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EXISTENCE OF SOLUTIONS TO A PHASE-FIELD MODEL WITH PHASE-DEPENDENT HEAT ABSORPTION

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Abstract. We consider a phase-field model for a phase change process with phase-dependent heat absorption. This model describes the behaviour of films exposed to radiative heating, where the film can change reversibly between amorphous and crystalline states. Existence and uniqueness of solutions as well as stability are established. Moreover, a maximum principle is proved for the phase-field equation.

1. Introduction

In recent years the phase-field method has emerged as a powerful computational approach to modelling and predicting a range of phase transitions and complex growth structures occurring during solidification. This has spurred many articles using this approach and proposing several mathematical models. Phase-field models have also been developed to treat both pure materials and binary alloys; as examples of papers where mathematical analysis of such models is performed, we single out [2, 3, 4, 5, 8, 11, 12, 14], where existence of solutions is investigated for various types of nonlinearities.

We consider in this paper a phase-field model for a phase change process with phase-dependent heat absorption. Such a model was proposed by Blyuss et al. [1] to model the behaviour of films exposed to radiative heating, where the film can change reversibly between amorphous and crystalline states. The models adopted so far have neglected the difference in the response of different phases to external heat sources considering only external forces which do not depend neither on the phase-field nor on the temperature. To be specific, the model we consider incorporates illumination and phase-dependent absorption in the equation for the temperature, and this influences the phase change process, as described by the phase-field equation.

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This model can be expressed as the following coupled system

$$\phi_t - \epsilon^2 \Delta\phi = \phi - \phi^3 + \frac{(\theta_M - \theta)}{\delta}\hat{\lambda}(1-\phi^2)^2 \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

$$\theta_t - K\Delta\theta = \frac{\delta}{2}\phi_t + \big[a_1 + a_{-1} + (a_1 - a_{-1})\phi\big]\frac{I}{2} + b(\theta_a - \theta) \quad \text{in } \Omega \times (0,T), \tag{1.2}$$

$$\frac{\partial\phi}{\partial n} = 0, \quad \frac{\partial\theta}{\partial n} = 0, \quad \text{on } \partial\Omega \times (0,T), \tag{1.3}$$

$$\phi(0) = \phi_0, \quad \theta(0) = \theta_0 \quad \text{in } \Omega. \tag{1.4}$$

Here Ω is an open bounded domain of $\mathbb{R}^N$, $N = 2,3$, with smooth boundary $\partial\Omega$ and $T > 0$. The order parameter (phase-field) ϕ is the state variable characterizing the different phases; the convection adopted is that $\phi \in [-1,1]$, with the lower limit $\phi = -1$ corresponding to pure melt while $\phi = 1$ represents solid. The function θ represents the temperature of a material which melts at $\theta = \theta_M$. The interface thickness ϵ is a small parameter and $\hat{\lambda}$ is a measure of the strength of coupling between the phase field and a dimensionless temperature field $(\theta - \theta_M)/\delta$, where δ is given by $\delta = L/C_p$, being $L > 0$ the latent heat and $C_p > 0$ the specific heat at constant pressure. For simplicity of exposition it will be assumed $\hat{\lambda} = 1$. The constant $K > 0$ denotes a thermal diffusivity; $a_{\pm 1}$ are the radiative absorption coefficients for the solid and molten phases; I is the rate of incident heating; b is a thermal emission coefficient and θ_a is the ambient temperature.

Our aim is to prove existence and uniqueness of solutions as well as stability. Moreover, a maximum principle is established for the phase-field equation which ensures that ϕ stays between -1 and 1 as long as the initial data ϕ_0 does. We observe that this bound on the phase-field will allow us to show a stability result and, subsequently, the uniqueness of the solution. Existence of solutions will be obtained by using an auxiliary problem. The approach is to modify the problem by introducing an appropriate truncation of $(1-\phi^2)^2$. This auxiliary problem will then be studied by using fixed point arguments.

Standard notation will be used. For a given fixed $T > 0$, we denote $Q = \Omega \times (0,T)$ and we consider the following spaces, for $q \geq 1$,

$$W_q^{2,1}(Q) = \{w \in L^q(Q) : D_x w, D_x^2 w \in L^q(Q), w_t \in L^q(Q)\}.$$

The outline of this paper is as follows. In the next section we study an auxiliary problem. The last section is devoted to prove the well-posedness of problem (1.1)-(1.4). First, we study the existence of solutions, secondly we establish a stability result which will give us uniqueness at the same time and, finally, a result of regularity of the solution will be obtained by applying L^p-theory of parabolic linear equations together with bootstrapping arguments.

2. An auxiliary problem

In this section, we introduce an auxiliary problem related to (1.1)-(1.4) for which we will prove a result of existence of solutions by using Leray-Schauder's fixed point theorem [6].

Let Π be the function

$$\Pi(r) = \begin{cases} -1, & r < -1 \\ r, & -1 \leq r \leq 1 \\ 1, & r > 1\,. \end{cases}$$

Consider the problem

$$\phi_t - \epsilon^2 \Delta\phi = \phi - \phi^3 + \frac{(\theta_M - \theta)}{\delta}(1 - \Pi(\phi)^2)^2 \quad \text{in } Q, \tag{2.1}$$

$$\theta_t - K\Delta\theta + b\theta = \frac{\delta}{2}\phi_t + \alpha\phi + \beta \quad \text{in } Q, \tag{2.2}$$

$$\frac{\partial\phi}{\partial n} = 0, \quad \frac{\partial\theta}{\partial n} = 0, \quad \text{on } \partial\Omega \times (0, T), \tag{2.3}$$

$$\phi(0) = \phi_0, \quad \theta(0) = \theta_0, \quad \text{in } \Omega, \tag{2.4}$$

where $\alpha = (a_1 - a_{-1})\frac{I}{2}$ and $\beta = (a_1 + a_{-1})\frac{I}{2} + b\theta_a$.

