

**ONE-PHASE STEFAN PROBLEM WITH A LATENT HEAT
DEPENDING ON THE POSITION OF THE FREE BOUNDARY
AND ITS RATE OF CHANGE**

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Communicated by Jesus Ildefonso Diaz

ABSTRACT. From the one-dimensional consolidation of fine-grained soils with threshold gradient, it can be derived a special type of Stefan problems where the seepage front, because of the presence of this threshold gradient, exhibits the features of a moving boundary. In this type of problems, in contrast with the classical Stefan problem, the latent heat is considered to depend inversely to the rate of change of the seepage front. In this paper, we study a one-phase Stefan problem with a latent heat that depends on the rate of change of the free boundary and on its position. The aim of this analysis is to extend prior results, finding an analytical solution that recovers, by specifying some parameters, the solutions that have already been examined in the literature regarding Stefan problems with variable latent heat. Computational examples are presented to examine the effect of this parameters on the free boundary.

1. INTRODUCTION

This article is a continuation of the work done by Zhou et al. in [18], where a one-dimensional consolidation process with a threshold gradient is studied. This problem is essentially a moving boundary problem where the seepage front, which moves downward gradually, plays the role of the free boundary due to the presence of this threshold gradient. This kind of problems are known in the literature as Stefan problems. They have been widely studied in the last century due to the fact that they arise in many significant areas of engineering, geoscience and industry [1]-[6], [8, 9, 12, 16]. A review of the literature on this topic was presentend in [15].

The classical Stefan problem intends to describe the process of a material undergoing a phase change like, for example, the melting process on an ice bar. Finding a solution to this problem consists in solving the heat-conduction equation in an unknown region which has also to be determined, imposing an initial condition, boundary conditions and the Stefan condition at the interface.

In this article, it will be considered the one-phase Stefan problem in a semi-infinite material with variable latent heat. The reduction to a one-phase problem is referred to the case in which it is assumed that one of the phases is at the

2010 *Mathematics Subject Classification.* 35R35, 80A22, 35C05, 35C06.

Key words and phrases. Stefan problem; threshold gradient; variable latent heat; one-dimensional consolidation; explicit solution; similarity solution.

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Submitted May 26, 2017. Published January 8, 2018.

phase-change temperature. The new mathematical feature of the problem to be solved is concerned with the fact that the latent heat is assumed to depend on the position of the free boundary as well as on its rate of change which is our novelty, i.e. $L = L(s(t), \dot{s}(t))$ (where $s(t)$ is the free front). It is known that in the classical formulation of the Stefan problem the latent heat is constant. The idea of a variable latent heat is motivated by the following previous works:

- In [11] it was considered a Stefan problem with a latent heat given as a function of the position of the interface $L = \varphi(s(t))$. Such assumption corresponds to the practical case when the influence of phenomena such as surface tension, pressure gradients and nonhomogeneity of materials are taken into account. Sufficient conditions that ensure the existence and uniqueness of solution were studied in this paper.

- In [17] it was provided an analytical solution to the one-phase Stefan problem with a latent heat defined as a linear function of the position, i.e. $L = \gamma s(t)$ (with γ a given constant). This hypothesis makes physical sense in the study of shoreline movement in a sedimentary basin. The extension to the two-phase problem was done in [13].

- In [19] it was considered a one-phase Stefan problem, with temperature and flux boundary condition at the fixed face, where the latent heat was not constant but, rather a power function of the position, i.e. $L = \gamma s^n(t)$ (with γ a given constant and n an arbitrary non-negative integer). The extension to a non-integer exponent was done in [20]. Moreover, in [2] the same problem with a convective (Robin) condition at the fixed face was considered, obtaining the results of [19] and [20] as a limit case.

- In [18] it was studied the one-dimensional consolidation problem with a threshold gradient. This problem is reduced to a one-phase Stefan problem with a latent heat expressed as $L = \frac{\gamma}{\dot{s}(t)}$. That is to say the latent heat depends on the rate of the moving boundary. It must be noted that the case considered in [18] is not properly a Stefan problem because the velocity of the moving boundary disappears, and it has to be treated as a free boundary problem with implicit conditions [7, 14].

