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GROUND STATE AND MULTIPLE SOLUTIONS FOR CRITICAL FRACTIONAL SCHRÖDINGER-POISSON EQUATIONS WITH PERTURBATION TERMS

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Abstract. In this article, we study a class of critical fractional Schrödinger-Poisson system with two perturbation terms. By using variational methods and Lusternik-Schnirelman category theory, the existence of ground state and two nontrivial solutions are established.

1. Introduction

In this article, we consider the nonlinear fractional Schrödinger-Poisson system with critical nonlinearity

$$\begin{gathered}(-\Delta)^s u+u+K(x)\phi u=a(x)|u|^{p-2}u+\mu b(x)|u|^{q-2}u+|u|^{2_s^*-2}u, \quad \text{in } \mathbb{R}^3,\\ (-\Delta)^t\phi=K(x)u^2, \quad \text{in } \mathbb{R}^3,\end{gathered} \tag{1.1}$$

where $(-\Delta)^\alpha$ is the fractional Laplacian operator for $\alpha=s,t$. $p,q\in(4,2_s^*)$, $s\in(\frac{3}{4},1)$, $2s+2t>3$, $\mu>0$ is a parameter, $K(x)$, $a(x)$ and $b(x)$ satisfy the following conditions:

- (A1) $K(x)\in C(\mathbb{R}^3)$, $K(x)\geq 0$ and $\lim_{|x|\to\infty}K(x)=K_\infty>0$;
- (A2) there exist $C_0>0$ and $k>0$ such that $K(x)\leq K_\infty-\frac{C_0}{(1+|x|)^k}$ for all $x\in\mathbb{R}^3$;
- (A3) there exist $C_1>0$ and $d>0$ such that $K(x)\leq K_\infty+\frac{C_1}{(1+|x|)^d}$ for all $x\in\mathbb{R}^3$;
- (A4) $a(x)\in C(\mathbb{R}^3)$, $a(x)\geq 0$ and $\lim_{|x|\to\infty}a(x)=a_\infty>0$;
- (A5) there exist $C_2>0$ and $a>0$ such that $a(x)\geq a_\infty-\frac{C_2}{(1+|x|)^a}$ for all $x\in\mathbb{R}^3$;
- (A6) $b(x)\in C(\mathbb{R}^3)$, $b(x)\geq 0$ and $\lim_{|x|\to\infty}b(x)=0$;
- (A7) there exist $C_3>0$ and $b>0$ such that $b(x)\geq\frac{C_3}{(1+|x|)^b}$ for all $x\in\mathbb{R}^3$.

Since the first equation in (1.1) is of fractional Schrödinger equation with a potential ϕ satisfying the fractional Poisson equation, we call system (1.1) a fractional Schrödinger-Poisson system. In recent years, equations or systems with fractional Laplace operators have been studied extensively because they are widely used in fractional quantum mechanics, physics, chemistry, obstacle problems, optimization and finance, we refer to see [12, 16, 20, 21, 23] and so on. It is also well applied in the mathematical theory of conformal geometry and minimal surface, see [9].

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As far as we know, there are a few papers considering (1.1) after it was introduced in [15]. In [15], the author studied the local and global well-posedness of the Cauchy problem

$$\begin{gathered} i\partial_t \Psi + \frac{1}{2}\Delta_x \Psi = A_0 \Psi + \alpha |\Psi|^{\gamma-1}\Psi, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \\ (-\Delta)_x^{\sigma/2} A_0 = |\Psi|^2, \\ \Psi(\cdot, x) = f, \end{gathered}$$

where $\sigma \in (0,1)$, $\alpha = \pm 1$, $1 < \gamma \le 5$. Recently, Zhang, Do ó and Squassina [38] established the existence of radial ground state solution to the following fractional Schrödinger-Poisson system with a general subcritical or critical nonlinearity

$$\begin{gathered} (-\Delta)^s u + \lambda \phi u = f(u), \quad \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = \lambda u^2, \quad \text{in } \mathbb{R}^3. \end{gathered}$$

Teng [33] studied the existence of a nontrivial ground state solution through using the method of Pohozaev-Nehari manifold, the monotonic trick and global compactness Lemma for the system

$$\begin{gathered} (-\Delta)^s u + V(x)u + \phi u = |u|^{p-1}u, \quad \text{in } \mathbb{R}^3, \\ (-\Delta)^s \phi = u^2, \quad \text{in } \mathbb{R}^3. \end{gathered}$$

Using a similar argument, Teng in [32] also studied the existence of ground state solutions for the critical problem with a perturbation term

$$\begin{gathered} (-\Delta)^s u + V(x)u + \phi u = \mu |u|^{q-1}u + |u|^{2_s^*-2}u, \quad \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, \quad \text{in } \mathbb{R}^3. \end{gathered}$$

For other related works, see [22, 28] and their references.

On the other hand, when $s = t = 1$, system (1.1) reduces to classical Schrödinger-Poisson system written by a more general form

$$\begin{gathered} -\Delta u + V(x)u + K(x)\phi u = f(x,u), \quad \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, \quad \text{in } \mathbb{R}^3. \end{gathered} \tag{1.2}$$

This is called the system of Schrödinger-Poisson equations because it consists of a Schrödinger equation coupled with a Poisson term. In the previous decades, there has been a lot of work dealing with the system (1.2) under different assumptions on V, K and f, see [2, 3, 4, 8, 10, 11, 15, 17, 19, 24, 27, 29, 35, 37, 39, 40] and the references therein. For example, in [3], the authors proved the existence of ground state solutions for the subcritical $3 < p < 6$ and the critical case $f = |u|^{p-2}u + u^5$ with $4 < p < 6$. For the case $p \le 2$ or $p \ge 6$, the reader may see [11] and for the case $2 < p < 6$, can see [2, 3, 8, 10, 24]. In the case of V being non-radial, $K \equiv 1$ and $f = |u|^{p-2}u$, the existence of ground state solution for system (1.2) was obtained in [3, 40] for $4 < p < 6$ and $3 < p \le 4$; In [5], the authors proved the existence of ground state and bound states for the case when $V \equiv 1$ and $f = a(x)|u|^{p-2}u$ with $4 < p < 6$. In [37], the author considered a general critical situation with two perturbation term and obtained the existence and multiplicity of solutions via using Lusternik-Schnirelman category due to [1, 6].

To the best of our knowledge, there are few papers on the multiplicity solutions for system (1.1). Inspired by [1, 6, 37], we construct two mappings:

$$F_R : S^2 = \{y \in \mathbb{R}^3 : |y| = 1\} \to \{u \in M : I(u) \le m_\infty - \varepsilon(R)\},$$
$$G : \{u \in M : I(u) < m_\infty\} \to S^2,$$

so that $G \circ F_R$ homotopic to the identity. Using the theory of Lusternik-Schnirelman category, we will establish the existence of two nontrivial solutions for system (1.1).

Our main results are stated as follows.

Theorem 1.1. *Assume that K, a and b satisfy* (A1), (A2), (A4)–(A6) *with $0 < k < \alpha$, where $\alpha = \min\{a, (3+2s)p\}$. Then problem* (1.1) *admits a positive ground state solution.*

Theorem 1.2. *Suppose that* (A1), (A3)–(A7) *hold with $b < \min\{\alpha, \beta\}$, where $\alpha = \min\{a, (3+2s)p\}$ and $\beta = \min\{d, 6+4s\}$. Then problem* (1.1) *admits a positive ground state solution.*

Theorem 1.3. *Assume that $K \in C^1(\mathbb{R}^3)$, $a \in C^1(\mathbb{R}^3)$ and $b(x)$ satisfy* (A1), (A3)–(A7) *with $K(x) \ge K_\infty$, $a(x) \le a_\infty$ and $\operatorname{meas}\{x \in \mathbb{R}^3 : K(x) \ge K_\infty\} > 0$. Then there exists $\mu_0 > 0$ small such that for any $\mu \in (0, \mu_0)$, problem* (1.1) *admits at least two nontrivial solutions.*

The rest of the paper is organized as follows: In Section 2, we give some preliminaries. In Section 3, we prove Theorem 1.1 and Theorem 1.2. Section 4 devotes to proving Theorem 1.3.

2. Preliminary lemmas

In the sequel, we use the following notation:

- $H^s(\mathbb{R}^3)$ denotes the fractional sobolev space with norm
$$\|u\|^2 := \int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u|^2 + u^2) dx$$
and
$$D^{s,2}(\mathbb{R}^3) := \{u \in L^{2_s^*}(\mathbb{R}^3) : (-\Delta)^{\frac{s}{2}} u \in L^2(\mathbb{R}^3)\}$$
denotes the homogeneous fractional sobolev space with the norm
$$\|u\|_{D^{s,2}}^2 := \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx.$$
- C denotes a universal positive constant (possibly different).
- It is well known that $H^\alpha(\mathbb{R}^3)$ is continuously embedded into $L^p(\mathbb{R}^3)$ for $2 \le p \le 2_\alpha^*$ ($2_\alpha^* = \frac{6}{3-2\alpha}$), and for any $\alpha \in (0,1)$, there exists a best constant $S_\alpha > 0$ such that
$$S_\alpha = \inf_{u \in D^{\alpha,2}} \frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} u|^2 dx}{\left(\int_{\mathbb{R}^3} |u(x)|^{2_\alpha^*} dx\right)^{\frac{2}{2_\alpha^*}}}$$
- For simplicity, we assume $K_\infty = 1$ and $a_\infty = 1$. Denote $H = H^s(\mathbb{R}^3)$ and $D^{s,2} = D^{s,2}(\mathbb{R}^3)$.

In this section, we assume (A1), (A4) and (A6) hold. Similar to the argument in [24], we know the function ϕ_u^t has the following properties.

Lemma 2.1. *For any $u \in H$, we have*

(i) $\phi_u^t \geq 0$;
(ii) $\phi_{\hbar u}^t = \hbar^2 \phi_u^t$, $\forall \hbar > 0$;
(iii) $\|\phi_u^t\|_{D^{t,2}} \leq C\|u\|^2_{\frac{12}{3+2t}} \leq C\|u\|^2$, $\int_{\mathbb{R}^3} \phi_u^t u^2 dx \leq C\|u\|^4_{\frac{12}{3+2t}} \leq C\|u\|^4$.

