(u_n, v_n) \to 0. \tag{3.1}$$

From Lemma 2.1(a), we obtain that (u_n, v_n) is bounded in $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$. Since $(u_n, v_n) \rightharpoonup (0,0)$ and $(u_n, v_n) \nrightarrow (0,0)$, by the Lemma 2.1(c) it follows that

$$c \geq \frac{s}{N}S_H^{N/2s}.$$

Note that

$$c + o_n(1) = \mathcal{J}_\infty(u_n, v_n) - \frac{1}{2_s^*}\mathcal{J}_\infty'(u_n, v_n)(u_n, v_n)$$

$$= \frac{s}{N}\int_{\mathbb{R}^N} \big[|(-\Delta)^{s/2}u_n|^2 + |(-\Delta)^{s/2}v_n|^2\big]dx,$$

which implies

$$\lim_{n \to +\infty} \frac{s}{N}\int_{\mathbb{R}^N} \big[|(-\Delta)^{s/2}u_n|^2 + |(-\Delta)^{s/2}v_n|^2\big]dx \geq S_H^{N/2s}. \tag{3.2}$$

Let L be a number such that $B_2(0)$ is covered by L balls of radius 1, $(R_n) \subset \mathbb{R}$, $(x_n) \subset \mathbb{R}^N$ such that

$$\sup_{y \in \mathbb{R}^N} \int_{B_{R_n^{-1}}(y)} \big[|(-\Delta)^{s/2}u_n|^2 + |(-\Delta)^{s/2}u_n|^2\big]dx$$

$$= \int_{B_{R_n^{-1}}(x_n)} \big[|(-\Delta)^{s/2}u_n|^2 + |(-\Delta)^{s/2}v_n|^2\big]dx$$

$$= \frac{S_H^{N/2s}}{2L}.$$

We define the sequence

$$(w_n(x), z_n(x)) = \Big(R_n^{\frac{2s-N}{2}} u_n\Big(\frac{x}{R_n} + x_n\Big), R_n^{\frac{2s-N}{2}} v_n\Big(\frac{x}{R_n} + x_n\Big)\Big).$$

Using a change of variable, we can prove that

$$\int_{B_1(0)} \left[|(-\Delta)^{s/2} w_n|^2 + |(-\Delta)^{s/2} z_n|^2\right] dx$$

$$= \frac{S_H^{N/2s}}{2L}$$

$$= \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} \left[|(-\Delta)^{s/2} w_n|^2 + |(-\Delta)^{s/2} z_n|^2\right] dx.$$

Now, for each $(\Phi_1, \Phi_2) \in \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$, we define

$$(\tilde{\Phi}_{1,n}, \tilde{\Phi}_{2,n})(x) = \left(R_n^{\frac{N-2s}{2}} \Phi_1(R_n(x - x_n)), R_n^{\frac{N-2s}{2}} \Phi_2(R_n(x - x_n))\right)$$

which satisfies

$$\int_{\mathbb{R}^N} \left[(-\Delta)^{s/2} u_n (-\Delta)^{s/2} \tilde{\Phi}_{1,n} + (-\Delta)^{s/2} v_n (-\Delta)^{s/2} \tilde{\Phi}_{2,n}\right] dx$$

$$= \int_{\mathbb{R}^N} \left[(-\Delta)^{s/2} w_n (-\Delta)^{s/2} \Phi_1 + (-\Delta)^{s/2} z_n (-\Delta)^{s/2} \Phi_2\right] dx \tag{3.3}$$

and

$$\int_{\mathbb{R}^N} [H_u(u_n, v_n) \tilde{\Phi}_{1,n} + H_v(u_n, v_n) \tilde{\Phi}_{2,n}] dx$$

$$= \int_{\mathbb{R}^N} [H_w(w_n, z_n) \Phi_1 + H_{z_n}(w_n, z_n) \Phi_2] dx. \tag{3.4}$$

These limits yield that

$$\mathcal{J}_\infty(w_n, z_n) \to c \quad \text{and} \quad \mathcal{J}'_\infty(w_n, z_n) \to 0. \tag{3.5}$$

From Lemma 3.1, there exists $(u_0, v_0) \in \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$ such that, up to a subsequence, $(u_n, v_n) \rightharpoonup (u_0, v_0)$ in $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$ and $\mathcal{J}'_\infty(u_0, v_0) = 0$.

As a consequence from following variant of the Concentration-Compactness Lions's Lemma [3, Lemma 4.3], we obtain

$$\int_{\mathbb{R}^N} H(w_n, z_n) \phi \, dx \to \int_{\mathbb{R}^N} H(u_0, v_0) \phi \, dx + \sum_{j \in J} \phi(x_j) \nu_j, \quad \forall \phi \in C_0^\infty(\mathbb{R}^N) \tag{3.6}$$

and

$$|(-\Delta)^{s/2} w_n|^2 + |(-\Delta)^{s/2} z_n|^2$$
$$\rightharpoonup \mu + \sigma$$
$$\geq |(-\Delta)^{s/2} u_0|^2 + |(-\Delta)^{s/2} v_0|^2 + \sum_{j \in J} \phi(x_j) \mu_j + \sum_{j \in J} \phi(x_j) \sigma_j, \forall \phi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$$

for some $\{x_j\}_{j \in J} \subset \mathbb{R}^N$ and for some $\{\nu_j\}_{j \in J}$, $\{\mu_j\}_{j \in J}, \{\sigma_j\}_{j \in J} \subset \mathbb{R}^+$ with $S_H \nu_j^{2/2_s^*} \leq \mu_j + \sigma_j$, where J is at most a countable set. Indeed, J is finite. To see this, consider $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \phi(x) \leq 1$, for all $x \in \mathbb{R}^N$, $\phi(x) = 0$ for all $x \in B_2^c(0)$ and $\phi(x) = 1$ for all $x \in B_1(0)$. Now fix $x_j \in \mathbb{R}^N$, $j \in J$ and define $\phi_\rho(x) = \phi(\frac{x - x_j}{\rho})$, for each $\rho > 0$. Thus, $0 \leq \phi_\rho(x) \leq 1$, for all $x \in \mathbb{R}^N$, $\phi_\rho(x) = 0$

for all $x \in B_{2\rho}^c(x_j)$ and $\phi_\rho(x) = 1$ for all $x \in B_\rho(x_j)$. We have that $(w_n\phi_\rho, z_n\phi_\rho)$ is bounded in $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$ and $\mathcal{J}'_\infty(w_n, z_n)(w_n\phi_\rho, z_n\phi_\rho) = o_n(1)$. Then

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2}w_n(-\Delta)^{s/2}(w_n\phi_\rho)dx + \int_{\mathbb{R}^N} (-\Delta)^{s/2}z_n(-\Delta)^{s/2}(z_n\phi_\rho)dx$$
$$= \int_{\mathbb{R}^N} H_w(w_n, z_n)(w_n\phi_\rho)dx + \int_{\mathbb{R}^N} H_z(w_n, z_n)(z_n\phi_\rho)dx + o_n(1). \tag{3.7}$$

As

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2}w_n(-\Delta)^{s/2}(w_n\phi_\rho)\,dx\,dy + \int_{\mathbb{R}^N} (-\Delta)^{s/2}z_n(-\Delta)^{s/2}(z_n\phi_\rho)\,dx\,dy$$
$$= \int_{\mathbb{R}^{2N}} \frac{(w_n(x) - w_n(y))^2\phi_\rho(y)}{|x - y|^{N+2s}}\,dx\,dy$$
$$+ \int_{\mathbb{R}^{2N}} \frac{(w_n(x) - w_n(y))(\phi_\rho(x) - \phi_\rho(y))w_n(x)}{|x - y|^{N+2s}}\,dx\,dy \tag{3.8}$$
$$+ \int_{\mathbb{R}^{2N}} \frac{(z_n(x) - z_n(y))^2\phi_\rho(y)}{|x - y|^{N+2s}}\,dx\,dy$$
$$+ \int_{\mathbb{R}^{2N}} \frac{(z_n(x) - z_n(y))(\phi_\rho(x) - \phi_\rho(y))z_n(x)}{|x - y|^{N+2s}}\,dx\,dy,$$

it is easy to verify that

$$\int_{\mathbb{R}^{2N}} \frac{(w_n(x) - w_n(y))^2\phi_\rho(y)}{|x - y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^{2N}} \frac{(z_n(x) - z_n(y))^2\phi_\rho(y)}{|x - y|^{N+2s}}\,dx\,dy$$
$$= \int_{\mathbb{R}^N} |(-\Delta)^{s/2}w_n|^2\phi_\rho(y)dy + \int_{\mathbb{R}^N} |(-\Delta)^{s/2}z_n|^2\phi_\rho(y)dy \tag{3.9}$$
$$\to \int_{\mathbb{R}^N} \phi_\rho(y)d\mu + \int_{\mathbb{R}^N} \phi_\rho(y)d\sigma \quad \text{as } n \to +\infty$$

and

$$\int_{\mathbb{R}^N} \phi_\rho(y)d\mu + \int_{\mathbb{R}^N} \phi_\rho(y)d\sigma \to \mu(\{x_j\}) + \sigma(\{x_j\}) = \mu_j + \sigma_j \quad \text{as } \rho \to 0. \tag{3.10}$$

Also, by Hölder inequality

$$\left| \int_{\mathbb{R}^{2N}} \frac{(w_n(x) - w_n(y))(\phi_\rho(x) - \phi_\rho(y))w_n(x)}{|x - y|^{N+2s}}\,dx\,dy \right|$$
$$\le \int_{\mathbb{R}^{2N}} \frac{|w_n(x) - w_n(y)||\phi_\rho(x) - \phi_\rho(y)||w_n(x)|}{|x - y|^{N+2s}}\,dx\,dy \tag{3.11}$$
$$\le C_1 \left(\int_{\mathbb{R}^{2N}} \frac{|\phi_\rho(x) - \phi_\rho(y)|^2|w_n(x)|^2}{|x - y|^{N+2s}}\,dx\,dy \right)^{1/2}$$

and

$$\left| \int_{\mathbb{R}^{2N}} \frac{(z_n(x) - z_n(y))(\phi_\rho(x) - \phi_\rho(y))z_n(x)}{|x - y|^{N+2s}}\,dx\,dy \right|$$
$$\le \int_{\mathbb{R}^{2N}} \frac{|z_n(x) - z_n(y)||\phi_\rho(x) - \phi_\rho(y)||z_n(x)|}{|x - y|^{N+2s}}\,dx\,dy \tag{3.12}$$
$$\le C_2 \left(\int_{\mathbb{R}^{2N}} \frac{|\phi_\rho(x) - \phi_\rho(y)|^2|z_n(x)|^2}{|x - y|^{N+2s}}\,dx\,dy \right)^{1/2}.$$

Arguing as in [32, Lemma 3.6], we see that

$$\lim_{\rho \to 0} \lim_{n \to +\infty} \int_{\mathbb{R}^{2N}} \frac{|\phi_\rho(x) - \phi_\rho(y)|^2 |w_n(x)|^2}{|x - y|^{N+2s}} \, dx \, dy = 0, \tag{3.13}$$

$$\lim_{\rho \to 0} \lim_{n \to +\infty} \int_{\mathbb{R}^{2N}} \frac{|\phi_\rho(x) - \phi_\rho(y)|^2 |z_n(x)|^2}{|x - y|^{N+2s}} \, dx \, dy = 0. \tag{3.14}$$

On the other hand, by (2.1) we have

$$\int_{\mathbb{R}^N} H_w(w_n, z_n)(w_n\phi_\rho)dx + \int_{\mathbb{R}^N} H_z(w_n, z_n)(z_n\phi_\rho)dx$$

$$= \int_{\mathbb{R}^N} \nabla H(w_n, z_n) \cdot (w_n\phi_\rho, z_n\phi_\rho) \tag{3.15}$$

$$= 2_s^* \int_{\mathbb{R}^N} H(w_n, z_n)\phi_\rho dx \to 2_s^* \int_{\mathbb{R}^N} \phi_\rho(y)d\nu$$

and

$$\int_{\mathbb{R}^N} \phi_\rho(y)d\nu \to \nu(\{x_j\}) = \nu_j \quad \text{as } \rho \to 0. \tag{3.16}$$

From (3.8), (3.9), (3.10), (3.11), (3.12), (3.13), (3.14), (3.15) and (3.16), it follows that

$$S_H \nu_j^{2/2_s^*} \le \mu_j + \sigma_j \le 2_s^* \nu_j.$$

Since that $\nu_j > 0$, we see that $S_H^{N/2s} \le (\mu_j + \sigma_j)^{N/2s} \le C\nu_j$, $\sum_{j \in J} \nu_j^{2/2_s^*} < \infty$ and so ν_j does not converge to zero, which means that J is finite.

From now on, we denote by $J = \{1, 2, \ldots, m\}$ and $\Gamma \subset \mathbb{R}^N$ the set given by

$$\Gamma = \{x_j \in \{x_j\}_{j \in J}; |x_j| > 1\},$$

with (x_j given by (3.6). Note that we can consider x_j, $j = 1, \ldots, m$, belonging to Γ, otherwise, we choose the smallest distance point for zero in this set. We are going to show that $(u_0, v_0) \neq (0, 0)$. Suppose, by contradiction, that $(u_0, v_0) = (0, 0)$. Then, by (3.6) we have

$$\int_{\mathbb{R}^N} H(w_n, z_n)\phi \, dx \to 0, \quad \forall \phi \in \mathcal{C}_0^\infty(\mathbb{R}^N \setminus \{x_1, x_2, \ldots, x_m\}). \tag{3.17}$$

Since $(\phi_{1,n}, \phi_{2,n}) = (\phi w_n, \phi z_n)$ with $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^N \setminus \{x_1, x_2, \ldots, x_m\})$ is bounded, we obtain $\mathcal{J}'_\infty(w_n, z_n)(\phi_{1,n}, \phi_{2,n}) = o_n(1)$; that is,

$$\int_{\mathbb{R}^N} [(-\Delta)^{s/2} w_n (-\Delta)^{s/2} \phi_{1,n} + (-\Delta)^{s/2} z_n (-\Delta)^{s/2} \phi_{2,n}]dx$$

$$- \frac{1}{2_s^*} \int_{\mathbb{R}^N} H_w(w_n, z_n)\phi_{1,n}dx - \frac{1}{2_s^*} \int_{\mathbb{R}^N} H_z(w_n, z_n)\phi_{2,n}dx = o_n(1), \tag{3.18}$$

or

$$\int_{\mathbb{R}^{2N}} \frac{(w_n(x) - w_n(y))(\phi_{1,n}(x)w_n(x) - \phi_{1,n}(y)w_n(y))}{|x - y|^{N+2s}} \, dx \, dy$$

$$+ \int_{\mathbb{R}^{2N}} \frac{(z_n(x) - z_n(y))(\phi_{2,n}(x)(z_n(x) - \phi_{2,n}(y)z_n(y))}{|x - y|^{N+2s}} \, dx \, dy \tag{3.19}$$

$$- \frac{1}{2_s^*} \int_{\mathbb{R}^N} H_w(w_n, z_n)\phi_{1,n}dx - \frac{1}{2_s^*} \int_{\mathbb{R}^N} H_z(w_n, z_n)\phi_{2,n}dx = o_n(1).$$

But the above equality is equivalent to

$$
\int_{\mathbb{R}^{2N}} w_n(x) \frac{(w_n(x) - w_n(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}} \, dx \, dy
$$
$$
+ \int_{\mathbb{R}^{2N}} \phi(y) \frac{(w_n(x) - w_n(y))^2}{|x - y|^{N+2s}} \, dx \, dy
$$
$$
+ \int_{\mathbb{R}^{2N}} z_n(x) \frac{(z_n(x) - z_n(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}} \, dx \, dy
$$
$$
+ \int_{\mathbb{R}^{2N}} \phi(y) \frac{(z_n(x) - z_n(y))^2}{|x - y|^{N+2s}} \, dx \, dy - \int_{\mathbb{R}^N} H(w_n, z_n)\phi \, dx = o_n(1).
$$

Then

$$
\Big| \int_{\mathbb{R}^{2N}} \phi(y) \frac{|w_n(x) - w_n(y)|^2}{|x - y|^{N+2s}} \, dx \, dy + \int_{\mathbb{R}^{2N}} \phi(y) \frac{|z_n(x) - z_n(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \Big|
$$
$$
= \Big| \int_{\mathbb{R}^N} H(w_n, z_n)\phi \, dx - \int_{\mathbb{R}^{2N}} w_n(x) \frac{(w_n(x) - w_n(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}} \, dx \, dy
$$
$$
- \int_{\mathbb{R}^{2N}} z_n(x) \frac{(z_n(x) - z_n(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}} \, dx \, dy + o_n(1) \Big|
$$
$$
\leq \Big| \int_{\mathbb{R}^N} H(w_n, z_n)\phi \, dx \Big| + \Big| \int_{\mathbb{R}^{2N}} w_n(x) \frac{(w_n(x) - w_n(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}} \, dx \, dy \Big|
$$
$$
+ \Big| \int_{\mathbb{R}^{2N}} z_n(x) \frac{(z_n(x) - z_n(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}} \, dx \, dy \Big| + o_n(1)
$$
$$
\leq \Big| \int_{\mathbb{R}^N} H(w_n, v_n)\phi \, dx \Big| + \|w_n\| \Big(\int_{\mathbb{R}^{2N}} |w_n(x)|^2 \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \Big)^{1/2}
$$
$$
+ \|z_n\| \Big(\int_{\mathbb{R}^{2N}} |z_n(x)|^2 \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \Big)^{1/2} + o_n(1).
$$
$$(3.20)$$

