Electronic Journal of Differential Equations, Vol. 2005(2005), No. 147, pp. 1–25.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

QUASISTATIC EVOLUTION OF DAMAGE IN AN ELASTIC-VISCOPLASTIC MATERIAL

KENNETH L. KUTTLER

Abstract. The mathematical theory of quasistatic elastic viscoplastic models with damage is studied. The existence of the unique local weak solution is established by using approximate problems and a priori estimates. Pointwise estimates on the damage are obtained using a new comparison technique which removes the necessity of including a subgradient term in the equation for damage.

1. Introduction

This work deals with quasistatic evolution of the macroscopic mechanical state of an elastic viscoplastic body and the development of microscopic material damage which results from internal compression or tension. The damage of the material is caused by the opening and growth of micro-cracks and micro-cavities which lead to the decrease in the load carrying capacity of the body and, eventually, to the possible failure of the system in which the body is situated. The model for the stress used here is given as a solution to an initial value problem.

$$\sigma' = \mathcal{A}(\zeta\varepsilon(\mathbf{u}))' + G(\sigma, \varepsilon(\mathbf{u}), \zeta), \quad \sigma(0) = \sigma_0.$$

Without the damage parameter ζ, this is the model of elastic viscoplastic material. For a discussion of the mathematical theory of these models, see [16]and also [17]. In this formula for the stress the damage parameter has values between 0 and 1. The above formula for the stress differs from [7] by allowing the damage to affect the elastic part of the stress and not just the viscoplastic part.

The novel idea of modelling material damage by the introduction of the damage field originated in the works of Frémond [9, 10, 11] and was motivated by the evolution of damage in concrete structures. These ideas have been extended recently in [1, 2, 3, 4, 6, 7, 12, 13, 15, 19] and in the references therein. Additional results and references can be found in the two new monographs [21, 22]. In this approach the damage field ζ varies between one and zero at each point in the body. When $\zeta = 1$ the material is damage-free, when $\zeta = 0$ the material is completely damaged, and for $0 < \zeta < 1$ it is partially damaged.

2000 *Mathematics Subject Classification.* 74D10, 74R99, 74C10, 35K50, 35K65, 35Q72, 35B05.
Key words and phrases. Existence and uniqueness; damage; comparison theorems; elastic viscoplastic materials.

Submitted September 16, 2005. Published December 12, 2005.

The evolution of the damage field is usually described by a parabolic inclusion with a damage source function which depends on the mechanical compression or tension. The reason it is an inclusion and not an equation is that a subgradient is included in the model to force the damage parameter to remain within the desired interval. It is interesting to find conditions on the damage source function which remove the necessity for using this subgradient term in the model.

I will show in this paper that the subgradient is not necessary when physically reasonable conditions are made on the damage source function which are sufficient to show the damage parameter remains in the desired interval. This makes possible considerable improvements in the regularity of the solutions although this aspect of the the elastic viscoplastic problem will be postponed for another paper. The goal in this paper is to consider the weak solutions under minimal regularity and compatibility conditions for the data. The argument which allows pointwise estimates on the damage is most impressive in the context of very weak solutions. It is based on a parabolic comparison principle which is easy to prove for classical solutions but is not obvious for the weak solutions discussed here.

The main result is Theorem 4.10 which is an existence and uniqueness theorem. It is seen that the equation for damage is solved in the classical sense because all the derivatives in the partial differential equation exist but the balance of momentum equation is only solved weakly. In later papers, more regularity will be obtained. Also, other types of mechanical situations will be considered such as problems with contact, wear, friction and adhesion.

2. The model

The body which occupies a domain $\Omega \in \mathbb{R}^d$ $(d = 1, 2, 3)$ with outer surface $\partial\Omega = \Gamma$ assumed to be sufficiently smooth, at least $C^{2,1}$ which means the second derivatives of the parameterizations defining $\partial\Omega$ are Lipshitz continuous. Volume forces of density $\mathbf{f}_B$ act in $\Omega_T = \Omega \times (0, T)$, for $T > 0$.

Denote by $\mathbf{u}$ the displacement field, σ the stress tensor, and $\varepsilon(\mathbf{u})$ the small or linearized strain tensor. Let ζ denote the *damage field*, which is defined in Ω_T and measures the fractional decrease in the strength of the material, to be described shortly. Integrating the differential equation, for the stress, σ is the solution of the integral equation

$$\sigma(t) = \zeta(t)\mathcal{A}\varepsilon(\mathbf{u}(t)) - \zeta_0\mathcal{A}\varepsilon(\mathbf{u}_0) - \sigma_0 + \int_0^t G(\sigma, \varepsilon(\mathbf{u}), \zeta)ds \tag{2.1}$$

where $\mathbf{u}_0$ is an initial displacement and σ_0 is an initial stress.

Assume $\mathcal{A} = \{\mathcal{A}_{ijkl}(\mathbf{x})\}$ satisfies the usual symmetries

$$\mathcal{A}_{ijkl}(\mathbf{x}) = \mathcal{A}_{klij}(\mathbf{x}), \mathcal{A}_{ijkl}(\mathbf{x}) = \mathcal{A}_{jikl}(\mathbf{x}).$$

Also it is always assumed

$$\mathcal{A}(\mathbf{x})\tau\cdot\tau \geq m_{\mathcal{A}}|\tau|^2_{\mathbb{S}_d}, \text{ for all } \tau \in \mathbb{S}_d.$$

Here and in the rest of the paper, $\mathbf{x}$ will denote a material point.

As a result of the tensile or compressive stresses in the body, micro-cracks and micro-cavities open and grow and this, in turn, causes the load bearing capacity of the material to decrease. This reduction in the strength of an isotropic material is modelled by introducing the damage field $\zeta = \zeta(\mathbf{x}, t)$ as the ratio

$$\zeta = \zeta(\mathbf{x}, t) = \frac{E_{eff}}{E}$$

between the effective modulus of elasticity E_{eff} and that of the damage-free material E. It follows from this definition that the damage field should only have values between 0 and 1.

Following the derivation in Frémond and Nedjar [9, 10] (see [11] for full details, and also [21]), the evolution of the microscopic cracks and cavities responsible for the damage is described by the differential inclusion

$$\zeta' - \kappa\,\Delta\zeta \in \phi(\varepsilon(\mathbf{u}), \zeta) - \partial I_{[0,1]}(\zeta). \tag{2.2}$$

However, in this paper, I will show that the subgradient term is not necessary provided physically reasonable assumptions are made on the source term, $\phi(\varepsilon(\mathbf{u}), \zeta)$. This assumption is essentially that whenever $\zeta \geq 1, \phi(\varepsilon(\mathbf{u}), \zeta) \leq 0$. This makes perfect sense because there should be no way the source term for damage to produce damage greater than 1. Thus in this paper the damage is governed by the equation

$$\zeta' - \kappa\,\Delta\zeta = \phi(\varepsilon(\mathbf{u}), \zeta)$$

rather than the inclusion (2.2). Here, the prime denotes the time derivative, Δ is the Laplace operator, $\kappa > 0$ is the damage diffusion constant, ϕ is the damage source function. There have been many different formulas proposed for ϕ but in this paper I will only assume the following Lipshitz continuity of ϕ.

$$|\phi(\varepsilon_1, \zeta_1) - \phi(\varepsilon_2, \zeta_2)| \leq K(|\varepsilon_1 - \varepsilon_2| + |\zeta_1 - \zeta_2|) \tag{2.3}$$

This may seem restrictive but one can give good physical reasons for making this assumption [18]. In addition, it is shown in this reference that in the elastic case the above assumption can be completely eliminated in the presence of suitable compatibility conditions on the initial data and other assumptions which allow the use of elliptic regularity theorems. Probably similar considerations will eventually apply to this elastic viscoplastic problem but at present this is not known.

The classical form of the problem is: Find a displacement field $\mathbf{u} : \Omega_T \to \mathbb{R}^d$, a stress field $\sigma : \Omega_T \to \mathbb{S}_d$, and a damage field $\zeta : \Omega_T \to \mathbb{R}$, such that

$$\begin{gathered}
-\operatorname{div}\sigma = \mathbf{f}_B \quad \text{in } \Omega_T,\\
\sigma(t) = \zeta(t)\mathcal{A}\varepsilon(\mathbf{u}(t)) - \zeta_0\mathcal{A}\varepsilon(\mathbf{u}_0) - \sigma_0 + \int_0^t G(\sigma, \varepsilon(\mathbf{u}), \zeta)ds \quad \text{in } \Omega_T,\\
\zeta' - \kappa\Delta\zeta = \phi(\varepsilon(\mathbf{u}), \zeta) \quad \text{in } \Omega_T,\\
\partial\zeta/\partial n = 0 \text{on } \partial\Omega \times (0, T),\\
\mathbf{u} = \mathbf{0} \text{ on } \Gamma_D \times (0, T),\ \sigma\mathbf{n} = \mathbf{f}_N \text{ on } \Gamma_N \times (0, T),\\
\zeta(0) = \zeta_0
\end{gathered}$$

First consider a truncated problem which depends on the truncation operator η_* which is a nondecreasing C^3 function satisfying

$$\eta_*(\zeta) \equiv \begin{cases} b & \text{if } \zeta > 1 + \epsilon,\\ \zeta & \text{if } \zeta_* \leq \zeta \leq 1,\\ a & \text{if } \zeta < \zeta_*/2. \end{cases} \tag{2.4}$$

where $b \geq 1$, $1 > \zeta_* > 0$, and $0 < a$. Note that as long as $\zeta \in [\zeta_*, 1]$ it makes no difference whether one writes ζ or $\eta_*(\zeta)$. The purpose for using η_* is to allow the study of global solutions. Then, starting with initial condition ζ_0 such that $\zeta_* < \zeta_0 \leq 1$ I will establish pointwise estimates which show that ζ remains in an interval on which $\eta_*(\zeta) = \zeta$. Replacing ζ with $\eta_*(\zeta)$, yields the classical form of the truncated problem.

Problem P. Find a displacement field $\mathbf{u} : \Omega_T \to \mathbb{R}^d$, a stress field $\sigma : \Omega_T \to \mathbb{S}_d$, and a damage field $\zeta : \Omega_T \to \mathbb{R}$, such that

$$-\operatorname{div}\sigma = \mathbf{f}_B \text{ in } \Omega_T, \tag{2.5}$$

$$\sigma(t) = \eta_*(\zeta)(t)\mathcal{A}\varepsilon(\mathbf{u}(t)) - \eta_*(\zeta_0)\mathcal{A}\varepsilon(\mathbf{u}_0) - \sigma_0 + \int_0^t G(\sigma, \varepsilon(\mathbf{u}), \eta_*(\zeta))ds \text{ in } \Omega_T, \tag{2.6}$$

$$\zeta' - \kappa\Delta\zeta = \phi(\varepsilon(\mathbf{u}), \eta_*(\zeta)) \text{ in } \Omega_T,\ \zeta(0) = \zeta_0, \tag{2.7}$$

$$\partial\zeta/\partial n = 0 \text{ on } \partial\Omega \times (0, T), \tag{2.8}$$

$$\mathbf{u} = \mathbf{0} \text{ on } \Gamma_D \times (0, T),\ \sigma\mathbf{n} = \mathbf{f}_N \text{ on } \Gamma_N \times (0, T). \tag{2.9}$$

Here, ϕ satisfies (2.3). I will also assume a Lipschitz condition on the function G which might depend on $\mathbf{x}$ although this dependence is not shown explicitly.

$$|G(\sigma_1, \varepsilon_1, \zeta_1) - G(\sigma_2, \varepsilon_2, \zeta_2)| \leq K(|\sigma_1 - \sigma_2|_{\mathbb{S}_d} + |\varepsilon_1 - \varepsilon_2|_{\mathbb{S}_d} + |\zeta_1 - \zeta_2|) \tag{2.10}$$

$$G(\mathbf{0}, \mathbf{0}, 0) \in L^2(\Omega; \mathbb{S}_d) \equiv Q \tag{2.11}$$

where $\mathbb{S}_d$ is the space of symmetric $d \times d$ matrices with the usual notion of inner product. Also Γ_D and Γ_N are disjoint subsets of $\partial\Omega$ whose union equals $\partial\Omega$ and Γ_D has positive surface measure.

3. Abstract formulation, existence and uniqueness

I will make use of the following two theorems found in Lions [20] and Simon [23], respectively.

Theorem 3.1. *Let $p \geq 1$, $q > 1$, $X_1 \subseteq X_2 \subseteq X_3$ with compact inclusion map $X_1 \to X_2$ and continuous inclusion map $X_2 \to X_3$, and let*

$$S_R = \{\mathbf{u} \in L^p(0, T; X_1) : \ \mathbf{u}' \in L^q(0, T; X_3),\ \|\mathbf{u}\|_{L^p(0,T;X_1)} + \|\mathbf{u}'\|_{L^q(0,T;X_3)} < R\}.$$

Then S_R is precompact in $L^p(0, T; X_2)$.

Theorem 3.2. *Let X_1, X_2 and X_3 be as above and let*

$$S_{RT} = \{\mathbf{u} : \|\mathbf{u}(t)\|_{X_1} + \|\mathbf{u}'\|_{L^q(0,T;X_3)} \leq R, \quad t \in [0, T]\},$$

for some $q > 1$. Then S_{RT} is precompact in $C(0, T; X_2)$.

