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STABILITY OF A LINEAR OSCILLATOR WITH VARIABLE PARAMETERS

A. O. Ignatyev

Abstract. A criterion of asymptotic stability for a linear oscillator with variable parameters is obtained. It is shown that this criterion is close to a necessary and sufficient conditions of asymptotic stability. An instability theorem is proved, and a mechanical example is considered.

1. Introduction

Consider an oscillator described by the following differential equation

$$\ddot{x} + f(t)\dot{x} + g(t)x = 0\,, \tag{1}$$

where the damping and rigidity coefficients $f(t)$ and $g(t)$ are continuous and bounded functions of the time t. Most of the theories examining a stability problem of the zero solution are based on the Lyapunov stability and instability theorems and the corresponding Lyapunov function is assumed as an energy-type function

$$V = \frac{1}{2}c_1(t)\dot{x}^2 + \frac{1}{2}c_2(t)x^2\,,$$

where $c_1(t), c_2(t)$ are time variable functions. In [6], A. P. Merkin considered the case $c_1(t) = c_2(t) = 1$ and stability conditions were obtained only for constant f and g. An extension was done in [15] for periodic functions $f(t)$ and $g(t)$. By means of a Lyapunov function which is a quadratic form with respect to x and $\dot{x}$, V. M. Starzhinsky [10] (assuming that $0 < l \le f(t) \le L,\ 0 < m \le g(t) \le M$) obtained sufficient conditions of asymptotical stability for the solution

$$x = 0, \quad \dot{x} = 0 \tag{2}$$

of equation (1). They are written as restrictions to the constants l, L, m, M. We note that for a linear n-dimensional system $\dot{x} = A(t)x$, the problem of asymptotical stability has been considered by many of authors [1-5, 11-14], but the obtained conditions on the elements of the matrix $A(t)$ are rather restrictive.

In this paper sufficient asymptotic stability conditions of the solution (2) are obtained which are close to necessary and sufficient conditions of stability.

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2. Criterion of the asymptotic stability

We suppose that $g(t)$ is continuously differentiable and that the inequalities

$$|f(t)| < M_1, \quad |g(t)| < M_2, \quad |\dot g(t)| < M_3 \tag{3}$$

hold for $t \in R_+ = [0;\infty)$.

Theorem 1. *If the conditions*

$$g(t) > \alpha_1 > 0, \qquad p(t) = \frac{1}{2}\frac{\dot g(t)}{g(t)} + f(t) > \alpha_2 > 0 \tag{4}$$

are fulfilled, then the solution (2) of the differential equation (1) is uniformly asymptotically stable.

Proof. Let us consider the function

$$V_1 = \frac{1}{2}\left(x^2 + 2\beta\frac{x\dot x}{\sqrt{g(t)}} + \frac{\dot x^2}{g(t)}\right) \qquad (\beta = const).$$

Its time derivative along the solutions of equation (1) has the form

$$\dot V_1 = \frac{1}{\sqrt{g(t)}}\left(\left(-\frac{p(t)}{\sqrt{g(t)}} + \beta\right)\dot x^2 - \beta p(t)x\dot x - \beta g(t)x^2\right)$$

If we take $\beta > 0$ sufficiently small, then V_1 is positive definite ($V_1 > 0$) and $\dot V_1$ is negative definite. Carrying out conditions $V_1 > 0, \dot V_1 < 0$, we can take

$$0 < \beta < \min\left\{1,\ \frac{\alpha_2}{2\sqrt{M_2}},\ \frac{8\alpha_1^3\alpha_2}{(M_3 + 2\alpha_1 M_1)^2\sqrt{M_2}}\right\}.$$

Thus all conditions of Lyapunov theorem [6, 9] are fulfilled and the zero solution of equation (1) is uniformly asymptotically stable.

Corollary 1. *If conditions (4) are fulfilled, then there exist positive numbers B, α such, that for $t > t_0 \geq 0$ inequalities*

$$|x(t) < B\exp\left[-\alpha(t - t_0)\right], \qquad |\dot x(t)| < B\exp\left[-\alpha(t - t_0)\right]$$

hold.

Proof. The results of Theorem 1 and [8] imply Corollary 1.