By the Lax-Milgram theorem, there exists a unique $\phi_u^t \in D^{t,2}(\mathbb{R}^3)$ such that $(-\Delta)^t \phi_u^t = K(x)u^2$. Thus, we can rewrite (1.1) as

$$(-\Delta)^s u + u + K(x)\phi_u^t u = a(x)|u|^{p-2}u + \mu b(x)|u|^{q-2}u + |u|^{2_s^*-2}u. \tag{2.1}$$

To find weak solutions to (2.1), we look for critical points of the functional $I(u): H \to \mathbb{R}$ associated with (2.1) which is defined by

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u|^2 + u^2) dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx \\ &\quad - \frac{1}{p}\int_{\mathbb{R}^3} a(x)|u|^p dx - \frac{\mu}{q}\int_{\mathbb{R}^3} b(x)|u|^q dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} dx. \end{aligned}$$

To prove the compactness, we need to consider the following problem at infinity associated with (2.1):

$$(-\Delta)^s u + u + \hat{\phi}_u^t u = |u|^{p-2}u + |u|^{2_s^*-2}u, \quad u > 0, \tag{2.2}$$

where $\hat{\phi}_u^t \in \mathcal{D}^{t,2}(\mathbb{R}^3)$ is the unique solution to problem

$$(-\Delta)^t \phi = u^2.$$

The functional associated with (2.2) is

$$\begin{aligned} I_\infty(u) &= \frac{1}{2}\int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} u|^2 + u^2) dx + \frac{1}{4}\int_{\mathbb{R}^3} \hat{\phi}_u^t u^2 dx \\ &\quad - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} dx. \end{aligned}$$

Let

$$m = \inf_{u \in M} I(u), \quad m_\infty = \inf_{u \in M_\infty} I_\infty(u),$$

where

$$\begin{gathered} M = \{u \in H^s(\mathbb{R}^3)\backslash\{0\} : [I'(u), u] = 0\}, \\ M_\infty = \{u \in H^s(\mathbb{R}^3)\backslash\{0\} : [I'_\infty(u), u] = 0\} \end{gathered}$$

are Nehari manifolds correspond to the functionals I and I_∞, respectively. Similar argument as [22, Proposition 3.4], we can obtain the following Lemma.

Lemma 2.2. *By using* [22, Proposition 3.4]*, problem* (2.2) *has a positive ground state solution* $u_\infty \in C^{1,2s+\sigma-1}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$*, where* $\sigma \in (0,1)$ *and* $2s + \sigma > 1$*.*

From $u \in H \cap C^{1,2s+\sigma-1}(\mathbb{R}^3)$, we see that $\lim_{|x|\to\infty} u_\infty(x) = 0$. Similar as the proof of [32, Proposition 3.8], we conclude that there exists $C > 0$ such that

$$0 < u_\infty(x) \leq \frac{C}{(1+|x|)^{3+2s}}, \quad \forall x \in \mathbb{R}^3.$$

Moreover, in [22], the authors showed that $m_\infty = c_\infty = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\infty(\gamma(t))$, where $\Gamma = \{\gamma \in C([0,1], H) : \gamma(0) = 0, I_\infty(\gamma(1)) < 0\}$ and

$$m_\infty = c_\infty = \inf_{u \in H\backslash\{0\}} \max_{t\geq 0} I_\infty(tu). \tag{2.3}$$

To prove the $(PS)_c$ condition, we need the following function and its estimates (see [32])

$$v_\varepsilon(x) = \psi(x)U_\varepsilon(x), \quad x \in \mathbb{R}^3,$$

where $U_\varepsilon(x) = \varepsilon^{-\frac{3-2s}{2}} u^*(x/\varepsilon)$,

$$u^*(\frac{x}{\varepsilon}) = \frac{\tilde{u}(x/S_s^{\frac{1}{2s}})}{\|\tilde{u}\|_{2_s^*}},$$

$\kappa \in \mathbb{R}\backslash\{0\}$, $\mu > 0$, and $x_0 \in \mathbb{R}^3$ are fixed constants, $\tilde{u}(x) = \kappa(\mu^2 + |x - x_0|^2)^{-\frac{3-2s}{2}}$, and $\psi \in C^\infty(\mathbb{R}^3)$ such that $0 \le \psi \le 1$ in $\mathbb{R}^3$, $\psi(x) \equiv 1$ in B_δ and $\psi(x) \equiv 0$ in $\mathbb{R}^3\backslash B_{2\delta}$. We know that

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_\varepsilon(x)|^2 dx \le S_s^{\frac{3}{2s}} + O(\varepsilon^{3-2s}), \tag{2.4}$$

$$\int_{\mathbb{R}^3} |v_\varepsilon(x)|^{2_s^*} dx = S_s^{\frac{3}{2s}} + O(\varepsilon^3), \tag{2.5}$$

$$\int_{\mathbb{R}^3} |v_\varepsilon(x)|^p dx = \begin{cases} O(\varepsilon^{\frac{(2-p)3+2sp}{2}}), & p > \frac{3}{3-2s}; \\ O(\varepsilon^{\frac{(2-p)3+2sp}{2}} |\log \varepsilon|), & p = \frac{3}{3-2s}; \\ O(\varepsilon^{\frac{3-2s}{2}p}), & p < \frac{3}{3-2s}. \end{cases} \tag{2.6}$$

Lemma 2.3. *Let $\{u_n\} \subset H$ be a bounded sequence such that $I(u_n) \to c \in (0, m_\infty)$ and $I'(u_n) \to 0$. Then $\{u_n\}$ admits a strongly convergent subsequence in H.*

Proof. First we show that $m_\infty < \frac{s}{3} S_s^{\frac{3}{2s}}$. By (2.3), we see that $c_\infty \le \sup_{t \ge 0} I_\infty(tv_\varepsilon)$. Thus we only need to prove $\sup_{t\ge0} I_\infty(tv_\varepsilon) < \frac{s}{3} S_s^{\frac{3}{2s}}$ for $\varepsilon > 0$ small. By Lemma 2.1, we have

$$I_\infty(tv_\varepsilon) \le \frac{1}{2} t^2 \|v_\varepsilon\|^2 + Ct^4 \|v_\varepsilon\|^4 - \frac{1}{2_s^*} t^{2_s^*} \|v_\varepsilon\|_{2_s^*}^{2_s^*}. \tag{2.7}$$

Form (2.4)-(2.6), there exists $\varepsilon_1 > 0$ small enough such that

$$\|v_\varepsilon\|^2 := \int_{\mathbb{R}^3} (|(-\Delta)^{\frac{s}{2}} v_\varepsilon|^2 + v_\varepsilon^2) dx \le S_s^{\frac{3}{2s}} + O(\varepsilon^{3-2s}) + O(\varepsilon^{3-2s}) \le \frac{3}{2} S_s^{\frac{3}{2s}}, \tag{2.8}$$

$$\|v_\varepsilon\|_{2_s^*}^{2_s^*} = S_s^{\frac{3}{2s}} + O(\varepsilon^3) \ge \frac{1}{2} S_s^{\frac{3}{2s}}, \tag{2.9}$$

for $\varepsilon \in (0, \varepsilon_1)$. Thus, form (2.7)-(2.9), we have

$$I_\infty(tv_\varepsilon) \le \frac{3}{4} t^2 S_s^{\frac{3}{2s}} + C\frac{9}{4} t^4 S_s^{\frac{3}{s}} - \frac{1}{2_s^*} t^{2_s^*} \frac{1}{2} S_s^{\frac{3}{2s}}. \tag{2.10}$$

By $2 < 4 < 2_s^*$, there exist a small $t_1 > 0$ and a large $t_2 > 0$ independent of $\varepsilon \in (0, \varepsilon_1)$ such that

$$\sup_{t \in [0,t_1] \cup [t_2, +\infty)} I_\infty(tv_\varepsilon) < \frac{s}{3} S_s^{\frac{3}{2s}}. \tag{2.11}$$

Form Lemma 2.1 and (2.4)-(2.6), we obtain

$$\begin{aligned} \sup_{t \in [t_1, t_2]} I_\infty(tv_\varepsilon) &\le \sup_{t \ge 0} [\frac{1}{2} t^2 \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_\varepsilon|^2 dx - \frac{1}{2_s^*} t^{2_s^*} \int_{\mathbb{R}^3} |v_\varepsilon(x)|^{2_s^*} dx] \\ &\quad + C\|v_\varepsilon\|_2^2 + C\|v_\varepsilon\|_{\frac{12}{3+2t}}^4 - C\|v_\varepsilon\|_p^p \\ &= \frac{s}{3} S_s^{\frac{3}{2s}} + O(\varepsilon^{3-2s}) - C\varepsilon^{\frac{(2-p)3+2sp}{2}}. \end{aligned} \tag{2.12}$$

In view of $p \in (4, 2_s^*)$, $s \in (\frac{3}{4}, 1)$, so we see that $\frac{(2-p)3+2sp}{2} < 3-2s$. By choosing $\varepsilon \in (0, \varepsilon_1)$ small, we obtain

$$\sup_{t\in[t_1,t_2]} I_\infty(tv_\varepsilon) < \frac{s}{3} S_s^{\frac{3}{2s}}. \tag{2.13}$$