Now, we show that

$$
\int_{\mathbb{R}^{2N}} |w_n(x)|^2 \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} \, dx \, dy = o_n(1), \tag{3.21}
$$

$$
\int_{\mathbb{R}^{2N}} |z_n(x)|^2 \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} \, dx \, dy = o_n(1). \tag{3.22}
$$

For this, let R be a positive number such that $\operatorname{supp}(\phi) \subset B_R(0)$ and write $\mathbb{R}^{2N}$ as

$$
\mathbb{R}^{2N} = \big[(\mathbb{R}^N \setminus B_R(0)) \times (\mathbb{R}^N \setminus B_R(0)) \big] \cup \big[B_R(0) \times (\mathbb{R}^N \setminus B_R(0)) \big]
$$
$$
\cup \big[(\mathbb{R}^N \setminus B_R(0)) \times B_R(0) \big]
$$
$$
= \Omega_1 \cup \Omega_2 \cup \Omega_3.
$$

Thus, we have

$$
\int_{\mathbb{R}^{2N}} |w_n(x)|^2 \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} \, dx \, dy
$$
$$
= \int_{\Omega_1} |w_n(x)|^2 \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} \, dx \, dy + \int_{\Omega_2} |w_n(x)|^2 \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \tag{3.23}
$$
$$
+ \int_{\Omega_3} |w_n(x)|^2 \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} \, dx \, dy
$$

and

$$
\int_{\mathbb{R}^{2N}} |z_n(x)|^2 \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} \, dx \, dy
$$

$$
= \int_{\Omega_1} |z_n(x)|^2 \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} \, dx \, dy + \int_{\Omega_2} |z_n(x)|^2 \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \qquad (3.24)
$$

$$
+ \int_{\Omega_3} |z_n(x)|^2 \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} \, dx \, dy.
$$

We will prove (3.23), the case (3.24) follows in an analogous way. To do this we estimate each integral in (3.23). Since $\phi = 0$ in $\mathbb{R}^N \setminus B_R(0)$, we have

$$
\int_{\Omega_1} |w_n(x)|^2 \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} \, dx \, dy = 0. \qquad (3.25)
$$

Using $|\phi| \le C_1$, $|\nabla \phi| \le C_2$ and using the mean value theorem, we infer that

$$
\int_{\Omega_2} |w_n(x)|^2 \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} \, dx \, dy
$$

$$
= \int_{B_R(0)} |w_n(x)|^2 dx \int_{\{y \in \mathbb{R}^N : |x-y| \le R\}} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dy
$$

$$
+ \int_{B_R(0)} |w_n(x)|^2 dx \int_{\{y \in \mathbb{R}^N : |x-y| > R\}} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dy \qquad (3.26)
$$

$$
\le C |\nabla \phi|^2_{L^\infty(\mathbb{R}^N)} \int_{B_R(0)} |w_n(x)|^2 dx \int_{\{y \in \mathbb{R}^N : |x-y| \le R\}} \frac{1}{|x - y|^{N+2s-2}} dy
$$

$$
+ C \int_{B_R(0)} |w_n(x)|^2 dx \int_{\{y \in \mathbb{R}^N : |x-y| > R\}} \frac{1}{|x - y|^{N+2s}} dy
$$

$$
= C R^{2-2s} \int_{B_R(0)} |w_n(x)|^2 dx + C R^{-2s} \int_{B_R(0)} |w_n(x)|^2 dx = o_n(1).
$$

Moreover, for the integral on Ω_3, we have

$$
\int_{\Omega_3} |w_n(x)|^2 \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} \, dx \, dy
$$

$$
= \int_{\mathbb{R}^N \setminus B_R(0)} |w_n(x)|^2 dx \int_{\{y \in B_R(0) : |x-y| \le R\}} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dy \qquad (3.27)
$$

$$
= \int_{\mathbb{R}^N \setminus B_R(0)} |w_n(x)|^2 dx \int_{\{y \in B_R(0) : |x-y| > R\}} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dy =: \mathcal{T}_R^1 + \mathcal{T}_R^2.
$$

It is not difficult to verify that if $(x, y) \in (\mathbb{R}^N \setminus B_R(0)) \times B_R(0)$ and $|x - y| \le R$, then $|x| \le 2R$, thus

$$
\mathcal{T}_R^1 = \int_{\mathbb{R}^N \setminus B_R(0)} |w_n(x)|^2 dx \int_{\{y \in B_R(0) : |x-y| \le R\}} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dy
$$

$$
\le C |\nabla \phi|^2_{L^\infty(\mathbb{R}^N)} \int_{B_{2R}(0)} |w_n(x)|^2 dx \int_{\{z \in B_R(0) : |z| \le R\}} \frac{1}{|z|^{N+2s-2}} dz \qquad (3.28)
$$

$$
= C R^{2-2s} \int_{B_{2R}(0)} |w_n(x)|^2 dx = o_n(1).
$$

Note that, there exists $k > 4$ such that

$$\Omega_3 = \left[(\mathbb{R}^N \setminus B_R(0)) \times (B_R(0))\right] \cup \left[B_{kR}(0) \times (B_R(0))\right] \cup \left[(\mathbb{R}^N \setminus B_{kR}(0)) \times B_R(0)\right].$$

Therefore,

$$
\begin{aligned}
&\int_{B_{kR}(0)} |w_n(x)|^2 dx \int_{\{y \in B_R(0) : |x-y| > R\}} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{N+2s}} dy \\
&\le C \int_{B_{kR}(0)} |w_n(x)|^2 dx \int_{\{z \in \mathbb{R}^N : |z| > R\}} \frac{1}{|z|^{N+2s}} dz \\
&= CR^{-2s} \int_{B_{kR}(0)} |w_n(x)|^2 dx = o_n(1).
\end{aligned}
\tag{3.29}
$$

If $(x,y) \in (\mathbb{R}^N \setminus B_{kR}(0)) \times B_R(0)$, then

$$|x-y| \ge |x| - |y| \ge \frac{kR}{2} - R > \frac{|x|}{2},$$

and using Hölder's inequality, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N \setminus B_{kR}(0)} |w_n(x)|^2 dx \int_{\{y \in B_R(0) : |x-y| > R\}} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{N+2s}} dy \\
&\le C \int_{\mathbb{R}^N \setminus B_{kR}(0)} dx \int_{\{y \in B_R(0) : |x-y| > R\}} \frac{|w_n(x)|^2}{|x-y|^{N+2s}} dy \\
&\le CR^N \int_{\mathbb{R}^N \setminus B_{kR}(0)} \frac{|w_n(x)|^2}{|x|^{N+2s}} dx \\
&\le \Big(\int_{\mathbb{R}^N \setminus B_{kR}(0)} |w_n(x)|^{2_s^*} dx \Big)^{2/2_s^*} \Big(\int_{\mathbb{R}^N \setminus B_{kR}(0)} |x|^{-(N+2s)\frac{2_s^*}{2_s^*-2}} \Big)^{\frac{2_s^*-2}{2_s^*}} \\
&\le Ck^{-N} \Big(\int_{\mathbb{R}^N \setminus B_{kR}(0)} |w_n(x)|^{2_s^*} dx \Big)^{2/2_s^*} \le Ck^{-N}.
\end{aligned}
\tag{3.30}
$$

From (3.29) and (3.30), we obtain

$$\mathcal{T}_R^2 \le Ck^{-N} + o_n(1). \tag{3.31}$$

Combining (3.20)-(3.28) and (3.31), we deduce that

$$
\begin{aligned}
&\limsup_{n \to +\infty} \int_{\mathbb{R}^N} |w_n(x)|^2 \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \\
&= \lim_{k \to +\infty} \limsup_{n \to +\infty} \int_{\mathbb{R}^N} |w_n(x)|^2 \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{N+2s}} \, dx \, dy = 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\limsup_{n \to +\infty} \int_{\mathbb{R}^N} |z_n(x)|^2 \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \\
&= \lim_{k \to +\infty} \limsup_{n \to +\infty} \int_{\mathbb{R}^N} |z_n(x)|^2 \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{N+2s}} \, dx \, dy = 0.
\end{aligned}
$$

Combining (3.18), (3.20), (3.21), (3.22), and (3.17), we conclude that

$$\int_{\mathbb{R}^{2N}} \phi(y) \frac{|w_n(x) - w_n(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + \int_{\mathbb{R}^{2N}} \phi(y) \frac{|z_n(x) - z_n(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \to 0 \tag{3.32}$$

for all $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^N \setminus \{x_1, \ldots, x_m\})$, which leads to

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2} w_n|^2 \phi \, dx + \int_{\mathbb{R}^N} |(-\Delta)^{s/2} z_n|^2 \phi \, dx = o_n(1). \qquad (3.33)$$

Let $\rho \in \mathbb{R}$ be a number that satisfies $0 < \rho < \min\{\operatorname{dist}(\Gamma, \bar{B}_1(0), 1)\}$. We will show that

$$\int_{B_{1+\rho}(0) \setminus B_{1+\frac{\rho}{3}}(0)} [|(-\Delta)^{s/2} w_n|^2 + |(-\Delta)^{s/2} z_n|^2] dx \to 0. \qquad (3.34)$$

To do this, we consider $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ such that $0 \le \phi(x) \le 1$ and $\phi(x) = 1$ if $x \in B_{1+\rho}(0)$. If $\tilde{\phi} = \phi|_{\mathbb{R}^N \setminus \{x_1, \ldots, x_m\}}$, follows by (3.33) that

$$\begin{aligned}
0 &\le \int_{B_{1+\rho}(0) \setminus B_{1+\frac{\rho}{3}}(0)} [|(-\Delta)^{s/2} w_n|^2 + |(-\Delta)^{s/2} z_n|^2] dx \\
&\le \int_{B_{1+\rho}(0)} [|(-\Delta)^{s/2} w_n|^2 + |(-\Delta)^{s/2} z_n|^2] dx \\
&= \int_{B_{1+\rho}(0)} [|(-\Delta)^{s/2} w_n|^2 + |(-\Delta)^{s/2} z_n|^2] \tilde{\phi} dx \\
&\le \int_{\mathbb{R}^N} [|(-\Delta)^{s/2} w_n|^2 + |(-\Delta)^{s/2} z_n|^2] \tilde{\phi} dx \to 0,
\end{aligned}$$

which implies that (3.34) occurs.

Let $\Phi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ be such that $0 \le \Phi(x) \le 1$, $|\nabla \Phi| \le 2$ for all $x \in \mathbb{R}^N$ and

$$\Phi(x) = \begin{cases} 1, & x \in B_{1+\frac{\rho}{3}}(0), \\ 0, & x \in B_{1+\frac{2\rho}{3}}^c(0) \end{cases}$$

and consider the sequence $(\Phi_{1,n}, \Phi_{2,n})$ given by

$$(\Phi_{1,n}(x), \Phi_{2,n}(x)) = (\Phi(x) w_n(x), \Phi(x) z_n(x)).$$

Using (3.21) and (3.22), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N \setminus B_{1+\rho}(0)} |(-\Delta)^{s/2} \Phi_{1,n}|^2 dx + \int_{\mathbb{R}^N \setminus B_{1+\rho}(0)} |(-\Delta)^{s/2} \Phi_{2,n}|^2 dx \\
&\le 2 \int_{(\mathbb{R}^N \setminus B_{1+\rho}(0)) \times \mathbb{R}^N} \frac{|w_n(y)|^2 |\Phi(x) - \Phi(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \\
&+ 2 \int_{(\mathbb{R}^N \setminus B_{1+\rho}(0)) \times \mathbb{R}^N} \frac{|\Phi(y)|^2 |w_n(x) - w_n(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \\
&\le 2 \int_{(\mathbb{R}^N \setminus B_{1+\rho}(0)) \times \mathbb{R}^N} \frac{|z_n(y)|^2 |\Phi(x) - \Phi(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \\
&+ 2 \int_{(\mathbb{R}^N \setminus B_{1+\rho}(0)) \times \mathbb{R}^N} \frac{|\Phi(y)|^2 |z_n(x) - z_n(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \\
&= o_n(1) + 2 \int_{\mathbb{R}^N \setminus B_{1+\rho}(0)} \Phi(x)^2 |(-\Delta)^{s/2} w_n(x)|^2 dx \\
&+ o_n(1) + 2 \int_{\mathbb{R}^N \setminus B_{1+\rho}(0)} \Phi(x)^2 |(-\Delta)^{s/2} z_n(x)|^2 dx \\
&= o_n(1).
\end{aligned} \qquad (3.35)$$

Similarly, we can obtain the estimate

$$\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} |(-\Delta)^{s/2}\Phi_{1,n}|^2 dx + \int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} |(-\Delta)^{s/2}\Phi_{2,n}|^2 dx$$

$$\leq 2 \int_{(B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0))\times\mathbb{R}^N} \frac{w_n(x)^2|\Phi(x)-\Phi(y)|^2}{|x-y|^{N+2s}}\, dx\, dy$$

$$+ 2 \int_{(B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0))\times\mathbb{R}^N} \frac{\Phi(y)^2|w_n(x)-w_n(y)|^2}{|x-y|^{N+2s}}\, dx\, dy$$

$$+ 2 \int_{(B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0))\times\mathbb{R}^N} \frac{z_n(x)^2|\Phi(x)-\Phi(y)|^2}{|x-y|^{N+2s}}\, dx\, dy$$

$$+ 2 \int_{(B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0))\times\mathbb{R}^N} \frac{\Phi(y)^2|z_n(x)-z_n(y)|^2}{|x-y|^{N+2s}}\, dx\, dy$$

$$\leq 2 \int_{B_{1+\rho}(0)\times\mathbb{R}^N} \frac{w_n(x)^2|\Phi(x)-\Phi(y)|^2}{|x-y|^{N+2s}}\, dx\, dy$$

$$+ 2 \int_{(B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0))\times\mathbb{R}^N} \frac{|w_n(x)-w_n(y)|^2}{|x-y|^{N+2s}}\, dx\, dy \tag{3.36}$$

$$+ 2 \int_{B_{1+\rho}(0)\times\mathbb{R}^N} \frac{z_n(x)^2|\Phi(x)-\Phi(y)|^2}{|x-y|^{N+2s}}\, dx\, dy$$

$$+ 2 \int_{(B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0))\times\mathbb{R}^N} \frac{|z_n(x)-z_n(y)|^2}{|x-y|^{N+2s}}\, dx\, dy$$

$$= 2 \int_{B_{1+\rho}(0)\times\mathbb{R}^N} \frac{w_n(x)^2|\Phi(x)-\Phi(y)|^2}{|x-y|^{N+2s}}\, dx\, dy$$

$$+ 2 \int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} |(-\Delta)^{s/2}w_n|^2 dx$$

$$+ 2 \int_{B_{1+\rho}(0)\times\mathbb{R}^N} \frac{z_n(x)^2|\Phi(x)-\Phi(y)|^2}{|x-y|^{N+2s}}\, dx\, dy$$

$$+ 2 \int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} |(-\Delta)^{s/2}z_n|^2 dx$$

$$= o_n(1),$$

where in the last equality we made use of estimates (3.21), (3.22), and (3.34).