Let $H \equiv (L^2(\Omega))^d$, $H_1 \equiv (H^1(\Omega))^d$, and

$$V \equiv \{\mathbf{v} \in H_1 : \mathbf{v} = \mathbf{0} \quad \text{on} \ \ \Gamma_D\}.$$

It follows from Korn's inequality that an equivalent norm on V is

$$\|\mathbf{u}\|_V \equiv |\varepsilon(\mathbf{u})|_Q$$

and I will use this as the norm on V. Also, $E \equiv H^1(\Omega)$.

Denote by $\mathcal{V}, \mathcal{E}, \mathcal{H}$, and $\mathcal{Y}$ the spaces

$$L^2(0, T; V), \quad L^2(0, T; E), \quad L^2(0, T; H), \quad L^2(0, T; L^2(\Omega)),$$

respectively. Since V is dense in H one can identify H with its dual H' and write

$$V \subseteq H = H' \subseteq V'.$$

Also let $Y \equiv L^2(\Omega)$ and in a similar way

$$E \subseteq Y = Y' \subseteq E'.$$

I will use the standard notation for the dual spaces and duality pairings. Recall that σ satisfies the identity

$$\sigma(t) = \eta_*(\zeta)(t)\mathcal{A}\varepsilon(\mathbf{u}(t)) - \eta_*(\zeta_0)\mathcal{A}\varepsilon(\mathbf{u}_0) - \sigma_0 + \int_0^t G(\sigma, \varepsilon(\mathbf{u}), \eta_*(\zeta))ds\,.$$

For fixed $\zeta \in \mathcal{Y}$ and $\tau \in L^2(0,T;Q)$, define $\Psi_{\zeta\tau} : L^2(0,T;Q) \to L^2(0,T;Q)$ by

$$\Psi_{\zeta\tau}(\sigma)(t) \equiv \eta_*(\zeta)(t)\mathcal{A}\tau - \eta_*(\zeta_0)\mathcal{A}\varepsilon(\mathbf{u}_0) - \sigma_0 + \int_0^t G(\sigma, \tau, \eta_*(\zeta))ds$$

Lemma 3.3. *The operator $\Psi_{\zeta\tau}$ has a unique fixed point in $L^2(0,T;Q)$.*

Proof. Let $\sigma_i, i = 1,2$ be two elements of $L^2(0,T;Q)$. Then since G is Lipschitz continuous, (2.10) holds, and

$$\begin{aligned}|\Psi_{\zeta\tau}(\sigma_1)(t) - \Psi_{\zeta\tau}(\sigma_2)(t)|_Q &= \Big|\int_0^t (G(\sigma_1, \tau, \eta_*(\zeta)) - G(\sigma_2, \tau, \eta_*(\zeta)))ds\Big| \\ &\le K\int_0^t |\sigma_1(s) - \sigma_2(s)|_Q ds.\end{aligned}$$

Therefore, letting $\lambda > 0$,

$$\int_0^T e^{-\lambda t}|\Psi_{\zeta\tau}(\sigma_1)(t) - \Psi_{\zeta\tau}(\sigma_2)(t)|_Q^2 dt \le \int_0^T e^{-\lambda t}(K\int_0^t |\sigma_1(s) - \sigma_2(s)|_Q ds)^2 dt.$$

Using Jensen's inequality,

$$\begin{aligned}\int_0^T e^{-\lambda t}|\Psi_{\zeta\tau}(\sigma_1)(t) - \Psi_{\zeta\tau}(\sigma_2)(t)|_Q^2 dt &\le \int_0^T K^2 te^{-\lambda t}\int_0^t |\sigma_1(s) - \sigma_2(s)|^2 dsdt \\ &= K^2\int_0^T\int_s^T te^{-\lambda t}dt|\sigma_1(s) - \sigma_2(s)|^2 ds \\ &\le K^2\frac{1+T\lambda}{\lambda^2}\int_0^T e^{-\lambda s}|\sigma_1(s) - \sigma_2(s)|^2 ds.\end{aligned}$$

Note that

$$\|f\|_\lambda^2 \equiv \int_0^T e^{-\lambda s}|f|_Q^2 ds$$

is an equivalent norm on $L^2(0,T;Q)$ is $\|\cdot\|_\lambda$ and that the above inequality shows that for λ large enough, $\Psi_{\zeta\tau}$ is a contraction mapping on $L^2(0,T;Q)$ with respect to $\|\cdot\|_\lambda$. Therefore, it has a unique fixed point in $L^2(0,T;Q)$. This proves the lemma. □

Denote this fixed point by

$$S(\zeta,\tau) = \sigma. \tag{3.1}$$

Thus

$$\begin{aligned}S(\zeta,\tau)(t) =& \eta_*(\zeta)(t)\mathcal{A}\tau - \eta_*(\zeta_0)\mathcal{A}\varepsilon(\mathbf{u}_0) \\ &- \sigma_0 + \int_0^t G(S(\zeta(s),\tau(s)),\tau,\eta_*(\zeta(s)))ds\end{aligned} \tag{3.2}$$

The next lemma shows the dependence of $S(\zeta,\tau)$ on τ and ζ.

Lemma 3.4. *The following inequalities hold for C and δ independent of τ_i in $L^2(0,T;Q)$ and ζ.*

$$\|S(\zeta,\tau_1)-S(\zeta,\tau_2)\|_{L^2(0,t;Q)} \le C\|\tau_1-\tau_2\|_{L^2(0,t;Q)}, \tag{3.3}$$

$$|S(\zeta,\mathbf{0})|_Q \le C, \tag{3.4}$$

$$(S(\zeta(t),\tau(t)),\tau(t))_Q \ge \delta|\tau(t)|_Q^2 - C - C\int_0^t |\tau(s)|_Q^2 ds, \tag{3.5}$$

$$\begin{aligned} &(S(\zeta(t),\tau_1(t))-S(\zeta(t),\tau_2(t)),\tau_1(t)-\tau_2(t))_Q \\ &\ge \delta|\tau_1(t)-\tau_2(t)|_Q^2 - C\int_0^t |\tau_1(s)-\tau_2(s)|_Q^2, \end{aligned} \tag{3.6}$$

$$\begin{aligned} |S(\zeta_1(t),\tau(t))-S(\zeta_2(t),\tau(t))|_Q^2 &\le C\big(\int_\Omega |\eta_*(\zeta_1(t))-\eta_*(\zeta_2(t))|^2|\tau(t)|_{\mathbb{S}_d}^2 dx\big) \\ &\quad + C\big(\int_0^t |\eta_*(\zeta_1(s))-\eta_*(\zeta_2(s))|_Y^2 ds\big). \end{aligned} \tag{3.7}$$

Proof. Let

$$\begin{gathered} \sigma_i(s) = \eta_*(\zeta)(s)\mathcal{A}\tau_i - \eta_*(\zeta_0)\mathcal{A}\varepsilon(\mathbf{u}_0), \\ -\sigma_0 + \int_0^s G(\sigma_i,\tau_i,\eta_*(\zeta))dr = S(\zeta(s),\tau_i(s)). \end{gathered}$$

Then since η_* is bounded, an inequality of the following form in which C is independent of τ_i holds.

$$|\sigma_1(s)-\sigma_2(s)|_Q \le C(|\tau_1(s)-\tau_2(s)| + \int_0^s (|\sigma_1(r)-\sigma_2(r)|_Q + |\tau_1(r)-\tau_2(r)|_Q)dr)$$

Now Gronwall's inequality implies that after adjusting C,

$$|\sigma_1(s)-\sigma_2(s)|_Q \le C\big(|\tau_1(s)-\tau_2(s)| + \int_0^s |\tau_1(r)-\tau_2(r)|_Q dr\big).$$

This implies (3.3). Next we consider (3.4). From the definition of $S(\zeta,\tau)$,

$$S(\zeta(t),\mathbf{0}) = -\eta_*(\zeta_0)\mathcal{A}\varepsilon(\mathbf{u}_0) - \sigma_0 + \int_0^t G(S(\zeta,\mathbf{0}),\mathbf{0},\eta_*(\zeta))ds$$

Now from (2.11) and the boundedness of η_*,

$$\begin{aligned} &|S(\zeta(t),\mathbf{0})|_Q \\ &\le |(\eta_*(\zeta_0)\mathcal{A}\varepsilon(\mathbf{u}_0)+\sigma_0)|_Q + \int_0^t |G(S(\zeta,\mathbf{0}),\mathbf{0},\eta_*(\zeta))|_Q ds \\ &\le |(\eta_*(\zeta_0)\mathcal{A}\varepsilon(\mathbf{u}_0)+\sigma_0)|_Q + \int_0^t K(|S(\zeta,\mathbf{0})|_Q+2)ds + \int_0^t |G(\mathbf{0},\mathbf{0},0)|_Q ds \end{aligned}$$

and so by Gronwall's inequality and (2.10),

$$|S(\zeta(t),\mathbf{0})|_Q \le (|(\eta_*(\zeta_0)\mathcal{A}\varepsilon(\mathbf{u}_0)+\sigma_0)|_Q + 2 + \int_0^T |G(\mathbf{0},\mathbf{0},0)|_Q ds)e^{KT} \equiv C$$

Consider (3.5). From the identity solved by $S(\zeta,\tau)$,

$$(S(\zeta(t),\tau(t)),\tau(t))_Q \ge m_{\mathcal{A}}\zeta_*|\tau(t)|_Q^2 - |(\eta_*(\zeta_0)\mathcal{A}\varepsilon(\mathbf{u}_0)+\sigma_0)|_Q|\tau(t)|_Q$$

$$- \int_0^t |G(S(\zeta,\tau),\tau,\eta_*(\zeta))|_Q ds|\tau(t)|_Q \,.$$

So,

$$(S(\zeta(t),\tau(t)),\tau(t))_Q \geq 3\delta|\tau(t)|_Q^2 - C - |\tau(t)|_Q \int_0^t |G(S(\zeta,\tau),\tau,\eta_*(\zeta))|_Q ds\,.$$

Now from (2.11), (2.10), (3.3), and (3.4) and adjusting constants as needed,

$$\begin{aligned}
&(S(\zeta(t),\tau(t)),\tau(t))_Q \\
&\geq 2\delta|\tau(t)|_Q^2 - C - K|\tau(t)|_Q \int_0^t (|S(\zeta(s),\tau(s))|_Q + |\tau(s)|_Q + 2)ds \\
&\quad - K|\tau(t)|_Q \int_0^t |G(\mathbf{0},\mathbf{0},0)|_Q ds \\
&\geq \delta|\tau(t)|_Q^2 - C - C\int_0^t |\tau(s)|_Q^2 ds - C\int_0^t |S(\zeta(s),\tau(s))|_Q^2 ds \\
&\geq \delta|\tau(t)|_Q^2 - C - C\int_0^t |\tau(s)|_Q^2 ds - C\int_0^t |\tau(s)|_Q^2 ds - \int_0^t |S(\zeta(s),0)|_Q^2 ds \\
&\geq \delta|\tau(t)|_Q^2 - C - C\int_0^t |\tau(s)|_Q^2 ds.
\end{aligned}$$

Next consider (3.6). From the assumptions on G and the definition of S, along with (3.3),

$$\begin{aligned}
&(S(\zeta(t),\tau_1(t)) - S(\zeta(t),\tau_2(t)),\tau_1(t)-\tau_2(t))_Q \\
&\geq m_{\mathcal{A}}\zeta_*|\tau_1(t)-\tau_2(t)|_Q^2 - |\tau_1(t)-\tau_2(t)|_Q \\
&\quad \times \int_0^t K\big(|S(\zeta(s),\tau_1(s)) - S(\zeta(s),\tau_2(s))|_Q + |\tau_1(s)-\tau_2(s)|_Q\big) \\
&\geq \delta|\tau_1(t)-\tau_2(t)|_Q^2 - C\int_0^t \big(|\tau_1(s)-\tau_2(s)|_Q^2\big).
\end{aligned}$$

It only remains to prove (3.7).

$$\begin{aligned}
&S(\zeta_1(t),\tau(t)) - S(\zeta_2(t),\tau(t)) \\
&= (\eta_*(\zeta_1(t)) - \eta_*(\zeta_2(t)))\mathcal{A}\tau(t) \\
&\quad + \int_0^t (G(S(\zeta_1,\tau),\tau,\eta_*(\zeta_1)) - G(S(\zeta_2,\tau),\tau,\eta_*(\zeta_2)))ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&|S(\zeta_1(t),\tau(t)) - S(\zeta_2(t),\tau(t))|_Q^2 \\
&\leq C\big(\int_\Omega |\eta_*(\zeta_1(t)) - \eta_*(\zeta_2(t))|^2|\tau(t)|_{\mathbb{S}_d}^2 dx\big) \\
&\quad + C\big(\int_0^t\int_\Omega (|S(\zeta_1,\tau) - S(\zeta_2,\tau)|_{\mathbb{S}_d}^2 + |\eta_*(\zeta_1(s)) - \eta_*(\zeta_2(s))|^2)\,dx\,ds\big)
\end{aligned}$$

Now by Gronwall's inequality and adjusting the constants,

$$|S(\zeta_1(t),\tau(t)) - S(\zeta_2(t),\tau(t))|_Q^2 \leq C\big(\int_\Omega |\eta_*(\zeta_1(t)) - \eta_*(\zeta_2(t))|^2|\tau(t)|_{\mathbb{S}_d}^2 dx\big)$$

$$+ C\big(\int_0^t \int_\Omega |\eta_*(\zeta_1(s)) - \eta_*(\zeta_2(s))|^2 dx\, ds\big)$$

This proves the lemma. □

Before continuing with the abstract formulation, here is a summary of the assumptions on the functions involved in the model and the data.

$$\mathcal{A}(\mathbf{x})\tau\cdot\tau \geq m_{\mathcal{A}}|\tau|^2_{\mathbb{S}_d}, \quad \text{for all } \tau \in \mathbb{S}_d. \tag{3.8}$$

$$\text{The mapping } \mathbf{x} \to \mathcal{A}(\mathbf{x}) \text{ is measurable and bounded.} \tag{3.9}$$

$$\mathcal{A}(\mathbf{x}) \text{ is symmetric.} \tag{3.10}$$

Here, $m_{\mathcal{A}}$ is a positive constant.