By (2.11) and (2.13), we have

$$m_\infty < \frac{s}{3} S_s^{\frac{3}{2s}}. \tag{2.14}$$

Since $\{u_n\}$ is bounded in H, up to a subsequence, we may assume that $u_n \rightharpoonup u$ weakly in H, $u_n \to u$ in $L^r_{\rm loc}(\mathbb{R}^3)$ for $1 \le r < 2_s^*$ and $u_n \to u$ a.e. $\mathbb{R}^3$. Thus by standard argument, we can show that $I'(u) = 0$. Set $v_n = u_n - u$. By the Brezis-Lieb Lemma in [36], we have that

$$\begin{gathered} \|v_n\|^2 = \|u_n\|^2 - \|u\|^2 + o(1), \\ \|v_n\|_{2_s^*}^{2_s^*} = \|u_n\|_{2_s^*}^{2_s^*} - \|u\|_{2_s^*}^{2_s^*} + o(1), \end{gathered} \tag{2.15}$$

and

$$\begin{gathered} \int_{\mathbb{R}^3} a(x)|v_n|^p dx = \int_{\mathbb{R}^3} a(x)|u_n|^p - \int_{\mathbb{R}^3} a(x)|u|^p + o(1), \\ \int_{\mathbb{R}^3} b(x)|v_n|^q dx = \int_{\mathbb{R}^3} b(x)|u_n|^q - \int_{\mathbb{R}^3} b(x)|u|^q + o(1). \end{gathered}$$

From $\lim_{|x|\to\infty} a(x) = 1$, $\lim_{|x|\to\infty} b(x) = 0$, and $v_n \to 0$ in $L^r_{\rm loc}(\mathbb{R}^{3})$ for any $r \in [1, 2_s^*)$, we deduce that

$$\begin{gathered} \int_{\mathbb{R}^3} a(x)|u_n|^p - \int_{\mathbb{R}^3} a(x)|u|^p = \int_{\mathbb{R}^3} |v_n|^p dx + o(1), \\ \int_{\mathbb{R}^3} b(x)|u_n|^q - \int_{\mathbb{R}^3} b(x)|u|^q = o(1). \end{gathered} \tag{2.16}$$

By [33, Lemma 2.5], we can see that

$$\int_{\mathbb{R}^3} K(x)\phi^t_{v_n} v_n^2 dx = \int_{\mathbb{R}^3} K(x)\phi^t_{u_n} u_n^2 dx - \int_{\mathbb{R}^3} K(x)\phi^t_u u^2 dx + o(1).$$

From $\lim_{|x|\to\infty} K(x) = 1$ and Hölder's inequality, it is easy to deduce that

$$\int_{\mathbb{R}^3} \hat{\phi}^t_{v_n} v_n^2 \,\mathrm{d}x = \int_{\mathbb{R}^3} K(x)\phi^t_{u_n} u_n^2 dx - \int_{\mathbb{R}^3} K(x)\phi^t_u u^2 dx + o(1). \tag{2.17}$$

Thus, from (2.15)-(2.17), it follows that

$$c - I(u) = I_\infty(v_n) + o(1). \tag{2.18}$$

By using [13, Proposition 5.1.1], we see that $u \in L^\infty(\mathbb{R}^3)$. Then by [36, Lemmas 8.1 and 8.9], we have that

$$\begin{gathered} |\int_{\mathbb{R}^3} (u_n^{2_s^*-1} - u^{2_s^*-1} - v_n^{2_s^*-1})\varphi dx| = o(1)\|\varphi\|, \quad \forall \varphi \in H, \\ |\int_{\mathbb{R}^3} a(x)(|u_n|^{p-2}u_n - |u|^{p-2}u - |v_n|^{p-2}v_n)\varphi dx| = o(1)\|\varphi\|, \quad \forall \varphi \in H, \\ |\int_{\mathbb{R}^3} b(x)(|u_n|^{q-2}u_n - |u|^{q-2}u - |v_n|^{q-2}v_n)\varphi dx| = o(1)\|\varphi\|, \quad \forall \varphi \in H. \end{gathered} \tag{2.19}$$

Together with $\lim_{|x|\to\infty} a(x)=1$, $\lim_{|x|\to\infty} b(x)=0$, we deduce that

$$\begin{gathered}\Big|\int_{\mathbb{R}^3}[a(x)(|u_n|^{p-2}u_n-|u|^{p-2}u)-|v_n|^{p-2}v_n]\varphi dx\Big|=o(1)\|\varphi\|,\quad \forall\varphi\in H,\\ \Big|\int_{\mathbb{R}^3}b(x)(|u_n|^{q-2}u_n-|u|^{q-2}u)\varphi dx\Big|=o(1)\|\varphi\|,\quad \forall\varphi\in H.\end{gathered}\tag{2.20}$$

Using [33, Lemma 2.5], we have

$$|\int_{\mathbb{R}^3}K(x)(\phi^t_{u_n}u_n-\phi^t_u u-\phi^t_{v_n}v_n)\varphi dx|=o(1)\|\varphi\|,\quad \forall\varphi\in H.$$

From $\lim_{|x|\to\infty} K(x)=1$, and similar to the of proof of (2.17), we obtain

$$\Big|\int_{\mathbb{R}^3}K(x)(\phi^t_{u_n}u_n-\phi^t_u u)\varphi dx-\int_{\mathbb{R}^3}\hat\phi^t_{v_n}v_n\varphi dx\Big|=o(1)\|\varphi\|,\quad \forall\varphi\in H.\tag{2.21}$$

Hence, by (2.19)-(2.21), it holds

$$I'_\infty(v_n)=o(1).\tag{2.22}$$

We claim $v_n\to 0$ in H. Two cases occur: either

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}_3}\int_{B_1(y)}|v_n|^2dx=0,$$

or there exists $\gamma>0$ such that

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}_3}\int_{B_1(y)}|v_n|^2dx\geq\gamma.$$

Thus, either $\|v_n\|_r\to 0$ for any $r\in(2,2^*_s)$ through using vanishing Lemma, or there $y_n\in\mathbb{R}^3$ with $|y_n|\to\infty$ such that $v_n(.+y_n)\rightharpoonup v\neq 0$ weakly in H. If $v_n(.+y_n)\rightharpoonup v\neq 0$ weakly in H, from (2.18) and (2.22), it follows that $c-I(u)=I_\infty(v_n(.+y_n))+o(1)$ and $I'_\infty(v_n(.+y_n))=o(1)$. Thus $I'_\infty(v)=0$ and

$$\begin{aligned}c-I(u)&=I_\infty(v_n(.+y_n))-\frac14[I'_\infty(v_n(.+y_n)),v_n(.+y_n)]\\&=\frac14\|(v_n(.+y_n)\|^2+(\frac14-\frac1p)\int_{\mathbb{R}^3}|(v_n(.+y_n)|^pdx\\&\quad+(\frac14-\frac1{2^*_s})\int_{\mathbb{R}^3}|(v_n(\cdot+y_n)|^{2^*_s}dx+o(1),\end{aligned}$$

form which we obtain

$$\begin{aligned}c&\geq I(u)+\frac14\|v\|^2+(\frac14-\frac1p)\int_{\mathbb{R}^3}|v|^pdx+(\frac14-\frac1{2^*_s})\int_{\mathbb{R}^3}|v|^{2^*_s}dx\\&=I(u)+I_\infty(v)-\frac14[I'_\infty(v),v)]=I(u)+I_\infty(v).\end{aligned}$$

By the definition of m_∞, we have $I_\infty(v)\geq m_\infty$. Since $I'(u)=0$, we have

$$\begin{aligned}I(u)&=I(u)-\frac14[I'(u),u]\\&=\frac14\|u\|^2+(\frac14-\frac1p)\int_{\mathbb{R}^3}a(x)|u|^pdx\\&\quad+(\frac14-\frac1q)\mu\int_{\mathbb{R}^3}b(x)|u|^qdx+(\frac14-\frac1{2^*_s})\int_{\mathbb{R}^3}|u|^{2^*_s}dx\geq 0,\end{aligned}$$

which leads to a contradiction with $c < m_\infty$. Thus $\|v_n\|_{L^r} \to 0$ for any $r \in (2, 2_s^*)$. By (2.18) and (2.22), we have

$$c - I(u) = \frac{1}{2}\|v_n\|^2 - \frac{1}{2_s^*}\|v_n\|_{2_s^*}^{2_s^*} + o(1),$$
$$\|v_n\|^2 - \|v_n\|_{2_s^*}^{2_s^*} = o(1).$$

Up to a subsequence, we may assume that $\|v_n\|^2 \to l$. Thus $\|v_n\|_{2_s^*}^{2_s^*} \to l$. If $l > 0$, by the definition of S_s, we obtain $l \geq (S_s)^{\frac{3}{2s}}$. Hence,

$$c = I(u) + \frac{1}{2}\|v_n\|^2 - \frac{1}{2_s^*}\|v_n\|_{2_s^*}^{2_s^*} = I(u) + (\frac{1}{2} - \frac{1}{2_s^*})l \geq \frac{s}{3}S_s^{\frac{3}{2s}},$$

which contradicts with $c < m_\infty < \frac{s}{3}S_s^{\frac{3}{2s}}$. Thus $l = 0$ and we complete the proof. $\square$

Lemma 2.4. *Suppose that $\alpha, \beta > n$, $f, g \in L^\infty(\mathbb{R}^n)$ and*

$$f(x) \leq \frac{C_1}{(1+|x|)^\alpha}, \quad g(x) \leq \frac{C_2}{(1+|x|)^\beta}.$$

Then there exits $C > 0$ such that

$$|f * g(x)| \leq \frac{C}{(1+|x|)^\gamma},$$

where $\gamma = \min\{\alpha, \beta\}$.