Since $(\Phi_{1,n}, \Phi_{2,n})$ is bounded in $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$, we derive that

$$\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} (-\Delta)^{s/2}w_n(-\Delta)^{s/2}\Phi_{1,n}dx + \int_{B_{1+\frac{\rho}{3}}(0)} (-\Delta)^{s/2}w_n(-\Delta)^{s/2}\Phi_{1,n}dx$$

$$+ \int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} (-\Delta)^{s/2}z_n(-\Delta)^{s/2}\Phi_{2,n}dx + \int_{B_{1+\frac{\rho}{3}}(0)} (-\Delta)^{s/2}z_n(-\Delta)^{s/2}\Phi_{2,n}dx$$

$$- \frac{1}{2_s^*}\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} \Phi_{1,n}H_w(w_n,z_n)dx - \frac{1}{2_s^*}\int_{B_{1+\frac{\rho}{3}}(0)} \Phi_{1,n}H_w(w_n,z_n)dx$$

$$-\frac{1}{2_s^*}\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)}\Phi_{2,n}H_z(w_n,z_n)dx - \frac{1}{2_s^*}\int_{B_{1+\frac{\rho}{3}}(0)}\Phi_{2,n}H_z(w_n,z_n)dx = o_n(1),$$

which implies

$$\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)}(-\Delta)^{s/2}w_n(-\Delta)^{s/2}\Phi_{1,n}dx + \int_{B_{1+\frac{\rho}{3}}(0)}|(-\Delta)^{s/2}\Phi_{1,n}|^2dx$$

$$+\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)}(-\Delta)^{s/2}z_n(-\Delta)^{s/2}\Phi_{2,n}dx + \int_{B_{1+\frac{\rho}{3}}(0)}|(-\Delta)^{s/2}\Phi_{2,n}|^2\,dx$$

$$-\frac{1}{2_s^*}\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)}\Phi_{1,n}H_w(w_n,z_n)dx - \frac{1}{2_s^*}\int_{B_{1+\frac{\rho}{3}}(0)}\Phi_{1,n}H_w(\Phi_{1,n},\Phi_{1,n})dx$$

$$-\frac{1}{2_s^*}\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)}\Phi_{2,n}H_z(w_n,z_n)dx - \frac{1}{2_s^*}\int_{B_{1+\frac{\rho}{3}}(0)}\Phi_{2,n}H_z(\Phi_{2,n},\Phi_{2,n})dx$$

$$= o_n(1).$$

(3.37)

Note that from Hölder inequality, (3.35) and (3.36) we obtain

$$\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)}\big[(-\Delta)^{s/2}w_n(-\Delta)^{s/2}\Phi_{1,n} + (-\Delta)^{s/2}z_n(-\Delta)^{s/2}\Phi_{2,n}\big]dx$$

$$= o_n(1).$$

(3.38)

Moreover, combining (2.1) and (3.17) we deduce

$$\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)}\Phi_{1,n}H_w(w_n,z_n)dx + \int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)}\Phi_{2,n}H_z(w_n,z_n)dx$$

$$= o_n(1).$$

(3.39)

From (3.37), (3.38), and (3.39), we obtain

$$\int_{B_{1+\frac{\rho}{3}}(0)}|(-\Delta)^{s/2}\Phi_{1,n}|^2dx + \int_{B_{1+\frac{\rho}{3}}(0)}|(-\Delta)^{s/2}\Phi_{2,n}|^2\,dx$$

$$-\frac{1}{2_s^*}\int_{B_{1+\frac{\rho}{3}}(0)}\Phi_{1,n}H_w(\Phi_{1,n},\Phi_{1,n})dx - \frac{1}{2_s^*}\int_{B_{1+\frac{\rho}{3}}(0)}\Phi_{2,n}H_z(\Phi_{2,n},\Phi_{2,n})dx = o_n(1).$$

Note that

$$\int_{\mathbb{R}^N}[|(-\Delta)^{s/2}\Phi_{1,n}|^2 + |(-\Delta)^{s/2}\Phi_{2,n}|^2]dx$$

$$= \int_{B_{1+\frac{\rho}{3}}(0)}[|(-\Delta)^{s/2}\Phi_{1,n}|^2 + |(-\Delta)^{s/2}\Phi_{2,n}|^2]dx$$

$$= \int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)}[|(-\Delta)^{s/2}\Phi_{1,n}|^2 + |(-\Delta)^{s/2}\Phi_{2,n}|^2]dx$$

$$\quad + \int_{B_{1+\frac{\rho}{3}}(0)}[|(-\Delta)^{s/2}\Phi_{1,n}|^2 + |(-\Delta)^{s/2}\Phi_{2,n}|^2]dx$$

$$= o_n(1) + \int_{B_{1+\frac{\rho}{3}}(0)}[|(-\Delta)^{s/2}\Phi_{1,n}|^2 + |(-\Delta)^{s/2}\Phi_{2,n}|^2]dx\,.$$

Using (2.1), we obtain

$$\int_{\mathbb{R}^N} H(\Phi_{1,n}, \Phi_{2,n})dx$$

$$= \int_{B_{1+\rho}(0)} H(\Phi_{1,n}, \Phi_{2,n})dx$$

$$= \int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} H(\Phi_{1,n}, \Phi_{2,n})dx + \int_{B_{1+\frac{\rho}{3}}(0)} H(\Phi_{1,n}\Phi_{2,n})dx,$$

from where we deduce

$$\int_{\mathbb{R}^N}[|(-\Delta)^{s/2}\Phi_{1,n}|^2 + |(-\Delta)^{s/2}\Phi_{2,n}|^2]dx - \int_{\mathbb{R}^N} H(\Phi_{1,n}, \Phi_{2,n})dx = o_n(1),$$

i.e.,

$$\|\Phi_{1,n}\|^2 + \|\Phi_{2,n}\|^2 - \int_{\mathbb{R}^N} H(\Phi_{1,n}, \Phi_{2,n})dx = o_n(1).$$

From the definition of S_H, we have

$$(\|\Phi_{1,n}\|^2 + \|\Phi_{2,n}\|^2)\Big[1 - \frac{1}{S_H^{2^*_s/2}}[\|\Phi_{1,n}\|^2 + \|\Phi_{2,n}\|^2]^{2^*_s-2}\Big]$$

$$= \|\Phi_{1,n}\|^2 + \|\Phi_{2,n}\|^2 - \frac{1}{S^{2^*_s/2}}[\|\Phi_{1,n}\|^2 + \|\Phi_{2,n}\|^2]^{2^*_s} \tag{3.40}$$

$$\le \int_{\mathbb{R}^N}[|(-\Delta)^{s/2}\Phi_{1,n}|^2 + |(-\Delta)^{s/2}\Phi_{2,n}|^2]dx - \int_{\mathbb{R}^N} H(\Phi_{1,n}, \Phi_{2,n})dx$$

$$= o_n(1).$$

On the other hand,

$$\|\Phi_{1,n}\|^2 + \|\Phi_{2,n}\|^2$$

$$= \int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} |(-\Delta)^{s/2}\Phi_{1,n}|^2 dx + \int_{B_{1+\frac{\rho}{3}}(0)} |(-\Delta)^{s/2}\Phi_{1,n}|^2 dx$$

$$+ \int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} |(-\Delta)^{s/2}\Phi_{2,n}|^2 dx + \int_{B_{1+\frac{\rho}{3}}(0)} |(-\Delta)^{s/2}\Phi_{2,n}|^2 dx$$

$$= o_n(1) + \int_{B_{1+\frac{\rho}{3}}(0)}[|(-\Delta)^{s/2}\Phi_{1,n}|^2 + |(-\Delta)^{s/2}\Phi_{2,n}|^2]dx.$$

Since $\Phi_{1,n} = w_n$, $\Phi_{2,n} = z_n$ in $B_{1+\frac{\rho}{3}}(0)$ and that $B_{1+\frac{\rho}{3}}(0) \subset B_2(0)$, we obtain

$$\|\Phi_{1,n}\|^2 + \|\Phi_{2,n}\|^2 \le o_n(1) + \int_{B_2(0)}[|(-\Delta)^{s/2}\Phi_{1,n}|^2 + |(-\Delta)^{s/2}\Phi_{2,n}|^2]dx,$$

which implies

$$\|\Phi_{1,n}\|^2 + \|\Phi_{2,n}\|^2 \le o_n(1) + \int_{\bigcup_{k=1}^L B_1(y_k)}[|(-\Delta)^{s/2}z_n|^2 + |(-\Delta)^{s/2}w_n|^2]dx$$

$$\le o_n(1) + \sum_{k=1}^L \int_{B_1(y_k)}[|(-\Delta)^{s/2}w_n|^2 + |(-\Delta)^{s/2}z_n|^2]dx$$

$$\le o_n(1) + L\sup_{y\in\mathbb{R}^N} \int_{B_1(y)}[|(-\Delta)^{s/2}w_n|^2 + |(-\Delta)^{s/2}w_n|^2]dx$$

$$\leq o_n(1) + \frac{S_H^{N/2s}}{2}.$$

Then

$$\left(\|\Phi_{1,n}\|^2 + \|\Phi_{2,n}\|^2 \right)^{1/2} \leq o_n(1) + \frac{S^{N/4s}}{\sqrt{2}},$$

i.e.,

$$(\|\Phi_{1,n}\| + \|\Phi_{2,n}\|)^{2_s^* - 2} \leq o_n(1) + \left(\frac{S^{N/4s}}{\sqrt{2}} \right)^{2_s^* - 2}$$

or

$$o_n(1) - \left(\frac{S^{N/4s}}{\sqrt{2}} \right)^{2_s^* - 2} \leq -(\|\Phi_{1,n}\|^2 + \|\Phi_{2,n}\|^2)^{2_s^* - 2}. \tag{3.41}$$

Using (3.40) and (3.41), we have that

$$(\|\Phi_{1,n}\|^2 + \|\Phi_{2,n}\|^2)\left[1 + o_n(1) - \frac{1}{S_H^{2_s^*/2}} \left(\frac{S_H^{N/4s}}{\sqrt{2}} \right)^{2_s^* - 2} \right]$$

$$= (\|\Phi_{1,n}\|^2 + \|\Phi_{2,n}\|^2)\left\{ 1 + \frac{1}{S_H^{2_s^*/2}} \left[o_n(1) - \left(\frac{S_H^{N/4s}}{\sqrt{2}} \right)^{2_s^* - 2} \right] \right\}$$

$$\leq (\|\Phi_{1,n}\|^2 + \|\Phi_{2,n}\|^2)\left[1 - \frac{1}{S_H^{2_s^*/2}} (\|\Phi_{1,n}\|^2 + \|\Phi_{2,n}\|^2)^{2_s^* - 2} \right] = o_n(1).$$

But the equality

$$\frac{N}{4s}(2_s^* - 2) - \frac{2_s^*}{2} = \frac{N}{4s}\left(\frac{4s}{N - 2s} \right) - \frac{N}{N - 2s} = 0$$

implies

$$(\|\Phi_{1,n}\|^2 + \|\Phi_{2,n}\|^2)\left[1 - \left(\frac{1}{2} \right)^{(2_s^* - 2)/2} \right] \leq o_n(1),$$

and then $(\Phi_{1,n}, \Phi_{2,n}) \to (0,0)$ in $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$. Since $w_n = \Phi_{1,n}, z_n = \Phi_{2,n}$ in $B_1(0)$, we deduce that

$$0 \leq \int_{B_1(0)} [|(-\Delta)^{s/2} w_n|^2 + |(-\Delta)^{s/2} z_n|^2] dx = \|\Phi_{1,n}\|^2 + \|\Phi_{2,n}\|^2,$$

which implies

$$\int_{B_1(0)} \left[|(-\Delta)^{s/2} w_n|^2 + |(-\Delta)^{s/2} z_n|^2 \right] dx \to 0 \quad \text{as } n \to \infty.$$

But this convergence contradicts that

$$\int_{B_1(0)} \left[|(-\Delta)^{s/2} w_n|^2 + |(-\Delta)^{s/2} z_n|^2 \right] dx = \frac{S_H^{N/2s}}{2L}, \quad \forall n \in \mathbb{N}.$$

Therefore, $(u_0, v_0) \neq (0,0)$.

Now we show that there is $(\tau_n, \zeta_n) \in \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$ such that (τ_n, ζ_n) is a $(PS)_{\tilde{c}}$ sequence for $\mathcal{J}_\infty$ satisfying

$$\tau_n(x) = u_n(x) - R_n^{(N-2s)/2} u_0(R_n(x - x_n)) + o_n(1),$$

$$\zeta_n(x) = v_n(x) - R_n^{(N-2s)/2} v_0(R_n(x - x_n)) + o_n(1),$$

up to a subsequence of (u_n, v_n). For this, we consider $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ such that $0 \le \psi(x) \le 1$ for all $x \in \mathbb{R}^N$ and

$$\psi(x) = \begin{cases} 1, & \text{if } x \in B_1(0), \\ 0, & \text{if } x \in B_2^c(0) \end{cases}$$

and consider (τ_n, ζ_n) a sequence defined by

$$\tau_n(x) = u_n(x) - R_n^{(N-2s)/2} u_0(R_n(x - x_n))\psi(\bar{R}_n(x - x_n)), \qquad (3.42)$$

$$\zeta_n(x) = v_n(x) - R_n^{(N-2s)/2} v_0(R_n(x - x_n))\psi(\bar{R}_n(x - x_n)), \qquad (3.43)$$

where $(\bar{R}_n)$ satisfies $\tilde{R}_n = \frac{R_n}{\bar{R}_n} \to \infty$. From (3.42) and (3.43), we obtain

$$R_n^{(2s-N)/2} \tau_n(x) = R_n^{(2s-N)/2} u_n(x) - u_0(R_n(x - x_n))\psi(\bar{R}_n(x - x_n)),$$

$$R_n^{(2s-N)/2} \zeta_n(x) = R_n^{(2s-N)/2} v_n(x) - v_0(R_n(x - x_n))\psi(\bar{R}_n(x - x_n)).$$

Making a change of variable, we conclude that

$$R_n^{(2s-N)/2} \tau_n\Big(\frac{z}{R_n} + x_n\Big) = R_n^{(2s-N)/2} u_n\Big(\frac{z}{R_n} + x_n\Big) - u_0\psi\Big(\frac{z}{\tilde{R}_n}\Big),$$

$$R_n^{(2s-N)/2} \zeta_n\Big(\frac{z}{R_n} + x_n\Big) = R_n^{(2s-N)/2} v_n\Big(\frac{z}{R_n} + x_n\Big) - v_0\psi\Big(\frac{z}{\tilde{R}_n}\Big).$$

Now we define

$$\tilde{\tau}_n = R_n^{(2s-N)/2} \tau_n\Big(\frac{z}{R_n} + x_n\Big) \quad \text{and} \quad \tilde{\zeta}_n = R_n^{(2s-N)/2} \zeta_n\Big(\frac{z}{R_n} + x_n\Big).$$

Since

$$w_n(x) = R_n^{(2s-N)/2} u_n\Big(\frac{x}{R_n} + x_n\Big) \quad \text{and} \quad z_n(x) = R_n^{(2s-N)/2} v_n\Big(\frac{x}{R_n} + x_n\Big)$$

implies

$$\tilde{\tau}_n(z) = w_n(z) - u_0(z)\psi\Big(\frac{z}{\tilde{R}_n}\Big), \qquad (3.44)$$

$$\tilde{\zeta}_n(z) = z_n(z) - v_0(z)\psi\Big(\frac{z}{\tilde{R}_n}\Big). \qquad (3.45)$$

If

$$\psi_n(z) = \psi\Big(\frac{z}{\tilde{R}_n}\Big) \qquad (3.46)$$

then

$$\psi_n(z) = \begin{cases} 1, & \text{if } z \in B_{\tilde{R}_n}(0), \\ 0, & \text{if } z \in B_{2\tilde{R}_n}^c(0). \end{cases}$$

From (3.45), (3.44) and (3.46), we derive that

$$\tilde{\tau}_n(z) = w_n(z) - u_0(z)\psi_n(z),$$

$$\tilde{\zeta}_n(z) = z_n(z) - v_0(z)\psi_n(z).$$

The result is proved if we show that $u_0\psi_n \to u_0$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ and $v_0\psi_n \to v_0$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, and that (w_n, z_n) is a $(PS)_{\tilde{c}}$ sequence for $\mathcal{J}_\infty$. For this, we note that

$$
\begin{aligned}
&\|u_0\psi_n - u_0\|^2 \\
&= \int_{\mathbb{R}^N} |(-\Delta)^{s/2}(u_0\psi_n - u_0)|^2 dx \\
&= \int_{\mathbb{R}^{2N}} \frac{|u_0(x)\psi_n(x) - u_0(x) - u_0(y)\psi_n(y) + u_0(y)|^2}{|x-y|^{N+2s}}\, dx\, dy \\
&= \int_{\mathbb{R}^{2N}} \frac{|u_0(x)(\psi_n(x) - \psi_n(y)) + (\psi_n(y) - 1)(u_0(x) - u_0(y))|^2}{|x-y|^{N+2s}}\, dx\, dy \\
&\leq 4 \int_{\mathbb{R}^{2N}} \frac{|u_0(x)|^2|\psi_n(x) - \psi_n(y)|^2}{|x-y|^{N+2s}}\, dx\, dy \\
&\quad + 4 \int_{\mathbb{R}^{2N}} \frac{|\psi_n(y) - 1|^2|u_0(x) - u_0(y)|^2}{|x-y|^{N+2s}}\, dx\, dy.
\end{aligned}
\tag{3.47}
$$

Arguing as in the proof of (3.21), if we replace w_n by u_0, and ϕ by ψ_n, since $\operatorname{supp}(\psi_n) \subset B_{2\tilde{R}}(0)$, we can see that

$$
\int_{\mathbb{R}^{2N}} \frac{|u_0(x)|^2|\psi_n(x) - \psi_n(y)|^2}{|x-y|^{N+2s}}\, dx\, dy = o_n(1).
\tag{3.48}
$$

Moreover, taking into account that $|\psi_n - 1| \leq 2$, $|\psi_n - 1| \to 0$ a.e. in $\mathbb{R}^N$ and $u_0 \in \mathcal{D}^{s,2}(\mathbb{R}^N)$, the Dominated Convergence Theorem implies that

$$
\int_{\mathbb{R}^{2N}} \frac{|\psi_n(y) - 1|^2|u_0(x) - u_0(y)|^2}{|x-y|^{N+2s}}\, dx\, dy = o_n(1).
\tag{3.49}
$$

Combining (3.47), (3.48), and (3.49), we obtain $u_0\psi_n \to u_0$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. Similarly arguing, we obtain $v_0\psi_n \to v_0$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. Hence,

$$
\begin{aligned}
\tilde{\tau}_n(z) &= w_n(z) - u_0(z) + o_n(1), \\
\tilde{\zeta}_n(z) &= z_n(z) - v_0(z) + o_n(1).
\end{aligned}
$$

Since $w_n \to u_0$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, $z_n \to v_0$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, $w_n \to u_0$ in $\mathbb{R}^N$ and $z_n \to v_0$, by [10, Lemma 2.2],

$$
\int_{\mathbb{R}^N} |(-\Delta)^{s/2}w_n|^2 dx = \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u_0|^2 dx + \int_{\mathbb{R}^N} |(-\Delta)^{s/2}(w_n - u_0)|^2 dx + o_n(1),
$$

$$
\int_{\mathbb{R}^N} |(-\Delta)^{s/2}z_n|^2 dx = \int_{\mathbb{R}^N} |(-\Delta)^{s/2}v_0|^2 dx + \int_{\mathbb{R}^N} |(-\Delta)^{s/2}(z_n - v_0)|^2 dx + o_n(1).
$$

By [3, Lemma 7.2], we have

$$
\int_{\mathbb{R}^N} H(w_n, z_n) dx = \int_{\mathbb{R}^N} H(u_0, v_0) dx + \int_{\mathbb{R}^N} H(w_n - u_0, z_n - v_0) dx + o_n(1)
$$

which implies that

$$
\mathcal{J}_\infty(\tau_n, \zeta_n) = \mathcal{J}_\infty(w_n, z_n) - \mathcal{J}_\infty(u_0, v_0) + o_n(1).
$$

Therefore, $\mathcal{J}_\infty(\tau_n, \zeta_n) \to \tilde{c}$ as $n \to +\infty$, where $\tilde{c} = c - \mathcal{J}_\infty(u_0, v_0)$. Moreover, using Hölder's inequality and [3, Lemma 7.2] a direct calculation gives us

$$
\|\mathcal{J}'_\infty(\tilde{\tau}_n, \tilde{\zeta}_n) - \mathcal{J}'_\infty(w_n, z_n) + \mathcal{J}'_\infty(u_0, v_0)\|_{(\mathcal{D} \times \mathcal{D})'} \to 0.
$$

Since (u_0, v_0) is a nontrivial critical point of $\mathcal{J}_\infty$, we conclude that

$$\mathcal{J}'_\infty(\tilde{\tau}_n, \tilde{\zeta}_n) = \mathcal{J}'_\infty(w_n, z_n) + \mathcal{J}'_\infty(u_0, v_0) + o_n(1) = \mathcal{J}'_\infty(w_n, z_n) + o_n(1).$$

Since

$$0 \leq \|\mathcal{J}'_\infty(\tau_n, \zeta_n)\|_{(\mathcal{D} \times \mathcal{D})'} \leq \|\mathcal{J}'_\infty(\tilde{\tau}_n, \tilde{\zeta}_n)\|_{(\mathcal{D} \times \mathcal{D})'},$$

it follows that $\mathcal{J}'_\infty(\tau_n, \zeta_n) \to 0$ and the proof of Lemma 3.1 is complete. $\qquad \square$

The next result is a version of nonlocal global compactness result for a fractional Laplacian system in $\mathbb{R}^N$ of the result due to Struwe that can be found in [26].