The *damage source function* $\phi : \Omega \times \mathbb{S}_d \times \mathbb{R} \to \mathbb{R}$ is Lipschitz and satisfies:

$$\begin{gathered} |\phi(\mathbf{x}, \varepsilon_1, \eta_*(\zeta_1)) - \phi(\mathbf{x}, \varepsilon_2, \eta_*(\zeta_2))| \leq L_\phi(|\ \varepsilon_1 - \varepsilon_2| + |\eta_*(\zeta_1) - \eta_*(\zeta_2)|) \\ \text{for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}_d, \ \ \zeta_1, \zeta_2 \in \mathbb{R}, \quad \text{a.e. } \mathbf{x} \in \Omega. \end{gathered} \tag{3.11}$$

$$\text{The function } \ \mathbf{x} \to \phi(\ \mathbf{x}, \varepsilon,\ \zeta) \text{ is measurable.} \tag{3.12}$$

$$\text{The mapping } \mathbf{x} \to \phi(\mathbf{x}, 0, 0) \text{ belongs to } L^2(\Omega). \tag{3.13}$$

$$\phi(\mathbf{x}, \varepsilon, \eta_*(\zeta)) \text{ is bounded} \tag{3.14}$$

Here, $L_\phi > 0$ is the Lipschitz constant. I will suppress the dependence of these functions on $\mathbf{x}$. Also it will eventually be assumed that for $0 < \zeta_* < 1$,

$$\phi(\varepsilon,\zeta) \leq 0 \quad \text{if} \quad \zeta \geq 1,\ \phi(\varepsilon,\zeta_*) \geq 0 \tag{3.15}$$

The first of these assumptions states the source term for damage is nonpositive whenever $\zeta = 1$. This makes perfect physical sense because it says the damage cannot be made to exceed 1. The omission of the second condition will not be fully explored in this paper. Based on an analogy with the elastic case, it is likely that if one leaves it out, the result will be local rather than global solutions to the problem.

As for the initial data and forcing function, the assumptions made in this paper are listed here. The body force and surface traction are assumed to satisfy

$$\mathbf{f}_B \in C([0,T]; H),\ \mathbf{f}_N \in C\big([0,T]; L^2(\Gamma_N)^d\big), \tag{3.16}$$

and $\mathbf{f} \in \mathcal{V}'$ is defined by

$$\langle \mathbf{f}(t), \mathbf{v}(t)\rangle_{V',V} = (\mathbf{f}_B(t), \mathbf{v}(t))_H + (\mathbf{f}_N(t), \mathbf{v}(t))_{L^2(\Gamma_N)^d}. \tag{3.17}$$

Thus

$$\mathbf{f} \in C([0,T]; V') \tag{3.18}$$

The initial conditions satisfy

$$\zeta_0 \in E, \quad \zeta_0(\mathbf{x}) \in (\zeta_*, 1], \quad 1 > \zeta_* > 0 \tag{3.19}$$

However, $\zeta_0 \in E$ will be used initially. Now, $L : E \to E'$ is defined by

$$\langle L\zeta, \xi\rangle \equiv \int_\Omega \nabla\zeta\cdot\nabla\xi\, dx. \tag{3.20}$$

Letting $\mathbf{w} \in \mathcal{V}$ and $\tau \in \mathcal{E}$, multiply (2.5) by $\mathbf{w}$ and integrate by parts. Using the boundary conditions for $\mathbf{u}$ this yields a variational formulation for (2.5) which is of the form

$$\int_\Omega \sigma_{ij}\varepsilon(\mathbf{w})_{ij} dx = \int_\Omega \mathbf{f}_B \cdot \mathbf{w} dx + \int_{\Gamma_N} \mathbf{f}_N \cdot \mathbf{w} d\alpha \tag{3.21}$$

where $\sigma = S(\zeta, \varepsilon(\mathbf{u}))$.

Now multiply (2.7) by τ and integrate by parts. With the boundary condition for ζ, this yields the variational formulation,

$$\zeta' + \kappa L\zeta = \phi(\varepsilon(\mathbf{u}), \eta_*(\zeta)), \quad \zeta(0) = \zeta_0 \tag{3.22}$$

Now define for $\zeta \in \mathcal{Y}$, $A : \mathcal{Y} \times \mathcal{V} \to \mathcal{V}'$ by

$$\langle A(\zeta, \mathbf{u}), \mathbf{w}\rangle \equiv \int_0^T \int_\Omega S(\zeta, \varepsilon(\mathbf{u}))_{ij}\varepsilon(\mathbf{w})_{ij} dx\, dt \tag{3.23}$$

The abstract version of Problem P is to find $\zeta \in \mathcal{E}, \zeta' \in \mathcal{E}'$ and $\mathbf{u} \in \mathcal{V}$ such that

$$\zeta' + \kappa L\zeta = \phi(\varepsilon(\mathbf{u}), \eta_*(\zeta)),\ \zeta(0) = \zeta_0, \tag{3.24}$$

$$A(\zeta, \mathbf{u}) = \mathbf{f} \text{ in } \mathcal{V}'. \tag{3.25}$$

I will denote this problem as P_V. It turns out that P_V is too difficult to study directly so I will consider a simpler problem and then obtain the solution to P_V as a fixed point. Fix $\zeta_1 \in \mathcal{Y}$. Then $P_{V\zeta_1}$ denotes the following problem. Find $\zeta \in \mathcal{E}, \zeta' \in \mathcal{E}'$ and $\mathbf{u} \in \mathcal{V}$ such that

$$\zeta' + \kappa L\zeta = \phi(\varepsilon(\mathbf{u}), \eta_*(\zeta)),\ \zeta(0) = \zeta_0 \in E, \tag{3.26}$$

$$A(\zeta_1, \mathbf{u}) = \mathbf{f} \text{ in } \mathcal{V}'. \tag{3.27}$$

For λ a positive constant, define new dependent variables, ζ_λ and $\mathbf{u}_\lambda$ by

$$\zeta_\lambda(t)e^{\lambda t} = \zeta(t), \quad \mathbf{u}_\lambda(t)e^{\lambda t} = \mathbf{u}(t).$$

Lemma 3.5. *For $\zeta_1 \in \mathcal{Y}$ there exists a unique solution to Problem $P_{V\zeta_1}$ which satisfies $\zeta, \zeta' \in \mathcal{Y}$.*

Proof. There exists a unique solution, $\mathbf{u}$ to 3.27 if and only if there exists a unique solution to

$$e^{-\lambda(\cdot)}A(\zeta_1, \mathbf{u}_\lambda e^{\lambda(\cdot)}) = e^{-\lambda(\cdot)}\mathbf{f} \text{ in } \mathcal{V}'. \tag{3.28}$$

This is equivalent to

$$\int_0^T \int_\Omega e^{-\lambda t} S(\zeta, e^{\lambda t}\varepsilon(\mathbf{u}_\lambda))_{ij}\varepsilon(\mathbf{w})_{ij} dx\, dt = \int_0^T \langle e^{-\lambda(\cdot)}\mathbf{f}, \mathbf{w}\rangle dt \tag{3.29}$$

for all $\mathbf{w} \in \mathcal{V}$. Now recall the definition of S in terms of a fixed point of an operator found in (3.2). Using this definition, (3.29) occurs if and only if

$$\begin{aligned}
&\int_0^T \int_\Omega Big(\eta_*(\zeta_1)(t)\mathcal{A}\varepsilon(\mathbf{u}_\lambda) - e^{-\lambda t}\eta_*(\zeta_0)\mathcal{A}\varepsilon(\mathbf{u}_0) - e^{-\lambda t}\sigma_0 \\
&+ \int_0^t e^{-\lambda t} G(S(\zeta_1, e^{\lambda s}\varepsilon(\mathbf{u}_\lambda)), e^{\lambda s}\varepsilon(\mathbf{u}_\lambda), \eta_*(\zeta_1))ds\Big)_{ij}\varepsilon(\mathbf{w})_{ij} dx\, ds \\
&= \int_0^T \langle e^{-\lambda(\cdot)}\mathbf{f}, \mathbf{w}\rangle dt
\end{aligned}$$

To simplify the notation denote the left side of the above equation by

$$\int_0^T \langle N_\lambda(t, \mathbf{u}_\lambda), \mathbf{w}\rangle dt$$

Then $N_\lambda : \mathcal{V} \to \mathcal{V}'$ given by

$$\langle N_\lambda \mathbf{u}, \mathbf{w}\rangle \equiv \int_0^T \langle N_\lambda(t, \mathbf{u}), \mathbf{w}\rangle dt$$

is obviously hemicontinuous and bounded. I will now show that if λ is large enough, then N_λ is also monotone and satisfies an inequality of the form

$$\langle N_\lambda \mathbf{u}_1 - N_\lambda \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2\rangle \geq \delta \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathcal{V}}^2$$

where $\delta > 0$ and does not depend on the $\mathbf{u}_i$. Let $\mathbf{u}_1, \mathbf{u}_2$ be two elements of $\mathcal{V}$. Then from Lemma 3.4,

$$\begin{aligned}
&\langle N_\lambda \mathbf{u}_1 - N_\lambda \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2\rangle \\
&\geq \zeta_* m_{\mathcal{A}} ||\mathbf{u}_1 - \mathbf{u}_2\|_{\mathcal{V}}^2 - K\int_0^T e^{-\lambda t}\int_0^t (|S\Big(\zeta_1, e^{\lambda s}\varepsilon(\mathbf{u}_1)\Big) \\
&- S(\zeta_1, e^{\lambda s}\varepsilon(\mathbf{u}_2))| + |e^{\lambda s}(\varepsilon(\mathbf{u}_1) - \varepsilon(\mathbf{u}_2))|)ds|\varepsilon(\mathbf{u}_1(t)) - \varepsilon(\mathbf{u}_2(t))|dt \\
&\geq \zeta_* m_{\mathcal{A}} \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathcal{V}}^2 - C\int_0^T e^{-\lambda t}\int_0^t e^{\lambda s}\big|\varepsilon(\mathbf{u}_1(s)) \\
&\quad - \varepsilon(\mathbf{u}_2(s))|ds|\varepsilon(\mathbf{u}_1(t)) - \varepsilon(\mathbf{u}_2(t))\big|dt
\end{aligned}$$

Using Holder's inequality and Jensen's inequality, the last term is dominated by

$$\begin{aligned}
&C\Big(\int_0^T |\varepsilon(\mathbf{u}_1(t)) - \varepsilon(\mathbf{u}_2(t))|^2 dt\Big)^{1/2} \\
&\times \Big(\int_0^T e^{-2\lambda t}\Big(\int_0^t e^{\lambda s}|\varepsilon(\mathbf{u}_1(s)) - \varepsilon(\mathbf{u}_2(s))|ds\Big)^2 dt\Big)^{1/2} \\
&\leq C\Big(\int_0^T |\varepsilon(\mathbf{u}_1(t)) - \varepsilon(\mathbf{u}_2(t))|^2 dt\Big)^{1/2} \\
&\quad \times \Big(\int_0^T e^{-2\lambda t}(\frac{e^{\lambda t}}{\lambda} - \frac{1}{\lambda})\Big(\int_0^t e^{\lambda s}|\varepsilon(\mathbf{u}_1(s)) - \varepsilon(\mathbf{u}_2(s))|^2 ds\Big)dt\Big)^{1/2} \\
&\leq C\Big(\int_0^T |\varepsilon(\mathbf{u}_1(t)) - \varepsilon(\mathbf{u}_2(t))|^2 dt\Big)^{1/2} \\
&\quad \times \Big(\int_0^T (\frac{e^{-\lambda t}}{\lambda})(\int_0^t e^{\lambda s}|\varepsilon(\mathbf{u}_1(s)) - \varepsilon(\mathbf{u}_2(s))|^2 ds)dt\Big)^{1/2} \\
&= \frac{C}{\sqrt{\lambda}}\Big(\int_0^T |\varepsilon(\mathbf{u}_1(t)) - \varepsilon(\mathbf{u}_2(t))|^2 dt\Big)^{1/2} \\
&\quad \times \Big(\int_0^T (\int_0^t e^{-\lambda(t-s)}|\varepsilon(\mathbf{u}_1(s)) - \varepsilon(\mathbf{u}_2(s))|^2 ds)dt\Big)^{1/2} \\
&= \frac{C}{\sqrt{\lambda}}\Big(\int_0^T |\varepsilon(\mathbf{u}_1(t)) - \varepsilon(\mathbf{u}_2(t))|^2 dt\Big)^{1/2} \\
&\quad \times \Big(\int_0^T |\varepsilon(\mathbf{u}_1(s)) - \varepsilon(\mathbf{u}_2(s))|^2 \int_s^T e^{-\lambda(t-s)} dt\, ds\Big)^{1/2} \\
&\leq \frac{C}{\sqrt{\lambda}}\Big(\int_0^T |\varepsilon(\mathbf{u}_1(t)) - \varepsilon(\mathbf{u}_2(t))|^2 dt\Big)^{1/2}\Big(\int_0^T |\varepsilon(\mathbf{u}_1(s)) - \varepsilon(\mathbf{u}_2(s))|^2 \frac{1}{\lambda} ds\Big)^{1/2} \\
&= \frac{C}{\lambda}\Big(\int_0^T |\varepsilon(\mathbf{u}_1(t)) - \varepsilon(\mathbf{u}_2(t))|^2 dt\Big).
\end{aligned}$$

Then letting $\delta = m_{\mathcal{A}}\zeta_*/2$, it follows that for λ large enough, the desired inequality holds.