Proof. By direct computations,

$$\begin{aligned}
&|f * g(x)| \\
&= \Big| \int_{\mathbb{R}^n} \frac{C_1}{(1+|x-y|)^\alpha} \frac{C_2}{(1+|y|)^\beta} \,\mathrm{d}y \Big| \\
&= \int_{|x-y| \geq \frac{|x|}{2}} \frac{C_1}{(1+|x-y|)^\alpha} \frac{C_2}{(1+|y|)^\beta} \,\mathrm{d}y + \int_{|x-y| < \frac{|x|}{2}} \frac{C_1}{(1+|x-y|)^\alpha} \frac{C_2}{(1+|y|)^\beta} \,\mathrm{d}y \\
&\leq \frac{C_1}{(1+\frac{|x|}{2})^\alpha} \int_{|x-y| \geq \frac{|x|}{2}} \frac{C_2}{(1+|y|)^\beta} \,\mathrm{d}y + \int_{\frac{|x|}{2} < |y| < \frac{3}{2}|x|} \frac{C_1}{(1+|x-y|)^\alpha} \frac{C_2}{(1+|y|)^\beta} \,\mathrm{d}y \\
&\leq \frac{C_1}{(1+\frac{|x|}{2})^\alpha} \int_{\mathbb{R}^n} \frac{C_2}{(1+|y|)^\beta} \,\mathrm{d}y + \frac{C_2}{(1+\frac{|x|}{2})^\beta} \int_{\frac{|x|}{2} < |y| < \frac{3}{2}|x|} \frac{C_1}{(1+|x-y|)^\alpha} \,\mathrm{d}y \\
&\leq \frac{C_1 C_2 2^\alpha}{(2+|x|)^\alpha} \int_{\mathbb{R}^n} \frac{1}{(1+|y|)^\beta} \,\mathrm{d}y + \frac{C_1 C_2 2^\beta}{(2+|x|)^\beta} \int_{|x-y| < \frac{5}{2}|x|} \frac{1}{(1+|x-y|)^\alpha} \,\mathrm{d}y \\
&\leq \frac{C_1 C_2 2^\alpha}{(2+|x|)^\alpha} \int_{\mathbb{R}^n} \frac{1}{(1+|y|)^\beta} \,\mathrm{d}y + \frac{C_1 C_2 2^\beta}{(2+|x|)^\beta} \int_{\mathbb{R}^n} \frac{1}{(1+|x-y|)^\alpha} \,\mathrm{d}y \\
&\leq C(\frac{1}{(2+|x|)^\alpha} + \frac{1}{(2+|x|)^\beta}) \\
&\leq \frac{C}{(1+|x|)^\gamma},
\end{aligned}$$

where $\gamma = \min\{\alpha, \beta\}$. $\square$

Now we recall the definition of Lusternik-Schnirelman category.

Definition 2.5. (i) For a topological space X, we say a non-empty, closed subset $A \subset X$ is contractible to a point in X if and only if there exist a continuous mapping $\eta : [0,1] \times A \to X$ such that for some $x_0 \in X$,

(a) $\eta(0,x) = x$ for all $x \in A$,
(b) $\eta(1,x) = x_0$ for all $x \in A$.

(ii) We define

$$\begin{aligned} \mathrm{cat}(X) = \min \Big\{ & k \in \mathbb{N} : \text{there exist closed subsets } A_1, \dots, A_k \subset X \text{ such that} \\ & A_i \text{ is contractible to a point in X for all } i \text{ and} \\ & \cup_{i=1}^k A_i = X \Big\}. \end{aligned}$$

We say $\mathrm{cat}(X) = \infty$ if do not exist finitely many closed subsets $A_1, \dots, A_k \subset X$ such that A_i is contractible to a point in X for all i and $\cup_{i=1}^k A_i = X$.

We need the following two important lemmas. See [1, Proposition 2.4 and Lemma 2.5].

Lemma 2.6. *Suppose that $\mathcal{M}$ is a Hilbert manifold and $\Psi \in C^1(\mathcal{M}, \mathbb{R})$. Assume that there exist $c_0 \in \mathbb{R}$ and $k \in \mathbb{N}$ such that $\Psi(u)$ satisfies the Palais-Smale condition for $c \le c_0$ and $\mathrm{cat}(\{u \in \mathcal{M} : \Psi(u) \le c_0\}) \ge k$. Then $\Psi(u)$ has at least k critical points in $\{u \in \mathcal{M} : \Psi(u) \le c_0\}$.*

Lemma 2.7. *Let X be a topological space. Suppose that there exist two continuous mappings $F : S^2 = \{y \in \mathbb{R}^3 : |y| = 1\} \to X$ and $G : X \to S^2$, such that $G \circ F$ is homotopic to identity $\mathrm{id} : S^2 \to S^2$, that is, there is a continuous mapping $\zeta : [0,1] \times S^2 \to S^2$ such that $\zeta(0,x) = (G \circ F)(x)$ for all $x \in S^2$ and $\zeta(1,x) = x$ for all $x \in S^2$. Then $\mathrm{cat}(X) \ge 2$.*

3. Proof of main results

Proof of Theorem 1.1. Let $\{u_n\} \subset M$ be a minimizing sequence for functional I, that is, $\{u_n\} \subset M$ and $I(u_n) \to m$, where

$$M = \{u \in H \backslash \{0\} : G(u) = [I'(u), u] = 0\}.$$

We claim $I'(u_n) \to 0$. By the Lagrange multiplier Theorem, there exists $\lambda_n \in \mathbb{R}$ such that

$$I'(u_n) - \lambda_n G'(u_n) \to 0.$$

Since $u_n \subset M$, we have

$$m + o(1) = I(u_n) - \frac{1}{4}(I'(u_n), u_n) \ge \frac{1}{4}\|u_n\|^2,$$

which implies that $\{u_n\}$ is bounded in H. Hence

$$\lambda_n [G'(u_n), u_n] \to 0. \tag{3.1}$$

By (A4) and (A6), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$a(x)|u|^p + b(x)|u|^q + |u|^{2_s^*} \le \varepsilon |u|^2 + C_\varepsilon |u|^{2_s^*}.$$

Taking $\varepsilon = 1/2$ and recalling the definition of S_s, we have

$$\|u_n\|^2 \le \int_{\mathbb{R}^3} a(x)|u_n|^p dx + \int_{\mathbb{R}^3} \mu b(x)|u_n|^q dx + \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx$$

$$\begin{aligned}
&\le \frac{1}{2}\int_{\mathbb{R}^3}|u_n|^2dx + C_{1/2}\int_{\mathbb{R}^3}|u_n|^{2_s^*}dx\\
&\le \frac{1}{2}\int_{\mathbb{R}^3}|u_n|^2dx + C_{1/2}\frac{\|u_n\|^{2_s^*}}{S_s^{\frac{3}{3-2s}}},
\end{aligned}$$

which implies that

$$\|u_n\|^2 \ge \frac{S_s^{\frac{3}{2s}}}{(2C_{1/2})^{\frac{3-2s}{2s}}}. \tag{3.2}$$

By (3.2), we obtain

$$\begin{aligned}
&[G'(u_n),u_n]\\
&= [G'(u_n),u_n] - 4[I'(u_n),u_n]\\
&= 2\|u_n\|^2 + 4\int_{\mathbb{R}^3}K(x)\phi_{u_n}^t u_n^2dx - p\int_{\mathbb{R}^3}a(x)|u_n|^pdx - q\int_{\mathbb{R}^3}\mu b(x)|u_n|^qdx\\
&\quad - 2_s^*\int_{\mathbb{R}^3}|u_n|^{2_s^*}dx - 4[\|u_n\|^2 + \int_{\mathbb{R}^3}K(x)\phi_{u_n}^t u_n^2dx - \int_{\mathbb{R}^3}a(x)|u_n|^pdx\\
&\quad - \int_{\mathbb{R}^3}\mu b(x)|u_n|^qdx - \int_{\mathbb{R}^3}|u_n|^{2_s^*}dx]\\
&= -2\|u_n\|^2 + (4-p)\int_{\mathbb{R}^3}a(x)|u_n|^pdx\\
&\quad + (4-q)\int_{\mathbb{R}^3}\mu b(x)|u_n|^qdx + (4-2_s^*)\int_{\mathbb{R}^3}|u_n|^{2_s^*}dx\\
&\le -2\|u_n\|^2 \le -2\frac{S_s^{\frac{3}{2s}}}{(2C_{1/2})^{\frac{3-2s}{2s}}}.
\end{aligned}$$

From (3.1), we have $\lambda_n \to 0$. Thus $I'(u_n) \to 0$. This means that $\{u_n\}$ is a $(PS)_m$ sequence for I, that is, $I(u_n) \to m$ and $I'(u_n) \to 0$. By Lemma 2.2, if $m \in (0, m_\infty)$, then $u_n \to u$ in H and thus $I(u) = m$ and $I'(u) = 0$. Hence, m is attained by $u \in H\backslash\{0\}$. For this purpose, it is sufficient to prove $m < m_\infty$.

Similar argument as (2.3), we can obtain the equivalent characterization of the least energy m:

$$m = \inf_{u\in H\backslash\{0\}}\max_{t\ge 0} I(tu). \tag{3.3}$$

Let $R > 0$ and $\gamma \in \mathbb{R}^3$ with $|\gamma| = 1$. By (3.3), clearly, we have

$$m \le \sup_{t\ge 0} I(tu_\infty(x - R\gamma)),$$

where u_∞ is a positive ground state solution for limit problem (2.2). Since

$$\begin{aligned}
&I(tu_\infty(x-R\gamma))\\
&\le \frac{t^2}{2}\|u_\infty(x-R\gamma)\|^2 + Ct^4\|u_\infty(x-R\gamma)\|^4 - \frac{t^{2_s^*}}{2_s^*}\|u_\infty(x-R\gamma)\|_{2_s^*}^{2_s^*}\\
&= \frac{t^2}{2}\|u_\infty\|^2 + Ct^4\|u_\infty\|^4 - \frac{t^{2_s^*}}{2_s^*}\|u_\infty\|_{2_s^*}^{2_s^*},
\end{aligned}$$

there exist a small $t' > 0$ and a large $t'' > 0$ independent of R and γ such that

$$\sup_{t\in[0,t']\cup[t'',+\infty)} I(tu_\infty(x-R\gamma)) < m_\infty. \tag{3.4}$$

On the other hand, by (A6), for any $u \in H$, we have

$$
\begin{aligned}
I(tu) &\le I_\infty(tu) + \frac{t^4}{4}\int_{\mathbb{R}^3}(K(x)-1)\phi_u^t u^2 dx - \frac{1}{p}t^p\int_{\mathbb{R}^3}(a(x)-1)|u|^p dx \\
&\quad + \frac{t^4}{4}\int_{\mathbb{R}^3}(\phi_u^t - \hat{\phi}_u^t)u^2\,\mathrm{d}x \\
&= I_\infty(tu) + \frac{t^4}{4}\int_{\mathbb{R}^3}(K(x)-1)\phi_u^t u^2 dx + \frac{t^4}{4}\int_{\mathbb{R}^3}(K(x)-1)\hat{\phi}_u^t u^2\,\mathrm{d}x \\
&\quad - \frac{1}{p}t^p\int_{\mathbb{R}^3}(a(x)-1)|u|^p dx.
\end{aligned}
$$