Theorem 3.2 (A global compactness result). *Let (u_n, v_n) be a $(PS)_c$ sequence for $\mathcal{J}$ with $(u_n, v_n) \rightharpoonup (u_0, v_0)$ in $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$. Then, up to a subsequence, (u_n, v_n) satisfies either,*

(a) *$(u_n, v_n) \to (u_0, v_0)$ in $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$ or,*
(b) *there exists $k \in \mathbb{N}$ and nontrivial solutions $(z_0^1, \zeta_0^1), (z_0^2, \zeta_0^2), \dots, (z_0^k, \zeta_0^k)$ for the system (2.2), such that*

$$\|(u_n, v_n)\|^2 \to \|(u_0, v_0)\|^2 + \sum_{j=1}^k \|(z_0^j, \zeta_0^j)\|^2,$$

$$\mathcal{J}(u_n, v_n) \to \mathcal{J}(u_0, v_0) + \sum_{j=1}^k \mathcal{J}_\infty(z_0^j, \zeta_0^j).$$

Proof. From the weak convergence and a density argument, we have that (u_0, v_0) is a critical point of $\mathcal{J}$. Suppose that $(u_n, v_n) \not\to (u_0, v_0)$ in $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$ and let $(w_n^1, z_n^1) \subset \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$ be the sequence given by $(w_n^1, z_n^1) = (u_n - u_0, v_n - v_0)$. Then by hypothesis, $(w_n^1, z_n^1) \rightharpoonup (0,0)$ in $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$ and $(w_n^1, v_n^1) \not\to (0,0)$. Applying [16, Lema 4.6] and [3, Lemma 7.2], we obtain

$$\mathcal{J}_\infty(w_n^1, z_n^1) = \mathcal{J}(u_n, v_n) - \mathcal{J}(u_0, v_0) + o_n(1), \tag{3.50}$$

$$\mathcal{J}'_\infty(w_n^1, z_n^1) = \mathcal{J}'(u_n, v_n) - \mathcal{J}'(u_0, v_0) + o_n(1). \tag{3.51}$$

Then, we conclude from (3.50) and (3.51) that (w_n^1, z_n^1) is a $(PS)_{c_1}$ sequence for $\mathcal{J}_\infty$. Hence, by Lemma 3.1, there are sequences $R_{n,1} \subset \mathbb{R}$, $x_{n,1} \subset \mathbb{R}^N$, $(z_0^1, \zeta_0^1) \in \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$ nontrivial solution for the system (2.2) and a $(PS)_{c_2}$ sequence $(w_n^2, z_n^2) \subset \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$ for $\mathcal{J}_\infty$ such that

$$w_n^2(x) = w_n^1(x) - R_{n,1}^{(N-2s)/2} z_0^1(R_{n,1}(x - x_{n,1})) + o_n(1),$$

$$z_n^2(x) = z_n^1(x) - R_{n,1}^{(N-2s)/2} \zeta_0^1(R_{n,1}(x - x_{n,1})) + o_n(1).$$

If we define

$$\Phi_n^1(x) = R_{n,1}^{(2s-N)/2} w_n^1 \left(\frac{x}{R_{n,1}} + x_{n,1} \right)$$

$$\Psi_n^1(x) = R_{n,1}^{(2s-N)/2} z_n^1 \left(\frac{x}{R_{n,1}} + x_{n,1} \right),$$

$$\widetilde{w}_n^2(x) = R_{n,1}^{(2s-N)/2} w_n^2 \left(\frac{x}{R_{n,1}} + x_{n,1} \right),$$

$$\widetilde{z}_n^2(x) = R_{n,1}^{(2s-N)/2} z_n^2 \left(\frac{x}{R_{n,1}} + x_{n,1} \right),$$

then we have

$$\widetilde{w}_n^2(x) = \Phi_n^1(x) - z_0^1(x) + o_n(1), \tag{3.52}$$

$$\widetilde{z}_n^2(x) = \Psi_n^1(x) - \zeta_0^1(x) + o_n(1), \tag{3.53}$$

$$\|\Phi_n^1\| = \|w_n^1\|, \quad \|\Psi_n^1\| = \|z_n^1\|, \tag{3.54}$$

$$\int_{\mathbb{R}^N} H(\Phi_n^1, \Psi_n^1)dx = \int_{\mathbb{R}^N} H(w_n^1, z_n^1)dx. \tag{3.55}$$

Hence,

$$\mathcal{J}_\infty(\Phi_n^1, \Psi_n^1) = \mathcal{J}_\infty(w_n^1, z_n^1), \tag{3.56}$$

$$\mathcal{J}_\infty'(\Phi_n^1, \Psi_n^1) \to 0 \quad \text{in } (\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N))'. \tag{3.57}$$

By (3.56) and (3.57) and from item (a) of Lemma 2.1, we have that (Φ_n^1, Ψ_n^1) is a bounded sequence in $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$ and, up to a subsequence, we have

$$(\Phi_n^1, \Psi_n^1) \rightharpoonup (z_0^1, \zeta_0^1) \quad \text{in } \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N) \tag{3.58}$$

Again, using [16, Lema 4.6] and [3, Lemma 7.2], we obtain

$$\begin{aligned}
\mathcal{J}_\infty(\widetilde{w}_n^2, \widetilde{z}_n^2) &= \mathcal{J}_\infty(\Phi_n^1, \Psi_n^1) - \mathcal{J}_\infty(z_0^1, \zeta_0^1) + o_n(1) \\
&= \mathcal{J}(u_n, v_n) - \mathcal{J}(u_0, v_0) - \mathcal{J}_\infty(z_0^1, \zeta_0^1) + o_n(1),
\end{aligned} \tag{3.59}$$

$$\mathcal{J}_\infty'(\widetilde{w}_n^2, \widetilde{z}_n^2) = \mathcal{J}_\infty'(\Phi_n^1, \Psi_n^1) - \mathcal{J}_\infty'(z_0^1, \zeta_0^1) + o_n(1). \tag{3.60}$$

If $(\widetilde{w}_n^2, \widetilde{z}_n^2) \to (0,0)$ in $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$ the proof is complete for $k = 1$, because in this case, we have

$$\|(u_n, v_n)\|^2 \to \|(u_0, v_0)\|^2 + \|(z_0^1, \zeta_0^1)\|^2.$$

Moreover, using continuity of $\mathcal{J}_\infty$, we obtain

$$\mathcal{J}(u_n, v_n) \to \mathcal{J}(u_0, v_0) + \mathcal{J}_\infty(z_0^1, \zeta_0^1).$$

If $(\widetilde{w}_n^2, \widetilde{z}_n^2) \not\to (0,0)$ in $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$, by (3.52)-(3.53) and (3.58) we have $(\widetilde{w}_n^2, \widetilde{z}_n^2) \rightharpoonup (0,0)$ in $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$, and using (3.59)$and$(3.60) we conclude that $(\widetilde{w}_n^2, \widetilde{z}_n^2)$ is a $(PS)_{c_2}$ sequence for $\mathcal{J}_\infty$.

By Lemma 3.1, there are sequences $(R_{n,2}) \subset \mathbb{R}$, $(x_{n,2}) \subset \mathbb{R}^N$, $(z_0^2, \zeta_0^2) \in \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$ nontrivial solutions of (2.2), and a $(PS)_{c_3}$ sequence $(w_n^3, z_n^3) \subset \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$ for $\mathcal{J}_\infty$ such that

$$w_n^3(x) = \widetilde{w}_n^2(x) - R_{n,2}^{(N-2s)/2} z_0^2 (R_{n,2}(x - x_{n,2})) + o_n(1)$$

$$z_n^3(x) = \widetilde{z}_n^2(x) - R_{n,2}^{(N-2s)/2} \zeta_0^2 (R_{n,2}(x - x_{n,2})) + o_n(1).$$

If

$$\Phi_n^2(x) = R_{n,2}^{(2s-N)/2} \widetilde{w}_n^2 \Big(\frac{x}{R_{n,2}} + x_{n,2} \Big),$$

$$\Psi_n^2(x) = R_{n,2}^{(2s-N)/2} \widetilde{z}_n^2 \Big(\frac{x}{R_{n,2}} + x_{n,2} \Big),$$

$$\widetilde{w}_n^3(x) = R_{n,2}^{(2s-N)/2} w_n^3 \Big(\frac{x}{R_{n,2}} + x_{n,2} \Big),$$

$$\widetilde{z}_n^3(x) = R_{n,2}^{(2s-N)/2} z_n^3 \Big(\frac{x}{R_{n,2}} + x_{n,2} \Big),$$

then

$$\widetilde{w}_n^3(x) = \Phi_n^2(x) - z_0^2(x) + o_n(1), \tag{3.61}$$

$$\widetilde{z}_n^3(x) = \Psi_n^2(x) - \zeta_0^2(x) + o_n(1). \tag{3.62}$$

Arguing as before, we conclude that

$$\|(\widetilde{w}_n^3, \widetilde{z}_n^3)\|^2 = \|(u_n, v_n)\|^2 - \|(u_0, v_0)\|^2 - \|(z_0^1, \zeta_0^1)\|^2 - \|(z_0^2, \zeta_0^2)\|^2 + o_n(1), \tag{3.63}$$

$$\mathcal{J}_\infty(\widetilde{w}_n^3, \widetilde{z}_n^3) = \mathcal{J}(u_n, v_n) - \mathcal{J}(u_0, v_0) - \mathcal{J}_\infty(z_0^1, \zeta_0^1) - \mathcal{J}_\infty(z_0^2, \zeta_0^2) + o_n(1), \tag{3.64}$$

$$\mathcal{J}_\infty'(\widetilde{w}_n^3, \widetilde{z}_n^3) = \mathcal{J}_\infty'(\Phi_n^2, \Psi_n^2) - \mathcal{J}_\infty'(z_0^2, \zeta_0^2) + o_n(1). \tag{3.65}$$

If $(\widetilde{w}_n^3, \widetilde{z}_n^3) \to (0,0)$ in $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$, the proof is complete for $k = 2$, because in this case $\|(\widetilde{w}_n^3, \widetilde{z}_n^3)\|^2 \to 0$ and from (3.63), we have

$$\|(u_n, v_n)\|^2 \to \|(u_0, v_0)\|^2 + \sum_{j=1}^{2} \|(z_0^j, \zeta_0^j)\|^2.$$

Similarly, if $(\widetilde{w}_n^3, \widetilde{z}_n^3) \to (0,0)$ in $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$ then the continuity of $\mathcal{J}_\infty$ assures us that $\mathcal{J}_\infty(\widetilde{w}_n^3, \widetilde{z}_n^3) \to 0$, and by (3.64) we obtain

$$\mathcal{J}(u_n, v_n) \to \mathcal{J}(u_0, v_0) + \sum_{j=1}^{2} \mathcal{J}_\infty(z_0^j, \zeta_0^j).$$

If $(\widetilde{w}_n^3, \widetilde{z}_n^3) \nrightarrow (0,0)$ in $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$, we can repeat the same arguments before and we can find $(z_0^1, \zeta_0^1), (z_0^2, \zeta_0^2), \ldots, (z_0^{k-1}, \zeta_0^{k-1})$ nontrivial solutions for the system (2.2) satisfying

$$\|(\widetilde{w}_n^k, \widetilde{z}_n^k)\|^2 = \|(u_n, v_n)\|^2 - \|(u_0, v_0)\|^2 - \sum_{j=1}^{k-1} \|(z_0^j, \zeta_0^j)\|^2 + o_n(1), \tag{3.66}$$

$$\mathcal{J}_\infty(\widetilde{w}_n^k, \widetilde{z}_n^k) = \mathcal{J}(u_n, v_n) - \mathcal{J}(u_0, v_0) - \sum_{j=1}^{k-1} \mathcal{J}_\infty(z_0^j, \zeta_0^j) + o_n(1). \tag{3.67}$$

From the definition of constant S_H, we obtain

$$\left(\int_{\mathbb{R}^N} H(z_0^j, \zeta_0^j) dx \right)^{2/2_s^*} S_H \le \|(z_0^j, \zeta_0^j)\|^2, \quad j = 1, 2, \ldots, k-1. \tag{3.68}$$

Since (z_0^j, ζ_0^j) is a nontrivial solution of (2.2), for $j = 1, 2, \ldots, k-1$, we have

$$\|(z_0^j, \zeta_0^j)\|^2 = \int_{\mathbb{R}^N} H(z_0^j, \zeta_0^j) dx.$$

Hence,

$$- \|(z_0^j, \zeta_0^j)\| \le -S_H^{N/2s}, \quad j = 1, 2, \ldots, k-1. \tag{3.69}$$

From (3.66) and (3.69), we have

$$\|(\widetilde{w}_n^k, \widetilde{z}_n^k)\|^2 = \|(u_n, v_n)\|^2 - \|(u_0, v_0)\|^2 - \sum_{j=1}^{k-1} \|(z_0^j, \zeta_0^j)\|^2 + o_n(1)$$

$$\le \|(u_n, v_n)\|^2 - \|(u_0, v_0)\|^2 - \sum_{j=1}^{k-1} S_H^{N/2s} + o_n(1)$$

$$= \|(u_n, v_n)\|^2 - \|(u_0, v_0)\|^2 - (k-1)S_H^{N/2s} + o_n(1).$$

Since (u_n, v_n) is bounded in $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$ for k sufficient large, we conclude that $(\widetilde{w}_n^k, \widetilde{z}_n^k) \to (0,0)$ in $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$ and the proof is complete. $\qquad \square$

The following two corollaries are immediate consequences of Theorem 3.2.

Corollary 3.3. *Let (u_n, v_n) be a $(PS)_c$ sequence for $\mathcal{J}$ with $c \in \left(0, \frac{s}{N} S_H^{N/2s}\right)$. Then, up to a subsequence, (u_n, v_n) converges strongly in $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$.*

Proof. Since (u_n, v_n) is bounded in $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$, we have

$$(u_n, v_n) \rightharpoonup (u_0, v_0) \quad \text{in } \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$$

and a denseness argument implies that $\mathcal{J}'(u_0, v_0) = 0$. Suppose, by contradiction, that

$$(u_n, v_n) \not\to (u_0, v_0) \quad \text{in } \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N).$$

From Theorem 3.2, there are nontrivial solutions $(z_0^1, \zeta_0^1), (z_0^2, \zeta_0^2), \ldots, (z_0^k, \zeta_0^k)$ of system (2.2) and $k \in \mathbb{N}$ such that

$$\|(u_n, v_n)\|^2 \to \|(u_0, v_0)\|^2 + \sum_{j=1}^k \|(z_0^j, \zeta_0^j)\|^2,$$

$$\mathcal{J}(u_n, v_n) \to \mathcal{J}(u_0, v_0) + \sum_{j=1}^k \mathcal{J}_\infty(z_0^j, \zeta_0^j).$$

By (2.1), we have

$$
\begin{aligned}
&\mathcal{J}(u_0, v_0)\\
&= \frac{1}{2}\|(u_0, v_0)\|^2 + \frac{1}{2}\int_{\mathbb{R}^N}(a(x)u_0^2 + b(x)v_0^2)dx - \frac{1}{2_s^*}\int_{\mathbb{R}^N} H(u_0, v_0)dx\\
&= \frac{1}{2}\|(u_0, v_0)\|^2 + \frac{1}{2}\Big(\int_{\mathbb{R}^N} H(u_0, v_0)dx - \|(u_0, v_0)\|^2\Big) - \frac{1}{2_s^*}\int_{\mathbb{R}^N} H(u_0, v_0)dx\\
&= \Big(\frac{1}{2} - \frac{1}{2_s^*}\Big)\int_{\mathbb{R}^N} H(u_0, v_0)dx\\
&= \frac{s}{N}\int_{\mathbb{R}^N} H(u_0, v_0)dx \geq 0.
\end{aligned}
$$

Then

$$c = \mathcal{J}(u_0, v_0) + \sum_{j=1}^k \mathcal{J}_\infty(z_0^j, \zeta_0^j) \geq \sum_{j=1}^k \mathcal{J}_\infty(z_0^j, \zeta_0^j) \geq \frac{ks}{N} S_H^{N/2s} \geq \frac{s}{N} S_H^{N/2s}$$

which contradicts $c \in (0, \frac{s}{N} S_H^{N/2s})$. $\qquad \square$

The next corollary tells us that the functional $\mathcal{J}$ satisfies the Palais-Smale condition.