It follows that there exists a unique solution, $\mathbf{u}$ to (3.27). Now using this $\mathbf{u}$ in the equation of (3.26), one notes that the right side of the equation is Lipschitz in ζ and so it follows by standard results there exists a unique ζ solving (3.26) which satisfies $\zeta, \zeta' \in \mathcal{Y}$. The way this can be done is to consider this equation with the ζ in the right side replaced with $\widehat{\zeta}$, a fixed element of $\mathcal{Y}$. Then since the operator on the left comes as a subgradient of a convex lower semicontinuous functional, there exists a solution having the desired regularity. [5] Then one shows a high enough power of the map taking $\widehat{\zeta}$ to ζ is a contraction. The unique fixed point is the desired solution. This proves the lemma. □

Now I will continue the consideration of problem P_V which is listed here again.
Problem P_V. Find a displacement field $\mathbf{u} : [0,T] \to V$ and a damage field ζ such that

$$A(\zeta, \mathbf{u}) = \mathbf{f} \quad \text{in } \mathcal{V}', \tag{3.30}$$

where

$$\begin{gathered}\langle A(\zeta, \mathbf{u}), \mathbf{w}\rangle \equiv \int_0^T \int_\Omega S\big(\zeta, \varepsilon(\mathbf{u})\big)_{ij} \varepsilon(\mathbf{w})_{ij} dx\, dt \\ \zeta' + \kappa L\zeta = \phi(\varepsilon(\mathbf{u}),\ \eta_*(\zeta)), \qquad \zeta(0) = \zeta_0. \end{gathered} \tag{3.31}$$

To simplify notation, let

$$|\cdot| = |\cdot|_{L^2(\Omega)}, \quad \|\cdot\| = \|\cdot\|_E.$$

For $\zeta \in \mathcal{Y}$, let $\mathbf{u}_\zeta \in \mathcal{V}$ denote the unique solution of the problem

$$A(\zeta, \mathbf{u}_\zeta) = \mathbf{f} \quad \text{in } \mathcal{V}'. \tag{3.32}$$

The following is a fundamental convergence result.

Lemma 3.6. *If $\zeta_n \to \zeta$ in $\mathcal{Y}$ as $n \to \infty$, then $\mathbf{u}_{\zeta_n} \to \mathbf{u}_\zeta$ in $\mathcal{V}$.*

Proof. Recall (3.2), listed here for convenience,

$$\begin{aligned} S(\zeta, \varepsilon(\mathbf{u}))(t) &= \eta_*(\zeta)(t)\mathcal{A}\varepsilon(\mathbf{u}) - \eta_*(\zeta_0)\mathcal{A}\varepsilon(\mathbf{u}_0) \\ &\quad - \sigma_0 + \int_0^t G(S(\zeta(s), \varepsilon(\mathbf{u})(s)), \varepsilon(\mathbf{u}), \eta_*(\zeta(s)))ds \end{aligned}$$

Then let

$$m(\zeta, \varepsilon(\mathbf{u}))(t) = -\eta_*(\zeta_0)\mathcal{A}\varepsilon(\mathbf{u}_0) - \sigma_0 + \int_0^t G(S(\zeta(s), \varepsilon(\mathbf{u})(s)), \varepsilon(\mathbf{u}), \eta_*(\zeta(s)))ds.$$

For short, let $\mathbf{u}_{\zeta_n} = \mathbf{u}_n$ and $\mathbf{u}_\zeta = \mathbf{u}$. Then

$$\begin{aligned} &\big|m(\zeta, \varepsilon(\mathbf{u}))(t) - m(\zeta_n, \varepsilon(\mathbf{u}_n))(t)\big|_Q^2 \\ &= \Big| \int_0^t \Big(G(S(\zeta_n(s), \varepsilon(\mathbf{u}_n)(s)), \varepsilon(\mathbf{u}_n), \eta_*(\zeta_n(s))) \\ &\quad - G(S(\zeta(s), \varepsilon(\mathbf{u})(s)), \varepsilon(\mathbf{u}), \eta_*(\zeta(s)))\Big) ds\Big|_Q^2 \\ &\le C \int_0^t \Big(|S(\zeta_n(s), \varepsilon(\mathbf{u}_n)(s)) - S(\zeta(s), \varepsilon(\mathbf{u})(s))|_Q^2 \\ &\quad + |\varepsilon(\mathbf{u}_n)(s) - \varepsilon(\mathbf{u})(s)|_Q^2 + |\eta_*(\zeta_n(s)) - \eta_*(\zeta(s))|_Y^2 \Big) ds \end{aligned}$$

$$\begin{aligned}
&\leq C\int_0^t |\eta_*(\zeta_n(s))-\eta_*(\zeta(s))|_Y^2 ds + C\int_0^t |\varepsilon(\mathbf{u}_n)(s)-\varepsilon(\mathbf{u})(s)|_Q^2 ds\\
&\quad + C\int_0^t |S(\zeta_n(s),\varepsilon(\mathbf{u}_n)(s))-S(\zeta_n(s),\varepsilon(\mathbf{u})(s))|_Q^2 ds\\
&\quad + C\int_0^t |S(\zeta_n(s),\varepsilon(\mathbf{u})(s))-S(\zeta(s),\varepsilon(\mathbf{u})(s))|_Q^2 ds.
\end{aligned}$$

Now from Lemma 3.4, (3.3) and adjusting constants, this is dominated by

$$\begin{aligned}
&C\int_0^t |\eta_*(\zeta_n(s))-\eta_*(\zeta(s))|_Y^2 ds + C\int_0^t |\varepsilon(\mathbf{u}_n)(s)-\varepsilon(\mathbf{u})(s)|_Q^2 ds\\
&+ C\int_0^t |S(\zeta_n(s),\varepsilon(\mathbf{u})(s))-S(\zeta(s),\varepsilon(\mathbf{u})(s))|_Q^2 ds.
\end{aligned}$$

Now using (3.7) of Lemma 3.4 this is dominated by

$$\begin{aligned}
&C\int_0^t |\eta_*(\zeta_n(s))-\eta_*(\zeta(s))|_Y^2 ds + C\int_0^t |\varepsilon(\mathbf{u}_n)(s)-\varepsilon(\mathbf{u})(s)|_Q^2 ds\\
&+ C\int_0^t\int_\Omega |\eta_*(\zeta_n(s))-\eta_*(\zeta(s))|^2 |\varepsilon(\mathbf{u}(s))|_{\mathbb{S}_d}^2 dx\,ds\\
&+ C\int_0^t\int_0^s |\eta_*(\zeta_n(r))-\eta_*(\zeta(r))|_Y^2 dr ds
\end{aligned}$$

which, after adjusting the constants, implies

$$\begin{aligned}
&|m(\zeta,\varepsilon(\mathbf{u}))(t)-m(\zeta_n,\varepsilon(\mathbf{u}_n))(t)|_Q^2\\
&\leq C\int_0^t |\eta_*(\zeta_n(s))-\eta_*(\zeta(s))|_Y^2 ds + C\int_0^t |\varepsilon(\mathbf{u}_n)(s)-\varepsilon(\mathbf{u})(s)|_Q^2 ds\\
&\quad + C\int_0^t\int_\Omega |\eta_*(\zeta_n(s))-\eta_*(\zeta(s))|^2 |\varepsilon(\mathbf{u}(s))|_{\mathbb{S}_d}^2 dx\,ds.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&|m(\zeta,\varepsilon(\mathbf{u}))(t)-m(\zeta_n,\varepsilon(\mathbf{u}_n))(t)|_Q\\
&\leq C\Big(\int_0^t |\eta_*(\zeta_n(s))-\eta_*(\zeta(s))|_Y^2 ds\Big)^{1/2} + C\Big(\int_0^t |\varepsilon(\mathbf{u}_n)(s)-\varepsilon(\mathbf{u})(s)|_Q^2 ds\Big)^{1/2}\\
&\quad + C\Big(\int_0^t\int_\Omega |\eta_*(\zeta_n(s))-\eta_*(\zeta(s))|^2 |\varepsilon(\mathbf{u}(s))|_{\mathbb{S}_d}^2 dx\,ds\Big)^{1/2}.
\end{aligned} \tag{3.33}$$

Then from $A(\zeta_n,\mathbf{u}_n)=\mathbf{f}$ and $A(\zeta,\mathbf{u})=\mathbf{f}$ it follows

$$\begin{aligned}
0 &= \int_0^t (\eta_*(\zeta_n)\mathcal{A}\varepsilon(\mathbf{u}_n)-\eta_*(\zeta)\mathcal{A}\varepsilon(\mathbf{u}),\varepsilon(\mathbf{u}_n)-\varepsilon(\mathbf{u}))_Q ds\\
&\quad + \int_0^t (m(\zeta_n,\varepsilon(\mathbf{u}_n))-m(\zeta,\varepsilon(\mathbf{u})),\varepsilon(\mathbf{u}_n)-\varepsilon(\mathbf{u}))_Q
\end{aligned}$$

and so from the above estimate in (3.33),

$$\int_0^t (\eta_*(\zeta_n)\mathcal{A}\varepsilon(\mathbf{u}_n)-\eta_*(\zeta)\mathcal{A}\varepsilon(\mathbf{u}),\varepsilon(\mathbf{u}_n)-\varepsilon(\mathbf{u}))_Q ds$$

$$\begin{aligned}
&\le \int_0^t |m(\zeta_n,\varepsilon(\mathbf{u}_n)) - m(\zeta,\varepsilon(\mathbf{u}))|_Q |\varepsilon(\mathbf{u}_n)-\varepsilon(\mathbf{u})|_Q ds \\
&\le \int_0^t \Big[C\Big(\int_0^s |\eta_*(\zeta_n(r)) - \eta_*(\zeta(r))|_Y^2 dr\Big)^{1/2} + C\Big(\int_0^s |\varepsilon(\mathbf{u}_n)(r) - \varepsilon(\mathbf{u})(r)|_Q^2 dr\Big)^{1/2} \\
&\quad + \Big(\int_0^s \int_\Omega |\eta_*(\zeta_n(r)) - \eta_*(\zeta(r))|^2 |\varepsilon(\mathbf{u}(r))|_{\mathbb{S}_d}^2 dx\, dr\Big)^{1/2}\Big] |\varepsilon(\mathbf{u}_n(s)) - \varepsilon(\mathbf{u}(s))|_Q ds.
\end{aligned}$$

Considering the left side of this inequality and manipulating the right side some more, one obtains an inequality of the following form.

$$\begin{aligned}
&\int_0^t \zeta_* m_{\mathcal{A}} \|\mathbf{u}_n - \mathbf{u}\|_V^2 ds \\
&\le C\int_0^t \int_\Omega |\eta_*(\zeta_n(s)) - \eta_*(\zeta(s))|^2 |\varepsilon(\mathbf{u}(s))|_{\mathbb{S}_d}^2 dx\, ds \\
&\quad + C\int_0^t \int_0^s |\eta_*(\zeta_n(r)) - \eta_*(\zeta(r))|_Y^2 + \|\mathbf{u}_n(r) - \mathbf{u}(r)\|_V^2 dr ds \\
&\quad + C\int_0^t \int_0^s \int_\Omega |\eta_*(\zeta_n(r)) - \eta_*(\zeta(r))|^2 |\varepsilon(\mathbf{u}(r))|_{\mathbb{S}_d}^2 dx dr ds \\
&\quad + \frac{\zeta_* m_{\mathcal{A}}}{2}\int_0^t \|\mathbf{u}_n - \mathbf{u}\|_V^2 ds.
\end{aligned}$$

Therefore, after adjusting constants and using Gronwall's inequality,

$$\begin{aligned}
\int_0^t \|\mathbf{u}_n - \mathbf{u}\|_V^2 ds &\le C\int_0^t \int_0^s |\eta_*(\zeta_n(r)) - \eta_*(\zeta(r))|_Y^2 dr\, ds \\
&\quad + C\int_0^t \int_\Omega |\eta_*(\zeta_n(s)) - \eta_*(\zeta(s))|^2 |\varepsilon(\mathbf{u}(s))|_{\mathbb{S}_d}^2 dx\, ds.
\end{aligned}$$

If the conclusion of the lemma is not true, then there exists $\varepsilon > 0$ and $\zeta_n \to \zeta$ in $\mathcal{Y}$ but $\|\mathbf{u}_n - \mathbf{u}\|_{\mathcal{V}} \ge \varepsilon$. Taking a subsequence, one can assume that the convergence of ζ_n to ζ is pointwise a.e. But now an application of the dominated convergence theorem in (3.34) yields a contradiction because the right side of the above inequality converges to 0. This proves the lemma. □

Now define the operator $\Phi : \mathcal{Y} \to \mathcal{Y}$ as follows. Let $\zeta \in \mathcal{Y}$, then $\Phi(\zeta)$ is the solution of

$$\Phi(\zeta)' + \kappa L\Phi(\zeta) = \phi(\varepsilon(\mathbf{u}_\zeta), \eta_*(\Phi(\zeta))), \quad \Phi(\zeta)(0) = \zeta_0. \tag{3.34}$$

Lemma 3.7. *The operator Φ is continuous.*

Proof. This is clear from the preceding lemma and routine Gronwall inequality arguments exploiting the Lipschitz continuity of ϕ. □

Lemma 3.8. $\Phi(\mathcal{Y})$ *lies in a compact and convex subset of* $\mathcal{Y}$.