Thus, choosing $u = u_\infty(x - R\gamma)$ in the inequality above and using (K_2), (a_1), we obtain

$$
\begin{aligned}
&I(tu_\infty(x-R\gamma)) \\
&\le I_\infty(tu_\infty) - \frac{t^4}{4}C_0\int_{\mathbb{R}^3}\frac{1}{(1+|x+R\gamma|)^k}\int_{\mathbb{R}^3}\frac{K(y+R\gamma)u_\infty^2(y)}{|x-y|^{3-2t}}dy|u_\infty(x)|^2\,dx\,dy \\
&\quad - \frac{t^4}{4}C_0\int_{\mathbb{R}^3}\frac{1}{(1+|x+R\gamma|)^k}\hat{\phi}_{u_\infty}^t(x)|u_\infty(x)|^2 dx \\
&\quad + \frac{1}{p}t^p C_2\int_{\mathbb{R}^3}\frac{1}{(1+|x+R\gamma|)^a}|u_\infty(x)|^p dx \\
&\le I_\infty(tu_\infty) - \frac{t^4}{4}C_0\int_{\mathbb{R}^3}\frac{1}{(1+|x+R\gamma|)^k}\int_{\mathbb{R}^3}\frac{K(y+R\gamma)u_\infty^2(y)}{|x-y|^{3-2t}}|u_\infty(x)|^2\,dx\,dy \\
&\quad - \frac{t^4}{4}C_0\int_{\mathbb{R}^3}\frac{1}{(1+|x+R\gamma|)^k}\hat{\phi}_{u_\infty}^t(x)|u_\infty(x)|^2 dx \\
&\quad + \frac{1}{p}t^p C_2 C_\sigma^p\int_{\mathbb{R}^3}\frac{1}{(1+|x+R\gamma|)^a}\frac{1}{(1+|x|)^{(3+2s)p}}dx \\
&\le I_\infty(tu_\infty) - \frac{t^4}{4}C_0\int_{\mathbb{R}^3}\frac{1}{(1+|x+R\gamma|)^k}\hat{\phi}_{u_\infty}^t(x)|u_\infty(x)|^2 dx \\
&\quad + \frac{1}{p}t^p C_2 C_\sigma^p\int_{\mathbb{R}^3}\frac{1}{(1+|x+R\gamma|)^a}\frac{1}{(1+|x|)^{(3+2s)p}}dx.
\end{aligned}
$$

Set $l(t) = I_\infty(tu_\infty)$, $t \in (0,\infty)$. It is easy to verify that $\sup_{t\ge 0} l(t) = I_\infty(u_\infty) = m_\infty$. Moreover, we have

$$
\begin{aligned}
\int_{\mathbb{R}^3}\frac{1}{(1+|x+R\gamma|)^k}\hat{\phi}_{u_\infty}^t(x)|u_\infty(x)|^2 dx &\ge \int_{|x|\le 1}\frac{1}{(1+|x+R\gamma|)^k}\hat{\phi}_{u_\infty}^t(x)|u_\infty(x)|^2 dx \\
&\ge C\int_{|x|\le 1}\frac{1}{(2+R)^k}\hat{\phi}_{u_\infty}^t(x)|u_\infty(x)|^2 dx \\
&\ge \frac{\widetilde{C}}{(2+R)^k}.
\end{aligned}
$$

By Lemma 2.4, we have

$$
\int_{\mathbb{R}^3}\frac{1}{(1+|x+R\gamma|)^a}\frac{1}{(1+|x|)^{(3+2s)p}}dx \le \frac{C}{(1+R)^\alpha},
$$

where $\alpha = \min\{a, (3+2s)p\}$. Thus

$$\sup_{t' \le t \le t''} I(tu_\infty(x - R\gamma)) \le m_\infty - \frac{(t')^4}{4} C_0 \widetilde{C} \frac{1}{(2+R)^k} + \frac{1}{p}(t'')^p C_2 C_\sigma^p C \frac{1}{(1+R)^\alpha}.$$

By $0 < k < \alpha$, there exists $\hat{R} > 0$ large such that for $R > \hat{R}$,

$$\sup_{t' \le t \le t''} I(tu_\infty(x - R\gamma)) < m_\infty, \quad \forall\, |\gamma| = 1.$$

Thus, combing with (3.4), for $R > \hat{R}$, we have

$$\sup_{t \ge 0} I(tu_\infty(x - R\gamma)) < m_\infty, \quad \forall\ |\gamma| = 1,$$

which yields $m < m_\infty$. The remaining of the proof of Theorem 1.1 is to show that the solution $u \in H$ is positive. □

Proof of Theorem 1.2. The argument is similar to the on in Theorem 1.1, we only need to prove for $R > 0$ large, $\sup_{t \ge 0} I(tu_\infty(x - R\gamma)) < m_\infty$ uniformly in γ. Clearly, there exist $0 < t' < t''$ independent of R and γ such that

$$\sup_{t \in [0,t'] \cup [t'', +\infty)} I(tu_\infty(x - R\gamma)) < m_\infty.$$

On the other hand, by (A3), (A5), (A7), and we have for any $\sigma > 0$, there exist $C_\sigma > 0$ such that

$$\begin{aligned}
&\sup_{t \in [t', t'']} I(tu_\infty(x - R\gamma)) \\
&\le \sup_{t \ge 0} I_\infty(tu_\infty) + \frac{C_1 (t'')^4}{4} \int_{\mathbb{R}^3} \frac{1}{(1 + |x + R\gamma|)^d} \int_{\mathbb{R}^3} \frac{K(y + R\gamma) u_\infty^2(y)}{|x-y|^{3-2t}} dy |u_\infty(x)|^2 dx \\
&\quad + \frac{C_1 (t'')^4}{4} \int_{\mathbb{R}^3} \frac{1}{(1 + |x + R\gamma|)^d} \hat{\phi}^t_{u_\infty}(x) |u_\infty(x)|^2 dx \\
&\quad + \frac{1}{p}(t'')^p C_2 C_\sigma^P \int_{\mathbb{R}^3} \frac{1}{(1 + |x + R\gamma|)^a} \frac{1}{(1+|x|)^{(3+2s)p}} dx \\
&\quad - \mu \frac{C_3 (t')^q}{q} \int_{\mathbb{R}^3} \frac{1}{(1 + |x + R\gamma|)^b} |u_\infty(x)|^q dx.
\end{aligned}$$

By calculations, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} \frac{1}{(1 + |x + R\gamma|)^b} |u_\infty(x)|^q dx \\
&\ge \int_{|x| \le 1} \frac{1}{(1 + |x + R\gamma|)^b} |u_\infty(x)|^q dx \\
&\ge \int_{|x| \le 1} \frac{1}{|2 + R|^b} |u_\infty(x)|^q dx \ge C \frac{1}{(2+R)^b}.
\end{aligned} \tag{3.5}$$

From Lemma 2.4, we obtain

$$\int_{\mathbb{R}^3} \frac{1}{(1 + |x + R\gamma|)^a} \frac{1}{(1+|x|)^{(3+2s)p}} dx \le \frac{\alpha}{(1+R)^\alpha}, \tag{3.6}$$

where $\alpha = \min\{a, (3+2s)p\}$.

By Hölder's inequality, (A3) and (3.6), we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \frac{1}{(1+|x+R\gamma|)^d} \int_{\mathbb{R}^3} \frac{K(y+R\gamma)u_\infty^2(y)}{|x-y|^{3-2t}} dy |u_\infty(x)|^2 dx \\
&\le (1+C_1) \int_{\mathbb{R}^3} \frac{1}{(1+|x+R\gamma|)^d} \int_{\mathbb{R}^3} \frac{u_\infty^2(y)}{|x-y|^{3-2t}} dy |u_\infty(x)|^2 dx \\
&= (1+C_1) \int_{\mathbb{R}^3} \frac{1}{(1+|x+R\gamma|)^d} \hat{\phi}^t_{u_\infty}(x) |u_\infty(x)|^2 dx \\
&\le C \|\hat{\phi}^t_{u_\infty}(x)\|_{2_s^*} \Big[\int_{\mathbb{R}^3} \Big(\frac{1}{(1+|x+R\gamma|)^d} \frac{1}{(1+|x|)^{6+4s}} \Big)^{\frac{6}{3+2s}} dx \Big]^{\frac{3+2s}{6}} \\
&\le C \Big[\frac{1}{(1+R)^m} \Big]^{\frac{3+2s}{6}} \le C \frac{1}{(1+R)^\beta},
\end{aligned} \tag{3.7}
$$

where $m = \min\{\frac{6d}{3+2s}, 12\}$. Similar as the above argument, we obtain

$$
\int_{\mathbb{R}^3} \frac{1}{(1+|x+R\gamma|)^d} \hat{\phi}^t_{u_\infty}(x) |u_\infty(x)|^2 dx \le C \frac{1}{(1+R)^\beta}, \tag{3.8}
$$

where $\beta = \min\{d, 6+4s\}$. By (3.5)-(3.8), we have

$$
\sup_{t\in[t',t'']} I(tu_\infty(x-R\gamma)) \le m_\infty - \overline{C}_1 \frac{1}{(2+R)^b} + \overline{C}_2 \frac{1}{(1+R)^\alpha} + \overline{C}_3 \frac{1}{(1+R)^\beta},
$$

where $\overline{C}_1, \overline{C}_2, \overline{C}_3$ are positive constants. Since $b < \min\{\alpha, \beta\}$, we obtain that there exists $R_0 > 0$ such that for $R > R_0$, $\sup_{t\ge 0} I(tu_\infty(x-R\gamma)) < m_\infty$ uniformly in γ. The proof is complete. $\square$

4. Proof of Theorem 1.3

Let $h(t) = I(tu_\infty(x-R\gamma))$, $t \in (0,\infty)$, $\gamma \in \mathbb{R}^3$ with $|\gamma| = 1$. Form the proof of Theorem 1.2, we know there exists $R_0 > 0$ such that for $R > R_0$, there exists $\varepsilon(R) > 0$ satisfying

$$
\sup_{t\ge 0} h(t) \le m_\infty - \varepsilon(R) < m_\infty \quad \text{uniformly in } \gamma.
$$

For any fixing R and γ, it is easy to check that $h(t)$ attains its maximum at a unique point $t = t_\infty$. Hence, we define a mapping $F_R : S^2 = \{\gamma \in \mathbb{R}^3 : |\gamma| = 1\} \to M$ by

$$
F_R(\gamma) = t_\infty u_\infty(x - R\gamma).
$$

Immediately we have the following Lemma.