We then have the following existence result.

Proposition 2.1. *Let $(\phi_0, \theta_0) \in H^{1+\gamma}(\Omega) \times H^{1+\gamma}(\Omega)$, $1/2 < \gamma \le 1$, satisfying the compatibility condition $\frac{\partial\phi_0}{\partial n} = \frac{\partial\theta_0}{\partial n} = 0$ a.e. on $\partial\Omega$. Then there exists $(\phi, \theta) \in W_2^{2,1}(Q) \times W_2^{2,1}(Q)$ solution to problem (2.1)-(2.4) for any fixed $T > 0$, which verifies the estimate*

$$\|\phi\|_{W_2^{2,1}(Q)} + \|\theta\|_{W_2^{2,1}(Q)} \le C\big(\|\phi_0\|_{H^1(\Omega)} + \|\theta_0\|_{H^1(\Omega)} + 1\big), \tag{2.5}$$

where C depends on Ω, and some physical parameters.

Proof. In order to apply Leray-Schauder's fixed point theorem we consider the following family of operators, indexed by the parameter $0 \le \lambda \le 1$,

$$\mathcal{T}_\lambda : B \to B,$$

where B is the Banach space

$$B = L^2(Q) \times L^2(Q),$$

and is defined as follows: given $(\hat\phi, \hat\theta) \in B$, let $\mathcal{T}_\lambda(\hat\phi, \hat\theta) = (\phi, \theta)$, where (ϕ, θ) is obtained by solving the problem

$$\phi_t - \epsilon^2 \Delta\phi - (\phi - \phi^3) = \lambda\frac{(\theta_M - \hat\theta)}{\delta}(1 - \Pi(\hat\phi)^2)^2 \quad \text{in } Q, \tag{2.6}$$

$$\theta_t - K\Delta\theta + b\theta = \frac{\delta}{2}\phi_t + \alpha\phi + \beta \quad \text{in } Q, \tag{2.7}$$

$$\frac{\partial\phi}{\partial n} = 0, \quad \frac{\partial\theta}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{2.8}$$

$$\phi(0) = \phi_0, \quad \theta(0) = \theta_0 \quad \text{in } \Omega. \tag{2.9}$$

Before we prove that $\mathcal{T}_\lambda$ is well defined, we observe that clearly (ϕ, θ) is a solution of (2.1)-(2.4) if and only if it is a fixed point of the operator $\mathcal{T}_1$.

To verify that the operator $\mathcal{T}_\lambda$ is well defined, observe that since $\hat\theta \in L^2(Q)$ and $|(1 - \Pi(\hat\phi)^2)^2| \le 1$, we infer from [8, Theorem 2.1] that there is a unique solution ϕ of equation (2.6) with $\phi \in W_2^{2,1}(Q)$ satisfying the first of the boundary conditions (2.8).

Since ϕ and $\phi_t \in L^2(Q)$, according to L^p-theory of parabolic equations [9, Theorem 9.1] there is a unique solution θ of equation (2.7) with $\theta \in W_2^{2,1}(Q)$ satisfying the second of the boundary conditions (2.8).

Therefore, for each $\lambda \in [0, 1]$, the mapping $\mathcal{T}_\lambda$ is well defined from B into B.

To prove continuity of $\mathcal{T}_\lambda$, let $(\hat\phi_n,\hat\theta_n)\in B$ strongly converging to $(\hat\phi,\hat\theta)\in B$; for each n, let (ϕ_n,θ_n) the corresponding solution of problem

$$\phi_{nt}-\epsilon^2\Delta\phi_n-(\phi_n-\phi_n^3)=\lambda\frac{(\theta_M-\hat\theta_n)}{\delta}(1-\Pi(\hat\phi_n)^2)^2 \quad \text{in } Q, \tag{2.10}$$

$$\theta_{nt}-K\Delta\theta_n+b\theta_n=\frac{\delta}{2}\phi_{nt}+\alpha\phi_n+\beta \quad \text{in } Q, \tag{2.11}$$

$$\frac{\partial\phi_n}{\partial n}=0,\quad \frac{\partial\theta_n}{\partial n}=0 \quad \text{on } \partial\Omega\times(0,T), \tag{2.12}$$

$$\phi_n(0)=\phi_0,\quad \theta_n(0)=\theta_0 \quad \text{in } \Omega. \tag{2.13}$$

Next, we show that the sequence (ϕ_n,θ_n) converges strongly to $(\phi,\theta)=\mathcal{T}_\lambda(\hat\phi,\hat\theta)$ in B. For that purpose, we will obtain estimates, uniformly with respect to n, for (ϕ_n,θ_n). We denote by C_i any positive constant independent of n.