Corollary 3.4. *The functional $\mathcal{J} : \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N) \to \mathbb{R}$ satisfies the Palais-Smale condition in $\left(\frac{s}{N} S_H^{N/2s}, \frac{2s}{N} S_H^{N/2s}\right)$.*

Proof. Let $(u_n, v_n) \subset \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$ be a sequence such that

$$\mathcal{J}(u_n, v_n) \to c \quad \text{and} \quad \mathcal{J}'(u_n, v_n) \to 0.$$

Since (u_n, v_n) is bounded in $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$, up to a subsequence, we have

$$(u_n, v_n) \rightharpoonup (u_0, v_0) \quad \text{in } \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N).$$

Moreover, $\mathcal{J}(u_0, v_0) \geq 0$. Suppose, by contradiction, that

$$(u_n, v_n) \not\rightarrow (u_0, v_0) \quad \text{in } \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N).$$

From Theorem 3.2, there are nontrivial solutions $(z_0^1, \zeta_0^1), (z_0^2, \zeta_0^2), .., (z_0^k, \zeta_0^k)$ of system (2.2), and a $k \in \mathbb{N}$ such that

$$\|(u_n, v_n)\|^2 \rightarrow \|(u_0, v_0)\|^2 + \sum_{j=1}^{k} \|(z_0^j, \zeta_0^j)\|^2,$$

$$\mathcal{J}(u_n, v_n) \rightarrow \mathcal{J}(u_0, v_0) + \sum_{j=1}^{k} \mathcal{J}_\infty(z_0^j, \zeta_0^j) = c.$$

Since $\mathcal{J}(u_0, v_0) \geq 0$, it follows that $k = 1$ and (z_0^1, ζ_0^1) cannot change sign. Hence,

$$c = \mathcal{J}(u_0, v_0) + \mathcal{J}_\infty(u_0, v_0) = \mathcal{J}(u_0, v_0) + \frac{s}{N} S_H^{N/2s}.$$

By the definition of S_H, $\mathcal{J}'(u_0, v_0) = 0$, and

$$\mathcal{J}(u_0, v_0) = \frac{s}{N} \int_{\mathbb{R}^N} H(u_0, v_0) dx$$

we have

$$\frac{2s}{N} S_H^{N/2s} \leq \mathcal{J}(u_0, v_0) + \frac{s}{N} S_H^{N/2s} = c,$$

which contradicts $c \in (\frac{s}{N} S_H^{N/2s}, \frac{2s}{N} S_H^{N/2s})$. $\qquad \square$

Corollary 3.5. Let (u_n, v_n) be a $(PS)_c$ sequence for $\mathcal{J}$ with

$$c \in \Big(\frac{ks}{N} S_H^{N/2s}, \frac{(k+1)s}{N} S_H^{N/2s} \Big),$$

where $k \in \mathbb{N}$. Then the weak limit (u_0, v_0) of (u_n, v_n) is not trivial.

Proof. Suppose, by contradiction, that $(u_0, v_0) \equiv (0,0)$. Since $c > 0$, it follows that $(u_n, v_n) \not\rightarrow (0,0)$ in $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$. By Theorem 3.2, up to a subsequence, we obtain

$$\|(u_n, v_n)\|^2 \rightarrow \|(u_0, v_0)\|^2 + \sum_{j=1}^{k} \|(z_0^j, \zeta_0^j)\|^2 = \sum_{j=1}^{k} \|(z_0^j, \zeta_0)\|^2,$$

$$\mathcal{J}(u_n, v_n) \rightarrow \mathcal{J}(u_0, v_0) + \sum_{j=1}^{k} \mathcal{J}_\infty(z_0^j, \zeta_0^j) = \sum_{j=1}^{k} \mathcal{J}_\infty(z_0^j, \zeta_0^j) = c \geq \frac{(k+1)s}{N} S_H^{N/2s}$$

which contradicts $c \in (\frac{ks}{N} S_H^{N/2s}, \frac{(k+1)s}{N} S_H^{N/2s})$. $\qquad \square$

Next we consider the functional $f : \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N) \rightarrow \mathbb{R}$ given by

$$f(u, v) := \|(u, v)\|^2 + \int_{\mathbb{R}^N} (a(x)u^2 + b(x)v^2) dx$$

and the manifold $\mathcal{M} \subset \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$ given by

$$\mathcal{M} := \Big\{ (u, v) \in \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} H(u, v) dx = 1 \Big\}.$$

Remark 3.6. Note that if $(u_n, v_n) \subset \mathcal{M}$ satisfies

$$f(u_n, v_n) \to c \quad \text{and} \quad f'|_{\mathcal{M}}(u_n, v_n) \to 0,$$

then the sequence $(w_n, z_n) \subset \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$, where

$$(w_n, z_n) = \left(c^{(N-2s)/4s} u_n, c^{(N-2s)/4s} v_n\right)$$

satisfies $\mathcal{J}(w_n, z_n) \to \frac{s}{N} c^{N/2s}$ and $\mathcal{J}'(w_n, z_n) \to 0$.

The remark above combined with Corollary 3.4 leads us to the following result.

Corollary 3.7. *Suppose that there are a sequence $(u_n, v_n) \subset \mathcal{M}$, and a number $c \in \left(S_H, 2^{2s/N} S_H\right)$ such that $f(u_n, v_n) \to c$ and $f'|_{\mathcal{M}}(u_n, v_n) \to 0$. Then, up to a subsequence, $(u_n, v_n) \to (u_0, v_0)$ in $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$ for some*

$$(u_0, v_0) \in \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N).$$

rom Corollaries 3.4 and 3.7 we have the following result.

Corollary 3.8. *Suppose that there are a sequences $(u_n, v_n) \subset \mathcal{M}$ and a number $c \in \left(S_H, 2^{2s/N} S_H\right)$ such that $f(u_n, v_n) \to c$ and $f'(u_n, v_n) \to 0$. Then $\mathcal{J}$ has a critical point $(w_0, z_0) \in \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$ with $\mathcal{J}(w_0, z_0) = \frac{s}{N} c^{N/2s}$.*

4. TECHNICAL LEMMAS

In this subsection, we prove some properties of the function $\Phi_{\delta,b}$ given in (1.6). Note that

$$(\Phi_{\delta,b}, \Phi_{\delta,b}) \subset \Sigma := \{(u, v) \in \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N) : u, v \geq 0\}. \quad (4.1)$$

Moreover, making a change of variable we can prove that

$$\Phi_{\delta,b} \in L^q(\mathbb{R}^N) \quad \text{for } q \in \left(\frac{N}{N-2s}, 2_s^*\right], \quad \forall \delta > 0, \ \forall b \in \mathbb{R}^N. \quad (4.2)$$

Lemma 4.1. *For each $b \in \mathbb{R}^N$, we have*

    (i) $\|\Phi_{\delta,b}\|_{H^{1,\infty}(\mathbb{R}^N)} \to 0$ *as $\delta \to +\infty$;*
    (ii) $\|\Phi_{\delta,b}\|_{H^{1,\infty}(\mathbb{R}^N)} \to +\infty$ *as $\delta \to 0$;*
    (iii) $|\Phi_{\delta,b}|_q \to 0$ *as $\delta \to 0$, for all $q \in \left(\frac{N}{N-2s}, 2_s^*\right)$;*
    (iv) $|\Phi_{\delta,b}|_q \to +\infty$ *as $\delta \to \infty$, for all $q \in \left(\frac{N}{N-2s}, 2_s^*\right)$.*

Proof. Using the definition of $\Phi_{\delta,b}$, we have

$$|\nabla \Phi_{\delta,b}(x)| = \frac{C \delta^{\frac{N-2s}{2}}}{[\delta^2 + |x - b|^2]^{\frac{N-2s+2}{2}}},$$

where C is a positive constant. Thus

$$\|\Phi_{\delta,b}\|_{H^{1,\infty}(\mathbb{R}^N)} = \tilde{C} \delta^{-\frac{N+2-2s}{2}}, \quad \tilde{C} > 0$$

and consequently (i) and (ii) follow. Now, note that

$$|\Phi_{\delta,b}|_q^q = \hat{C}^q \delta^{\frac{q(2s-N)}{2} + N} \int_{\mathbb{R}^N} \left(\frac{1}{1 + |z|^2}\right)^{\frac{q(N-2s)}{2}} dz, \quad \hat{C} > 0,$$

and so, for all $q \in \left(\frac{N}{N-2s}, 2_s^*\right)$, (iii) and (iv) follow. $\qquad\square$

Lemma 4.2. *For each $\epsilon > 0$, we have*

$$\int_{\mathbb{R}^N \setminus B_\epsilon(0)} |(-\Delta)^{s/2}\Phi_{\delta,0}|^2 dx \to 0, \quad as \ \delta \to 0.$$

The proof of the above lemma can be found in [13, Lemma 4.2].

Lemma 4.3. *Assume condition* (H7). *Then:*

(i) *for each $\epsilon > 0$, there are $\underline{\delta} = \underline{\delta}(\epsilon) > 0$ and $\bar{\delta} = \bar{\delta}(\epsilon) > 0$ such that*

$$\sup_{b \in \mathbb{R}^N} f(\ell_0 \Phi_{\delta,b}, t_0 \Phi_{\delta,b}) < S_H + \epsilon, \quad \forall \delta \in (0, \underline{\delta}] \cup [\bar{\delta}, \infty);$$

(ii) *for each $\delta > 0$, we have*

$$\lim_{|b| \to +\infty} f(\ell_0 \Phi_{\delta,b}, t_0 \Phi_{\delta,b}) = S_H.$$

Proof. (i) Consider $b \in \mathbb{R}^N$, $q \in (\frac{N}{2s}, p_2]$ and $t \in (1, +\infty)$ with $\frac{1}{q} + \frac{1}{t} = 1$. By a simple calculations,

$$\frac{N}{N-2s} < 2t < 2_s^*.$$

Since $\Phi_{\delta,b} \in L^d(\mathbb{R}^N)$ for all $d \in \left(\frac{N}{N-2s}, 2_s^*\right)$, we obtain $|\Phi_{\delta,b}|^2 \in L^t(\mathbb{R}^N)$. Then, using Hölder's inequality and a change of variable, we have

$$\int_{\mathbb{R}^N} a(x)|\Phi_{\delta,b}|^2 dx \leq |a|_q \Big(\int_{\mathbb{R}^N} \Big| \frac{c\delta^{(N-2s)/2}}{[\delta^2 + |x - b|^2]^{(N-2s)/2}} \Big|^{2t} dx \Big)^{1/t}$$

$$= |a|_q \Big(\int_{\mathbb{R}^N} \Big| \frac{c\delta^{(N-2s)/2}}{[\delta^2 + |z|^2]^{(N-2s)/2}} \Big|^{2t} dz \Big)^{1/t}$$

$$= |a|_q \Big(\int_{\mathbb{R}^N} |\Phi_{\delta,0}|^{2t} dz \Big)^{1/t}$$

$$= |a|_q |\Phi_{\delta,0}|_{2t}^2, \quad \forall b \in \mathbb{R}^N.$$

Arguing in the same way, we have

$$\int_{\mathbb{R}^N} b(x)|\Phi_{\delta,b}|^2 dx \leq |b|_q |\Phi_{\delta,0}|_{2t}^2, \quad \forall b \in \mathbb{R}^N.$$

From Lemma 4.1(iii), given $\epsilon > 0$ there exists $\underline{\delta} = \underline{\delta}(\epsilon) > 0$ such that

$$\sup_{b \in \mathbb{R}^N} f(\ell_0 \Phi_{\delta,b}, t_0 \Phi_{\delta,b}) \leq S_H + \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq S_H + \epsilon, \quad \forall \delta \in (0, \underline{\delta}].$$

On the other hand, suppose $q \in [p_1, \frac{N}{2s})$ with $t \in (1, +\infty)$ and $\frac{1}{q} + \frac{1}{t} = 1$. In these conditions we have $2t - 2_s^* > 0$, $|\Phi_{\delta,y}|^{2_s^*} \in L^1(\mathbb{R}^N)$ and for $\delta > 1$, $|\Phi_{\delta,y}| \in L^\infty(\mathbb{R}^N)$, and so $|\Phi_{\delta,y}|^2 \in L^t(\mathbb{R}^N)$. Thus, using Hölder's inequality with exponents q and t and remembering that $\|\Phi_{\delta,0}\|_{2_s^*} = 1$, we deduce

$$\ell_0^2 \int_{\mathbb{R}^N} a(x)|\Phi_{\delta,y}|^2 dx \leq \ell_0^2 |a|_q \Big(\int_{\mathbb{R}^N} |\Phi_{\delta,0}|^{2t} dz \Big)^{1/t}$$

$$= \ell_0^2 |a|_q |\Phi_{\delta,0}|_\infty^{\frac{2t-2_s^*}{t}} \Big(\int_{\mathbb{R}^N} |\Phi_{\delta,0}|^{2_s^*} dx \Big)^{1/t} \qquad (4.3)$$

$$\leq \ell_0^2 |a|_q C \delta^{\frac{2s-N}{2} \frac{2t-2_s^*}{t}}, \quad \forall b \in \mathbb{R}^N.$$

Once $\left(\frac{2s-N}{2}\right)\left(\frac{2t-2_s^*}{t}\right) < 0$, given $\epsilon > 0$, there is $\bar{\delta} = \bar{\delta}(\epsilon) > 1$ such that

$$\delta^{\frac{2s-N}{2}\frac{2t-2_s^*}{t}} < \frac{\epsilon}{2\ell_0^2 C|a|_q} \quad \forall \delta \in [\bar{\delta}, \infty). \tag{4.4}$$

Arguing in the same way, we have

$$t_0^2 \int_{\mathbb{R}^N} b(x)|\Phi_{\delta,b}|^2 dx \le t_0^2 |a|_q C \delta^{\frac{2s-N}{2}\frac{2t-2_s^*}{t}}, \quad \forall b \in \mathbb{R}^N. \tag{4.5}$$

Combining (4.3), (4.4), and (4.5), we obtain

$$\ell_0^2 \sup_{b \in \mathbb{R}^N} \int_{\mathbb{R}^N} a(x)|\Phi_{\delta,b}|^2 dx < \frac{\epsilon}{2}, \quad \forall \delta \in [\bar{\delta}, \infty),$$

$$t_0^2 \sup_{b \in \mathbb{R}^N} \int_{\mathbb{R}^N} b(x)|\Phi_{\delta,b}|^2 dx < \frac{\epsilon}{2}, \quad \forall \delta \in [\bar{\delta}, \infty).$$

Therefore,

$$f(\ell_0 \Phi_\delta, t_0 \Phi_\delta) = \int_{\mathbb{R}^N} |(-\Delta)^{s/2} \ell_0 \Phi_{\delta,b}|^2 dx + \int_{\mathbb{R}^N} |(-\Delta)^{s/2} t_0 \Phi_{\delta,b}|^2 dx$$

$$+ \ell_0^2 \int_{\mathbb{R}^N} a(x)|\Phi_{\delta,b}|^2 dx + t_0^2 \int_{\mathbb{R}^N} b(x)|\Phi_{\delta,b}|^2 dx$$

$$< S_H + \epsilon, \quad \forall b \in \mathbb{R}^N, \ \forall \delta \in [\bar{\delta}, \infty).$$

(ii) Since

$$f(\ell_0 \Phi_{\delta,b}, t_0 \Phi_{\delta,b}) = \int_{\mathbb{R}^N} |(-\Delta)^{s/2} \ell_0 \Phi_{\delta,b}|^2 dx + \int_{\mathbb{R}^N} |(-\Delta)^{s/2} t_0 \Phi_{\delta,b}|^2 dx$$

$$+ \ell_0^2 \int_{\mathbb{R}^N} a(x)|\Phi_{\delta,b}|^2 dx + t_0^2 \int_{\mathbb{R}^N} b(x)|\Phi_{\delta,b}|^2 dx$$

$$= S_H + \ell_0^2 \int_{\mathbb{R}^N} a(x)|\Phi_{\delta,b}|^2 dx + t_0^2 \int_{\mathbb{R}^N} b(x)|\Phi_{\delta,b}|^2 dx,$$

it suffices to prove that

$$\lim_{|b| \to \infty} \left(\ell_0^2 \int_{\mathbb{R}^N} a(x)|\Phi_{\delta,b}|^2 dx + t_0^2 \int_{\mathbb{R}^N} b(x)|\Phi_{\delta,b}|^2 dx \right) = 0, \quad \forall \delta > 0. \tag{4.6}$$