Based on the bibliography mentioned above it is quite natural from a mathematical point of view to define a one-phase Stefan problem with a latent heat given by $L = \gamma s^\beta(t) \dot{s}^\delta(t)$ (with γ a given constant and β and δ arbitrary real constants).

It is worth pointing out that this formulation constitutes a mathematical generalization of the one-phase classical Stefan problem and the problems studied in [17, 18, 20].

The aim of this article is to prove in Section 3 the existence and uniqueness of the explicit solution of the problem given by equations (2.1)-(2.5) in Section 2. Moreover, in Section 4 we will consider some special cases and computational examples for different values of the parameters involved in the problem (2.1)-(2.5). The analytical solution that will be obtained in this work will recover in one formula, by choosing different values for β and δ , the solutions obtained in: the classical Stefan problem ($\beta = \delta = 0$), the problem considered in [17] ($\beta = 1, \delta = 0$), the problems solved in [18, 20] ($\beta \in \mathbb{R}_0^+, \delta = 0$) and the problem studied in [18] ($\beta = 0, \delta = -1$).

2. FORMULATION OF THE PROBLEM

This article is intended to study the one-dimensional one-phase Stefan problem for the fusion of a semi-infinite material $x > 0$ in which it is involved a variable latent heat. From a mathematical point of view the problem can be formulated as follows: Find the free boundary $s = s(t)$ (separation between phases) and the temperature $T = T(x, t)$ in the liquid portion of the material that satisfy the one-dimensional heat conduction equation:

$$a^2 T_{xx}(x, t) = T_t(x, t), \quad 0 < x < s(t), \quad (2.1)$$

subject to the boundary condition

$$T(0, t) = t^{\alpha/2} T_0, \quad t > 0 \quad (2.2)$$

the temperature condition at the interface

$$T(s(t), t) = 0, \quad t > 0 \quad (2.3)$$

the Stefan condition at the interface

$$-kT_x(s(t), t) = L(s(t), \dot{s}(t))\dot{s}(t), \quad t > 0 \quad (2.4)$$

and the initial condition

$$s(0) = 0. \quad (2.5)$$

Here the parameters a^2 (diffusion coefficient) and $k > 0$ (thermal conductivity) are known constants. The phase change temperature is 0 and the imposed temperature at the fixed face $x = 0$ is given by $t^{\alpha/2} T_0 > 0$, where we assume that α is a non-negative real exponent.

The remarkable feature of this problem is related to the condition at the interface given by the Stefan condition (2.4), where the latent heat by unit of volume will be defined by

$$L(s(t), \dot{s}(t)) = \gamma s(t)^\beta \dot{s}(t)^\delta, \quad (2.6)$$

where γ is a given constant, and β and δ are arbitrary real constants.

3. EXPLICIT SOLUTION OF THE PROBLEM

To solve problem (2.1)-(2.5), we use a similarity transformation to the one given in [19, 20]:

$$T(x, t) = t^{\alpha/2} \varphi(\eta), \quad \text{with} \quad \eta = \frac{x}{2a\sqrt{t}}. \quad (3.1)$$

Computing the derivatives of T :

$$T_{xx}(x, t) = \frac{t^{(\alpha/2-1)}}{4a^2} \varphi''(\eta), \quad (3.2)$$

$$T_t(x, t) = \frac{\alpha}{2} t^{(\alpha/2-1)} \varphi(\eta) - t^{(\alpha/2-1)} \varphi'(\eta) \frac{\eta}{2}, \quad (3.3)$$

we obtain that the temperature given by (3.1) satisfies the heat equation (2.1) if and only if φ is the solution of the ordinary differential equation

$$\varphi''(\eta) + 2\eta\varphi'(\eta) - 2\alpha\varphi(\eta) = 0, \quad (3.4)$$