Lemma 4.1. *There exists $R_0 > 0$ such that for $R > R_0$, there exists $\varepsilon(R) > 0$ satisfying $F_R(S^2) \subset \{u \in M : I(u) \le m_\infty - \varepsilon(R)\}$ uniformly in $\gamma \in S^2$.*

For $u \in H$, we define a map $\Phi : H \to H$ by

$$
\Phi(u)(x) := \frac{1}{|B_1(x)|} \int_{B_1(x)} |u(y)| dy, \quad \forall x \in \mathbb{R}^3,
$$

where $|B_1(x)|$ is the Lebesgue measure of $B_1(x)$. Let

$$
\hat{u}(x) = [\Phi(u)(x) - \frac{1}{2} \max_{x\in\mathbb{R}^3} \Phi(u)(x)]^+,
$$

and $\beta : H\backslash\{0\} \to \mathbb{R}^3$ given by

$$\beta(u) = \frac{1}{\|\hat{u}\|_1} \int_{\mathbb{R}^3} x\hat{u}(x)dx.$$

Obviously, $\beta(u)$ is well defined for all $u \in H\backslash\{0\}$ and $\beta(u)$ has a compact support in $\mathbb{R}^3$. Moreover, $\beta(u)$ is continuous in $H\backslash\{0\}$ and satisfies the following properties.

Lemma 4.2. *(i) For any $t \neq 0$ and $u \in H\backslash\{0\}$, $\beta(tu) = \beta(u)$.*
(ii) For any $z \in \mathbb{R}^3$ and $u \in H\backslash\{0\}$, $\beta(u(x-z)) = \beta(u) + z$.

Define a functional $J : H \to \mathbb{R}$ given as follows

$$J(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} a(x)|u|^p dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} dx, \quad u \in H.$$

Lemma 4.3. *$m_0 := \inf_{M_0} J(u) = m_\infty$ is not attained, where*

$$M_0 = \{u \in H\backslash\{0\} : [J'(u), u] = 0\}.$$

Proof. First, we show that for any $u \in M_0$, there exists a unique $0 < \tau \leq 1$ such that $\tau u \in M_\infty$. Indeed, by $u \in M_0$ and $\tau u \in M_\infty$, we have

$$\|u\|^2 + \int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx = \int_{\mathbb{R}^3} a(x)|u|^p dx + \int_{\mathbb{R}^3} |u|^{2_s^*} dx, \tag{4.1}$$

and then

$$\begin{aligned} \tau^p \int_{\mathbb{R}^3} a(x)|u|^p dx + \tau^{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} dx &\leq \tau^p \int_{\mathbb{R}^3} |u|^p dx + \tau^{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} dx \\ &= \tau^2\|u\|^2 + \tau^4 \int_{\mathbb{R}^3} \hat{\phi}_u^t u^2 dx. \end{aligned} \tag{4.2}$$

From (A3) and $K(x) \geq 1$ for any $x \in \mathbb{R}^3$, it follows that

$$\int_{\mathbb{R}^3} \hat{\phi}_u^t u^2 dx \leq \int_{\mathbb{R}^3} K(x)\hat{\phi}_u^t u^2 dx \leq \int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx. \tag{4.3}$$

If $\tau > 1$, by (4.1), (4.2) and (4.3), we deduce that

$$\begin{aligned} \tau^4(\|u\|^2 + \int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx) &\geq \tau^4(\|u\|^2 + \int_{\mathbb{R}^3} \hat{\phi}_u^t u^2 dx) \\ &\geq \tau^p(\int_{\mathbb{R}^3} a(x)|u|^p dx + \int_{\mathbb{R}^3} |u|^{2_s^*} dx) \\ &= \tau^p(\|u\|^2 + \int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx), \end{aligned}$$

which yields $\tau \leq 1$, this achieves a contradiction. Hence $\tau \leq 1$ and the claim is true.

For $u \in M_0$, using (4.3), we have

$$\begin{aligned} J(u) &= J(u) - \frac{1}{p}[J'(u), u] \\ &= (\frac{1}{2} - \frac{1}{p})\|u\|^2 + (\frac{1}{4} - \frac{1}{p})\int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx + (\frac{1}{p} - \frac{1}{2_s^*})\int_{\mathbb{R}^3} |u|^{2_s^*} dx \\ &\geq (\frac{1}{2} - \frac{1}{p})\|\tau u\|^2 + (\frac{1}{4} - \frac{1}{p})\tau^4\int_{\mathbb{R}^3} \hat{\phi}_u^t u^2 dx + (\frac{1}{p} - \frac{1}{2_s^*})\int_{\mathbb{R}^3} |\tau u|^{2_s^*} dx \\ &= I_\infty(\tau u) - \frac{1}{p}[I'_\infty(\tau u), \tau u] \end{aligned}$$

$$= I_\infty(\tau u) \geq m_\infty,$$

which implies that $m_0 \geq m_\infty$.

Next we prove $m_0 \leq m_\infty$. Let $w_n = u_\infty(. - z_n)$, where $z_n \in \mathbb{R}^3$ with $|z_n| \to \infty$. We claim that for $w_n \in M_\infty$, there exists $\tau_n \geq 1$ such that $\tau_n w_n \in M_0$. In fact, from $w_n \in M_\infty$ and $\tau_n w_n \in M_0$, it holds

$$\|w_n\|^2 + \int_{\mathbb{R}^3} \hat{\phi}^t_{w_n} w_n^2 dx = \int_{\mathbb{R}^3} |w_n|^p dx + \int_{\mathbb{R}^3} |w_n|^{2_s^*} dx,$$

and then

$$\begin{aligned}
&\tau_n^p \int_{\mathbb{R}^3} |w_n|^p dx + \tau_n^{2_s^*} \int_{\mathbb{R}^3} |w_n|^{2_s^*} dx \\
&\geq \tau_n^p \int_{\mathbb{R}^3} a(x)|w_n|^p dx + \tau_n^{2_s^*} \int_{\mathbb{R}^3} |w_n|^{2_s^*} dx \\
&= \tau_n^2 \|w_n\|^2 + \tau_n^4 \int_{\mathbb{R}^3} K(x)\phi^t_{w_n} w_n^2 dx.
\end{aligned}$$

If $\tau_n < 1$, then

$$\begin{aligned}
\tau_n^p\Big(\int_{\mathbb{R}^3} |w_n|^p dx + \int_{\mathbb{R}^3} |w_n|^{2_s^*} dx\Big) &\geq \tau_n^4\Big(\|w_n\|^2 + \int_{\mathbb{R}^3} K(x)\phi^t_{w_n} w_n^2 dx\Big) \\
&\geq \tau_n^4\Big(\|w_n\|^2 + \int_{\mathbb{R}^3} \hat{\phi}^t_{w_n} w_n^2 dx\Big) \\
&= \tau_n^4\Big(\int_{\mathbb{R}^3} |w_n|^p dx + \int_{\mathbb{R}^3} |w_n|^{2_s^*} dx\Big),
\end{aligned}$$

which leads to a contradiction with $\tau_n < 1$. Hence $\tau_n \geq 1$ and the claim holds.

By the definition of m_0 and $\tau_n u_n \in M_0$, we have

$$\begin{aligned}
m_0 \leq J(\tau_n w_n) &= \frac{1}{2}\tau_n^2 \|u_\infty\|^2 + \frac{1}{4}\tau_n^4 \int_{\mathbb{R}^3} K(x)\phi^t_{w_n} w_n^2 dx \\
&\quad - \frac{1}{p}\tau_n^p \int_{\mathbb{R}^3} a(x)|w_n|^p dx - \frac{1}{2_s^*}\tau_n^{2_s^*} \int_{\mathbb{R}^3} |u_\infty(x)|^{2_s^*} dx.
\end{aligned}$$

By Lebesgue dominated convergence Theorem, we deduce that

$$\begin{gathered}
\begin{aligned}
&\lim_{n\to\infty} \int_{\mathbb{R}^3} K(x)\phi^t_{w_n} w_n^2 dx \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^3} K(x+z_n) \int_{\mathbb{R}^3} \frac{K(y+z_n)u_\infty^2(y)}{|x-y|^{3-2t}} dy |u_\infty(x)|^2 \, dx \, dy \\
&= \int_{\mathbb{R}^3} \hat{\phi}^t_{u_\infty}(x)|u_\infty(x)|^2 dx,
\end{aligned} \\
\begin{aligned}
\lim_{n\to\infty} \int_{\mathbb{R}^3} a(x)|w_n|^p dx &= \lim_{n\to\infty} \int_{\mathbb{R}^3} a(x+z_n)|u_\infty(x)|^p dx \\
&= \int_{\mathbb{R}^3} |u_\infty(x)|^p dx.
\end{aligned}
\end{gathered} \tag{4.4}$$