We multiply (2.10) successively by ϕ_n, ϕ_{nt} and $-\Delta\phi_n$, and integrate over $\Omega\times(0,t)$. After integration by parts and the use of Hölder and Young inequalities, we obtain the following three estimates

$$\begin{aligned}\frac12\int_\Omega|\phi_n|^2dx+&\int_0^t\int_\Omega\left(\epsilon^2|\nabla\phi_n|^2+|\phi_n|^4\right)dx\,ds\\ &\le C_1+C_2\int_0^t\int_\Omega\left(|\hat\theta_n|^2+|\phi_n|^2\right)dx\,ds,\end{aligned} \tag{2.14}$$

$$\begin{aligned}\frac12\int_0^t\int_\Omega|\phi_{nt}|^2\,dx\,ds+&\int_\Omega\left(\frac{\epsilon^2}{2}|\nabla\phi_n|^2+\frac{|\phi_n|^4}{4}-\frac{|\phi_n|^2}{2}\right)dx\\ &\le C_1+C_2\int_0^t\int_\Omega|\hat\theta_n|^2\,dx\,ds,\end{aligned} \tag{2.15}$$

$$\begin{aligned}\frac12\int_\Omega|\nabla\phi_n|^2dx+&\frac{\epsilon^2}{2}\int_0^t\int_\Omega|\Delta\phi_n|^2\,dx\,ds\\ &\le C_1+C_2\int_0^t\int_\Omega\left(|\nabla\phi_n|^2+|\hat\theta_n|^2\right)dx\,ds.\end{aligned} \tag{2.16}$$

By multiplying (2.15) by $\frac12$ and adding the result to (2.14) we find

$$\int_\Omega\left(|\phi_n|^2+|\nabla\phi_n|^2+|\phi_n|^4\right)dx\le C_1+C_2\int_0^t\int_\Omega\left(|\hat\theta_n|^2+|\phi_n|^2\right)dx\,ds.$$

Since $\|\hat\theta_n\|_{L^2(Q)}$ is bounded independent of n, by using Gronwall's lemma we deduce that

$$\|\phi_n\|_{L^\infty(0,T;H^1(\Omega))}\le C_1. \tag{2.17}$$

Then, thanks to estimates (2.14)-(2.16) we arrive at

$$\|\phi_n\|_{L^2(0,T;H^2(\Omega))}+\|\phi_{nt}\|_{L^2(Q)}\le C_1. \tag{2.18}$$

Next, from L^p-theory of parabolic equations applied to equation (2.11) we have

$$\|\theta_n\|_{W_2^{2,1}(Q)}\le C_1(\|\theta_0\|_{H^1(\Omega)}+\|\phi_{nt}\|_{L^2(Q)}+\|\phi_n\|_{L^2(Q)}+1). \tag{2.19}$$

We now infer from (2.17),(2.18) and (2.19) that the sequence (ϕ_n,θ_n) is bounded in $W_2^{2,1}(Q)$ and in

$$W=\left\{v\in L^\infty(0,T;H^1(\Omega)),\, v_t\in L^2(0,T;L^2(\Omega))\right\}.$$

Since $W_2^{2,1}(Q)$ is compactly embedded in $L^2(0,T;H^1(\Omega))$ and W in $C([0,T];L^2(\Omega))$ [13, Corollary 4], it follows that there exist

$$\phi,\, \theta \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \quad \text{with } \phi_t,\, \theta_t \in L^2(Q),$$

and a subsequence of (ϕ_n,θ_n) (which we still denote by (ϕ_n,θ_n)), such that, as $n \to +\infty$,

$$\begin{gathered} (\phi_n,\theta_n) \to (\phi,\theta) \quad \text{in } \big(L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))\big)^2 \quad \text{strongly}, \\ (\phi_n,\theta_n) \rightharpoonup (\phi,\,\theta) \quad \text{in } \big(W_2^{2,1}(Q)\big)^2 \quad \text{weakly}. \end{gathered} \tag{2.20}$$

It now remains to pass to the limit as n tends to $+\infty$ in (2.10)-(2.13). Since the embedding of $W_2^{2,1}(Q)$ into $L^9(Q)$ is compact [10], we infer that ϕ_n^3 converges to ϕ^3 in $L^2(Q)$. Moreover, since $(1-\Pi(\cdot)^2)^2$ is a bounded Lipschitz continuous function and $\hat{\phi_n}$ converges to $\hat{\phi}$ in $L^2(Q)$, we have that $(1-\Pi(\hat{\phi_n})^2)^2$ converges to $(1-\Pi(\hat{\phi})^2)^2$ in $L^p(Q)$ for any $p \in [1,\infty)$. We then pass to the limit in (2.10) and get (2.6).

From convergence (2.20), it is easy to pass to the limit in (2.11) and conclude that (2.7) holds almost everywhere.

Therefore $\mathcal{T}_\lambda$ is continuous for all $0 \le \lambda \le 1$. At the same time, $\mathcal{T}_\lambda$ is bounded in $W_2^{2,1}(Q) \times W_2^{2,1}(Q)$ but, the embedding of this space in B is compact. Hence, $\mathcal{T}_\lambda$ is a compact operator for each $\lambda \in [0,1]$.

To prove that for $(\hat{\phi},\hat{\theta})$ in a bounded set of B, T_λ is uniformly continuous with respect to λ, let $0 \le \lambda_1, \lambda_2 \le 1$ and (ϕ_i,θ_i) $(i=1,2)$ be the corresponding solutions of (2.6)-(2.9). We observe that $\phi = \phi_1 - \phi_2$ and $\theta = \theta_1 - \theta_2$ satisfy the problem

$$\begin{aligned} \phi_t - \epsilon^2 \Delta\phi = &\phi(1 - (\phi_1^2 + \phi_1\phi_2 + \phi_2^2)) \\ &+ (\lambda_1 - \lambda_2)\Big(\frac{(\theta_M - \hat{\theta})}{\delta}(1 - \Pi(\hat{\phi})^2)^2\Big) \text{ in } Q, \end{aligned} \tag{2.21}$$