whose general solution, in this case, can be written as (see the proof in the Appendix A)

$$\varphi(\eta) = c_1 M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\eta^2\right) + c_2 \eta M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\eta^2\right), \quad (3.5)$$

where c_1 and c_2 are arbitrary constants. The function $M(a, b, z)$, called Kummer function, is defined by

$$M(a, b, z) = \sum_{s=0}^{\infty} \frac{(a)_s}{(b)_s s!} z^s, \quad (3.6)$$

in which b cannot be a non-positive integer, and where $(a)_s$ is the Pochhammer symbol defined by:

$$(a)_s = a(a+1)(a+2)\dots(a+s-1), \quad (a)_0 = 1 \quad (3.7)$$

All the properties of Kummer's functions to be used in this paper can be found in [10].

Therefore $T(x, t)$ is given by

$$T(x, t) = t^{\alpha/2} \left[c_1 M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\eta^2\right) + c_2 \eta M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\eta^2\right) \right]. \quad (3.8)$$

where c_1 and c_2 are constants that must be determined in order that $T(x, t)$ satisfies the conditions (2.2)-(2.4).

From equation (2.2), taking into account that $M(-\frac{\alpha}{2}, \frac{1}{2}, 0) = 1$, it is obtained

$$c_1 = T_0. \quad (3.9)$$

From condition (2.3), we have $\varphi(\frac{s(t)}{2a\sqrt{t}}) = 0$ for all $t > 0$. Then

$$s(t) = 2a\xi\sqrt{t}, \quad (3.10)$$

where ξ is a positive constant that has to be determined. Bearing in mind that the free boundary $s(t)$ is defined by (3.10), it can be deduced from (2.3) and (3.9) that

$$c_2 = \frac{-T_0 M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\xi^2\right)}{\xi M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\xi^2\right)}. \quad (3.11)$$

Until now, $s(t)$ and c_2 are given in function of ξ . To determine ξ , we apply the Stefan condition (2.4) which has not been considered yet. For that purpose, $T_x(x, t)$ must be calculated,

$$T_x(x, t) = \frac{t^{(\alpha-1)/2}}{a} \left[c_1 \alpha M\left(-\frac{\alpha}{2} + 1, \frac{3}{2}, -\eta^2\right) + \frac{c_2}{2} M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{1}{2}, -\eta^2\right) \right] \quad (3.12)$$

From the Stefan condition, taking into account (3.10) and (3.12), we obtain

$$\begin{aligned} & -\frac{kt^{(\alpha-1)/2}}{a} \left[c_1 \alpha M\left(-\frac{\alpha}{2} + 1, \frac{3}{2}, -\xi^2\right) + \frac{c_2}{2} M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{1}{2}, -\xi^2\right) \right] \\ & = \gamma 2^\beta \xi^{\beta+\delta+1} a^{\beta+\delta+1} t^{(\beta-\delta-1)/2}. \end{aligned} \quad (3.13)$$

As c_1 , c_2 and ξ does not depend on t , (3.13) makes sense if and only if $t^{(\alpha-1)/2} = t^{(\beta-\delta-1)/2}$. This leads to

$$\alpha = \beta - \delta. \quad (3.14)$$

Therefore, assuming that (3.14) holds, condition (2.4) leads to

$$\begin{aligned} & \frac{kT_0}{2a\xi M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\xi^2\right)} \left[-2\alpha\xi^2 M\left(-\frac{\alpha}{2} + 1, \frac{3}{2}, -\xi^2\right) M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\xi^2\right) \right. \\ & \quad \left. + M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\xi^2\right) M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{1}{2}, -\xi^2\right) \right] \\ & = \gamma 2^\beta a^{\beta+\delta+1} \xi^{\beta+\delta+1}. \end{aligned} \quad (3.15)$$