If $\tau_n \to 1$, we obtain $m_0 \leq \lim_{n\to\infty} J(t_n w_n) = I_\infty(u_\infty) = m_\infty$, form which we see that $m_0 = m_\infty$. Thus we only need to prove $\tau_n \to 1$. By $\tau_n w_n \in M_0$, with $\tau_n \geq 1$, we have

$$\tau_n^4\Big(\|w_n\|^2 + \int_{\mathbb{R}^3} K(x)\phi^t_{w_n} w_n^2 dx\Big)$$

$$
\begin{aligned}
&\geq \tau_n^2\|w_n\|^2+\tau_n^4\int_{\mathbb{R}^3}K(x)\phi_{w_n}^t w_n^2dx\\
&=\tau_n^p\int_{\mathbb{R}^3}a(x)|w_n|^pdx+\tau_n^{2_s^*}\int_{\mathbb{R}^3}|w_n|^{2_s^*}dx\\
&\geq \tau_n^p(\int_{\mathbb{R}^3}a(x)|w_n|^pdx+\int_{\mathbb{R}^3}|w_n|^{2_s^*}dx).
\end{aligned}
$$

Thus, by (4.4), we deduce that

$$
\begin{aligned}
1\leq \tau_n^{p-4}&\leq \frac{\|w_n\|^2+\int_{\mathbb{R}^3}K(x)\phi_{w_n}^t w_n^2dx}{\int_{\mathbb{R}^3}a(x)|w_n|^pdx+\int_{\mathbb{R}^3}|w_n|^{2_s^*}dx}\\
&=\frac{\|u_\infty\|^2+\int_{\mathbb{R}^3}\hat{\phi}_{u_\infty}^t(x)|u_\infty(x)|^2dx+o(1)}{\int_{\mathbb{R}^3}|u_\infty(x)|^pdx+o(1)+\int_{\mathbb{R}^3}|u_\infty(x)|^{2_s^*}dx},
\end{aligned}
$$

which yields $\tau_n\to 1$ by using $u_\infty\in M_\infty$.

Next we prove m_0 is not attained. Assume by contradiction that there exists $u_0\in M_0$ such that $m_0=J(u_0)$. We claim $J'(u_0)=0$. Set $\widetilde{G}(u)=[J'(u),u]$, By the Lagrange multipliers Theorem, we obtain $\lambda\in\mathbb{R}$ such that $J'(u_0)-\lambda\widetilde{G}'(u_0)\to 0$, similar to the of proof of Theorem 1.1, we have $J'(u_0)=0$. Note that if u_0 is sing-changing, by Remark 5.6 in [34], we see that $J(u_0)\geq 2m_0$, a contradiction. Thus we may assume that $u_0\geq 0$ in H and $u_0\not\equiv 0$, we claim $u_0>0$, by the definition of $\phi_{u_0}^t(x)$, there exists $C>0$ such that

$$
\begin{aligned}
\phi_{u_0}^t(x)&=\int_{|x-y|\geq 1}\frac{K(y)u_0^2(y)}{|x-y|^{3-2t}}dy+\int_{|x-y|<1}\frac{K(y)u_0^2(y)}{|x-y|^{3-2t}}dy\\
&\leq C\|u_0\|_2^2+C\int_{|x-y|<1}\frac{1}{|x-y|^{3-2t}}dy<+\infty,
\end{aligned}
$$

and $|g|\leq C(|u_0|+|u_0|^{q-1})$, where $g(x)=a(x)|u_0(x)|^{p-2}u_0(x)+|u_0(x)|^{2_s^*-2}u_0(x)-u_0(x)-K(x)\phi_{u_0}^t(x)u_0(x)$. Then it follows from [22, Proposition 3.4] that there exists $\sigma\in(0,1)$ such that $u_0\in C^{0,\sigma}$. Let ω satisfy $-\Delta\omega=-u_0-K(x)\phi_{u_0}^t u+a(x)|u_0|^{p-2}u_0+|u_0|^{2_s^*-2}u_0\in C^{0,\sigma}$. By the Hölder regularity theory for the Laplacian, we have $\omega\in C^{2,\sigma}$. It follows from $2s+\sigma>1$ that $(-\Delta)^{1-s}\omega\in C^{1,2s+\sigma-1}$. Then, since $(-\Delta)^s(u_0-(-\Delta)^{1-s}\omega)=0$, the function $u-(-\Delta)^{1-s}\omega$ is harmonic and we obtain u_0 has the same regularity as $(-\Delta)^{1-s}\omega$. That is, $u_0\in C^{1,2s+\sigma-1}$. The regularity obtained above implies that

$$
(-\Delta)^s u_0=-\int_{\mathbb{R}^3}\frac{u_0(x+y)+u_0(x-y)-2u_0(x)}{|y|^{3+2s}}dy.
$$

Assume that there exists $x_0\in\mathbb{R}^3$ such that $u_0(x_0)=0$, then by $u_0\not\equiv 0$ and $u_0\geq 0$,

$$
(-\Delta)^s u(x_0)=-\int_{\mathbb{R}^3}\frac{u(x_0+y)+u(x_0-y)}{|y|^{3+2s}}dy<0.
$$

However, noting that $-\Delta u_0=-u_0-K(x)\phi_{u_0}^t u_0+a(x)|u_0|^{p-2}u_0+|u_0|^{2_s^*-2}u_0$ we obtain $-\Delta u_0(x_0)=0$, which is a contradiction. Therefore, $u_0>0$.

From the above proof, we see that for $u_0\in M_0$, there exists a unique $\tau_0\leq 1$ such that $\tau_0u_0\in M_\infty$. Thus,

$$
\begin{aligned}
m_\infty&\leq I_\infty(\tau_0u_0)\\
&=I_\infty(\tau_0u_0)-\frac{1}{p}[I'_\infty(\tau_0u_0),\tau_0u_0]
\end{aligned}
$$

$$
\begin{aligned}
&= \big(\frac{1}{2}-\frac{1}{p}\big)\|\tau_0 u_0\|^2 + (\frac{1}{4}-\frac{1}{p})\tau_0^4\int_{\mathbb{R}^3}\hat{\phi}^t_{u_0}u_0^2dx + (\frac{1}{p}-\frac{1}{2_s^*})\int_{\mathbb{R}^3}|\tau_0 u_0|^{2_s^*}dx\\
&\le \big(\frac{1}{2}-\frac{1}{p}\big)\|u_0\|^2 + (\frac{1}{4}-\frac{1}{p})\int_{\mathbb{R}^3}K(x)\phi^t_{u_0}u_0^2dx + (\frac{1}{p}-\frac{1}{2_s^*})\int_{\mathbb{R}^3}|u_0|^{2_s^*}dx\\
&= J(u_0) - \frac{1}{p}[J'(u_0),u_0] = J(u_0) = m_0.
\end{aligned}
$$

From $m_0 = m_\infty$, it follows that

$$
\begin{aligned}
&(\frac{1}{2}-\frac{1}{p})\|\tau_0 u_0\|^2 + (\frac{1}{4}-\frac{1}{p})\tau_0^4\int_{\mathbb{R}^3}\hat{\phi}^t_{u_0}u_0^2dx + (\frac{1}{p}-\frac{1}{2_s^*})\int_{\mathbb{R}^3}|\tau_0 u_0|^{2_s^*}dx\\
&= \big(\frac{1}{2}-\frac{1}{p}\big)\|u_0\|^2 + (\frac{1}{4}-\frac{1}{p})\int_{\mathbb{R}^3}K(x)\phi^t_{u_0}u_0^2dx + (\frac{1}{p}-\frac{1}{2_s^*})\int_{\mathbb{R}^3}|u_0|^{2_s^*}dx,
\end{aligned}
$$

that is

$$
\begin{aligned}
&\tau_0^2\|u_0\|^2 + \tau_0^4\int_{\mathbb{R}^3}\hat{\phi}^t_{u_0}u_0^2dx + \tau_0^{2_s^*}\int_{\mathbb{R}^3}|u_0|^{2_s^*}dx\\
&= \|u_0\|^2 + \int_{\mathbb{R}^3}K(x)\phi^t_{u_0}u_0^2dx + \int_{\mathbb{R}^3}|u_0|^{2_s^*}dx.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&(1-\tau_0^2)\|u_0\|^2 + \int_{\mathbb{R}^3}(K(x)-1)\phi^t_{u_0}u_0^2dx + \int_{\mathbb{R}^3}(\phi^t_{u_0}-\hat{\phi}^t_{u_0})u_0^2dx\\
&+ (1-\tau_0^4)\int_{\mathbb{R}^3}\hat{\phi}^t_{u_0}u_0^2dx + (1-\tau_0^{2_s^*})\int_{\mathbb{R}^3}|u_0|^{2_s^*}dx\\
&= (1-\tau_0^2)\|u_0\|^2 + \int_{\mathbb{R}^3}(K(x)-1)(\phi^t_{u_0}+\hat{\phi}^t_{u_0})u_0^2dx\\
&\quad + (1-\tau_0^4)\int_{\mathbb{R}^3}\hat{\phi}^t_{u_0}u_0^2dx + (1-\tau_0^{2_s^*})\int_{\mathbb{R}^3}|u_0|^{2_s^*}dx = 0,
\end{aligned}
$$

by $\tau_0 \le 1$, so

$$
\int_{\mathbb{R}^3}(K(x)-1)(\phi^t_{u_0}+\hat{\phi}^t_{u_0})u_0^2dx = 0,
$$

this contradicts u_0 being positive, $K(x)\ge 1$ and $\operatorname{meas}\{x\in\mathbb{R}^3 : K(x)>1\}>0$. $\square$

Lemma 4.4. *There exists $\rho_0>0$ such that for $u\in M_0$ satisfying $J(u)\le m_\infty+\rho_0$, it holds $|\beta(u)|>0$.*

Proof. Assume by the contrary that there exists $\{u_n\}\subset M_0$ such that $J(u_n)\to m_\infty = m_0$ and $|\beta(u)|=0$. Similar to the proof Theorem 1.1, we can derive by the Lagrange multipliers Theorem that $J'(u_n)\to 0$. We omit the proof here. Similar to the proof Lemma 2.3, we obtain $u_n \rightharpoonup u$ weakly in H, $J'(u)=0$, and

$$
m_\infty - J(u) = I_\infty(v_n) + o(1) \text{ and } I'_\infty(v_n) = o(1), \tag{4.5}
$$

where $v_n = u_n - u$.