Note that given $\epsilon > 0$, there are $k_1, k_2 > 0$ such that

$$\left(\int_{\mathbb{R}^N \setminus B_\rho(0)} a(x)^{N/2s} dx \right)^{2s/N} < \epsilon, \quad \forall \rho > k_1, \tag{4.7}$$

$$\left(\int_{\mathbb{R}^N \setminus B_\rho(0)} |\Phi_{\delta,b}|^{2_s^*} dx \right)^{1/2_s^*} = \left(\int_{\mathbb{R}^N \setminus B_\rho(0)} |\Phi_{\delta,0}|^{2_s^*} dz \right)^{1/2_s^*} < \epsilon, \quad \forall \rho > k_2. \tag{4.8}$$

Let $k_0 = \max\{k_1, k_2\}$ and consider

$$k_0 < 2\rho < |b| \quad (\rho \text{ fixed}) \tag{4.9}$$

and note that

$$B_\rho(0) \cap B_\rho(b) = \emptyset. \tag{4.10}$$

Using Hölder's inequality with exponents $N/2s$ and $N/(N-2s)$, and taking into account (4.7), (4.8), (4.9), and (4.10), we obtain

$$\int_{\mathbb{R}^N} a(x)|\Phi_{\delta,b}|^2 dx$$

$$\leq \Big(\int_{\mathbb{R}^N \setminus (B_\rho(0) \cup B_\rho(b))} a^{N/2s} dx \Big)^{2s/N} \Big(\int_{\mathbb{R}^N \setminus (B_\rho(0) \cup B_\rho(b))} |\Phi_{\delta,b}|^{2_s^*} dx \Big)^{\frac{N-2s}{N}}$$

$$+ \Big(\int_{B_\rho(0)} a^{N/2s} dx \Big)^{2s/N} \Big(\int_{B_\rho(0)} |\Phi_{\delta,b}|^{2_s^*} dx \Big)^{\frac{N-2s}{N}}$$

$$+ \Big(\int_{B_\rho(b)} a^{N/2s} dx \Big)^{2s/N} \Big(\int_{B_\rho(b)} |\Phi_{\delta,b}|^{2_s^*} dx \Big)^{\frac{N-2s}{N}}$$

$$\leq \Big(\int_{\mathbb{R}^N \setminus B_\rho(0)} a^{N/2s} dx \Big)^{2s/N} \Big(\int_{\mathbb{R}^N \setminus B_\rho(b)} |\Phi_{\delta,b}|^{2_s^*} dx \Big)^{\frac{N-2s}{N}}$$

$$+ \Big(\int_{\mathbb{R}^N} a^{N/2s} dx \Big)^{2s/N} \Big(\int_{\mathbb{R}^N \setminus B_\rho(b)} |\Phi_{\delta,b}|^{2_s^*} dx \Big)^{\frac{N-2s}{N}}$$

$$+ \Big(\int_{\mathbb{R}^N \setminus B_\rho(0)} a^{N/2s} dx \Big)^{2s/N} \Big(\int_{\mathbb{R}^N} |\Phi_{\delta,b}|^{2_s^*} dx \Big)^{\frac{N-2s}{N}}$$

$$< \epsilon\epsilon^2 + \epsilon^2 |a|_{N/2s} + \epsilon.$$

Arguing similarly for the second part of (4.6), we obtain

$$\int_{\mathbb{R}^N} b(x) |\Phi_{\delta,b}|^2 dx < \epsilon\epsilon^2 + \epsilon^2 |b|_{N/2s} + \epsilon$$

and the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 4.4. *Under Assumption* (H8),

$$\sup_{\delta>0, b\in\mathbb{R}^N} f(\ell_0 \Phi_{\delta,b}, t_0 \Phi_{\delta,b}) < 2^{2s/N} S_H.$$

Proof. Using the definition of $\mathcal{F}$, Hölder's inequality with $N/2s$ and $N/(N-2s)$, and condition (H8), we obtain

$$\sup_{\delta>0, b\in\mathbb{R}^N} f(\ell_0 \Phi_{\delta,b}, t_0 \Phi_{\delta,b}) \leq S_H + \ell_0^2 |a|_{N/2s} + t_0^2 |b|_{N/2s}$$

$$< S_H + S_H(2^{2s/N} - 1) = 2^{2s/N} S_H. \qquad\square$$

In what follows, we consider the function

$$\xi(x) = \begin{cases} 0, & \text{if } |x| < 1 \\ 1, & \text{if } |x| \geq 1 \end{cases}$$

and define $\kappa : D^{s,2}(\mathbb{R}^N) \times D^{s,2}(\mathbb{R}^N) \to \mathbb{R}^{N+1}$ by

$$\kappa(u,v) = \frac{1}{S_H} \int_{\mathbb{R}^N} \Big(\frac{x}{|x|}, \xi(x) \Big) [\ell_0^2 |(-\Delta)^{s/2} u|^2 + t_0^2 |(-\Delta)^{s/2} v|^2] dx = (\beta(u,v), \gamma(u,v)),$$

where

$$\beta(u,v) = \frac{1}{S_H} \int_{\mathbb{R}^N} \frac{x}{|x|} [\ell_0^2 |(-\Delta)^{s/2} u|^2 + t_0^2 |(-\Delta)^{s/2} v|^2] dx,$$

$$\gamma(u,v) = \frac{1}{S_H} \int_{\mathbb{R}^N} \xi(x) [\ell_0^2 |(-\Delta)^{s/2} u|^2 + t_0^2 |(-\Delta)^{s/2} v|^2] dx.$$

Lemma 4.5. *If* $|b| \geq 1/2$, *then*

$$\beta(\Phi_{\delta,b}, \Phi_{\delta,b}) = \frac{b}{|b|} + o_\delta(1) \quad \text{as } \delta \to 0.$$

Proof. By Lemma 4.2, there is $\hat{\delta} > 0$ such that

$$\left| \beta(\Phi_{\delta,b}, \Phi_{\delta,b}) - \frac{1}{S_H} \int_{B_\epsilon(b)} \frac{x}{|x|} [\ell_0^2 |(-\Delta)^{s/2} \Phi_{\delta,b}|^2 + t_0^2 |(-\Delta)^{s/2} \Phi_{\delta,b}|^2 dx] \right|$$

$$= \left| \frac{\ell_0^2 + t_0^2}{S_H} \int_{\mathbb{R}^N \setminus B_\epsilon(b)} \frac{x}{|x|} |(-\Delta)^{s/2} \Phi_{\delta,b}|^2 \right| \qquad (4.11)$$

$$\le \frac{\ell_0^2 + t_0^2}{S_H} \int_{B_\epsilon(b)} |(-\Delta)^{s/2} \Phi_{\delta,b}|^2 dx < \epsilon,$$

for all $\delta \in (0, \hat{\delta})$. On the other hand, for $\epsilon > 0$ sufficiently small and $|b| > 1/2$, we have

$$\left| \frac{x}{|x|} - \frac{b}{|b|} \right| < 4\epsilon, \quad \forall x \in B_\epsilon(x),$$

and so

$$\left| \frac{b}{|b|} - \frac{\ell_0^2 + t_0^2}{S_H} \int_{B_\epsilon(b)} \frac{x}{|x|} |(-\Delta)^{s/2} \Phi_{\delta,b}|^2 dx \right| < 4\epsilon + \epsilon, \quad \forall \delta \in (0, \hat{\delta}). \qquad (4.12)$$

From (4.11) and (4.12), it follows that

$$\left| \beta(\Phi_{\delta,b}, \Phi_{\delta,b}) - \frac{b}{|b|} \right| \le \left| \beta(\Phi_{\delta,b}, \Phi_{\delta,b}) - \frac{\ell_0^2 + t_0^2}{S_H} \int_{B_\epsilon(b)} \frac{x}{|x|} |(-\Delta)^{s/2} \Phi_{\delta,b}|^2 dx \right|$$

$$+ \left| \frac{\ell_0^2 + t_0^2}{S_H} \int_{B_\epsilon(b)} \frac{x}{|x|} |(-\Delta)^{s/2} \Phi_{\delta,b}|^2 - \frac{b}{|b|} \right|$$

$$< C\epsilon, \quad \forall \delta \in (0, \hat{\delta}).$$

This completes the proof. $\qquad \square$

Now we define the set

$$\Im = \{(u,v) \in \mathcal{M}; \kappa(u,v) = (0, \frac{1}{2})\}.$$

Lemma 4.6. *The set $\Im$ is not empty.*

Proof. Since that $\Phi_{\delta,0}$ is an odd function and $B_r(0)$ is symmetric, we have that $\beta(\Phi_{\delta,0}, \Phi_{\delta,0}) = 0$. From Lemma 4.2, we see that

$$\gamma(\Phi_{\delta,0}, \Phi_{\delta,0}) \to 0 \quad \text{as } \delta \to 0. \qquad (4.13)$$

On the other hand

$$\gamma(\Phi_{\delta,0}, \Phi_{\delta,0}) = 1 - \frac{\ell_0^2 + t_0^2}{S_H} \int_{B_1(0)} |(-\Delta)^{s/2} \Phi_{\delta,0}|^2 dx, \qquad (4.14)$$

and moreover, by [24, Proposition 2.2], we see that

$$\int_{B_1(0)} |(-\Delta)^{s/2} \Phi_{\delta,0}|^2 dx \le \int_{B_1(0)} |\nabla \Phi_{\delta,0}|^2 dx$$

$$\le C\delta^{2s-2} \int_{B_1(0)} \frac{|z|^2}{[1 + |z|^2]^{N-2s+2}} dz \qquad (4.15)$$

$$\le \tilde{C}\delta^{2s-2} \to 0, \quad \text{as } \delta \to +\infty.$$

Combining (4.14) and (4.15) we have

$$\gamma(\Phi_{\delta,0}, \Phi_{\delta,0}) \to 1, \quad \text{as } \delta \to +\infty. \qquad (4.16)$$

By (4.13) and (4.16) there is $\delta_1 > 0$ such that $(\Phi_{\delta_1,0}, \Phi_{\delta_1,0}) \in \Im$. $\qquad \square$

Lemma 4.7. *The number $c_0 = \inf_{(u,v) \in \Im} f(u,v)$ satisfies the inequality $c_0 > S_H$.*

Proof. Since $\Im \subset \mathcal{M}$, we have
$$S_H \leq c_0. \tag{4.17}$$
Suppose, by contradiction, that $S_H = c_0$. By Ekeland variational principle [31], there is $(u_n, v_n) \in D^{s,2}(\mathbb{R}^N) \times D^{s,2}(\mathbb{R}^N)$ such that
$$\int_{\mathbb{R}^N} H(u_n, v_n)dx = 1, \quad \tau(u_n, v_n) \to \left(0, \frac{1}{2}\right), \tag{4.18}$$
$$f(u_n, v_n) \to S_H, \quad f'|_{\mathcal{M}}(u_n, v_n) \to 0. \tag{4.19}$$
Then, (u_n, v_n) is bounded in $D^{s,2}(\mathbb{R}^N) \times D^{s,2}(\mathbb{R}^N)$, and, up to a subsequence, $(u_n, v_n) \rightharpoonup (\bar{u}, \bar{v})$ in $D^{s,2}(\mathbb{R}^N) \times D^{s,2}(\mathbb{R}^N)$.

If $(w_n, z_n) = (S_H^{\frac{N-2s}{4s}} u_n, S_H^{\frac{N-2s}{4s}} v_n)$ and $(\bar{w}, \bar{z}) = (S_H^{\frac{N-2s}{4s}} \bar{u}, S_H^{\frac{N-2s}{4s}} \bar{v})$, we see that $(w_n, z_n) \rightharpoonup (\bar{w}, \bar{z})$ in $D^{s,2}(\mathbb{R}^N) \times D^{s,2}(\mathbb{R}^N)$, and so, by (4.19) and Remark 3.6, we obtain
$$\mathcal{J}(w_n, z_n) \to \frac{s}{N} S_H^{\frac{N}{2s}} \quad \text{and} \quad \mathcal{J}'(w_n, z_n) \to 0.$$

We are going to show that $(\bar{w}, \bar{z}) \equiv (0,0)$. First of all, note that
$$(u_n, v_n) \nrightarrow (\bar{u}, \bar{v}) \quad \text{in } D^{s,2}(\mathbb{R}^N) \times D^{s,2}(\mathbb{R}^N), \tag{4.20}$$
because otherwise $(\bar{u}, \bar{v}) \neq (0,0)$ and
$$\begin{aligned}
S_H &\leq \int_{\mathbb{R}^N} |(-\Delta)^{s/2}\bar{u}|^2 dx + \int_{\mathbb{R}^N} |(-\Delta)^{s/2}\bar{v}|^2 dx \\
&< \int_{\mathbb{R}^N} |(-\Delta)^{s/2}\bar{u}|^2 dx + \int_{\mathbb{R}^N} |(-\Delta)^{s/2}\bar{v}|^2 dx + \int_{\mathbb{R}^N} a(x)|\bar{u}|^2 dx + \int_{\mathbb{R}^N} b(x)|\bar{v}|^2 dx \\
&= S_H,
\end{aligned}$$
which is a contradiction. Thus, $(w_n, z_n) \nrightarrow (\bar{w}, \bar{z})$ in $D^{s,2}(\mathbb{R}^N) \times D^{s,2}(\mathbb{R}^N)$ and, since (w_n, z_n) is a $(PS)_c$ sequence for $\mathcal{J}$, by Theorem 3.2, we have
$$\mathcal{J}(w_n, z_n) \to \mathcal{J}(\bar{w}, \bar{z}) + \sum_{j=1}^{k} \mathcal{J}_\infty(u_0^j, v_0^j) = \frac{s}{N} S_H^{\frac{N}{2s}}.$$
Using $\mathcal{J}'(u_0^j, v_0^j) = 0$, we obtain
$$\mathcal{J}(\bar{w}, \bar{z}) = 0, \quad k = 1, \quad u_0^j, v_0^j > 0, \quad \mathcal{J}_\infty(\bar{w}, \bar{z}) = \frac{s}{N} \int_{\mathbb{R}^N} H(\bar{w}, \bar{z})dx, \tag{4.21}$$
which implies that $(\bar{w}, \bar{z}) = (0,0)$. Then (w_n, z_n) is a $(PS)_c$ sequence for $\mathcal{J}$ such that $(w_n, z_n) \rightharpoonup (0,0)$, $(w_n, z_n) \nrightarrow (0,0)$, $\int_{\mathbb{R}^N} a(x)|w_n|^2 dx = o_n(1)$ and $\int_{\mathbb{R}^N} b(x)|z_n|^2 dx = o_n(1)$. Therefore,
$$\begin{aligned}
\frac{s}{N} S_H^{\frac{N}{2s}} + o_n(1) &= \mathcal{J}(w_n, z_n) \\
&= \mathcal{J}_\infty(w_n, z_n) + \int_{\mathbb{R}^N} a(x)|w_n|^2 dx + \int_{\mathbb{R}^N} b(x)|z_n|^2 dx \\
&= \mathcal{J}_\infty(w_n, z_n) + o_n(1)
\end{aligned} \tag{4.22}$$
and
$$\|\mathcal{J}'_\infty(w_n, z_n)\|_{(\mathcal{D} \times \mathcal{D})'} \leq \|\mathcal{J}'(w_n, z_n)\|_{(\mathcal{D} \times \mathcal{D})'} + o_n(1). \tag{4.23}$$

From (4.22) and (4.23) we conclude that (w_n, z_n) is a $(PS)_c$ sequence for $\mathcal{J}_\infty$ and by Lemma 3.1, there are sequences $(R_n) \subset \mathbb{R}$, $(x_n) \subset \mathbb{R}^N$, u_0^1, v_0^1 nontrivial solution of (2.2) and (τ_n, ζ_n) a $(PS)_c$ sequence for $\mathcal{J}_\infty$ such that

$$w_n(x) = \tau_n(x) + R_n^{\frac{N-2s}{2}} u_0^1(R_n(x - x_n)) + o_n(1),$$

$$z_n(x) = \zeta_n(x) + R_n^{\frac{N-2s}{2}} v_0^1(R_n(x - x_n)) + o_n(1).$$

Setting

$$\widetilde{\tau}_n(x) = R_n^{\frac{N-2s}{2}} u_0^1(R_n(x - x_n)) \quad \text{and} \quad \widetilde{\zeta}_n(x) = R_n^{\frac{N-2s}{2}} v_0^1(R_n(x - x_n))$$

and making change of variable, we have

$$\mathcal{J}'_\infty(\widetilde{\tau}_n, \widetilde{\zeta}_n)(\varphi_1, \varphi_2) = \mathcal{J}'_\infty(u_0^1, v_0^1)(\varphi_{1,n}, \varphi_{2,n}) = 0,$$

for all $(\varphi_1, \varphi_2) \in \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$ and for all $n \in \mathbb{N}$; thus $(\widetilde{\tau}_n, \widetilde{\zeta}_n)$ is a solution of (2.2), for all $n \in \mathbb{N}$. Moreover, from definition of $(\widetilde{\tau}_n, \widetilde{\zeta}_n)$ and by (4.21), we obtain

$$\widetilde{\tau}_n(x) = \widetilde{\zeta}_n(x) = c\Big(\frac{\delta_n}{\delta_n^2 + |x - b_n|^2}\Big)^{\frac{N-2s}{2}}, \quad x \in \mathbb{R}^N.$$

Therefore,

$$u_n(x) = \overline{\tau_n}(x) + \Phi_{\delta_n, b_n}(x) + o_n(1) \quad \text{and} \quad v_n(x) = \overline{\zeta_n}(x) + \Phi_{\delta_n, b_n}(x) + o_n(1),$$

where

$$\overline{\tau_n}(x) = S_H^{\frac{4s}{N-2s}} \tau_n(x), \quad \overline{\zeta_n}(x) = S_H^{\frac{4s}{N-2s}} \zeta_n(x),$$

$$\Phi_{\delta_n, b_n}(x) = S_H^{\frac{4s}{N-2s}} \widetilde{\tau}_n(x) = S_H^{\frac{4s}{N-2s}} \widetilde{\zeta}_n(x).$$

By (4.21), we derive that $\tau_n \to 0$ and $\zeta_n \to 0$ in $D^{s,2}(\mathbb{R}^N)$, which implies that $\overline{\tau_n} \to 0$ and $\overline{\zeta_n} \to 0$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. Therefore, from (4.18), we have

$$\Big(0, \frac{1}{2}\Big) + o_n(1) = \kappa(u_n, v_n) = \kappa(\Phi_{\delta_n, b_n}, \Phi_{\delta_n, b_n})$$

which implies that

(i) $\beta(\Phi_{\delta_n, b_n}, \Phi_{\delta_n, b_n}) \to 0$,
(ii) $\gamma(\Phi_{\delta_n, b_n}, \Phi_{\delta_n, b_n}) \to 1/2$.