$$\theta_t - K\Delta\theta + b\theta = \frac{\delta}{2}\phi_t + \alpha\phi \quad \text{in } Q, \tag{2.22}$$

$$\frac{\partial\phi}{\partial n} = 0, \quad \frac{\partial\theta}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{2.23}$$

$$\phi(0) = 0, \quad \theta(0) = 0 \quad \text{in } \Omega. \tag{2.24}$$

We remark that $d := \phi_1^2 + \phi_1\phi_2 + \phi_2^2 \ge 0$. Now, multiply equation (2.21) by ϕ and integrate over $\Omega \times (0,t)$; after integration by parts and the use of Hölder and Young inequalities we obtain

$$\begin{aligned} &\int_\Omega |\phi|^2 dx + \int_0^t \int_\Omega |\nabla\phi|^2 \,dx\,ds \\ &\le C_1 \int_0^t \int_\Omega |\phi|^2 \,dx\,ds + C_2 |\lambda_1 - \lambda_2|^2 \int_0^t \int_\Omega (|\hat{\theta}|^2 + 1)\,dx\,ds. \end{aligned}$$

By applying Gronwall's lemma we arrive at

$$\|\phi\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\phi\|^2_{L^2(0,T;H^1(\Omega))} \le C_1\,|\lambda_1 - \lambda_2|^2. \tag{2.25}$$

Multiplying (2.21) by ϕ_t and using Hölder inequality, we get

$$\begin{aligned}\int_0^t\int_\Omega |\phi_t|^2\,dx\,ds+&\frac{\epsilon^2}{2}\int_\Omega|\nabla\phi|^2dx\\ &\le C_1\int_0^t\int_\Omega|\phi|^2\,dx\,ds+\frac{1}{2}\int_0^t\int_\Omega|\phi_t|^2\,dx\,ds\\ &\quad + C_2\Big(\int_0^t\int_\Omega|\phi|^{10/3}\,dx\,ds\Big)^{3/5}\Big(\int_0^t\int_\Omega|d|^5\,dx\,ds\Big)^{2/5}\\ &\quad + C_3|\lambda_1-\lambda_2|^2\int_0^t\int_\Omega(|\hat\theta|^2+1)\,dx\,ds.\end{aligned}$$

Since $W_2^{2,1}(Q)\hookrightarrow L^{10}(Q)$, the following interpolation inequality holds

$$\|\phi\|^2_{L^{10/3}(Q)}\le\eta\,\|\phi\|^2_{W_2^{2,1}(Q)}+\tilde C\,\|\phi\|^2_{L^2(Q)}\quad\text{for all }\eta>0.$$

Moreover, since $\|d\|_{L^5(Q)}\le C$, with C depending on $\|\phi_i\|_{L^{10}(Q)}$, $i=1,2$, by rearranging the terms in the last inequality, we obtain

$$\begin{aligned}\int_0^t\int_\Omega|\phi_t|^2\,dx\,ds+\int_\Omega|\nabla\phi|^2dx\le& C_1\int_0^t\int_\Omega|\phi|^2\,dx\,ds+C_2\eta\|\phi\|^2_{W_2^{2,1}(Q)}\\ &+C_3|\lambda_1-\lambda_2|^2\int_0^t\int_\Omega(|\hat\theta|^2+1)\,dx\,ds.\end{aligned}\tag{2.26}$$

Multiplying (2.21) by $-\Delta\phi$, we infer in a similar way that

$$\begin{aligned}\int_\Omega|\nabla\phi|^2dx+&\int_0^t\int_\Omega|\Delta\phi|^2dxds\\ &\le C_1\int_0^t\int_\Omega\big(|\phi|^2+|\nabla\phi|^2\big)\,dx\,ds+C_2\eta\|\phi\|^2_{W_2^{2,1}(Q)}\\ &\quad+C_3|\lambda_1-\lambda_2|^2\int_0^t\int_\Omega(|\hat\theta|^2+1)\,dx\,ds.\end{aligned}\tag{2.27}$$

By taking $\eta>0$ small enough and considering (2.25), we conclude from (2.26) and (2.27) that

$$\|\phi\|^2_{W_2^{2,1}(Q)}+\|\phi\|^2_{L^\infty(0,T;H^1(\Omega))}\le C_1|\lambda_1-\lambda_2|^2.\tag{2.28}$$

Next, by multiplying (2.22) by θ, integrating over $\Omega\times(0,t)$ and using Hölder inequality we have

$$\int_\Omega|\theta|^2dx+\int_0^t\int_\Omega|\nabla\theta|^2\,dx\,ds\le C_1\int_0^t\int_\Omega(|\phi_t|^2+|\phi|^2+|\theta|^2)\,dx\,ds.$$

Thus, by using Gronwall's lemma and (2.28), we infer that

$$\|\theta\|^2_{L^\infty(0,T;L^2(\Omega))}\le C_1\,|\lambda_1-\lambda_2|^2.\tag{2.29}$$

It follows from (2.28) and (2.29) that $\mathcal{T}_\lambda$ is uniformly continuous with respect to λ on bounded sets of B.