Taking note of the identity proved in [20],

$$e^{-\xi^2} = -2\alpha\xi^2 M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\xi^2\right) M\left(-\frac{\alpha}{2} + 1, \frac{3}{2}, -\xi^2\right) \\ + M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\xi^2\right) M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{1}{2}, -\xi^2\right),$$

and the relationship presented in [10],

$$M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\xi^2\right) = e^{-\xi^2} M\left(\frac{\alpha}{2} + 1, \frac{3}{2}, \xi^2\right), \quad (3.16)$$

equation (3.15) becomes

$$\frac{kT_0}{\gamma a^{\beta+\delta+2} 2^{\beta+1}} \frac{1}{\xi M\left(\frac{\alpha}{2} + 1, \frac{3}{2}, \xi^2\right)} = \xi^{\beta+\delta+1}. \quad (3.17)$$

That is to say ξ is a positive solution of the equation

$$\frac{kT_0}{\gamma a^{\beta+\delta+2} 2^{\beta+1}} f(z) = z^{\beta+\delta+1}, \quad z > 0, \quad (3.18)$$

where

$$f(z) = \frac{1}{z M\left(\frac{\alpha}{2} + 1, \frac{3}{2}, z^2\right)}. \quad (3.19)$$

Furthermore, from [10] knowing that

$$\frac{d}{dz} [z M\left(\frac{\alpha}{2} + 1, \frac{3}{2}, z^2\right)] = M\left(\frac{\alpha}{2} + 1, \frac{1}{2}, z^2\right) \quad (3.20)$$

it follows that

$$f'(z) = -f^2(z) M\left(\frac{\alpha}{2} + 1, \frac{1}{2}, z^2\right). \quad (3.21)$$

In this way, it can be said that the left-hand side of equation (3.18) given by $LE(z) = \frac{kT_0}{\gamma a^{\beta+\delta+2} 2^{\beta+1}} f(z)$ satisfies

$$(LE)'(z) = -\frac{kT_0}{\gamma a^{\beta+\delta+2} 2^{\beta+1}} f^2(x) M\left(\frac{\alpha}{2} + 1, \frac{1}{2}, z^2\right) < 0, \quad (3.22)$$

$$LE(0) = +\infty, \quad (3.23)$$

$$LE(+\infty) = 0, \quad (3.24)$$

meanwhile the right hand side of (3.18) given by $RI(z) = z^{\beta+\delta+1}$ satisfies

$$(RI)'(z) = (\beta + \delta + 1)z^{\beta+\delta} > 0, \quad (\text{if } \beta + \delta + 1 \geq 0) \quad (3.25)$$

$$RI(0) = 0, \quad (3.26)$$

$$RI(+\infty) = +\infty. \quad (3.27)$$

Thus, from (3.22)-(3.24) and (3.25)-(3.27), one can conclude that equation (3.18) has a unique positive solution ξ provided that $\beta + \delta + 1 \geq 0$.

It should be mentioned that because of (3.14), i.e $\alpha = \beta - \delta$, the fact that $\alpha \geq 0$, and the request that $\beta + \delta + 1 \geq 0$, the results obtained in this paper are valid only if $\beta \geq \max(\delta, -\delta - 1)$.

The above arguments can be summarized in the following theorem.

Theorem 3.1. *Let β and δ be arbitrary constants satisfying $\beta \geq \max(\delta, -\delta - 1)$. Taking $\alpha = \beta - \delta$, there exists a unique solution of a similarity type for the one-phase Stefan problem (2.1)-(2.5) given by*

$$T(x, t) = t^{\alpha/2} \left[c_1 M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\eta^2\right) + c_2 \eta M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\eta^2\right) \right], \quad (3.28)$$

$$s(t) = 2a\xi\sqrt{t}, \quad (3.29)$$

where $\eta = \frac{x}{2a\sqrt{t}}$ and the constants c_1 and c_2 are given by

$$c_1 = T_0, \quad c_2 = \frac{-T_0 M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\xi^2\right)}{\xi M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\xi^2\right)}, \quad (3.30)$$

and the dimensionless coefficient ξ is obtained as the unique positive solution of the equation

$$\frac{kT_0}{\gamma a^{\beta+\delta+2} 2^{\beta+1}} f(z) = z^{\beta+\delta+1}, \quad z > 0, \quad (3.31)$$

in which f is the real function defined by

$$f(z) = \frac{1}{z M\left(\frac{\alpha}{2} + 1, \frac{3}{2}, z^2\right)}, \quad z > 0. \quad (3.32)$$