Passing to a subsequence, one of the following cases must occur.

(a) $\delta_n \to +\infty$ when $n \to +\infty$;
(b) $\delta_n \to \overline{\delta} \neq 0$ when $n \to +\infty$;
(c) $\delta_n \to 0$ and $b_n \to \overline{b}$ when $n \to +\infty$ with $|\overline{b}| < 1/2$;
(d) $\delta_n \to 0$ when $n \to +\infty$ and $|b_n| \geq 1/2$ for n sufficiently large.

Suppose that (a) is true. Then

$$\gamma(\Phi_{\delta_n, b_n}, \Phi_{\delta_n, b_n}) = 1 - \frac{\ell_0^2 + t_0^2}{S_H} \int_{B_1(0)} |(-\Delta)^{s/2} \Phi_{\delta_n, b_n}|^2 dx$$

and by Lemma 4.1 we deduce that

$$|\gamma(\Phi_{\delta_n, b_n}, \Phi_{\delta_n, b_n}) - 1| = \frac{\ell_0^2 + t_0^2}{S_H} \int_{B_1(0)} |(-\Delta)^{s/2} \Phi_{\delta_n, b_n}|^2 dx = o_n(1)$$

which contradicts (ii).

Suppose that (b) is true. In this case we may suppose that $|b_n| \to +\infty$ because if $b_n \to \bar{b}$, we can prove that

$$\Phi_{\delta_n, b_n} \to \Phi_{\overline{\delta}, \overline{b}} \quad \text{in } \mathcal{D}^{s,2}(\mathbb{R}^N).$$

Since $\overline{\tau_n}, \overline{\zeta_n} \to 0$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ and $u_n = \overline{\tau_n} + \Phi_{\delta_n, b_n} + o_n(1)$, $v_n = \overline{\zeta_n} + \Phi_{\delta_n, b_n} + o_n(1)$, we see that (u_n, v_n) converges in $\mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathcal{R}^N)$ but this is a contradiction with (4.20). Hence,

$$\begin{aligned}
\gamma(\Phi_{\delta_n, b_n}, \Phi_{\delta_n, b_n}) &= \frac{\ell_0^2 + t_0^2}{S_H} \int_{B_1(0)} |(-\Delta)^{s/2} \Phi_{\delta_n, b_n}|^2 dx \\
&= \frac{\ell_0^2 + t_0^2}{S_H} \int_{\mathbb{R}^N \setminus B_1(0)} |(-\Delta)^{s/2} \Phi_{\delta_n, b_n}|^2 dx \qquad (4.24) \\
&= 1 - \frac{\ell_0^2 + t_0^2}{S_H} \int_{B_1(-b_n)} |(-\Delta)^{s/2} \Phi_{\delta_n, 0}|^2 dx.
\end{aligned}$$

Applying Lebesgue's theorem we can show that

$$\int_{B_1(-b_n)} |(-\Delta)^{s/2} \Phi_{\delta_n, 0}|^2 dx \to 0 \quad \text{as } n \to +\infty$$

and from (4.24) we obtain $\gamma(\Phi_{\delta_n, b_n}, \Phi_{\delta_n, b_n}) \to 1$, as $n \to +\infty$, which again contradicts (ii).

Suppose that (c) is true. Note that

$$\begin{aligned}
\gamma(\Phi_{\delta_n, b_n}, \Phi_{\delta_n, b_n}) &= \frac{\ell_0^2 + t_0^2}{S_H} \int_{B_1(0)} |(-\Delta)^{s/2} \Phi_{\delta_n, b_n}|^2 dx \\
&= \frac{\ell_0^2 + t_0^2}{S_H} \int_{\mathbb{R}^N \setminus B_1(0)} |(-\Delta)^{s/2} \Phi_{\delta_n, b_n}|^2 dx \qquad (4.25) \\
&= 1 - \frac{\ell_0^2 + t_0^2}{S_H} \int_{B_1(-b_n)} |(-\Delta)^{s/2} \Phi_{\delta_n, 0}|^2 dx.
\end{aligned}$$

Therefore, using again the Lebesgue theorem, we deduce that

$$\lim_{n \to +\infty} \frac{\ell_0^2 + t_0^2}{S_H} \int_{B_1(-b_n)} |(-\Delta)^{s/2} \Phi_{\delta_n, 0}|^2 dx = 1$$

From (4.25) we obtain $\gamma(\Phi_{\delta_n, b_n}, \Phi_{\delta_n, b_n}) \to 0$, which again contradicts (ii).

Suppose that (d) is true. Since $|b_n| \geq 1/2$ for n large, we have that $b_n \nrightarrow 0$ in $\mathbb{R}^N$. From Lemma 4.5 we have

$$\beta(\Phi_{\delta_n, b_n}, \Phi_{\delta_n, b_n}) = \frac{b_n}{|b_n|} + o_n(1).$$

Thus, $\beta(\Phi_{\delta_n, b_n}, \Phi_{\delta_n, b_n}) \nrightarrow 0$, which contradicts (i). So, $S_H < c_0$ and the proof is complete. $\square$

Lemma 4.8. *There is $\delta_1 \in (0, 1/2)$ such that*

(a) $f(\ell_0 \Phi_{\delta_1, b}, t_0 \Phi_{\delta_1, b}) < \frac{c_0 + S_H}{2}$, $\forall b \in \mathbb{R}^N$;
(b) $\gamma(\Phi_{\delta_1, b}, \Phi_{\delta_1, b}) < 1/2$ *for all* $b \in \mathbb{R}^N$ *such that* $|b| < 1/2$;
(c) $|\beta(\Phi_{\delta_1, b}, \Phi_{\delta_1, b}) - \frac{b}{|b|}| < 1/4$ *for all* $b \in \mathbb{R}^N$ *such that* $|b| \geq 1/2$.

Proof. From Lemma 4.3, we can choose $\varepsilon = \frac{c_0 - S_H}{2} > 0$, $\delta_2 < \min\{\underline{\delta}, 1/2\}$, and conclude that

$$f(\ell_0 \Phi_{\delta_3, b}, t_0 \Phi_{\delta_3, b}) \leq \sup_{b \in \mathbb{R}^N} f(\ell_0 \Phi_{\delta_3, b}, t_0 \Phi_{\delta_3, b}) < S_H + \frac{c_0 - S_H}{2} = \frac{c_0 + S_H}{2}, \quad (4.26)$$

for all $b \in \mathbb{R}^N$. Now by the definition of ξ and Lemma 4.1, we have

$$\gamma(\Phi_{\delta, b}, \Phi_{\delta, b}) = 1 - \frac{\ell_0 + t_0}{S_H} \int_{B_1(-b)} |(-\Delta)^{s/2} \Phi_{\delta, 0}|^2 dz.$$

and by Lebesgue Theorem

$$\frac{\ell_0 + t_0}{S_H} \int_{B_1(-b)} |(-\Delta)^{s/2} \Phi_{\delta, 0}|^2 dz \to 1,$$

thus $\gamma(\Phi_{\delta, b}, \Phi_{\delta, b}) \to 1$ as $\delta \to 0$. The above convergence assures us that there is $\hat{\delta} > 0$ such that $\gamma(\Phi_{\delta, b}, \Phi_{\delta, b}) < \frac{1}{2}$ for all $\delta \in (0, \hat{\delta})$. Choosing $\delta_4 < \min\{\hat{\delta}, 1/2\}$ we have

$$\gamma(\Phi_{\delta_4, b}, \Phi_{\delta_4, b}) < \frac{1}{2}, \quad \forall b \in \mathbb{R}^N \text{ with } |b| < \frac{1}{2}. \quad (4.27)$$

Furthermore, by Lemma 4.5, there is $\tilde{\delta} > 0$ such that

$$\left| \beta(\Phi_{\delta, b}, \Phi_{\delta, b}) - \frac{b}{|b|} \right| < \frac{1}{4}, \quad \forall \delta \in (0, \tilde{\delta}), \text{ with } |b| \geq \frac{1}{2}.$$

Thus, choosing $\delta_5 < \min\{\tilde{\delta}, 1/2\}$ we obtain

$$\left| \beta(\Phi_{\delta_5, b}, \Phi_{\delta_5, b}) - \frac{b}{|b|} \right| < \frac{1}{4}, \quad \forall b \in \mathbb{R}^N, \text{ with } |b| \geq \frac{1}{2}. \quad (4.28)$$

Finally, choosing $\delta_1 = \min\{\delta_3, \delta_4, \delta_5\}$ the result follows from (4.26), (4.27), and (4.28). $\qquad \square$

Lemma 4.9. *There is $\delta_2 > 0$ such that*

(a) $f(\ell_0 \Phi_{\delta_2, b}, t_0 \Phi_{\delta_2, b}) < \frac{c_0 + S_H}{2}$ *for all $b \in \mathbb{R}^N$;*
(b) $\gamma(\Phi_{\delta_2, b}, \Phi_{\delta_2, b}) > \frac{1}{2}$ *for all $b \in \mathbb{R}^N$.*

Proof. Given $\epsilon = \frac{c_0 - S_H}{2} > 0$, by Lemma 4.3, we can choose $\delta_3 > \max\{\overline{\delta}, 1/2\}$ such that

$$f(\ell_0 \Phi_{\delta_3}, t_0 \Phi_{\delta_3}) \leq \sup_{b \in \mathbb{R}^N} f(\ell_0 \Phi_{\delta_3}, t_0 \Phi_{\delta_3}) < \frac{S_H + c_0}{2}, \quad \forall b \in \mathbb{R}^N. \quad (4.29)$$

On other hand,

$$\gamma(\Phi_{\delta, b}, \Phi_{\delta, b}) = 1 - \frac{\ell_0^2 + t_0^2}{S_H} \int_{B_1(-b)} |(-\Delta)^{s/2} \Phi_{\delta, 0}|^2 dx$$

and applying [24, Proposition 2.2] and Lemma 4.1 see that

$$\int_{B_1(-b)} |(-\Delta)^{s/2} \Phi_{\delta, 0}|^2 dx \to 0 \quad \text{as } \delta \to +\infty.$$

Thus, for each $b \in \mathbb{R}^N$, $\gamma(\Phi_{\delta, b}, \Phi_{\delta, b}) \to 1$ as $\delta \to +\infty$; hence, there is $\hat{\delta} > 0$ such that

$$\gamma(\Phi_{\delta, b}, \Phi_{\delta, b}) > \frac{1}{2}, \quad \forall \delta \in (\hat{\delta}, +\infty).$$

Choosing $\delta_4 > \max\{\hat{\delta}, 1/2\}$, we have

$$\gamma(\Phi_{\delta_4,b}, \Phi_{\delta_4,b}) > \frac{1}{2}, \quad \forall b \in \mathbb{R}^N. \tag{4.30}$$

Now, choosing $\delta_2 = \max\{\delta_3, \delta_4\}$ the result follows of (4.29) and (4.30). □

Lemma 4.10. *There is $R > 0$ such that*
  (a) $f(\ell_0 \Phi_{\delta,b}, t_0 \Phi_{\delta,b}) < \frac{c_0 + S_H}{2}$ *for all b for which $|b| \geq R$ and $\delta \in [\delta_1, \delta_2]$;*
  (b) $(\beta(\Phi_{\delta,b}, \Phi_{\delta,b})|b)_{\mathbb{R}^N} > 0$ *for all b for which $|b| \geq R$ and $\delta \in [\delta_1, \delta_2]$.*

Proof. From Lemma 4.3 assuming $\delta = \frac{c_0 - S_H}{2} > 0$, we can find $R_1 > 0$, big enough, such that

$$f(\ell_0 \Phi_{\delta,b}, t_0 \Phi_{\delta,b}) < S_H + \delta = \frac{S_H + c_0}{2}, \quad \forall b : |b| \geq R_1, \text{ and } \delta \in [\delta_1, \delta_2], \tag{4.31}$$

and item (a) follows.

Now, for each $b \in \mathbb{R}^N$ we consider the sets $(\mathbb{R}^N)_b^+ = \{x \in \mathbb{R}^N; (x|b)_{\mathbb{R}^N} > 0\}$ and $(\mathbb{R}^N)_b^- = \mathbb{R}^N \setminus (\mathbb{R}^N)_b^+$. Since ϵ varies in the compact set $[\delta_1, \delta_2]$, we can prove there is $R_2 > 0$ big enough and $r \in (0, \frac{1}{4})$ such that the following things are true if $|b| \geq R_2$ and $|b - b_0| = \frac{1}{2}$,

$$B_r(b_0) = \{x \in \mathbb{R}^N; |b - b_0| < r\} \subset (\mathbb{R}^N)_b^+.$$

Initially, note that for every $x \in B_r(b_0)$, we have

$$|(-\Delta)^{s/2} \Phi_{\delta,b}|^2$$
$$= \int_{\mathbb{R}^N} \frac{|\Phi_{\delta,b}(x) - \Phi_{\delta,b}(y)|^2}{|x - y|^{N+2s}} dy$$
$$\geq \int_{B_{\frac{r}{2}}(b) \setminus B_{\frac{r}{4}}(b)} \frac{\left| \frac{c\delta^{\frac{N-2s}{2}}}{[\delta^2 + |x-b|^2]^{\frac{N-2s}{2}}} - \frac{c\delta^{\frac{N-2s}{2}}}{[\delta^2 + |y-b|^2]^{\frac{N-2s}{2}}} \right|^2}{|x - y|^{N+2s}} dy$$
$$\geq \int_{B_{\frac{r}{2}}(b) \setminus B_{\frac{r}{4}}(b)} \frac{c\delta_1^{N-2s}}{(\frac{7}{8})^{N+2s}} \left| \frac{1}{[\delta_2^2 + \frac{9}{16}]^{\frac{N-2s}{2}}} - \frac{1}{[\delta_1^2 + \frac{1}{64}]^{\frac{N-2s}{2}}} \right|^2 dy := H_1 > 0.$$

Thus,

$$(\beta(\Phi_{\delta,b}, \Phi_{\delta,b})|b)_{\mathbb{R}^N}$$
$$\geq \frac{\ell_0^2 + t_0^2}{S_H} \left\{ \int_{B_r(b_0)} \frac{(x|b)}{|x|} H_1 dx + \int_{(\mathbb{R}^N)_b^-} \frac{(x|b)}{|x|} |(-\Delta)^{s/2} \Phi_{\delta,b}|^2 dx \right\}$$
$$\geq \frac{\ell_0^2 + t_0^2}{S_H} \left\{ |b| \int_{B_r(b_0)} \frac{(x|b)}{|x||b|} H_1 dx - |b| \int_{(\mathbb{R}^N)_b^-} |(-\Delta)^{s/2} \Phi_{\delta,b}|^2 dx \right\} \tag{4.32}$$
$$\geq \frac{\ell_0^2 + t_0^2}{S_H} \left\{ |b| C_1 \int_{B_r(b_0)} \frac{H_1}{|x|} dx - |b| \int_{(\mathbb{R}^N)_b^-} |(-\Delta)^{s/2} \Phi_{\delta,b}|^2 dx \right\}$$
$$= \frac{\ell_0^2 + t_0^2}{S_H} \left\{ |b| H_2 - |b| \int_{(\mathbb{R}^N)_b^-} |(-\Delta)^{s/2} \Phi_{\delta,b}|^2 dx \right\},$$

where

$$H_2 := C_1 \int_{B_r(b_0)} \frac{H_1}{|x|} dx.$$

Moreover, using [24, Propositin 2.2] and spherical coordinates, we deduce that

$$\int_{(\mathbb{R}^N)_b^-} |(-\Delta)^{s/2}\Phi_{\delta,b}(x)|^2 dx \leq \int_{B_{R_2}^c(b)} |(-\Delta)^{s/2}\Phi_{\delta,b}(x)|^2 dx$$

$$\leq C_2 \int_{B_{R_2}^c(b)} |(-\Delta)^{s/2}\Phi_{\delta,b}(x)|^2 dx$$

$$\leq C_2 \int_{B_{R_2}^c(0)} |(-\Delta)^{s/2}\Phi_{\delta,0}(x)|^2 dx$$

$$\leq C_2 \delta^{2s-2} \int_{B_{R_2}^c(0)} \frac{|z|}{(1+|z|^2)^{N+2s+2}} dz$$

$$= C_3 \delta^{N-2s} R_2^{-(N+2-4s)},$$

where we can choose $R_2 > 0$ large, such that for all $b \in \mathbb{R}^N$ with $|b| > R_2$, we have

$$\int_{(\mathbb{R}^N)_b^-} |(-\Delta)^{s/2}\Phi_{\delta,b}(x)|^2 dx < H_2. \tag{4.33}$$

Therefore, from (4.32) and (4.33), it follows that

$$(\beta(\Phi_{\delta,b}, \Phi_{\delta,b})|b) \geq \frac{\ell_0^2 + t_0^2}{S_H}|b|\Big\{H_2 - \int_{(\mathbb{R}^N)_b^-} |(-\Delta)^{s/2}\Phi_{\delta,b}(x)|^2 dx\Big\} > 0, \tag{4.34}$$

for all $|b| > R_2$ and for all $\delta \in [\delta_1, \delta_2]$. Now, choosing $R = \max\{R_1, R_2\}$ the result of (4.31) and (4.34). $\qquad\square$

5. Proof of main theorem

To prove Theorem 1.1, we first fix some notation and give some more technical lemmas. Consider the set

$$\mathcal{V} := \{(b,\delta) \in \mathbb{R}^N \times (0,\infty) : |b| < R \text{ and } \delta \in (\delta_1, \delta_2)\},$$

where δ_1, δ_2 and R are given by Lemmas 4.8, 4.9, and 4.10, respectively.