Now we estimate the set of all fixed points of $\mathcal{T}_\lambda$. Let $(\phi,\theta)\in B$ be such a fixed point, i.e. a solution of the problem

$$\phi_t-\epsilon^2\Delta\phi-(\phi-\phi^3)=\lambda\frac{(\theta_M-\theta)}{\delta}(1-\Pi(\phi)^2)^2 \quad \text{in } Q, \tag{2.30}$$

$$\theta_t-K\Delta\theta+b\theta=\frac{\delta}{2}\phi_t+\alpha\phi+\beta \quad \text{in } Q, \tag{2.31}$$

$$\frac{\partial\phi}{\partial n}=0, \quad \frac{\partial\theta}{\partial n}=0 \quad \text{on } \partial\Omega\times(0,T), \tag{2.32}$$

$$\phi(0)=\phi_0, \quad \theta(0)=\theta_0 \quad \text{in } \Omega. \tag{2.33}$$

First, we multiply equation (2.30) successively by ϕ, ϕ_t and $-\Delta\phi$, and integrate over Ω. After integration by parts, using Hölder and Young inequalities we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|\phi|^2dx+\int_\Omega\left(\epsilon^2|\nabla\phi|^2+|\phi|^4\right)dx\le C_1+C_2\int_\Omega(|\theta|^2+|\phi|^2)dx, \tag{2.34}$$

$$\frac{1}{2}\int_\Omega|\phi_t|^2dx+\frac{d}{dt}\int_\Omega\left(\frac{\epsilon^2}{2}|\nabla\phi|^2+\frac{1}{4}|\phi|^4-\frac{1}{2}|\phi|^2\right)dx\le C_1+C_2\int_\Omega|\theta|^2dx, \tag{2.35}$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla\phi|^2dx+\int_\Omega\frac{\epsilon^2}{2}|\Delta\phi|^2dx\le C_1+C_2\int_\Omega(|\theta|^2+|\nabla\phi|^2)dx. \tag{2.36}$$

Next, by multiplying (2.31) with θ, arguments similar to the previous ones lead to the following estimate

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|\theta|^2dx+K\int_\Omega|\nabla\theta|^2dx\le\frac{1}{8}\int_\Omega|\phi_t|^2dx+C_1\int_\Omega(|\theta|^2+|\phi|^2)dx. \tag{2.37}$$

Now, multiply (2.35) by $\frac{1}{2}$ and add the result to (2.34), (2.36) and (2.37) to obtain

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\Big(\frac{1}{4}|\phi|^2+\big(\frac{\epsilon^2}{4}+\frac{1}{2}\big)|\nabla\phi|^2+\frac{1}{8}|\phi|^4+\frac{1}{2}|\theta|^2\Big)dx\\&\quad+\int_\Omega\Big(\epsilon^2|\nabla\phi|^2+|\phi|^4+\frac{1}{8}|\phi_t|^2+\frac{\epsilon^2}{2}|\Delta\phi|^2+K|\nabla\theta|^2\Big)dx\\&\qquad\le C_1+C_2\int_\Omega\left(|\theta|^2+|\phi|^2+|\nabla\phi|^2\right)dx.\end{aligned} \tag{2.38}$$

Integrating with respect t and using Gronwall's lemma we find

$$\|\phi\|_{L^\infty(0,T;H^1(\Omega))}+\|\theta\|_{L^\infty(0,T;L^2(\Omega))}\le C_1,$$

where C_1 is independent of λ. Therefore, all fixed points of $\mathcal{T}_\lambda$ in B are bounded independently of $\lambda\in[0,1]$.

Finally, observe that the equation $x-\mathcal{T}_0(x)=0$ is equivalent to say that problem (2.6)-(2.9) for $\lambda=0$ has a unique solution. This is concluded reasoning exactly as in the beginning of this proof, when we proved that $\mathcal{T}_\lambda$ was well defined.

Therefore, we can apply Leray-Schauder's fixed point theorem, and so there is at least one fixed point $(\phi,\theta)\in B\cap W_2^{2,1}(Q)\times W_2^{2,1}(Q)$ of the operator $\mathcal{T}_1$, i.e., $(\phi,\theta)=\mathcal{T}_1(\phi,\theta)$. This corresponds to a solution of problem (2.1)-(2.4).

To prove estimate (2.5), observe that from (2.38) it follows

$$\|\phi\|_{W_2^{2,1}(Q)}+\|\theta\|_{L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))}\le C\big(\|\phi_0\|_{H^1(\Omega)}+\|\theta_0\|_{L^2(\Omega)}+1\big). \tag{2.39}$$

To obtain an estimate for $\|\theta\|_{W_2^{2,1}(Q)}$, we apply L^p-theory of parabolic equations

$$\|\theta\|_{W_2^{2,1}(Q)}\le C\big(\|\theta_0\|_{H^1(\Omega)}+\|\phi_t\|_{L^2(Q)}+\|\phi\|_{L^2(Q)}+1\big).$$

Using (2.39) we deduce the desired estimate. The proof of Proposition 2.1 is thus complete. □

3. Existence and uniqueness

In this section, we prove the well-posedness of problem (1.1)-(1.4). We begin with the following existence result.

Theorem 3.1. *Let be given functions satisfying: ϕ_0, $\theta_0 \in H^{1+\gamma}(\Omega)$ with $1/2 < \gamma \leq 1$, the compatibility condition $\frac{\partial \phi_0}{\partial n} = \frac{\partial \theta_0}{\partial n} = 0$ a.e. on $\partial\Omega$ and such that $-1 \leq \phi_0 \leq 1$ a.e. in Ω. Then there exists $(\phi, \theta) \in W_2^{2,1}(Q) \times W_2^{2,1}(Q)$ solution to problem (1.1)-(1.4) which satisfies*

$$-1 \leq \phi \leq 1 \quad \textit{for all } t \in [0,T] \textit{ and a.e. in } \Omega.$$

In addition to that the following estimate

$$\|\phi\|_{W_2^{2,1}(Q)} + \|\theta\|_{W_2^{2,1}(Q)} \leq C\big(\|\phi_0\|_{H^1(\Omega)} + \|\theta_0\|_{H^1(\Omega)} + 1\big) \tag{3.1}$$

holds with C depending on Ω, T and the physical parameters.