4. SPECIAL CASES AND COMPUTATIONAL EXAMPLES

This section is meant to highlight the problems that are generalized in this work by showing that the solutions already reached in the literature can be obtained from the one we present by just choosing the appropriate parameters β , δ and thus α . For each case it is going to be done a computational example in order to see how the parameter ξ , that characterizes the free boundary, varies with respect to δ , for a fixed β .

Properties found in [10] and [20] will be helpful in the subsequent arguments:

$$M(0, b, z) = 1 \quad (4.1)$$

$$M(a, b, z) = e^z M(a, b, -z) \quad (4.2)$$

$$M\left(-\frac{n}{2}, \frac{1}{2}, -z^2\right) = 2^{n-1} \Gamma\left(\frac{n}{2} + 1\right) [i^n \operatorname{erfc}(z) + i^n \operatorname{erfc}(-z)] \quad (4.3)$$

$$zM\left(-\frac{n}{2} + \frac{1}{2}, \frac{3}{2}, -z^2\right) = 2^{n-2} \Gamma\left(\frac{n}{2} + \frac{1}{2}\right) [i^n \operatorname{erfc}(-z) - i^n \operatorname{erfc}(z)] \quad (4.4)$$

with $n \in \mathbb{N}$, and where $i^n \operatorname{erfc}(\cdot)$ is the repeated integral of the complementary error function defined by

$$i^0 \operatorname{erfc}(z) = \operatorname{erfc}(z) = 1 - \operatorname{erf}(z), \quad \operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2} du, \quad (4.5)$$

$$i^n \operatorname{erfc}(z) = \int_z^{+\infty} i^{n-1} \operatorname{erfc}(t) dt \quad (4.6)$$

Let us analyze the explicit solution achieved in each of the following problems:

Classical one-phase Stefan problem. The latent heat is given by $L = \gamma$ constant, so the solution in this case can be recovered from the solution given by (3.28)-(3.32) by taking $\beta = 0, \delta = 0$ and thus $\alpha = 0$. It must be pointed out that in this case, using that

$$M\left(0, \frac{1}{2}, -\eta^2\right) = 1, \quad \text{and} \quad M\left(\frac{1}{2}, \frac{3}{2}, -\eta^2\right) = \frac{\sqrt{\pi}}{2\eta} \operatorname{erf}(\eta), \quad (4.7)$$

we get a temperature with the form $T(x, t) = c_1 + c_2 \operatorname{erf}(\eta)$, like in the classical literature [1, 3, 5, 6, 8, 12, 16].

Problem of the shoreline movement in a sedimentary basin [17, 13]. In this problem it was considered a latent heat that varies linearly with the position, that is to say $L = \gamma s(t)$. Therefore the solution can be obtained from Theorem 3.1 by choosing $\beta = 1, \delta = 0$ and thus $\alpha = 1$. Using (4.3), and the properties: $i \operatorname{erf}(z) = \frac{e^{-z^2}}{\sqrt{\pi}} + z \operatorname{erfc}(z)$ and $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$, the temperature becomes:

$$T(x, t) = c_1 \left[\sqrt{t} e^{-\eta^2} + \frac{\sqrt{\pi}}{2} x \operatorname{erfc}(\eta) \right] + \frac{c_2}{2} x \quad (4.8)$$

in accordance to the solution shown in [17] and [13].

Problem with a latent heat defined as a power function of the position solved in [19] and [20]. In these papers, the latent heat L is defined as a power function of the position, i.e., $L = \gamma s(t)^\beta$ with γ constant and β a non-negative real exponent. Choosing $\beta \in \mathbb{R}_0^+, \delta = 0$, and then $\alpha = \beta$, the solutions given in [19] and [20] are automatically recovered.