Remark 1. Inequalities (4) are sufficient conditions for asymptotical stability of the trivial solution. They are close to necessary and sufficient ones in the following sense. If conditions (4) are changed to one of the next

$$g(t) > \alpha_1 > 0, \quad p(t) < -\alpha_2 < 0, \tag{5}$$

$$g(t) < -\alpha_1 < 0, \quad p(t) > \alpha_2 > 0, \tag{6}$$

$$g(t) < -\alpha_1 < 0, \quad p(t) < -\alpha_2 < 0, \tag{7}$$

then (2) is unstable.

Proof. Let (5) be fulfilled. Let us take $\beta < 0$ with $|\beta|$ so small that $V_1 > 0,\ \dot V_1 > 0$. This proves the instability of the zero solution. If one of the conditions (6), (7) is true, then consider the Lyapunov function

$$V_2 = \frac{1}{2}\left(x^2 + 2\beta\frac{x\dot x}{\sqrt{-g(t)}} + \frac{\dot x^2}{g(t)}\right),$$

whose time derivative along the solutions of equation (1) has the form

$$\dot V_2 = \frac{1}{\sqrt{-g(t)}}\left(\left(\frac{p(t)}{\sqrt{-g(t)}} + \beta\right)\dot x^2 - \beta p(t)x\dot x - \beta g(t)x^2\right).$$

Choosing $|\beta|$ small enough, one can make the function $\dot V_2$ of fixed sign (in the case (6) we suppose $\beta > 0$, in the case (7) $\beta < 0$). But V_2 changes its sign. Thus according to [9], the trivial solution of (1) is unstable.

Remark 2. If $f(t)$ and $g(t)$ are constants, then the conditions (4) amount to the usual Routh-Hurwitz criterion.

3. Instability of the zero solution

Now let us obtain instability conditions. Noting $\dot x = y$, we get the system

$$\dot x = y, \qquad \dot y = -g(t)x - f(t)y \tag{8}$$

which is equivalent to equation (1). It has the trivial solution

$$x = 0, \quad y = 0 \tag{9}$$

Theorem 2. *The solution (9) of the system (8) is unstable if there exists some t_0 such that, for each $t > t_0$, one of the following conditions*

$$D(t) = \frac{1}{4}f^2(t) + g(t) \le 0, \tag{10}$$

$$D(t) > 0, \quad 4f(t)D(t) + \frac{1}{2}\dot f(t)f(t) + \dot g(t) - (\dot f(t) + f^2(t) + 4D(t))\sqrt{D(t)} < 0, \tag{11}$$

$$D(t) > 0, \quad 4f(t)D(t) + \frac{1}{2}\dot f(t)f(t) + \dot g(t) + (\dot f(t) + f^2(t) + 4D(t))\sqrt{D(t)} < 0 \tag{12}$$

holds.

Proof. Let ϵ be an arbitrary positive number. We shall show that, for any sufficiently small $\delta > 0$, there exists some x_0, y_0 with

$$|x_0| < \delta, \qquad |y_0| < \delta \tag{13}$$

and some $T > 0$ such that, for $t = t_0 + T$, the trajectory $x(t), y(t)$ $(x(t_0) = x_0, y(t_0) = y_0)$ reaches the boundary of the domain

$$|x| < \epsilon, \quad |y| < \epsilon \tag{14}$$

Consider the function $V = xy$. Its time derivative along the solutions of (8) has the form

$$\dot{V} = y^2 - f(t)xy - g(t)x^2.$$

Take $x_0 > 0,\ y_0 > 0$ satisfying (13) and such that

$$\dot{V}(t_0, x_0, y_0) = y_0^2 - f(t_0)x_0y_0 - g(t_0)x_0^2 > 0.$$

Consider the trajectory $x(t), y(t)$ of (8) with initial data $x(t_0) = x_0, y(t_0) = y_0$. Without loss of generality we can assume $D(t_0) < 0$. Let $[t_0; t_1]$, $[t_2; t_3]$,..., $[t_{2n}; t_{2n+1}]$,... be segments on which condition (10) holds and $(t_1; t_2)$, $(t_3; t_4)$,...,$(t_{2n-1}; t_{2n})$, ... be the intervals on which inequalities (11) or (12) are valid. As $\dot{V} \geq 0$ on $[t_0; t_1]$, the trajectory is staying in the domain $xy \geq x_0y_0$ on this segment. Now let us consider $x(t), y(t)$ when $t \in (t_1; t_2)$. On this interval $\dot{V}$ changes its sign. $\dot{V} = 0$ if