For the sequence $\{v_n\}$, two cases may occur: $\|v_n\|_r\to 0$ for any $r\in(2,2_s^*)$, or there $y_n\in\mathbb{R}^3$ with $|y_n|\to\infty$ such that $v_n(.+y_n)\rightharpoonup v\ne 0$ weakly in H. By virtue of $J'(u)=0$, we can deduce that $J(u)\ge 0$. From Lemma 2.3, we see that $m_\infty < \frac{s}{3}S_s^{\frac{3}{2s}}$. Thus $m_\infty - J(u) < \frac{s}{3}S_s^{\frac{3}{2s}}$.

If $\|v_n\|_r\to 0$ for any $r\in(2,2_s^*)$, by (4.6), we have $m_\infty - J(u) = \frac{1}{2}\|v_n\|^2 - \frac{1}{2_s^*}\|v_n\|_{2_s^*}^{2_s^*} + o(1)$ and $\|v_n\|^2 - \|v_n\|_{2_s^*}^{2_s^*} = o(1)$. Up to a subsequence, we may assume that $\|v_n\|^2 \to l$ and then $\|v_n\|_{2_s^*}^{2_s^*} \to l$. If $l > 0$, by the definition of S_s, we obtain $l \geq S_s^{\frac{3}{2s}}$. So $m_\infty - J(u) = \frac{1}{2}\|v_n\|^2 - \frac{1}{2_s^*}\|v_n\|_{2_s^*}^{2_s^*} = \frac{s}{3}l \geq \frac{s}{3}S_s^{\frac{3}{2s}}$, a contradiction with $m_\infty - J(u) < \frac{s}{3}S_s^{\frac{3}{2s}}$. Thus, $l = 0$ and then $u_n \to u$ in H, we obtain $m_0 = J(u)$, a contradiction with m_0 is not attained. Therefore, $v_n(.+y_n) \rightharpoonup v \neq 0$ weakly in H. Similar to the proof Lemma 2.3, we can deduce that

$$
\begin{gathered}
m_\infty - J(u) = I_\infty(v_n(.+y_n)) + o(1), \\
I'_\infty(v_n(.+y_n)) = o(1).
\end{gathered}
$$

Hence, $I'_\infty(v) = 0$ and by using Fatou's Lemma, we have

$$
\begin{aligned}
m_\infty - J(u) &= I_\infty(v_n(.+y_n)) - \frac{1}{4}[I'_\infty(v_n(.+y_n)), v_n(.+y_n)] + o(1) \\
&= \frac{1}{4}\|(v_n(.+y_n)\|^2 + (\frac{1}{4} - \frac{1}{p})\int_{\mathbb{R}^3} |(v_n(.+y_n)|^p dx \\
&\quad + (\frac{1}{4} - \frac{1}{2_s^*})\int_{\mathbb{R}^3} |(v_n(.+y_n)|^{2_s^*} dx + o(1) \\
&\geq \frac{1}{4}\|v\|^2 + (\frac{1}{4} - \frac{1}{p})\int_{\mathbb{R}^3} |v|^p dx + (\frac{1}{4} - \frac{1}{2_s^*})\int_{\mathbb{R}^3} |v|^{2_s^*} dx \\
&= I_\infty(v) - \frac{1}{4}(I'_\infty(v), v)) = I_\infty(v) \geq m_\infty.
\end{aligned}
$$

Combining with $J(u) \geq 0$, we obtain $J(u) = 0$ and then $v_n(.+y_n) = u_n(.+y_n) \to v$ in H. By Lemma 4.2, we have

$$
\beta(v(x)) + o(1) = \beta(u_n(x+y_n)) = \beta(u_n) - y_n = -y_n.
$$

Which yields $|\beta(v(x))| = \infty$, this leads to a contradiction. □

Lemma 4.5. *There exists $\mu_0 > 0$ small such that for $\mu \in (0, \mu_0)$, we have $|\beta(u)| > 0$ for $u \in \{u \in M : I(u) < m_\infty\}$.*

Proof. Let $u \in M$ be such that $I(u) < m_\infty$, then we have

$$
m_\infty > I(u) = I(u) - \frac{1}{4}[I'(u), u] \geq \frac{1}{4}\|u\|^2. \tag{4.6}
$$

Using the conditions (A4), (A6) and $u \in M$, we obtain for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$
\begin{aligned}
\|u\|^2 &\leq \int_{\mathbb{R}^3} a(x)|u|^p dx + \mu\int_{\mathbb{R}^3} b(x)|u|^q dx + \int_{\mathbb{R}^3} |u|^{2_s^*} dx \\
&\leq (1+\mu)\Big[\varepsilon\int_{\mathbb{R}^3} |u|^2 dx + C_\varepsilon\int_{\mathbb{R}^3} |u|^{2_s^*} dx\Big].
\end{aligned} \tag{4.7}
$$

Choose $\varepsilon \in (0, 1/4)$, we have $\frac{1}{2}\|u\|^2 \leq (1+\mu)C_\varepsilon\int_{\mathbb{R}^3} |u|^{2_s^*} dx$ for $\mu \in (0,1)$. In fact, if $\varepsilon \in (0, \frac{1}{4})$ and $\mu \in (0,1)$, we obtain $0 < (1+\mu)\varepsilon < 1/2$, and then $\frac{1}{2} - (1+\mu)\varepsilon > 0$. Thus, it holds

$$
\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + (\frac{1}{2} - (1+\mu)\varepsilon)\int_{\mathbb{R}^3} |u|^2 dx \geq 0,
$$

that is

$$
\frac{1}{2}\|u\|^2 \leq \|u\|^2 - (1+\mu)\varepsilon\int_{\mathbb{R}^3} |u|^2 dx,
$$

by (4.7), we have

$$\|u\|^2 - (1+\mu)\varepsilon \int_{\mathbb{R}^3} |u|^2 dx \le (1+\mu)C_\varepsilon \int_{\mathbb{R}^3} |u|^{2_s^*} dx,$$

so

$$\frac{1}{2}\|u\|^2 \le (1+\mu)C_\varepsilon \int_{\mathbb{R}^3} |u|^{2_s^*} dx.$$

Thus, by the definition of S_s, there exists $L_0 > 0$ independent of $\mu \in (0,1)$ such that

$$\int_{\mathbb{R}^3} |u|^{2_s^*} dx \ge \frac{L_0}{(1+\mu)^{\frac{3}{2s}}}. \tag{4.8}$$

Similar to the argument of Lemma 4.3, we can deduce that for any $u \in M$, there exists a unique $\tau(u) \ge 1$ such that $\tau(u)u \in M_0$. Then

$$\begin{aligned}
&\tau^4(u)(\|u\|^2 + \int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx) \\
&\ge \tau^2(u)\|u\|^2 + \tau^4(u)\int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx \\
&= \tau^p(u)\int_{\mathbb{R}^3} a(x)|u|^p dx + \tau^{2_s^*}(u)\int_{\mathbb{R}^3} |u|^{2_s^*} dx \\
&\ge \tau^{2_s^*}(u)\int_{\mathbb{R}^3} |u|^{2_s^*} dx,
\end{aligned}$$

which implies that

$$\tau^{2_s^*-4}(u) \le \frac{\|u\|^2 + \int_{\mathbb{R}^3} K(x)\phi_u^t u^2 dx}{\int_{\mathbb{R}^3} |u|^{2_s^*} dx}.$$

Together with (4.6) and (4.8), we derive there exists $\overline{C} > 0$ independent of $\mu \in (0,1)$ such that

$$1 \le \tau^{2_s^*-4}(u) \le \overline{C}(1+\mu)^{\frac{3}{2s}}. \tag{4.9}$$

Note that for $u \in M$ with $I(u) < m_\infty$, thus

$$m_\infty > I(u) = \sup_{t\ge 0} I(tu) \ge I(t(u)u) = J(t(u)u) - \mu\frac{t^q(u)}{q}\int_{\mathbb{R}^3} b(x)|u|^q dx.$$

By (4.6) and (4.9), there exists a small $\mu_0 \in (0,1)$ such that $\mu \in (0,\mu_0)$,

$$J(t(u)u) < m_\infty + \mu\frac{t^q(u)}{q}\int_{\mathbb{R}^3} b(x)|u|^q dx \le m_\infty + \rho_0.$$

Form Lemma 4.4, we have $|\beta(t(u)u)| > 0$. Hence, Lemma 4.2 implies $|\beta(u)| > 0$. $\square$

Lemma 4.6. *For $\mu \in (0,\mu_0)$, define $G : \{u \in M : I(u) < m_\infty\} \to S^2$ by $G(u) = \frac{\beta(u)}{|\beta(u)|}$. Then for $R > R_0$ and $\mu \in (0,\mu_0)$, the map*

$$G \circ F_R : S^2 \to S^2; y \to G \circ (F_R(y))$$

is homotopic to the identity.

Proof. Similar to the argument of [1, 37, Proposition 2.9], we define the map $\zeta(\theta, y) : [0,1] \times S^2 \to S^2$ by

$$\zeta(\theta,y) = \begin{cases} G((1-2\theta)F_R(y) + 2\theta u_\infty(x - Ry)), & \theta \in [0,1/2), \\ G(u_\infty(x - \frac{R}{2(1-\theta)}y), & \theta \in [1/2,1), \\ y, & \theta = 1. \end{cases}$$

By the definition of G and Lemma 2.7, tt is not difficult to check that $\zeta(\theta, y) \in C([0,1] \times S^2, S^2)$, $\zeta(0, y) = G \circ (F_R(y))$ for $y \in S^2$ and $\zeta(1, y) = y$ for $y \in S^2$. The proof is complete. $\square$

Proof of Theorem 1.3. Form Lemma 2.7, Lemma 4.1 and Lemma 4.6, we have that for $R > R_0$ and $\mu \in (0, \mu_0)$, it holds

$$\text{cat}(\{u \in M : I(u) \leq m_\infty - \varepsilon(R)\}) \geq 2.$$

Then by Lemma 2.3 and Lemma 2.6, we see that I admits at least two nontrivial critical point in $\{u \in M : I(u) < m_\infty\}$. $\square$

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