One-dimensional consolidation problem with threshold gradient [18]. In this work, it is considered a one-dimensional consolidation problem with a threshold gradient which can be transformed into a one-phase Stefan problem with a latent heat that depends on the rate of change of the moving boundary. It is studied the case in which $L = \frac{\gamma}{s'(t)}$. We remark here that this problem is not a Stefan problem because the velocity of the free boundary does not appear but it is a free boundary problem for the heat equation with implicit free boundary conditions [7, 14]. Fixing $\beta = 0, \delta = -1$ and so $\alpha = 1$, the solution of this problem can be obtained from Theorem 3.1 the temperature can be expressed as (4.8). In addition, taking into account (4.2), it is obtained that

$$f(z) = \frac{1}{zM\left(\frac{3}{2}, \frac{3}{2}, z^2\right)} = \frac{1}{ze^{z^2}M\left(0, \frac{3}{2}, -z^2\right)} = \frac{e^{-z^2}}{z} \quad (4.9)$$

in agreement to the solution in [18].

Once it has been compared the solution obtained in this paper with the solutions presented in the literature, we are going to run some computational examples. To solve the Stefan problem (2.1)-(2.5) it is necessary to solve equation (3.31) which is equivalent to find the unique zero of the function

$$H(z) = \frac{kT_0}{\gamma a^{\beta+\delta+2} 2^{\beta+1}} f(z) - z^{\beta+\delta+1}, \quad z > 0, \quad (4.10)$$

in which f is the function defined by (3.32).

We are going to apply Newton's method with the iteration formula

$$z_k = z_{k-1} - \frac{H(z_{k-1})}{H'(z_{k-1})}. \quad (4.11)$$

to solve (4.10). For the computational examples we consider the corresponding thermal parameters for the water in liquid state i.e., $k = 0.58 [W/(m^\circ C)]$ and $a^2 = 1.39 \times 10^{-7} [m^2/s]$. Without loss of generality we assume $\gamma = 1$. Newton's Method will be implemented using Matlab software to find the unique positive solution of equation (3.31). It is worth pointing out that in this programming language, the Kummer function $M(a, b, z)$ is represented by the 'hypergeom' command. The stopping criterion to be used here is the boundedness of the absolute error $|z_k - z_{k-1}| < 10^{-15}$.

Figure 1 shows the variation of ξ (solution of (4.10)) with respect to δ , choosing different values for the coefficient that characterizes the temperature at the fixed face ($T_0 = 1, 5$ or $10 [^\circ C/s^{\alpha/2}]$) and fixing $\beta = 0$.

Looking at Figure 1, it is obvious that solution of ξ for the Classical Stefan problem is obtained for $\delta = 0$. For $\delta = 1$ the solution is given by [17], and for any other $\delta \in \mathbb{R}_0^+$ the solution is given by [19] and [20].

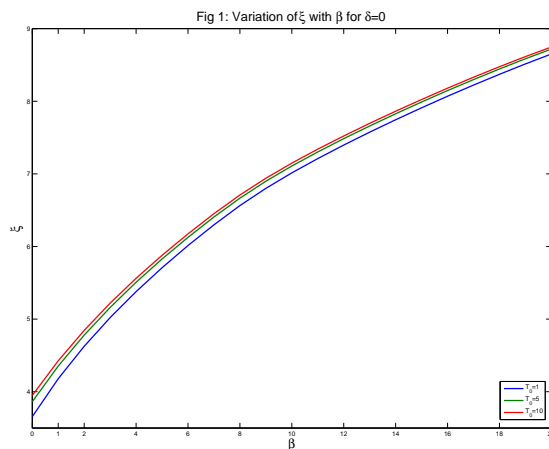


FIGURE 1. Variation of ξ with δ , for $\beta = 0$ and $T_0 = 1, 5, 10$.