Proof. Observe that it suffices to show that a solution $(\phi, \theta) \in W_2^{2,1}(Q) \times W_2^{2,1}(Q)$ to auxiliary problem (2.1)-(2.4) with $-1 \leq \phi_0 \leq 1$ a.e. in Ω satisfies $-1 \leq \phi \leq 1$. In fact, if $-1 \leq \phi \leq 1$ by definition of the operator Π we have that $\Pi(\phi) = \phi$ and, subsequently, (ϕ, θ) will be a solution of the original problem (1.1)-(1.4).

First, we prove that if $\phi_0 \leq 1$ a.e. in Ω then $\phi(t) \leq 1$ for all $t \in [0,T]$ and a.e. in Ω. Let us consider the positive part of $(\phi - 1)$ namely $(\phi - 1)^+ = \max(\phi - 1, 0)$. According to [7], we have that $\nabla(\phi - 1)^+ = \nabla\phi$ if $\phi - 1 \geq 0$ and $\nabla(\phi - 1)^+ = 0$ otherwise. Similarly, we have $(\phi - 1)^+_t = \phi_t$ if $\phi - 1 \geq 0$ and $(\phi - 1)^+_t = 0$ otherwise.

Multiplying equation (2.1) by $(\phi - 1)^+$ and integrating over $\Omega \times (0,t)$, for any $0 \leq t \leq T$, we obtain

$$\begin{aligned} &\|(\phi - 1)^+(t)\|^2_{L^2(\Omega)} + \epsilon^2 \int_0^t \|\nabla(\phi - 1)^+\|^2_{L^2(\Omega)} ds \\ &= \|(\phi_0 - 1)^+\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega \Big(\phi - \phi^3 + \frac{(\theta_M - \theta)}{\delta}(1 - \Pi(\phi)^2)^2\Big)(\phi - 1)^+ \,dx\,ds. \end{aligned}$$

Since $\phi_0 \leq 1$ one has that $\|(\phi_0 - 1)^+\|_{L^2(\Omega)} = 0$. Moreover, if $\phi < 1$ the last integral vanishes. Now, observe that if $\phi \geq 1$ we have that $(\phi - \phi^3)(\phi - 1)^+ = \phi(1 - \phi^2)(\phi - 1)^+ \leq 0$ and $\Pi(\phi) = 1$. Thus $(1 - \Pi(\phi)^2)^2 = 0$ and so we can conclude that

$$\|(\phi - 1)^+(t)\|^2_{L^2(\Omega)} \leq 0, \quad \text{for all } 0 \leq t \leq T.$$

Therefore, $(\phi - 1)^+(t) = 0$ for all $0 \leq t \leq T$ and a.e. in Ω, which implies that $\phi(t) \leq 1$ for all $0 \leq t \leq T$ and a.e. in Ω.

Next, we prove that if $\phi_0 \geq -1$ a.e. in Ω then $\phi(t) \geq -1$ for all $t \in [0,T]$ and a.e. in Ω. For this we consider the negative part of $(\phi + 1)$ namely $(\phi + 1)^- = \max(-(\phi + 1), 0)$. By multiplying equation (2.1) by $-(\phi + 1)^-$ we obtain

$$\begin{aligned} &\|(\phi + 1)^-(t)\|^2_{L^2(\Omega)} + \epsilon^2 \int_0^t \|\nabla(\phi + 1)^-\|^2_{L^2(\Omega)} ds \\ &= \|(\phi_0 + 1)^-\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega \big(\phi - \phi^3 + \frac{(\theta_M - \theta)}{\delta}(1 - \Pi(\phi)^2)^2\big)\big(-(\phi + 1)^-\big) \,dx\,ds. \end{aligned}$$

Similarly as before, since $\phi_0 \geq -1$ we have that $\|(\phi_0+1)^-\|_{L^2(\Omega)} = 0$. Moreover, if $\phi \geq -1$ the last integral vanishes. Now, observe that if $\phi < -1$ we have that $(\phi - \phi^3)\big(-(\phi+1)^-\big) = \phi(1-\phi)(1+\phi)\big(-(\phi+1)^-\big) \leq 0$ and $\Pi(\phi) = -1$. Thus $(1-\Pi(\phi)^2)^2 = 0$ and we deduce

$$\|(\phi+1)^-(t)\|^2_{L^2(\Omega)} \leq 0, \quad \text{for all } 0 \leq t \leq T.$$

Therefore, $(\phi+1)^-(t) = 0$ for all $0 \leq t \leq T$ and a.e. in Ω, which implies that $\phi(t) \geq -1$ for all $0 \leq t \leq T$ and a.e. in Ω. The proof is then complete. □

We will prove stability of the solutions which will give us uniqueness at the same time. We will denote by C a positive constant that may change from one relation to another.

Theorem 3.2. *Let be given functions satisfying: ϕ_0^i, $\theta_0^i \in H^{1+\gamma}(\Omega)$ with $1/2 < \gamma \leq 1$, $\frac{\partial \phi_0^i}{\partial n} = \frac{\partial \theta_0^i}{\partial n} = 0$ a.e. on $\partial\Omega$ and such that $-1 \leq \phi_0^i \leq 1$ a.e. in Ω, $i = 1, 2$. Let (ϕ_i, θ_i) be the corresponding solutions to problem* (1.1)-(1.4). *Then the following stability estimate holds*

$$\|\phi_1 - \phi_2\|_{W_2^{2,1}(Q)} + \|\theta_1 - \theta_2\|_{W_2^{2,1}(Q)} \leq C\big(\|\phi_0^1 - \phi_0^2\|_{H^1(\Omega)} + \|\theta_0^1 - \theta_0^2\|_{H^1(\Omega)}\big),$$

where C depends on $\|\phi_i\|_{W_2^{2,1}(Q)}$ and $\|\theta_i\|_{W_2^{2,1}(Q)}$.