$$y = (\frac{1}{2}f + \sqrt{D})x \tag{15}$$

or

$$y = (\frac{1}{2}f - \sqrt{D})x \tag{16}$$

and $\dot{V} > 0$ if

$$y > (\frac{1}{2}f + \sqrt{D})x \tag{17}$$

or

$$y < (\frac{1}{2}f - \sqrt{D})x. \tag{18}$$

Let $t_* \in (t_1; t_2)$ be such moment of time, that $y(t_*) = (\frac{1}{2}f(t_*) + \sqrt{D(t_*)})x(t_*)$, i.e. the point of the trajectory belongs to the straight line (15) when $t = t_*$.

We shall show that $x(t), y(t)$ satisfy the inequality (17) if $t \in (t_*; t_* + \Delta t)$ and $\Delta t > 0$ is sufficiently small. To this end, we write $\ddot{V}$ under the condition $\dot{V} = 0$:

$$\ddot{V}\Big|_{(15)} = -(4f(t)D(t) + \frac{1}{2}\dot{f}(t)f(t) + \dot{g}(t) + (\dot{f}(t) + f^2(t) + 4D(t))\sqrt{D(t)})x^2$$

Taking into account conditions (12), we obtain $\ddot{V} > 0$ under $\dot{V} = 0$, i.e. the trajectory belongs to the domain $\dot{V} > 0$ when $t \in (t_*; t_* + \Delta t)$.

If $t'_* \in [t_1; t_2]$ is such moment of time, that $y(t'_*) = (\frac{1}{2}f(t'_*) - \sqrt{D(t'_*)})x(t'_*)$ (i.e. the point of the trajectory belongs to the straight line (16) when $t = t'_*$), then, using conditions (11), we obtain that $x(t), y(t)$ satisfy the inequality (18) for $t \in (t'_*; t'_* + \Delta t)$ where $\Delta t > 0$ is sufficiently small. Thus it is proved the trajectory lies in the domain $\dot{V} \geq 0$ when $t \in [t_1; t_2]$.

One can show analogously that the point $x(t), y(t)$ belongs to the set $\dot{V} \geq 0$ when $t \in [t_n; t_{n+1}]$ $(n = 3, 4, ...)$. It means that for the trajectory the inequality $\dot{V}(x(t), y(t)) \geq 0$ holds for every $t \geq t_0$. But from the last inequality it follows, that $x(t)y(t) \geq x_0y_0$ for every $t \geq t_0$.

Let us show that the boundary of (14) is reached for the finite interval of time. Consider on the plane x, y the domain

$$\Omega = \{x, y :\ xy \geq x_0y_0,\ 0 < x < \epsilon,\ 0 < y < \epsilon\}$$

We shall estimate the time for which the trajectory $x(t), y(t)$ can stay in Ω. In this domain the inequality $y \geq \epsilon^{-1} x_0 y_0$ holds, so from the first of the equations (8) we obtain $x(t) \geq x_0 + \epsilon^{-1} x_0 y_0 (t - t_0)$. This relation implies that the interval of time, for which the trajectory lies in Ω, can be estimated by the number $T = \epsilon(\epsilon - x_0) x_0^{-1} y_0^{-1}$. The trajectory cannot leave Ω intersecting the hyperbola $xy = x_0 y_0$, hence one of the inequalities (14) is broken. This completes the proof.

Example. N. N. Moiseyev [7] wrote down a differential equation of small plane oscillations of a rocket, whose centre of mass moves rectilinearly with constant velocity. It has a form (1), where $f(t) = ae^{-\alpha t}$, $g(t) = be^{-\alpha t}$; $\alpha > 0$, a, b are positive constants, x is an angle of attack. The author obtained sufficient conditions for stability of the zero solution and showed that in the case of plane small oscillations of a rocket, these conditions are not fulfilled. But he did not prove instability of the small oscillations.

Let us apply Theorem 2 in order to prove instability of the solution (2). Actually, there exists $t_0 > 0$ such that inequalities (12) hold for $t \geq t_0$. This proves instability of the small oscillations.