Figure 2, shows the variation of ξ with respect to β ($-1 \leq \beta \leq 1$), choosing different values for the coefficient that characterizes the temperature at the fixed face ($T_0 = 1, 5$ or $10 [^\circ C/s^{\alpha/2}]$) and fixing $\delta = 0$. The case for $\beta = -1$ corresponds to the solution of the problem analyzed in [18].

The results obtained indicate that ξ increases with β increasing and $\delta = 0$, and the same happens when $\beta = 0$ is fixed and δ varies between -1 and 1 . Moreover, it can be assured that the greater the value of T_0 , the higher is the value obtained for ξ (parameter that characterizes the free boundary) implying that the fusion process occurs faster.

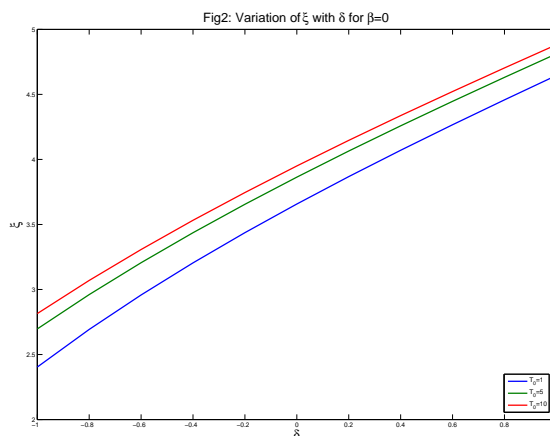


FIGURE 2. Variation of ξ with β , for $\delta = 0$ and $T_0 = 1, 5, 10$.

5. APPENDIX

We prove that the general solution of the ordinary differential equation

$$\varphi''(\eta) + 2\eta\varphi'(\eta) - 2\alpha\varphi(\eta) = 0. \quad (5.1)$$

is

$$\varphi(\eta) = c_1 M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\eta^2\right) + c_2 \eta M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\eta^2\right), \quad (5.2)$$

regardless of α being an integer, or a non-integer non-negative number, where c_1 and c_2 are arbitrary constants.

α non-negative, non-integer. Introducing the new variable $w(\eta) = -\eta^2$ as in [20] and defining $g(w) = \varphi(\eta(w))$ we obtain that (5.1) is equivalent to the Kummer's differential equation

$$wg''(w) + g'(w)\left(\frac{1}{2} - w\right) + \frac{\alpha}{2}g(w) = 0. \quad (5.3)$$

whose general solution, according to [10], is

$$g(w) = \widehat{c}_1 M\left(-\frac{\alpha}{2}, \frac{1}{2}, w\right) + \widehat{c}_2 U\left(-\frac{\alpha}{2}, \frac{1}{2}, w\right), \quad (5.4)$$

where \widehat{c}_1 and \widehat{c}_2 are arbitrary constants. Because U can be defined as

$$U(a, b, z) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} M(a, b, z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} M(a-b+1, 2-b, z). \quad (5.5)$$

we obtained that the general solution of (5.3) is

$$g(w) = \overline{c}_1 M\left(-\frac{\alpha}{2}, \frac{1}{2}, w\right) + \overline{c}_2 w^{1/2} M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, w\right), \quad (5.6)$$

where \overline{c}_1 and \overline{c}_2 are arbitrary constants, arriving in this way to a φ solution of (5.1) defined by (5.2).

$\alpha = n$ **non-negative integer**. According to [19], the general solution of (5.1) is

$$\varphi(\eta) = \widehat{c}_1 i^n \operatorname{erfc}(\eta) + \widehat{c}_2 i^n \operatorname{erfc}(-\eta). \quad (5.7)$$

where $i^n \operatorname{erfc}(\cdot)$ is the family of the repeated integrals of the complementary error function.