Proof. We observe that $\phi = \phi_1 - \phi_2$ and $\theta = \theta_1 - \theta_2$ verify the following problem

$$\begin{aligned}\phi_t - \epsilon^2 \Delta\phi =& \phi(1 - (\phi_1^2 + \phi_1\phi_2 + \phi_2^2)) \\ &+ \frac{(\theta_M - \theta_1)}{\delta}(1-\phi_1^2)^2 - \frac{(\theta_M - \theta_2)}{\delta}(1-\phi_2^2)^2 \quad \text{in } Q,\end{aligned} \tag{3.2}$$

$$\theta_t - K\Delta\theta + b\theta = \frac{\delta}{2}\phi_t + \alpha\phi \quad \text{in } Q, \tag{3.3}$$

$$\frac{\partial\phi}{\partial n} = 0, \quad \frac{\partial\theta}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{3.4}$$

$$\phi(0) = \phi_0^1 - \phi_0^2 = \phi_0, \quad \theta(0) = \theta_0^1 - \theta_0^2 = \theta_0 \quad \text{in } \Omega. \tag{3.5}$$

Now, using the identity $(1-\phi_1^2)^2 - (1-\phi_2^2)^2 = \phi(\phi_1+\phi_2)(\phi_1^2+\phi_2^2-2)$ equation (3.2) can be written as

$$\begin{aligned}\phi_t - \epsilon^2\Delta\phi = \phi\big(1 - (\phi_1^2 + \phi_1\phi_2 + \phi_2^2)\big) + \frac{\theta_M}{\delta}\phi(\phi_1+\phi_2)(\phi_1^2+\phi_2^2-2) \\ + \frac{1}{\delta}\theta_1\phi(\phi_1+\phi_2)(\phi_1^2+\phi_2^2-2) + \frac{1}{\delta}\theta(1-\phi_2^2)^2.\end{aligned}$$

Since $|\phi_i| \leq 1$, from L^p-theory of parabolic equations we have

$$\|\phi\|_{W_2^{2,1}(Q)} \leq C\big(\|\phi_0\|_{H^1(\Omega)} + \|\phi\|_{L^2(Q)} + \|\theta\|_{L^2(Q)} + \|\theta_1\phi\|_{L^2(Q)}\big)$$

and

$$\begin{aligned}\|\theta\|_{W_2^{2,1}(Q)} &\leq C\big(\|\theta_0\|_{H^1(\Omega)} + \|\phi_t\|_{L^2(Q)} + \|\phi\|_{L^2(Q)}\big) \\ &\leq C\big(\|\theta_0\|_{H^1(\Omega)} + \|\phi_0\|_{H^1(\Omega)} + \|\phi\|_{L^2(Q)} + \|\theta\|_{L^2(Q)} + \|\theta_1\phi\|_{L^2(Q)}\big).\end{aligned}$$

The L^2-norm of $\theta_1\phi$ can be bounded by using Hölder inequality and the Sobolev embedding

$$\begin{aligned}\|\theta_1\phi\|_{L^2(Q)} &\le \Big(\int_0^T \|\theta_1\|^2_{L^4(\Omega)}\|\phi\|^2_{L^4(\Omega)}dt\Big)^{1/2}\\ &\le C\|\theta_1\|_{L^\infty(0,T;H^1(\Omega))}\|\phi\|_{L^2(0,T;H^1(\Omega))}.\end{aligned}$$

Thus, we conclude that

$$\|\phi\|_{W_2^{2,1}(Q)} + \|\theta\|_{W_2^{2,1}(Q)} \le C\big(\|\phi_0\|_{H^1(\Omega)} + \|\theta_0\|_{H^1(\Omega)} + \|\theta\|_{L^2(Q)} + \|\phi\|_{L^2(0,T;H^1(\Omega))}\big). \tag{3.6}$$

To obtain estimates for ϕ and θ we return to equations (3.2)-(3.3) and use standard techniques. We first deduce that

$$\frac{1}{2}\frac{d}{dt}\|\phi\|^2_{L^2(\Omega)} + \epsilon^2\|\nabla\phi\|^2_{L^2(\Omega)} \le C\big(\|\phi\|^2_{L^2(\Omega)} + \|\theta\|^2_{L^2(\Omega)} + \int_\Omega |\theta_1|\,|\phi|^2dx\big),$$

where we used that $|\phi_i| \le 1$.