Theorem 3. *If in equation (1) the functions $f(t), g(t)$ are vanishing, i.e.*

$$\lim_{t\to\infty} f(t) = 0, \quad \lim_{t\to\infty} g(t) = 0,$$

then the equilibrium (2) cannot be uniformly stable.

Proof. Consider a system of differential equations (8) which has the trivial solution (9). Let us take arbitrary $\epsilon > 0$. We shall show that for every $\delta > 0$, there exist x_0, y_0, satisfying (13) and some $t_0 \geq 0$ such that the trajectory $x(t), y(t)$, where $x(t_0) = x_0,\ y(t_0) = y_0$, leaves the domain (14) with time increasing. Denote

$$\sigma(t) = \frac{1}{2}|f(t)| + \frac{1}{2}\sqrt{f^2(t) + 4|g(t)|}$$

The functions $f(t), g(t)$ are vanishing, hence $\sigma(t)$ is also vanishing. Let us choose such $t_0 > 0$ that $\sigma(t) < x_0 y_0 \epsilon^{-2}$ holds for $t \geq t_0$. Using the auxiliary function $V = xy$, the trajectory $x(t), y(t)$ will be disposed in the domain $\dot{V} > 0$ for $t \geq t_0$. Then, as it follows from Theorem 2 proof, there exists such time moment $t > t_0$, under which the trajectory leaves the domain (14). The proof is complete.

References

1. M.S.P. Eastham, *The Asymptotic Solution of Linear Differential Systems: Applications of the Levinson Theorem*, Oxford Science Publications, Oxford, 1989.
2. H.Ginfold, P.F.Hsieh, and Y.Sibuya, *Globally Analytic Simplification and the Levinson Theorem*, J. Math. Anal. Appl. **182** (1994), 269–286.
3. A.A. Lebedev, *On the method of Lyapunov functions construction*, J. of Appl. Math. and Mech. **21** (1957), 121–124. (Russian)
4. N. Levinson, *The asymptotic nature of solutions of linear ordinary differential equations*, Duke Math. J. **15** (1948), 111–126.
5. N.Ya. Lyaschenko, *On the asymptotic stability of solutions of the system of differential equations*, Dokl. Akad. Nauk SSSR **96** (1954), 237–239. (Russian)
6. A.P. Merkin, *Stability of Motion*, "Nauka", Moscow, 1976. (Russian)
7. N.N. Moiseyev, *Asymptotic Methods of Nonlinear Mechanics*, Nauka, Moscow, 1969. (Russian)

8. K.P. Persidsky, *On the stability theory of integrals of differential equations*, Izvestiya fiz.-mat. ob. Kaz. Univ. **8** (1936–1937), 47–85. (Russian)
9. N. Rouche, P. Habets, M. Laloy, *Stability Theory by Liapunov's Direct Method*, Springer-Verlag, New York, 1977.
10. V.M. Starzhinsky, *Sufficient conditions of stability of the mechanical system with one degree of freedom*, J. of Appl. Math. and Mech. **16** (1952), 369–374. (Russian)
11. W.T. Trench, *On t_∞ Quasi-Similarity of Linear Systems*, Annali di Matematica pura ed applicata **142, No.4** (1985), 293–302.
12. ______, *Asymptotic Behavior of Solutions of Asymptotically Constant Coefficient Systems of Linear Differential Equations*, Computers Math. Applic. **30, No.12** (1995), 111–117.
13. B.S. Razumikhin, *Estimates of solutions of a system of perturbed differential equations with variable parameters*, J. of Appl. Math. and Mech **21** (1957), 119–120. (Russian)
14. Van Dan-Chzhy, *On the stability of the zero solution of the system of linear differential equations with variable parameters*, Vestnik Moskov. Univ. Mat. **24** (1969), 86–92. (Russian)
15. V.A. Yakubovich and V.M. Starzhinsky, *Linear differential equations with periodic coefficients and their applications*, "Nauka", Moscow, 1972. (Russian)

Alexander O. Ignatyev
Institute for Applied Mathematics and Mechanics
R.Luxemburg Street 74, Donetsk-340114, Ukraine
E-mail address: ignat@iamm.ac.donetsk.ua