Proof. Let $\zeta \in \mathcal{Y}$. Then, it follows from (3.34) and the boundedness assumption on ϕ that

$$\begin{aligned}
&\frac{1}{2}|\Phi(\zeta)(t)|_{L^2(\Omega)}^2 - \frac{1}{2}|\zeta_0|_{L^2(\Omega)}^2 + \kappa\int_0^t \|\Phi(\zeta)(s)\|_{H^1(\Omega)}^2 ds \\
&\le C + \kappa\int_0^t |\Phi(\zeta)(s)|_{L^2(\Omega)}^2 ds + \int_0^t |\Phi(\zeta)(s)|_{L^2(\Omega)}^2 ds,
\end{aligned}$$

and so by Gronwall's inequality there is a positive constant C, independent of ζ, such that

$$|\Phi(\zeta)(t)|_{L^2(\Omega)}^2 + \|\Phi(\zeta)\|_{\mathcal{H}_1}^2 \le C.$$

It follows now from (3.34) that $\|\Phi(\zeta)'\|_{\mathcal{E}'} \le C$, for a positive constant C which is independent of ζ. Therefore, there exists another constant C such that

$$\|\Phi(\zeta)\|_{\mathcal{E}}^2 + \|\Phi(\zeta)'\|_{\mathcal{E}'}^2 \le C,$$

for all $\zeta \in \mathcal{Y}$, and the conclusion follows now from Theorem 3.1. □

The following lemma will be used to prove the uniqueness part in the next theorem.

Lemma 3.9. *Let $y, y' \in \mathcal{Y}$, $y(0) = 0$, and assume that $y \in L^2(0, T; H^2(\Omega))$ and it satisfies $\partial y/\partial n = 0$ on $\partial\Omega$. Then*

$$\int_0^t (y', -\Delta y)_{L^2(\Omega)} ds \ge 0.$$

Proof. Let $L : D(L) \subseteq \mathcal{Y} \to \mathcal{Y}$ be defined by (3.20), where $D(L) \equiv \{z \in \mathcal{Y} : Lz \in \mathcal{Y}\}$. Note that L was defined above as $L : \mathcal{E} \to \mathcal{E}'$. Then L is a maximal monotone operator and $Ly = -\Delta y$ for $y \in D(L)$. Also, since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, it follows that $D(L)$ is dense in $\mathcal{Y}$. Let

$$y_\varepsilon \equiv (I + \varepsilon L)^{-1} y,$$

for a small positive ε. Thus, $y_\varepsilon' = (I + \varepsilon L)^{-1} y' \in D(L)$ and so it is routine to verify that

$$\int_0^t (y_\varepsilon', (-\Delta y_\varepsilon))_{L^2(\Omega)} ds \ge 0.$$

Also, since $D(L)$ is dense in $\mathcal{Y}$, it follows from standard results on maximal monotone operators [5] that, as $\varepsilon \to 0$,

$$\begin{gathered} -\Delta y_\varepsilon = L y_\varepsilon = L(I + \varepsilon L)^{-1} y = (I + \varepsilon L)^{-1} L y \to L y = -\Delta y, \\ (I + \varepsilon L)^{-1} y' = y_\varepsilon' \to y' \quad \text{weakly in } \mathcal{Y}. \end{gathered}$$

Therefore,

$$0 \le \lim_{\varepsilon \to 0} \int_0^t (y_\varepsilon', -\Delta y_\varepsilon)_{L^2(\Omega)} ds = \int_0^t (y', -\Delta y)_{L^2(\Omega)} ds.$$

This proves the lemma. □

Finally, here is the existence and uniqueness theorem for Problem P_V.

Theorem 3.10. *Let $\zeta_0 \in E$ and $\mathbf{f} \in L^\infty(0, T; V')$. Then there exists a unique solution to the system* (3.30) *and* (3.31) *which satisfies*

$$\zeta' \in \mathcal{Y}, \quad L\zeta \in L^2(0, T; L^2(\Omega)), \quad \zeta \in L^2(0, T; H^2(\Omega)), \quad \mathbf{u} \in L^\infty(0, T; V).$$

Proof. The existence of a solution to (3.30) and (3.31) which satisfies $\zeta, \zeta' \in \mathcal{Y}$ and $\mathbf{u} \in \mathcal{V}$ follows from the Schauder fixed-point theorem. Consider the equation for $\mathbf{u}$. From Lemma 3.4 applied to $\mathbf{u}\mathcal{X}_{[t-h,t+h]}$,

$$\int_{t-h}^{t+h} \langle \mathbf{f}(t), \mathbf{u} \rangle ds = \int_{t-h}^{t+h} (S(\zeta, \varepsilon(\mathbf{u})), \varepsilon(\mathbf{u})) ds$$

$$\geq \delta \int_{t-h}^{t+h} |\varepsilon(\mathbf{u}(s))|_Q^2 ds - \int_{t-h}^{t+h} C \int_0^s |\varepsilon(\mathbf{u}(r))|^2 dr\, ds - 2hC$$

and so since $\mathbf{f} \in L^\infty(0,T;V')$, this implies

$$\frac{\delta}{2} \int_{t-h}^{t+h} |\varepsilon(\mathbf{u}(s))|_Q^2 ds \leq 2hC + \int_{t-h}^{t+h} C \int_0^s |\varepsilon(\mathbf{u}(r))|^2 dr ds.$$

Now divide by $2h$ and apply the fundamental theorem of calculus to obtain that for a.e. t,

$$|\varepsilon(\mathbf{u}(t))|_Q^2 \leq C + C \int_0^t |\varepsilon(\mathbf{u}(s))|^2 ds.$$

Then an application of Gronwall's inequality yields

$$|\varepsilon(\mathbf{u}(t))|_Q^2 \leq C \text{ a.e.} \tag{3.35}$$

which shows that $\mathbf{u} \in L^\infty(0,T;V)$.

The regularity of ζ follows from $\zeta' \in \mathcal{Y}$ which implies $\zeta + L\zeta \in \mathcal{Y}$ and then standard regularity results imply that $\zeta \in L^2(0,T;H^2(\Omega))$. See [14].

It remains to verify the uniqueness of the solution. Suppose then that $(\zeta_i, \mathbf{u}_i)$, for $i = 1, 2$, are two solutions with the specified regularity. Then,

$$\frac{1}{2}|\zeta_1(t) - \zeta_2(t)|_Y^2 + \kappa \int_0^t |\nabla(\zeta_1 - \zeta_2)(s)|^2 ds \leq C \int_0^t \big(||\mathbf{u}_1 - \mathbf{u}_2\|_V^2 + |\zeta_1 - \zeta_2|_Y^2\big) ds$$

Hence Gronwall's inequality yields

$$|\zeta_1(t) - \zeta_2(t)|_Y^2 + \int_0^t |\nabla(\zeta_1 - \zeta_2)(s)|^2 ds \leq C \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 ds \tag{3.36}$$

Also, from the equation for $\mathbf{u}$

$$\int_0^t (S(\zeta_1, \varepsilon(\mathbf{u}_1)) - S(\zeta_2, \varepsilon(\mathbf{u}_2)), \varepsilon(\mathbf{u}_1) - \varepsilon(\mathbf{u}_2))_Q ds = 0. \tag{3.37}$$

Now recall Lemma 3.4. Two of the formulas established there were

$$\begin{aligned}
&|S(\zeta_1(t), \tau(t)) - S(\zeta_2(t), \tau(t))|_Q^2 \\
&\leq C\Big(\int_\Omega |\eta_*(\zeta_1(t)) - \eta_*(\zeta_2(t))|^2 |\tau(t)|_{\mathbb{S}_d}^2 dx\Big) + C\Big(\int_0^t |\eta_*(\zeta_1(s)) - \eta_*(\zeta_2(s))|_Y^2 ds\Big)
\end{aligned}$$

and

$$\begin{aligned}
&(S(\zeta(t), \tau_1(t)) - S(\zeta(t), \tau_2(t)), \tau_1(t) - \tau_2(t))_Q \\
&\geq \delta|\tau_1(t) - \tau_2(t)|_Q^2 - C \int_0^t |\tau_1(s) - \tau_2(s)|_Q^2 ds.
\end{aligned}$$

The first of these inequalities implies

$$\begin{aligned}
&\big|(S(\zeta_1(t), \tau(t)) - S(\zeta_2(t), \tau(t)), \varepsilon)_Q\big| \\
&\leq C\Big(\int_\Omega |\eta_*(\zeta_1(t)) - \eta_*(\zeta_2(t))|^2 |\tau(t)|_{\mathbb{S}_d}^2 dx\Big)^{1/2} |\varepsilon|_Q \\
&\quad + C\Big(\int_0^t |\eta_*(\zeta_1(s)) - \eta_*(\zeta_2(s))|_Y^2 ds\Big)^{1/2} |\varepsilon|_Q \\
&\leq C|\varepsilon|_Q\Big(\|\zeta_1(t) - \zeta_2(t)\|_{L^\infty(\Omega)} |\tau(t)|_Q + \Big(\int_0^t |\zeta_1(s) - \zeta_2(s)|_Y^2 ds\Big)^{1/2}\Big)
\end{aligned}$$

Using these estimates in (3.37),

$$\begin{aligned}
0 &= \int_0^t (S(\zeta_1(s), \varepsilon(\mathbf{u}_1(s))) - S(\zeta_1(s), \varepsilon(\mathbf{u}_2(s))), \varepsilon(\mathbf{u}_1(s)) - \varepsilon(\mathbf{u}_2(s)))_Q ds \\
&\quad + \int_0^t (S(\zeta_1, \varepsilon(\mathbf{u}_2)) - S(\zeta_2, \varepsilon(\mathbf{u}_2)), \varepsilon(\mathbf{u}_1(s)) - \varepsilon(\mathbf{u}_2(s)))_Q ds \\
&\geq \int_0^t \Big(\delta |\varepsilon(\mathbf{u}_1)(s) - \varepsilon(\mathbf{u}_2)(s)|_Q^2 - C\int_0^s |\varepsilon(\mathbf{u}_1)(r) - \varepsilon(\mathbf{u}_2)(r)|_Q^2 dr\Big) ds \\
&\quad - C\int_0^t |\varepsilon(\mathbf{u}_1(s)) - \varepsilon(\mathbf{u}_2(s))|_Q \Big(\|\zeta_1(s) - \zeta_2(s)\|_{L^\infty(\Omega)} |\varepsilon(\mathbf{u}_2)(s)|_Q \\
&\quad + \Big(\int_0^s |\zeta_1(r) - \zeta_2(r)|_Y^2 dr\Big)^{1/2}\Big) ds.
\end{aligned}$$

Now letting $r \in (3/2, 2)$, so that $H^r(\Omega)$ imbeds compactly into $L^\infty(\Omega)$, and using (3.35) this implies after adjusting constants, an inequality of the form

$$\begin{aligned}
&\int_0^t \|\mathbf{u}_1 - \mathbf{u}_2\|_V^2 ds \\
&\leq C\int_0^t\int_0^s \|\mathbf{u}_1(r) - \mathbf{u}_2(r)\|_V^2 dr ds + C\int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V \|\zeta_1(s) - \zeta_2(s)\|_{H^r(\Omega)} ds \\
&\quad + C\int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V \Big(\int_0^s |\zeta_1(r) - \zeta_2(r)|_Y^2 dr\Big)^{1/2} ds
\end{aligned}$$

From (3.36),

$$\begin{aligned}
&\int_0^t \|\mathbf{u}_1 - \mathbf{u}_2\|_V^2 ds \\
&\leq C\int_0^t\int_0^s \|\mathbf{u}_1(r) - \mathbf{u}_2(r)\|_V^2 dr ds + C\int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V \|\zeta_1(s) - \zeta_2(s)\|_{H^r(\Omega)} ds \\
&\quad + C\int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V \Big(\int_0^s\int_0^r \|\mathbf{u}_1(p) - \mathbf{u}_2(p)\|^2 dp\, dr\Big)^{1/2} ds \\
&\leq C\int_0^t\int_0^s \|\mathbf{u}_1(r) - \mathbf{u}_2(r)\|_V^2 dr ds + \frac{1}{4}\int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 ds \\
&\quad + C\int_0^t \|\zeta_1(s) - \zeta_2(s)\|_{H^r(\Omega)}^2 ds \\
&\quad + \frac{1}{4}\int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 ds + C\int_0^t\int_0^s\int_0^r \|\mathbf{u}_1(p) - \mathbf{u}_2(p)\|^2 dp\, dr\, ds
\end{aligned}$$

Now an application of Gronwall's inequality yields

$$\int_0^t \|\mathbf{u}_1 - \mathbf{u}_2\|_V^2 ds \leq C\int_0^t \|\zeta_1(s) - \zeta_2(s)\|_{H^r(\Omega)}^2 ds\,. \tag{3.38}$$