The last term can be bounded by using Hölder and Young inequalities

$$\begin{aligned}\int_\Omega |\theta_1|\,|\phi|^2dx &\le \|\theta_1\|_{L^4(\Omega)}\|\phi\|_{L^4(\Omega)}\|\phi\|_{L^2(\Omega)}\\ &\le C\|\theta_1\|^2_{L^\infty(0,T;H^1(\Omega))}\|\phi\|^2_{L^2(\Omega)} + \frac{\epsilon^2}{2}\|\nabla\phi\|^2_{L^2(\Omega)}.\end{aligned}$$

By rearranging terms we arrive at

$$\frac{d}{dt}\|\phi\|^2_{L^2(\Omega)} + \epsilon^2\|\nabla\phi\|^2_{L^2(\Omega)} \le C\big(\|\phi\|^2_{L^2(\Omega)} + \|\theta\|^2_{L^2(\Omega)}\big).$$

Next, by multiplying equation (3.3) by θ, we obtain, for any $\eta > 0$,

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{L^2(\Omega)} + K\|\nabla\theta\|^2_{L^2(\Omega)} \le \eta\|\phi_t\|^2_{L^2(\Omega)} + C\big(\|\phi\|^2_{L^2(\Omega)} + \|\theta\|^2_{L^2(\Omega)}\big).$$

By integrating in time we deduce from the above relations that

$$\begin{aligned}&\|\phi\|^2_{L^2(\Omega)} + \|\theta\|^2_{L^2(\Omega)} + \int_0^t \big(\|\nabla\phi\|^2_{L^2(\Omega)} + \|\nabla\theta\|^2_{L^2(\Omega)}\big)ds\\ &\le C\big(\|\phi_0\|^2_{L^2(\Omega)} + \|\theta_0\|^2_{L^2(\Omega)} + \int_0^t (\|\phi\|^2_{L^2(\Omega)} + \|\theta\|^2_{L^2(\Omega)})ds\big) + \eta\|\phi_t\|^2_{L^2(Q)}\end{aligned}$$

Taking η small enough and using (3.6) yields

$$\begin{aligned}&\|\phi\|^2_{L^2(\Omega)} + \|\theta\|^2_{L^2(\Omega)} + \int_0^t \big(\|\nabla\phi\|^2_{L^2(\Omega)} + \|\nabla\theta\|^2_{L^2(\Omega)}\big)ds\\ &\le C\big(\|\phi_0\|^2_{H^1(\Omega)} + \|\theta_0\|^2_{H^1(\Omega)} + \int_0^t (\|\phi\|^2_{L^2(\Omega)} + \|\theta\|^2_{L^2(\Omega)})ds\big).\end{aligned}$$

Gronwall's lemma implies

$$\begin{aligned}&\|\phi\|^2_{L^2(\Omega)} + \|\theta\|^2_{L^2(\Omega)} + \int_0^t \big(\|\nabla\phi\|^2_{L^2(\Omega)} + \|\nabla\theta\|^2_{L^2(\Omega)}\big)ds\\ &\le C\big(\|\phi_0\|^2_{H^1(\Omega)} + \|\theta_0\|^2_{H^1(\Omega)}\big).\end{aligned}$$

By plugging this in (3.6) we obtain the desired stability result. □

Corollary 3.3. *Let assumptions in theorem 3.1 be fulfilled. Then there exists a unique solution $(\phi,\theta) \in W_2^{2,1}(Q) \times W_2^{2,1}(Q)$ to problem* (1.1)-(1.4).

Remark 3.4. The results stated in Theorems 3.1 and 3.2 still hold, exactly with the same proofs, for initial conditions ϕ_0 and θ_0 in any functional space including $H^1(\Omega)$ and for which it makes sense to require that $\frac{\partial \phi_0}{\partial n} = \frac{\partial \theta_0}{\partial n} = 0$ a.e. on $\partial\Omega$ in order to apply L^p-theory of the parabolic linear equations. Moreover, a weaker version of theorems hold, with a natural weaker formulation of (1.1)-(1.4), for initial conditions ϕ_0 and θ_0 just in $H^1(\Omega)$. For the proof, it is enough to take sequences in $H^{1+\gamma}(\Omega)$ with $1/2 < \gamma \leq 1$ satisfying the compatibility condition and converging to ϕ_0 and θ_0 in $H^1(\Omega)$, and then to consider a sequence of approximate problems with these initial conditions. Since the sequence of approximate solutions will satisfy estimate (3.1), it will be possible to pass to the limit and recover a solution of the original problem.

We will prove a regularity result under the additional assumption that the initial data are smooth enough by using L^p-theory of the parabolic linear equations together with bootstrapping arguments.

Theorem 3.5. *Let $p \geq 2$. Let be given functions satisfying: ϕ_0, $\theta_0 \in W_p^{2-\frac{2}{p}}(\Omega) \cap H^{1+\gamma}(\Omega)$ with $1/2 < \gamma \leq 1$, $\frac{\partial \phi_0}{\partial n} = \frac{\partial \theta_0}{\partial n} = 0$ a.e. on $\partial\Omega$ and such that $-1 \leq \phi_0 \leq 1$ a.e. in Ω. Then the unique solution to problem* (1.1)-(1.4) *satisfies*

$$(\phi, \theta) \in W_p^{2,1}(Q) \times W_p^{2,1}(Q).$$

Proof. According to theorem 3.1 and corollary 3.3 there exists a unique solution $(\phi, \theta) \in W_2^{2,1}(Q) \times W_2^{2,1}(Q)$ to problem (1.1)-(1.4). Since $|\phi| \leq 1$ and $W_2^{2,1}(Q) \hookrightarrow L^{10}(Q)$ from L^p-theory of parabolic equations applied to the phase-field equation we have that $\phi \in W_{10}^{2,1}(Q)$ and, subsequently, from the temperature equation we conclude $\theta \in W_{10}^{2,1}(Q)$. Now, since $W_{10}^{2,1}(Q) \hookrightarrow L^\infty(Q)$ by applying again L^p-theory of parabolic equations we conclude that $\phi \in W_p^{2,1}(Q)$ for any $p \geq 2$ and consequently $\theta \in W_p^{2,1}(Q)$ for any $p \geq 2$. $\square$

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