Let c_1 and c_2 be arbitrary constants. Taking:

$$\widehat{c}_1 = c_1 2^{n-1} \Gamma\left(\frac{n}{2} + 1\right) - c_2 2^{n-2} \Gamma\left(\frac{n}{2} + \frac{1}{2}\right) \quad (5.8)$$

$$\widehat{c}_2 = c_1 2^{n-1} \Gamma\left(\frac{n}{2} + 1\right) + c_2 2^{n-2} \Gamma\left(\frac{n}{2} + \frac{1}{2}\right) \quad (5.9)$$

in (5.7) leads to

$$\begin{aligned} \varphi(\eta) &= [c_1 2^{n-1} \Gamma\left(\frac{n}{2} + 1\right) - c_2 2^{n-2} \Gamma\left(\frac{n}{2} + \frac{1}{2}\right)] i^n \operatorname{erfc}(\eta) \\ &\quad + [c_1 2^{n-1} \Gamma\left(\frac{n}{2} + 1\right) + c_2 2^{n-2} \Gamma\left(\frac{n}{2} + \frac{1}{2}\right)] i^n \operatorname{erfc}(-\eta), \\ &= c_1 2^{n-1} \Gamma\left(\frac{n}{2} + 1\right) [i^n \operatorname{erfc}(\eta) + i^n \operatorname{erfc}(-\eta)] \\ &\quad + c_2 2^{n-2} \Gamma\left(\frac{n}{2} + \frac{1}{2}\right) [i^n \operatorname{erfc}(-\eta) - i^n \operatorname{erfc}(\eta)], \\ &= c_1 M\left(-\frac{n}{2}, \frac{1}{2}, -\eta^2\right) + c_2 \eta M\left(-\frac{n}{2} + \frac{1}{2}, \frac{3}{2}, -\eta^2\right). \end{aligned} \quad (5.10)$$

arriving to a solution of (5.1) given by a φ defined as (5.2), using the properties stated in (4.3)-(4.4).

Nomenclature	
a^2	Diffusivity coefficient ($a^2 = k/\rho c$), $[m^2/s]$
c	Specific heat capacity, $[m^2/^\circ C s^2]$
k	Thermal conductivity, $[W/(m^\circ C)]$
s	Position of the free front, $[m]$
t	Time, $[s]$
T	Temperature, $[^\circ C]$
T_0	Coefficient that characterizes the temperature at the fixed face, $[^\circ C/s^{\alpha/2}]$
x	Spatial coordinate, $[m]$
α	Power of the time that characterizes the temperature at the fixed boundary, dimensionless
β	Power of the position that characterizes the latent heat per unit of volume, dimensionless
δ	Power of the velocity that characterizes the latent heat per unit of volume, dimensionless
γ	Coefficient that characterizes the latent heat per unit of volume, $[s^{\delta-2} kg/(m^{\beta+\delta+1})]$
ρ	Density, $[kg/m^3]$
ξ	Coefficient that characterizes the free interface, dimensionless
η	Similarity variable in expression (3.1), dimensionless.

Conclusions. In this work we analyzed a Stefan problem with a latent heat that depends on the position of the free boundary as well as on its rate of change. An explicit solution has been found using the similarity technique and the theory

of Kummer functions. This exact solution gathers in one formula the solutions obtained in the previous papers [17, 18, 19, 20], constituting a generalization of them. The exact solution is worth finding it, since it can be used to provide a benchmark for verifying the accuracy of numerical methods that approximate the solution of Stefan problems.

We have also applied Newton's method to the problem (2.1)-(2.5) to estimate the parameter ξ that characterizes the free front numerically. The solutions given in the literature have also been recovered. In addition it was observed that this parameter increases with respect to the parameter β , fixing $\delta = 0$ and vice versa. Also, it can be noted that if the coefficient that characterizes the initial temperature T_0 becomes greater, ξ also does, meaning that the phase-change happens quicker, validating mathematically what it seems obvious from the physical point of view.

Acknowledgements. This work was sponsored by projects PIP No 0275 from CONICET-UA, and PICTO Austral 2016 No 0090, Rosario, Argentina. This work was also supported by grant AFOSR-SOARD FA9550-14-1-0122.

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