It follows from (3.36) that

$$|\zeta_1(t) - \zeta_2(t)|_Y^2 + \int_0^t |\nabla(\zeta_1 - \zeta_2)(s)|^2 ds \leq C\int_0^t \|\zeta_1(s) - \zeta_2(s)\|_{H^r(\Omega)}^2 ds\,. \tag{3.39}$$

The equations for ζ_1 and ζ_2 imply that

$$\begin{aligned}
&\int_0^t ((\zeta_1' - \zeta_2'), -\Delta(\zeta_1 - \zeta_2))_{L^2(\Omega)} ds + \kappa \int_0^t |\Delta(\zeta_1 - \zeta_2)|^2_{L^2(\Omega)} ds \\
&\le \int_0^t |(\phi(\varepsilon(\mathbf{u}_1), \eta_*(\zeta_1)) - \phi(\varepsilon(\mathbf{u}_2), \eta_*(\zeta_2)), \Delta(\zeta_1 - \zeta_2))_{L^2(\Omega)}| ds \\
&\le C \int_0^t (\|\mathbf{u}_1 - \mathbf{u}_2\|_V + |\zeta_1 - \zeta_2|_Y) |\Delta(\zeta_1 - \zeta_2)|_{L^2(\Omega)} ds.
\end{aligned}$$

It follows from Lemma 3.9 that the first term is nonnegative, thus from (3.38),

$$\begin{aligned}
\frac{\kappa}{2} \int_0^t |\Delta(\zeta_1 - \zeta_2)|^2_Y ds &\le C \int_0^t \big(\|\mathbf{u}_1 - \mathbf{u}_2\|^2_V + |\zeta_1 - \zeta_2|^2_Y\big) ds \\
&\le C \Big[\int_0^t \|\zeta_1 - \zeta_2\|^2_{H^r(\Omega)} + |\zeta_1 - \zeta_2|^2_Y \Big] ds
\end{aligned}$$

Then using regularity results, adjusting constants and using the compactness of the imbedding of $H^2(\Omega)$ into $H^r(\Omega)$,

$$\begin{aligned}
\int_0^t \|\zeta_1 - \zeta_2\|^2_{H^2(\Omega)} ds &\le C \Big[\int_0^t \|\zeta_1 - \zeta_2\|^2_{H^r(\Omega)} + |\zeta_1 - \zeta_2|^2_Y \Big] ds \\
&\le C \int_0^t |\zeta_1 - \zeta_2|^2_Y ds + \frac{1}{2} \int_0^t \|\zeta_1 - \zeta_2\|^2_{H^2(\Omega)} ds.
\end{aligned}$$

Therefore, an inequality of the following form holds

$$\int_0^t \|\zeta_1 - \zeta_2\|^2_{H^2(\Omega)} ds \le C \int_0^t |\zeta_1 - \zeta_2|^2_Y ds.$$

From (3.39) and the above inequality,

$$\begin{aligned}
&|\zeta_1(t) - \zeta_2(t)|^2_Y + \int_0^t \|\zeta_1 - \zeta_2\|^2_E ds \\
&\le C \int_0^t \|\zeta_1(s) - \zeta_2(s)\|^2_{H^r(\Omega)} ds + \int_0^t |\zeta_1 - \zeta_2|^2_Y ds \\
&\le C \int_0^t |\zeta_1(s) - \zeta_2(s)|^2_Y ds
\end{aligned}$$

and by Gronwall's inequality, $\zeta_1 = \zeta_2$. From this it follows immediately from Lemma 3.5 that $\mathbf{u}_1 = \mathbf{u}_2$ and this proves uniqueness. □

4. Removing η_*

This section considers how to remove η_* and involves only the assumptions

$$\zeta_0(\mathbf{x}) \in [\zeta_*, 1] \tag{4.1}$$

$$\phi(\varepsilon, \zeta_*) \ge 0, \quad \phi(\varepsilon, 1) \le 0. \tag{4.2}$$

It is based on some fundamental comparison theorems which apply to semilinear parabolic equations which are interesting for their own sake.

Definition 4.1. Let Ω be an open set. Then Ω has the interior ball condition at $\mathbf{x} \in \partial\Omega$ if there exists $\mathbf{z} \in \Omega$ and $r > 0$ such that $B(\mathbf{z}, r) \subseteq \Omega$ and $\mathbf{x} \in \partial B(\mathbf{z}, r)$.

With these definitions, the following is a special case of a famous lemma by Hopf, [8].

Lemma 4.2. *Let Ω be a bounded open set and suppose $\mathbf{x}_0 \in \partial\Omega$ and Ω has the interior ball condition at $\mathbf{x}_0$ with the ball being $B(\mathbf{z}, r)$. Suppose for $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$*

$$\Delta u \geq 0 \quad \text{in } \Omega. \tag{4.3}$$

Then if $u(\mathbf{x}_0) = \max\{u(\mathbf{x}) : \mathbf{x} \in \overline{\Omega}\}$ and $u(\mathbf{x}) < u(\mathbf{x}_0)$ for $\mathbf{x} \in \Omega$, it follows

$$\frac{\partial u}{\partial n}(\mathbf{x}_0) > 0 \tag{4.4}$$

where $\mathbf{n}$ is the exterior unit normal to the ball at the point $\mathbf{x}_0$.

Lemma 4.3. *If Ω has $C^{2,1}$ boundary then every point of $\partial\Omega$ has the interior ball condition. In addition, there exist at each point of $\partial\Omega$ arbitrarily small balls tangent to $\partial\Omega$ such that the exterior unit normal to the ball at that point coincides with the exterior unit normal to Ω.*

From now on, assume the boundary of Ω is $C^{2,1}$. Suppose the following holds for a measurable function f.

$$f : (0, T) \times \Omega \times \mathbb{R} \to \mathbb{R}, \tag{4.5}$$

$$|f(t, \mathbf{x}, \zeta) - f(t, \mathbf{x}, \xi)| \leq K|\zeta - \xi|, \tag{4.6}$$

$$f(t, \mathbf{x}, \zeta) \leq -2\varepsilon < 0 \text{ if } \zeta \geq b, \tag{4.7}$$

$$f(\cdot, \cdot, 0) \in L^2(0, T; L^2(\Omega)), \tag{4.8}$$

Also let $\Omega_T \equiv (0, T) \times \Omega, B_T \equiv (-T, 2T) \times (\Omega + B(\mathbf{0}, 1))$. In order to take a convolution, f is extended to $\widehat{f}$ as follows

$$\hat{f}(t, \mathbf{x}, \zeta) \equiv \begin{cases} f(t, \mathbf{x}, \zeta) & \text{if } (t, \mathbf{x}) \in [0, T] \times \Omega \\ -2\varepsilon & \text{if } (t, \mathbf{x}) \in B_T \setminus \Omega_T \\ 0 & \text{if } (t, \mathbf{x}) \notin B_T \end{cases}$$

If $\zeta \in \mathcal{Y}$,

$$\hat{\zeta}(t, \mathbf{x}) \equiv \begin{cases} \zeta(t, \mathbf{x}) & \text{if } (t, \mathbf{x}) \in [0, T] \times \Omega \\ 0 & \text{otherwise.} \end{cases}$$

Now define

$$f_n(t, \mathbf{x}, \zeta) \equiv \int_{\mathbb{R}^{d+2}} \hat{f}(t - s, \mathbf{x} - \mathbf{y}, \zeta - \xi)\psi_n(s, \mathbf{y}, \xi) ds\, dy\, d\xi$$

where ψ_n is a mollifier having support in $B(\mathbf{0}, \frac{1}{n}) \subseteq \mathbb{R}^{d+2}$. Thus $f_n \in C^\infty(\mathbb{R}^{d+2})$.

Lemma 4.4. *Let f_n be defined above. Then if n is large enough and $(t, \mathbf{x}) \in (0, T) \times \Omega$,*

$$f_n(t, \mathbf{x}, \zeta) \leq -\varepsilon \quad \text{if } \zeta \geq b. \tag{4.9}$$

For $\zeta \in \mathcal{Y}$ and $\delta > 0$ given and $\zeta_1 \in \mathcal{Y}$ arbitrary, it follows that for all n sufficiently large, depending only on δ and ζ,

$$\Big(\int_0^t \int_\Omega |f_n(s, \mathbf{x}, \zeta_1) - f(s, \mathbf{x}, \zeta)|^2 dx\, ds\Big)^{1/2} \leq \delta + \sqrt{2}K\Big(\int_0^t |\zeta_1 - \zeta|_Y^2 ds\Big)^{1/2}. \tag{4.10}$$

Proof. Since the integral is from 0 to t, change $\zeta_1(s)$ to equal $\zeta(s)$ for $s > t$. Then

$$\Big(\int_0^t \int_\Omega |f_n(s, \mathbf{x}, \zeta_1(s, \mathbf{x})) - f(s, \mathbf{x}, \zeta(s, \mathbf{x}))|^2 dx\, ds\Big)^{1/2}$$

$$
\begin{aligned}
&\le \Big(\int_0^T \int_\Omega |f_n(s,\mathbf{x},\zeta_1(s,\mathbf{x})) - f(s,\mathbf{x},\zeta(s,\mathbf{x}))|^2 dx\, ds\Big)^{1/2} \\
&= \Big(\int_0^T \int_\Omega \Big| \int_{\mathbb{R}^{d+2}} \big(\hat f(s-r,\mathbf{x}-\mathbf{y},\zeta_1(s,\mathbf{x})-\xi) \\
&\quad - f(s,\mathbf{x},\zeta(s,\mathbf{x}))\big)\psi_n(r,\mathbf{y},\xi) ds dy d\xi\Big|^2 dx\, ds\Big)^{1/2}
\end{aligned}
$$

By Minkowski's inequality, the above expression is bounded by

$$
\begin{aligned}
&\int_{B(\mathbf{0},\frac1n)} \psi_n(r,\mathbf{y},\xi)\Big(\int_0^T \int_\Omega |\hat f(s-r,\mathbf{x}-\mathbf{y},\zeta_1(s,\mathbf{x})-\xi) \\
&- f(s,\mathbf{x},\zeta(s,\mathbf{x}))|^2 dx\, ds\Big)^{1/2} dr\, dy\, d\xi \\
&\le \int_{B(\mathbf{0},\frac1n)} \psi_n(r,\mathbf{y},\xi)\cdot\Big(\int_{\mathbb{R}^{d+1}} |\hat f(s-r,\mathbf{x}-\mathbf{y},\hat\zeta_1(s,\mathbf{x})-\xi) \\
&\quad - \hat f(s,\mathbf{x},\hat\zeta(s,\mathbf{x}))|^2 dx\, ds\Big)^{1/2} dr\, dy\, d\xi
\end{aligned}
\tag{4.11}
$$

Now consider the inner integral for $(r,\mathbf{y},\xi)\in B(\mathbf{0},\frac1n)$.

$$
\begin{aligned}
&\Big(\int_{\mathbb{R}^{d+1}} \Big|\hat f\big(s-r,\mathbf{x}-\mathbf{y},\hat\zeta_1(s,\mathbf{x})-\xi\big) - \hat f\big(s,\mathbf{x},\hat\zeta(s,\mathbf{x})\big)\Big|^2 dx\, ds\Big)^{1/2} \\
&\le \Big(\int_{[r,T+r]\times\Omega+\mathbf{y}} \Big| f\big(s-r,\mathbf{x}-\mathbf{y},\hat\zeta_1(s,\mathbf{x})-\xi\big) - f\big(s-r,\mathbf{x}-\mathbf{y},\hat\zeta(s,\mathbf{x})\big)\Big|^2 dx\, ds\Big)^{1/2} \\
&\quad + \Big(\int_{\mathbb{R}^{d+1}} \Big|\hat f\big(s-r,\mathbf{x}-\mathbf{y},\hat\zeta(s,\mathbf{x})\big) - \hat f\big(s-r,\mathbf{x}-\mathbf{y},\hat\zeta(s-r,\mathbf{x}-\mathbf{y})\big)\Big|^2 dx\, ds\Big)^{1/2} \\
&\quad + \Big(\int_{\mathbb{R}^{d+1}} \Big|\hat f\big(s-r,\mathbf{x}-\mathbf{y},\hat\zeta(s-r,\mathbf{x}-\mathbf{y})\big) - \hat f\big(s,\mathbf{x},\hat\zeta(s,\mathbf{x})\big)\Big|^2 dx\, ds\Big)^{1/2} \\
&\le \sqrt2 K\Big(\int_{[r,T+r]\times\Omega+\mathbf{y}} |\hat\zeta_1(s,\mathbf{x}) - \hat\zeta(s,\mathbf{x})|^2 dx\, ds\Big)^{1/2} \\
&\quad + \sqrt2 K\Big(\int_{[r,T+r]\times\Omega+\mathbf{y}} |\xi|^2 dx\, ds\Big)^{1/2} \\
&\quad + K\Big(\int_{\mathbb{R}^{d+1}} |\hat\zeta(s,\mathbf{x}) - \hat\zeta(s-r,\mathbf{x}-\mathbf{y})|^2 dx\, ds\Big)^{1/2} \\
&\quad + \Big(\int_{\mathbb{R}^{d+1}} |\hat f(s-r,\mathbf{x}-\mathbf{y},\hat\zeta(s-r,\mathbf{x}-\mathbf{y})) - \hat f(s,\mathbf{x},\hat\zeta(s,\mathbf{x}))|^2 dx\, ds\Big)^{1/2}.
\end{aligned}
$$