Let $Q : \mathbb{R}^N \times (0,\infty) \to \mathcal{D}^{s,2}(\mathbb{R}^N)$ be the continuous function given by

$$Q(b,\delta) = \Phi_{\delta,b}.$$

With the above notation, we define the sets

$$\Theta := \{(Q(b,\delta), Q(b,\delta)) : (b,\delta) \in \overline{\mathcal{V}}\},$$

$$\mathcal{H} := \{h \in \mathcal{C}(\Sigma \cap \mathcal{M}) : h(u,v) = (u,v), \ \forall (u,v) \in \Sigma \cap \mathcal{M} : f(\ell_0 u, t_0 v) < \frac{c_0 + S_H}{2}\},$$

$$\Gamma := \{\mathcal{A} \subset \Sigma \cap \mathcal{M} : \mathcal{A} = h(\Theta), h \in \mathcal{H}\}.$$

Note that $\Theta \subset \Sigma \cap \mathcal{M}$, $\Theta = Q(\overline{\mathcal{V}}) \times Q(\overline{\mathcal{V}})$ is compact and $\mathcal{H} \neq 0$, because the identity function is in $\mathcal{H}$.

Lemma 5.1. *Let $\mathcal{F} : \overline{\mathcal{V}} \to \mathbb{R}^{N+1}$ be the function defined by*

$$\mathcal{F}(b,\delta) = (\kappa \circ (Q,Q))(b,\delta) = \frac{\ell_0^2 + t_0^2}{S_H} \int_{\mathbb{R}^N} \big(\frac{x}{|x|}, \xi(x)\big)|(-\Delta)^{s/2}\Phi_{b,\delta}|^2 \, dx.$$

Then the topological degree is $d(\mathcal{F}, \mathcal{V}, (0, 1/2)) = 1$.

Proof. Define $\mathcal{Z} : [0,1] \times \overline{\mathcal{V}} \to \mathbb{R}^{N+1}$ the homotopy by

$$\mathcal{Z}(t, (b, \delta)) = t\mathcal{F}(b, \delta) + (1-t)\mathcal{I}_{\overline{\mathcal{V}}}(b, \delta)$$

where $\mathcal{I}$ is the identity operator. Using Lemma 4.8 and Lemma 4.9, we can show that $(0, 1/2) \notin \mathcal{Z}([0,1] \times (\partial \mathcal{V}))$, i.e.,

$$t\beta(\Phi_{b,\delta}, \Phi_{b,\delta}) + (1-t)b \neq 0, \quad \forall t \in [0,1] \text{ and } \forall (b,\delta) \in \partial\mathcal{V} \tag{5.1}$$

or

$$t\gamma(\Phi_{b,\delta}, \Phi_{b,\delta}) + (1-t)\delta \neq \frac{1}{2}, \quad \forall t \in [0,1] \text{ and } \forall (b,\delta) \in \partial\mathcal{V}. \tag{5.2}$$

Since, $(0, 1/2) \notin \mathcal{Z}([0,1] \times (\partial\mathcal{V}))$, we have that $d(\mathcal{F}, \mathcal{V}, (0, 1/2))$, $d(\mathcal{Z}(t, \cdot), \mathcal{V}, (0.1/2))$ is well defined and by properties of the topological degree

$$d(\mathcal{F}, \mathcal{V}, (0, 1/2)) = d(\mathcal{I}_{\overline{\mathcal{V}}}, \mathcal{V}, (0, 1/2)).$$

Since $(0, 1/2) \in \mathcal{V}$, we deduce that

$$d(\mathcal{F}, \mathcal{V}, (0, 1/2)) = d(\mathcal{I}_{\overline{\mathcal{V}}}, \mathcal{V}, (0, 1/2)) = 1.$$

$\square$

Lemma 5.2. *If $\mathcal{A} \in \Gamma$, then $\mathcal{A} \cap \Im \neq \emptyset$.*

Proof. It is sufficient to prove that for all $h \in \mathcal{H}$, there exists (b_0, δ_0) such that

$$(\kappa \circ \mathcal{H} \circ (Q, Q))(b_0, \delta_0) = \Big(0, \frac{1}{2}\Big).$$

Given $h \in \mathcal{H}$, let $\mathcal{F}_h : \overline{\mathcal{V}} \to \mathbb{R}^{N+1}$ be the continuous function given by

$$\mathcal{F}_h(b, \delta) = (\kappa \circ h \circ (Q, Q))(b, \delta).$$

Now we show that $\mathcal{F}_h = \mathcal{F}$ in $\partial\mathcal{V}$. Note that

$$\partial\mathcal{V} = \Pi_1 \cup \Pi_2 \cup \Pi_3, \tag{5.3}$$

where

$$\Pi_1 := \{(b, \delta_1) : |b| \leq R\}, \quad \Pi_2 := \{(b, \delta_2) : |b| \leq R\},$$
$$\Pi_3 := \{(b, \delta_3) : |b| = R \text{ and } \delta \in [\delta_1, \delta_2]\}.$$

If $(b, \delta) \in \Pi_1$, then $(b, \delta) = (b, \delta_1)$. and by Lemma 4.8-(a), we have

$$\begin{aligned}
f(\ell_0 Q(b, \delta), t_0 Q(b, \delta)) &= f(\ell_0 Q(b, \delta_1), t_0 Q(b, \delta_1)) \\
&= f(\ell_0 \Phi_{\delta_1, b}, t_0 \Phi_{\delta_1, b}) \\
&< \frac{S_H + c_0}{2}, \quad \forall (b, \delta) \in \Pi_1.
\end{aligned} \tag{5.4}$$

If $(b, \delta) \in \Pi_2$, then $(b, \delta) = (b, \delta_2)$, and by Lemma 4.9(a), we have

$$\begin{aligned}
f(\ell_0 Q(b, \delta), t_0 Q(b, \delta)) &= f(\ell_0 Q(b, \delta_2), t_0 Q(b, \delta_2)) \\
&= f(\ell_0 \Phi_{\delta_2, b}, t_0 \Phi_{\delta_2, b}) \\
&< \frac{S_H + c_0}{2}, \quad \forall (b, \delta) \in \Pi_2.
\end{aligned} \tag{5.5}$$

If $(b, \delta) \in \Pi_3$, then $|b| = R$ and $\delta \in [\delta_1, \delta_2]$ and by Lemma 4.10(a), we obtain

$$f(\ell_0 Q(b, \delta), t_0 Q(b, \delta)) = f(\ell_0 \Phi_{\delta, b}, t_0 \Phi_{\delta, b}) < \frac{S_H + c_0}{2}, \quad \forall (b, \delta) \in \Pi_3. \tag{5.6}$$

Combining (5.3), (5.4), (5.5), and (5.6), we obtain

$$f(\ell_0\Phi_{\delta,b}, t_0\Phi_{\delta,b}) < \frac{S_H + c_0}{2}, \quad \forall (b,\delta) \in \partial\mathcal{V}.$$

Thus,

$$\begin{aligned}
\mathcal{F}_h(b,\delta) &= (\kappa \circ h \circ (Q,Q))(b,\delta)\\
&= (\kappa \circ h)(Q(b,\delta), Q(b,\delta))\\
&= \kappa(h((Q(b,\delta), Q(b,\delta))))\\
&= \kappa((Q(b,\delta), Q(b,\delta)))\\
&= (\kappa \circ (Q,Q))(b,\delta)\\
&= \mathcal{F}(b,\delta), \quad \forall (b,\delta) \in \partial\mathcal{V}.
\end{aligned}$$

Since $(0,1/2) \notin \mathcal{F}(\partial\mathcal{V})$, we obtain

$$d(\mathcal{F}_h, \mathcal{V}, (0,1/2)) = d(\mathcal{F}, \mathcal{V}, (0,1/2)).$$

By Lemma 5.1, we have

$$d(\mathcal{F}_h, \mathcal{V}, (0,1/2)) = d(\mathcal{F}, \mathcal{V}, (0,1/2)) = 1,$$

and there is $(b_0, \delta_0) \in \mathcal{V}$ such that

$$\mathcal{F}_h(b_0, \delta_0) = (\kappa \circ h \circ (Q,Q))(b_0, \delta_0) = \left(0, \frac{1}{2}\right)$$

and the proof is complete.   □

Proof of Theorem 1.1. We define the number

$$c = \inf_{\mathcal{A} \in \Gamma} \max_{(u,v) \in \mathcal{A}} f(u,v)$$

and for each $q \in \mathbb{R}$, we define the set

$$f^q := \{(u,v) \in \Sigma \cap \mathcal{M} : f(u,v) \leq q\}.$$

We start our analysis by noting that

$$S_H < c < 2^{2s/N} S_H. \tag{5.7}$$

In fact, by Lemma 4.4,

$$\begin{aligned}
c &= \inf_{\mathcal{A} \in \Gamma} \max_{(u,v) \in \mathcal{A}} f(u,v) \leq \max_{(u,v) \in \Theta} f(u,v)\\
&\leq \sup_{(b,\delta) \in \mathbb{R}^N \times (0,+\infty)} f(\ell_0\Phi_{\delta,b}, t_0\Phi_{\delta,b}) < 2^{2s/N} S_H.
\end{aligned}$$

On the other hand, by Lemmas 4.7 and 5.2, we obtain

$$\begin{aligned}
S_H < c_0 &= \inf_{(u,v) \in \Im} f(u,v)\\
&= \inf_{\mathcal{A} \in \Gamma} \max_{(u,v) \in \mathcal{A}} f(u,v)\\
&\leq \sup_{(b,\delta) \in \mathbb{R}^N \times (0,+\infty)} f(\ell_0\Phi_{\delta,b}, t_0\Phi_{\delta,b}) < 2^{2s/N} S_H,
\end{aligned} \tag{5.8}$$

from where it follows (5.7).

Using the definition of c, there exists the sequence $(u_n, v_n) \in \Sigma \cap \mathcal{M}$ such that $f(u_n, v_n) \to c$. Suppose, by contradiction, that $f'|_{\mathcal{M}}(u_n, v_n) \not\to 0$. Then, there exists $(u_{n_j}, v_{n_j}) \subset (u_n, v_n)$ such that

$$\|f'|_{\mathcal{M}}(u_{n_j}, v_{n_j})\|_* \geq C > 0, \quad \forall j \in \mathbb{N}.$$

By a deformation Lemma [31], there exists a continuous application $\eta : [0, 1] \times \Sigma \cap \mathcal{M} \to \Sigma \cap \mathcal{M}$ and $\epsilon_0 > 0$ such that

    (a) $\eta(0, (u, v)) = (u, v)$;
    (b) $\eta(t, (u, v)) = (u, v)$ for all $(u, v) \in f^{c-\varepsilon_0} \cup \{(\Sigma \cap \mathcal{M}) \setminus f^{c+\epsilon_0}\}$ and all $t \in [0, 1]$;
    (c) $\eta(1, f^{c+\epsilon_0/2}) \subset f^{c-\epsilon_0/2}$.

From the definition of c, there exists $\widetilde{\mathcal{A}} \in \Gamma$ such that

$$c \leq \max_{(u,v) \in \widetilde{\mathcal{A}}} f(u, v) < c + \frac{\epsilon_0}{2},$$

where

$$\widetilde{\mathcal{A}} \subset f^{c + \frac{\epsilon_0}{2}}. \tag{5.9}$$

Since $\widetilde{\mathcal{A}} \in \Gamma$ we have $\widetilde{\mathcal{A}} \subset (\Sigma \cap \mathcal{M})$ and there exists $\overline{h} \in \mathcal{H}$ such that

$$\overline{h}(\Theta) = \widetilde{\mathcal{A}}. \tag{5.10}$$

From the definition of η, we have

$$\eta(1, \widetilde{A}) \subset (\Sigma \cap \mathcal{M}). \tag{5.11}$$

Let $h_* : (\Sigma \cap \mathcal{M}) \to (\Sigma \cap \mathcal{M})$ be the function given by $h_*(u, v) = \eta(1, \overline{h}(1, v))$. Note that $h_* \in \mathcal{C}(\Sigma \cap \mathcal{M}, \Sigma \cap \mathcal{M})$.

We are going to show that

$$f^{c+\epsilon_0} \setminus f^{c-\epsilon_0} \subset f^{2^{2s/N} S_H} \setminus f^{(S_H+c_0)/2}. \tag{5.12}$$

Indeed, given $(u, v) \in f^{c+\epsilon_0} \setminus f^{c-\epsilon_0}$, we have

$$c - \epsilon_0 < f(u, v) \leq c + \epsilon_0$$

and by (5.7), for ϵ_0 sufficiently small, we obtain

$$c - \epsilon_0 < f(u, v) \leq c + \epsilon_0 < 2^{2s/N} S_H.$$

Now, combining Lemma 4.7 with (5.8), we have

$$\frac{S_H + c_0}{2} < c_0 - \epsilon_0 < c - \epsilon_0 < 2^{2s/N} S_H,$$

$$\frac{S_H + c_0}{2} < c_0 - \epsilon_0 \leq c - \epsilon_0 < f(u, v)$$

which implies $(u, v) \in f^{2^{2s/N} S_H} \setminus f^{(S_H+c_0)/2}$, from where it follows (5.12).

Consider $(u, v) \in (\Sigma \cap \mathcal{M})$ such that

$$f(u, v) < \frac{S_H + c_0}{2}. \tag{5.13}$$

Then $\overline{h}(u, v) = (u, v)$ and from (5.13), we have that

$$(u, v) \notin f^{2^{2s/N} S_H} \setminus f^{\frac{S_H+c_0}{2}}$$

and by (5.12), we have $(u, v) \notin f^{c+\epsilon_0} \setminus f^{c-\epsilon_0}$. Thus,

$$(u, v) \in f^{c-\epsilon_0} \cup \{(\Sigma \cap \mathcal{M}) \setminus f^{c+\epsilon_0}\}$$

and by (b), we obtain $\eta(1, (u, v)) = (u, v)$. Therefore,

$$h_*(u, v) = \eta(1, \overline{h}(u, v)) = \eta(1, (u, v)) = (u, v),$$

which shows that $h_* \in \mathcal{H}$, and so,

$$h_*(\Theta) = \eta(1, \overline{h}(\Theta)) = \eta(1, \tilde{\mathcal{A}}) \in \Gamma.$$

Hence,

$$c = \inf_{\mathcal{A}} \max_{(u,v) \in \mathcal{A}} f(u, v) \leq \max_{(u,v) \in \mathcal{A}} f(u, v). \tag{5.14}$$

On the other hand, by (c) and (5.9), we obtain

$$\eta(1, \tilde{\mathcal{A}}) \subset \eta(1, f^{c + \frac{\epsilon_0}{2}}) \subset f^{c - \frac{\epsilon_0}{2}}.$$

That is,

$$f(u, v) \leq c - \frac{\epsilon_0}{2}, \quad \forall (u, v) \in \eta(1, \tilde{\mathcal{A}}),$$

which implies that

$$\max_{(u,v) \in \eta(1, \tilde{\mathcal{A}})} f(u, v) \leq c - \frac{\epsilon_0}{2},$$

which is a contradiction. Therefore, we must have

$$f(u_n, v_n) \to c \quad \text{and} \quad f'|_{\mathcal{M}}(u_n, v_n) \to 0$$

and from Corollary 3.7, up to a subsequence, we have $u_n \to u_0$, $v_n \to v_0$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, and satisfies

$$f(u_0, v_0) = c \quad \text{and} \quad f'|_{\mathcal{M}}(u_0, v_0) = 0.$$

By Corollary 3.8, $\mathcal{J}$ has a critical point $(w_0, z_0) \in \mathcal{D}^{s,2}(\mathbb{R}^N) \times \mathcal{D}^{s,2}(\mathbb{R}^N)$ such that

$$\mathcal{J}(w_0, z_0) = \frac{s}{N} c^{N/2s},$$

and by (5.7), we obtain

$$\frac{s}{N} S_H^{\frac{N}{2s}} < \mathcal{J}(w_0, z_0) < \frac{2s}{N} S_H^{\frac{N}{2s}}.$$

The positivity of (w_0, z_0) is a consequence of maximum principle that can be see in [25, Proposition 2.17]. □

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