Using continuity of translation, in $L^2(\mathbb{R}^{d+1})$ and the above convention that $\zeta_1=\zeta$ on $[t,T]$, this is dominated by

$$
\begin{aligned}
&\delta + \sqrt2 K\Big(\int_{[r,T+r]\times\Omega+\mathbf{y}} |\hat\zeta_1(s,\mathbf{x}) - \hat\zeta(s,\mathbf{x})|^2 dx\, ds\Big)^{1/2} \\
&\le \delta + \sqrt2 K\Big(\int_0^t \int_\Omega |\zeta_1(s,\mathbf{x}) - \zeta(s,\mathbf{x})|^2 dx\, ds\Big)^{1/2}
\end{aligned}
$$

provided n is large enough. Therefore, from (4.11),

$$
\Big(\int_0^t \int_\Omega |f_n(s,\mathbf{x},\zeta_1) - f(s,\mathbf{x},\zeta)|^2 dx\, ds\Big)^{1/2}
$$

$$\leq \delta + \sqrt{2}K\Big(\int_0^t \int_\Omega |\zeta_1(s,\mathbf{x}) - \zeta(s,\mathbf{x})|^2 dx\, ds\Big)^{1/2}$$

This proves (4.10). It remains to verify (4.9) for $(t,\mathbf{x}) \in \Omega_T$. Recall $B_T \equiv (-T, 2T) \times (\Omega + B(\mathbf{0},1))$ and so

$$\begin{aligned}
(t,\mathbf{x}) - B_T &= (t - 2T, t + T) \times (\mathbf{x} - \Omega + B(\mathbf{0},1)) \\
&\supseteq (-T, T) \times (\mathbf{x} - \Omega + B(\mathbf{0},1))
\end{aligned}$$

Then letting $\zeta \geq b$,

$$\begin{aligned}
f_n(t,\mathbf{x},\zeta) &\equiv \int_{\mathbb{R}^{d+2}} \hat{f}(t-r, \mathbf{x}-\mathbf{y}, \zeta - \xi)\psi_n(r,\mathbf{y},\xi) dr\, dy\, d\xi \\
&= \int_{\mathbb{R}^{d+2}} \big(\hat{f}(t-r, \mathbf{x}-\mathbf{y}, \zeta - \xi) - \hat{f}(t-r, \mathbf{x}-\mathbf{y}, \zeta)\big)\psi_n(r,\mathbf{y},\xi) dr\, dy\, d\xi \\
&\quad + \int_{\mathbb{R}^{d+2}} \hat{f}(t-r, \mathbf{x}-\mathbf{y}, \zeta)\psi_n(r,\mathbf{y},\xi) dr\, dy\, d\xi \\
&\leq K \int_{B(\mathbf{0},\frac{1}{n})} |\xi|\psi_n(r,\mathbf{y},\xi) dr\, dy\, d\xi \\
&\quad + \int_{B(\mathbf{0},\frac{1}{n}) \cap ((t,\mathbf{x}) - B_T) \times \mathbb{R}} \hat{f}(t-r, \mathbf{x}-\mathbf{y}, \zeta)\psi_n(r,\mathbf{y},\xi) dr\, dy\, d\xi \\
&\leq \frac{K}{n} + (-2\varepsilon) \int_{B(\mathbf{0},\frac{1}{n}) \cap ((t,\mathbf{x}) - B_T) \times \mathbb{R}} \psi_n(r,\mathbf{y},\xi) dr\, dy\, d\xi
\end{aligned}$$

Letting n be such that $1/n < \min(\frac{T}{2}, 1)$, it follows

$$\begin{gathered}
B(\mathbf{0}, \frac{1}{n}) \cap ((t,\mathbf{x}) - B_T) \times \mathbb{R} \supseteq \\
B(\mathbf{0}, \frac{1}{n}) \cap (-T, T) \times (\mathbf{x} - \Omega + B(\mathbf{0},1)) \times \mathbb{R} \supseteq B(\mathbf{0}, \frac{1}{n})
\end{gathered}$$

and so for such n, the above conditions imply

$$f_n(t,\mathbf{x},\zeta) \leq \frac{K}{n} + (-2\varepsilon).$$

Now choosing n still larger, we obtain this is larger than $-\varepsilon$. It suffices to choose

$$n > \max(\frac{1}{\min(\frac{T}{2}, 1)}, \frac{K}{\varepsilon}).$$

This proves the lemma. $\square$

The next lemma is fairly routine and gives conditions under which weak solutions are actually classical solutions which are smooth enough to apply the reasoning of Lemma 4.2.

Lemma 4.5. *Let $\zeta_0 \in C_c^\infty(\Omega)$ and let ζ be the weak solution to*

$$\zeta' + \kappa L\zeta = f_n(\cdot,\cdot,\zeta), \quad \zeta(0) = \zeta_0.$$

where f_n is described above. Then ζ is C^1 in t and C^2 in $\mathbf{x}$.

Proof. I will give a brief argument for the sake of completeness. Using standard theory of maximal monotone operators, it is routine to obtain that this solution satisfies $\zeta' \in \mathcal{Y}$ and $L\zeta \in \mathcal{Y}$. Then by elliptic regularity theorems applied pointwise, it follows $\zeta \in L^2(0,T;H^2(\Omega))$ and $\frac{\partial \zeta_n}{\partial n} = 0$ on $\partial\Omega$. Thus

$$\zeta' - \kappa\Delta\zeta = f_n(\cdot,\cdot,\zeta), \quad \zeta(0) = \zeta_0 \tag{4.12}$$

Multiplying both sides by ζ and integrating from 0 to t yields after using the Lipschitz continuity of f in the last variable an estimate for $|\zeta(t)|_Y$ which is independent of t. Multiplying both sides by $-\Delta\zeta$ and integrating from 0 to t then gives an estimate for $\|\zeta(t)\|_E + \int_0^T \|\Delta\zeta(t)\|^2 dt$. Multiplying both sides by ζ' and integrating gives an estimate for $\|\zeta'\|_{\mathcal{Y}} + \|\zeta(t)\|_E$. One can also obtain the solution to (4.12) as a limit as $\varepsilon \to 0$ of solutions of systems of the form

$$(1+\varepsilon L)\zeta' + \kappa L\zeta = f_n(\cdot,\cdot,\zeta), \quad (1+\varepsilon L)\zeta(0) = (1+\varepsilon L)\zeta_{0n}.$$

obtaining similar estimates to those just mentioned for this regularized system. These solutions have $L\zeta' \in \mathcal{Y}$ and so $-\Delta\zeta' \in \mathcal{Y}$. Multiplying by $-\Delta\zeta'$ and integrating yields eventually an estimate of the form

$$\int_0^t \|\zeta'\|_E^2 ds + |\Delta\zeta(t)|_Y \le C$$

for C independent of ε,which is preserved when passing to the limit as $\varepsilon \to 0$. Thus using elliptic regularity theorems applied for a.e. t, the solutions to (4.12) satisfy $\zeta \in L^\infty(0,T;H^2(\Omega)), \zeta' \in \mathcal{E}$. Now from (4.15) and the assumption $\partial\Omega$ is $C^{2,1}$ it follows $\zeta \in L^2(0,T;H^3(\Omega))$.

Next differentiating the equation (4.12) with respect to t yields

$$-\kappa L\zeta_{0n} + f_n(0,\cdot,\zeta_{0n}(\cdot)) \in E$$

because $\zeta_0 \in C_c^\infty(\Omega)$ and so $L\zeta_0 \in Y$. This will be the new initial condition for $\xi \equiv \zeta'$ and we note that because of the regularity of ζ_0 this initial condition is in E as just claimed. Thus there exists a unique solution, ξ, to

$$\begin{gathered}\xi' + \kappa L\xi = f_{n,1}(\cdot,\cdot,\zeta) + f_{n,3}(\cdot,\cdot,\zeta)\xi,\\ \xi(0) = \kappa L\zeta_0 + f_n(0,\cdot,\zeta_0(\cdot))\end{gathered}$$

which has the same regularity as ζ. Thus $\zeta' \in L^\infty(0,T;H^2(\Omega)) \cap L^2(0,T;H^3(\Omega))$ and $\zeta'' \in \mathcal{E}$. By Theorem 3.2 this implies $\zeta' \in C(0,T;H^r(\Omega))$ where $r > 3/2$. Since the dimension is no larger than 3, this shows $t \to \zeta(t,\mathbf{x})$ is C^1. Now (4.12) and the fact just shown that $\zeta' \in L^\infty(0,T;H^2(\Omega))$ and elliptic regularity shows that $\zeta \in L^\infty(0,T;H^4(\Omega))$. Therefore, by Theorem 3.2 this shows $\zeta \in C([0,T];H^q(\Omega))$ where $q > 7/2$. It follows the partial derivatives of ζ up to order 2 are in $H^r(\Omega)$ where $r > 3/2$. Since the dimension is no larger than 3, this implies all these partial derivatives are continuous. To summarize, $\zeta(t,\mathbf{x})$ is C^1 in t and C^2 in $\mathbf{x}$. This proves the lemma. □

One could continue in this manner and using the Sobolev embedding theorems obtain the solution to (4.12) is in $C^\infty([0,T]\times\overline{\Omega})$ provided the boundary was C^∞ but it is not necessary. The purpose for this was only to obtain solutions which are sufficiently smooth to carry out the estimate of the following lemma which is based on the Hopf lemma.

Lemma 4.6. *Let f satisfy* (4.5)-(4.8) *and suppose*

$$\zeta' + \kappa L\zeta = f(\cdot,\cdot,\zeta), \quad \zeta(0) = \zeta_0 \in Y \tag{4.13}$$

where $\zeta_0(\mathbf{x}) \leq b$ *and* L *is the operator defined above mapping* E *to* E' *as*

$$\langle L\zeta, \xi\rangle \equiv \int_\Omega \nabla\zeta \cdot \nabla\xi dx \tag{4.14}$$

Then $\zeta(t)(\mathbf{x}) \leq b$ *a.e.* $\mathbf{x}$.

Proof. Let $\zeta_{0n} \in C_c^\infty(\Omega)$ such that $|\zeta_{0n} - \zeta_0|_Y \to 0$ and $\zeta_{0n}(\mathbf{x}) \leq b$. Also let f_n be defined as above so f_n is C^∞. Let ζ_n be the solution to

$$\zeta_n' + \kappa L\zeta_n = f_n(\cdot,\cdot,\zeta_n), \quad \zeta_n(0) = \zeta_{0n}. \tag{4.15}$$

By Lemma 4.5, ζ_n is C^1 in t and C^2 in $\mathbf{x}$.

First we show $\zeta_n \leq b$. Suppose the maximum value of ζ_n on $[0,T]\times\overline{\Omega}$ is achieved at $(t_0, \mathbf{x}_0)$. If $t_0 = 0$ nothing else needs to be done because it is assumed $\zeta_0 \leq b$. Suppose then that $t_0 > 0$. If $\zeta_n(t_0, \mathbf{x}_0) < b$, we are done again since this implies what was to be shown. Suppose then that $\zeta_n(t_0, \mathbf{x}_0) \geq b$. First suppose $\mathbf{x}_0 \in \Omega$. Then by the second derivative test, it must follow $\Delta\zeta_n(t_0, \mathbf{x}_0) \leq 0$. Therefore, from (4.12),

$$\zeta_n'(t_0, \mathbf{x}_0) = \kappa\Delta\zeta_n(t_0, \mathbf{x}_0) + f_n(t_0, \mathbf{x}_0, \zeta_n(t_0, \mathbf{x}_0)) < -\varepsilon < 0$$

which is a contradiction to the maximum occurring at $(t_0, \mathbf{x}_0)$. The only remaining case to consider is $\mathbf{x}_0 \in \partial\Omega$. Here we will use Lemma 4.4. Consider the interior balls tangent to $\partial\Omega$ at $\mathbf{x}_0$. If any of these balls has $\Delta\zeta_n \geq 0$ on that ball, then by Lemma 4.2 $\frac{\partial\zeta}{\partial n}(t_0, \mathbf{x}_0) > 0$ which does not occur because in fact $\frac{\partial\zeta}{\partial n}(t_0, \mathbf{x}_0) = 0$. Therefore, in every such ball, there are points, $\mathbf{x}_1$ where $\Delta\zeta_n(t_0, \mathbf{x}_1) < 0$. It follows by continuity of f_n there is one of these balls small enough that for $\mathbf{x}_1$ in it, $f_n(t_0, \mathbf{x}_1, \zeta_n(t_0, \mathbf{x}_1)) < -\frac{\varepsilon}{2}$. Therefore, passing to a limit, it follows

$$\begin{aligned}\zeta_n'(t_0, \mathbf{x}_0) &= \lim_{\mathbf{x}_1\to\mathbf{x}_0} \zeta_n'(t_0, \mathbf{x}_1)\\ &= \lim_{\mathbf{x}_1\to\mathbf{x}_0} (\kappa\Delta\zeta_n(t_0, \mathbf{x}_1) + f_n(t_0, \mathbf{x}_1, \zeta_n(t_0, \mathbf{x}_1))) \leq -\frac{\varepsilon}{2} < 0\end{aligned}$$

which is another contradiction. This proves $\zeta_n \leq b$. In fact, this shows $\zeta_n < b$ at points $(t, \mathbf{x})$ where $t > 0$.

Now consider the case where f is not regularized. Using (4.13) and Lemma 4.4 and letting $\delta > 0$ be given, the following is valid for all n large enough.

$$\begin{aligned}&\frac{1}{2}|\zeta(t) - \zeta_n(t)|_Y^2 - \frac{1}{2}|\zeta_0 - \zeta_{0n}|_Y^2\\ &\leq \int_0^t ((f_n(s,\mathbf{x},\zeta_n) - f(s,\mathbf{x},\zeta)), \zeta_n - \zeta)\\ &\leq \int_0^t |f_n(s,\mathbf{x},\zeta_n) - f(s,\mathbf{x},\zeta)|_Y |\zeta_n - \zeta| ds\\ &\leq \Big(\int_0^t |f_n(s,\mathbf{x},\zeta_n) - f(s,\mathbf{x},\zeta)|_Y^2 ds\Big)^{1/2} \Big(\int_0^t |\zeta_n - \zeta|^2 ds\Big)^{1/2}\\ &\leq \Big(\delta + K(\int_0^t |\zeta_n - \zeta|_Y^2 ds)^{1/2}\Big)\Big(\int_0^t |\zeta_n - \zeta|^2 ds\Big)^{1/2}\\ &\leq \delta^2 + (K^2+1)\int_0^t |\zeta_n - \zeta|_Y^2 ds\end{aligned}$$

and so by Gronwall's inequality,

$$\max\{|\zeta_n(t) - \zeta(t)|_Y : t \in [0,T]\} \leq (|\zeta_0 - \zeta_{0n}|_Y^2 + 2\delta^2)e^{2(K^2+1)T}.$$

Thus there exists an increasing sequence, $\{n_k\}$ such that

$$\max\{|\zeta_{n_k}(t)-\zeta(t)|_Y : t\in[0,T]\}\leq (|\zeta_0-\zeta_{0n_k}|_Y^2+2\frac{1}{k^2})e^{2(K^2+1)T}.$$

Taking a further subsequence, if necessary,

$$\max\{|\zeta_{n_k}(t)-\zeta_{n_{k+1}}(t)|_Y : t\in[0,T]\}\leq \frac{1}{2^k}$$

and so as in the usual proof of completeness of L^p, it follows that $\zeta_{n_k}(t)(\mathbf{x})\to\zeta(t)(\mathbf{x})$ a.e. $\mathbf{x}$. But $\zeta_{n_k}(t)(\mathbf{x})\leq b$ and so $\zeta(t)(\mathbf{x})\leq b$ a.e. This proves the lemma. □

The next corollary involves weakening the assumption that $f(t,\mathbf{x},\zeta)\leq -2\varepsilon$ when $\zeta\geq b$ to $f(t,\mathbf{x},\zeta)\leq 0$ when $\zeta\geq b$.

Corollary 4.7. *Let f satisfy*

$$\begin{gathered} f:(0,T)\times\Omega\times\mathbb{R}\to\mathbb{R},\\ |f(t,\mathbf{x},\zeta)-f(t,\mathbf{x},\xi)|\leq K|\zeta-\xi|,\\ f(t,\mathbf{x},\zeta)\leq 0\quad \text{if } \zeta\geq b,\\ f(\cdot,\cdot,0)\in L^2(0,T;L^2(\Omega)), \end{gathered}$$

and suppose $\zeta_0\in Y$ is such that $\zeta_0(\mathbf{x})\leq b$. Then the solution, ζ, to

$$\zeta'+\kappa L\zeta=f(\cdot,\cdot,\zeta),\quad \zeta(0)=\zeta_0$$

satisfies $\zeta(t)(\mathbf{x})\leq b$ a.e.

Proof. Let ζ_ε be the solution to

$$\zeta_\varepsilon'+\kappa L\zeta_\varepsilon=f(\cdot,\cdot,\zeta_\varepsilon)-\varepsilon,\quad \zeta_\varepsilon(0)=\zeta_0.$$

Thus from Lemma 4.6, $\zeta_\varepsilon(t)(\mathbf{x})\leq b$ a.e. $\mathbf{x}$. Furthermore, $\zeta_\varepsilon\to\zeta$ uniformly in $C([0,T];Y)$ and so as in the above, a subsequence has the property that for each t $\zeta_\varepsilon(t)(\mathbf{x})\to\zeta(t)(\mathbf{x})$ a.e. $\mathbf{x}$. Thus $\zeta(t)(\mathbf{x})\leq b$ a.e. $\mathbf{x}$. □

A similar set of arguments implies the following corollary.

Corollary 4.8. *Let f satisfy*

$$\begin{gathered} f:(0,T)\times\Omega\times\mathbb{R}\to\mathbb{R},\\ |f(t,\mathbf{x},\zeta)-f(t,\mathbf{x},\xi)|\leq K|\zeta-\xi|,\\ f(t,\mathbf{x},\zeta)\geq 0\quad \text{if } \zeta\leq\zeta_*<1,\\ f(\cdot,\cdot,0)\in L^2(0,T;L^2(\Omega)), \end{gathered}$$

and suppose $\zeta_0\in Y$ is such that $\zeta_0(\mathbf{x})\geq\zeta_*$. Then the solution, ζ to

$$\zeta'+\kappa L\zeta=f(\cdot,\cdot,\zeta),\quad \zeta(0)=\zeta_0$$

satisfies $\zeta(t)(\mathbf{x})\geq\zeta_*$ a.e.

Next the above comparison results are used to eliminate η_* under the assumption

$$\phi(\varepsilon,\zeta)\leq 0\quad \text{if } \zeta\geq 1,\quad \phi(\varepsilon,\zeta)\geq 0\quad \text{if } \zeta\leq\zeta_* \tag{4.16}$$

For example, a formula which has been proposed for ϕ in [18] is

$$\phi(\varepsilon,\zeta)=-(\frac{(1-m_\zeta)\zeta}{1-m_\zeta\zeta}(\lambda_u^+\Phi_{q^*}(\varepsilon^+)+\lambda_u^-\Phi_{q^*}(\varepsilon^-))-\lambda_w)_++H(\zeta) \tag{4.17}$$

in which ε^+ and ε^- are the positive and negative parts of the symmetric matrix, ε and $H(\zeta)$ is a Lipschitz function on $[\delta, 1]$ for each $\delta > 0$ which vanishes when $\zeta = 0$ and λ_w is a positive parameter. The function H represents self mending of the material as might take place in a bone. Now letting $f(t, \mathbf{x}, \zeta) \equiv \phi(\varepsilon(\mathbf{u}(t, \mathbf{x})), \eta_*(\zeta))$, it follows f satisfies the conditions needed for Corollary 4.7. If $H(\zeta_*)$ is large enough, then letting $f(t, \mathbf{x}, \zeta) \equiv \phi(\varepsilon(\mathbf{u}(t, \mathbf{x})), \eta_*(\zeta))$ it follows f satisfies the conditions for Corollary 4.8.

Now recall Theorem 3.10 listed here for convenience.

Theorem 4.9. *Let $\zeta_0 \in E$ and $\mathbf{f} \in L^\infty(0, T; V')$. Then there exists a unique solution to the system* (3.30) *and* (3.31) *which satisfies*

$$\zeta' \in \mathcal{Y}, \quad L\zeta \in L^2(0, T; L^2(\Omega)), \quad \zeta \in L^2(0, T; H^2(\Omega)), \quad \mathbf{u} \in L^\infty(0, T; V).$$

Define $A'(\mathbf{u}, \zeta) \in V'$ by replacing every occurrence of $\eta_*(\zeta)$ with ζ in the definition of $A(\mathbf{u}, \zeta)$.

Theorem 4.10. *Let $\zeta_0 \in E, \zeta_0(\mathbf{x}) \in [\zeta_*, 1]$, and $\mathbf{f} \in L^\infty(0, T; V')$. Also suppose* (4.16)*. Then there exists a unique solution to*

$$A'(\mathbf{u}, \zeta) = \mathbf{f} \quad \text{in } \mathcal{V}', \tag{4.18}$$

$$\zeta' + \kappa L\zeta = \phi(\varepsilon(\mathbf{u}), \zeta), \qquad \zeta(0) = \zeta_0, \tag{4.19}$$

$$\zeta' \in \mathcal{Y}, \quad L\zeta \in L^2(0, T; L^2(\Omega)), \quad \zeta \in L^2(0, T; H^2(\Omega)), \quad \mathbf{u} \in L^\infty(0, T; V). \tag{4.20}$$

This solution satisfies $\zeta(t)(\mathbf{x}) \in [\zeta_*, 1]$ a.e. for each t.

Proof. Letting $f(t, \mathbf{x}, \zeta) \equiv \phi(\varepsilon(\mathbf{u}(t, \mathbf{x})), \eta_*(\zeta))$ and $(\zeta, \mathbf{u})$ be the unique solution to Theorem 3.10, Corollaries 4.12 and 3.36 imply $\zeta(t)(\mathbf{x}) \in [\zeta_*, 1]$ for a.e. $\mathbf{x}$. Therefore, the solution to Theorem 3.10 is the solution to Theorem 4.10. □

This is the main theorem of the paper. Note that it gives global solutions to the problem of damage based on the assumption (4.16). If only the fist half of (4.16) holds it can be shown that a local solution to the problem in which the elastic viscoplastic part has ζ instead of $\eta_*(\zeta)$ is obtained. This requires the proof that ζ is continuous with values in a suitable Sobolev space. It follows from compatibility conditions on the initial data and further estimates. This has been carried out in [18] for the elastic case. However, for this elastic viscoplastic model, it remains to be established.

Acknowledgments. This problem was suggested to me by Professor Viaño when I visited the University of Santiago de Compostela. The support for this visit was greatly appreciated. I would also like to thank the referee who made many helpful suggestions.

References

[1] T. A. Angelov, *On a rolling problem with damage and wear,* Mech. Res. Comm., **26** (1999), 281–286.

[2] K. T. Andrews, K. L. Kuttler, M. Rochdi and M. Shillor, *One-dimensional dynamic thermoviscoelastic contact with damage*, J. Math. Anal. Appl., **272** (2002), 249–275.

[3] E. Bonetti and M. Frémond, *Damage theory: microscopic effects of vanishing macroscopic motions*, Comp. Appl. Math., **22** (3)(2003), 313–333.

[4] E. Bonetti and G. Schimperna, *Local existence for Frémond's model of damage in elastic materials*, Comp.Mech. Thermodyn. 16 (2004) no 4, 319-335.

[5] Brezis H, "Opérateurs maximaux monotones et semigroupes de contractions dans les espaces de Hilbert," Math Studies, 5, North Holland, 1973.

[6] M. Campo, J. R. Fernández, K. L. Kuttler, M. Shillor and J. M. Viaño, *Numerical analysis and simulations of a dynamic contact problem with damage*, preprint.

[7] O. Chau, J. R. Fernández, W. Han and M. Sofonea, *A frictionless contact problem for elastic-viscoplastic materials with normal compliance and damage*, Comput. Meth. Appl. Mech. Engrg., **191** (2002), 5007–5026.

[8] L.C. Evans *Partial Differential Equations,* Berkeley Mathematics Lecture Notes. 1993.

[9] M. Frémond and B. Nedjar, *Damage in concrete: the unilateral phenomenon*, Nuclear Engng. Design, **156** (1995), 323–335.

[10] M. Frémond and B. Nedjar, *Damage, gradient of damage and principle of virtual work,* Intl. J. Solids Structures, **33** (8) (1996), 1083–1103.

[11] M. Frémond, "Non-smooth Thermomechanics," Springer, Berlin, 2002.

[12] M. Frémond, K. L. Kuttler, B. Nedjar and M. Shillor, *One-dimensional models of damage,* Adv. Math. Sci. Appl., **8** (1998), 541–570.

[13] M. Frémond, K. L. Kuttler and M. Shillor, *Existence and uniqueness of solutions for a one-dimensional damage model,* J. Math. Anal. Appl., **229** (1999), 271–294.

[14] P. Grisvard, "Elliptic problems in non-smooth domains," Pittman, 1985.

[15] W. Han, M. Shillor and M. Sofonea, *Variational and numerical analysis of a quasistatic viscoelastic problem with normal compliance, friction and damage,* J. Comput. Appl. Math., **137** (2001), 377–398.

[16] I. R. Ionescu, M. Sofonea, Functional and Numerical Methods in Viscoplasticity, Oxford University Press, Oxford, 1993.

[17] W. Han and M. Sofonea, "Quasistatic contact problems in viscoelasticity and viscoplasticity," American Mathematical Society-Intl. Press, 2002.

[18] K.L.Kuttler and M. Shillor, *Quasistatic Material Damage in an Elastic Body,* in preparation.

[19] K. L. Kuttler, M. Shillor, *Quasistatic Evolution of Damage in an Elastic Body.* To appear in Nonlinear Analysis Real world applications.

[20] J. L. Lions, "Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires," Dunod, Paris, 1969.

[21] M. Shillor, M. Sofonea, and J. J. Telega "Models and Analysis of Quasistatic Contact," Lecture Notes in Physics **655**, Springer, Berlin, 2004.

[22] M. Sofonea, W. Han, and M. Shillor, "Analysis and Approximation of Contact problems with Adhesion or Damage," Pure and Applied Mathematics 276, Chapman and Hall/CRC Press. Boca Raton Florida, 2005.

[23] J. Simon, *Compact sets in the space $L^p(0,T;B)$,* Ann. Mat. Pura. Appl., **146** (1987), 65-96.

Kenneth Kuttler
Department of Mathematics, Brigham Young University, Provo, UT 84602, USA
E-mail address: `klkuttle@math.byu.edu`