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A NECESSARY AND SUFFICIENT CONDITION FOR THE EXISTENCE OF POSITIVE SOLUTIONS TO SINGULAR BOUNDARY-VALUE PROBLEMS OF HIGHER ORDER DIFFERENTIAL EQUATIONS

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Abstract. By constructing some special cones and using fixed point theorem of cone expansion and compression, this paper presents some necessary and sufficient conditions for the existence of C^{4n-2} positive solutions to a class of singular boundary-value problems. Some examples are presented to illustrate our main results.

1. Introduction and Preliminary

Singular boundary-value problems (SBVP) for ordinary differential equations arise in the field of gas dynamics, fluid mechanics, theory of boundary layer, and so on. These problems are also an important branch in the field of differential equations [1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. In recent years, the positive solutions of singular boundary-value problems for higher order nonlinear differential equations have been studied extensively; see for example [3, 7, 8, 9, 11, 12, 13, 14, 15, 16].

For instance, in the superlinear case, Shi [9] obtained some necessary and sufficient conditions for the existence of $C^2[0,1]$ or $C^3[0,1]$ positive solutions of differential equations under some conditions. In the sublinear case, Wei [12] gave a necessary and sufficient condition for the existence of C^2 and C^3 positive solutions by means of the method of lower and upper solutions with the maximum principle for

$$\begin{gathered} x^{(4)}(t) = f(t, x(t)), \quad \text{for all } 0 < t < 1, \\ x(0) = x(1) = x''(0) = x''(1) = 0. \end{gathered}$$

In [16], Zhang discussed the boundary-value problem

$$\begin{gathered} x^{(4n)}(t) = f(t, x(t)), \quad \text{for all } 0 < t < 1, \\ x^{(2k)}(0) = x^{(2k)}(1) = 0, \quad k = 0, 1, 2, \dots, 2n-1, \end{gathered} \tag{1.1}$$

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by using the method of lower and upper solutions, where $f \in C[(0,1) \times (0,+\infty), [0,+\infty)]$, $f(t,x) \not\equiv 0$, and there exist constants λ, μ, N, M with $-\infty < \lambda \leq 0 \leq \mu < 1$, $\frac{2(\mu-\lambda)}{1+\mu} < 1$, $0 < N \leq 1 \leq M$ such that for any $0 < t < 1$, $x \in (0,\infty)$, satisfying

$$\begin{gathered} c^{\mu} f(t,x) \leq f(t,cx) \leq c^{\lambda} f(t,x), \quad 0 \leq c \leq N, \\ c^{\lambda} f(t,x) \leq f(t,cx) \leq c^{\mu} f(t,x), \quad c \geq M. \end{gathered} \tag{1.2}$$

The main results of [16] are the following two theorems.

Theorem 1.1. *Under assumption* (1.2), (1.1) *has a* C^{4n-2} *positive solution if and only if*

$$\begin{gathered} 0 < \int_0^1 t(1-t) f(t, t(1-t)) dt < +\infty, \\ \lim_{t \to 0+} t \int_t^1 (1-s) f(s, s(1-s)) ds = 0, \\ \lim_{t \to 1-} t \int_t^1 (1-s) f(s, s(1-s)) ds = 0. \end{gathered}$$

Theorem 1.2. *Under assumption* (1.2), (1.1) *has a* C^{4n-1} *positive solution if and only if*

$$0 < \int_0^1 f(t, t(1-t)) dt < +\infty.$$

Note that when $n = 1$, Theorems 1.1 and 1.2 are the results in [9]. Inspired by above results, this paper investigates the boundary-value problem

$$\begin{gathered} u^{(4n)}(t) = f(t, u(t), u^{(4n-2)}(t)), \quad 0 < t < 1, \\ u(0) = u(1) = 0, \\ R_1(u) =: a u^{(2k)}(0) - b u^{(2k+1)}(0) = 0, \\ R_2(u) =: c u^{(2k)}(1) + d u^{(2k+1)}(1) = 0, \quad k = 1, 2, \dots, 2n-1. \end{gathered} \tag{1.3}$$

where $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$, $\Delta = ac+ad+bc > 0$, and $f \in C[(0,1) \times (0,+\infty) \times (-\infty,0), [0,+\infty)]$ is quasi-homogeneous with respect to the last two variables, that is, there are constants λ, μ, α, β; N_1, M_1, N_2, M_2 with $-\infty < \lambda \leq 0 \leq \mu < \infty$, $0 \leq \alpha \leq \beta < 1$, $\mu + \beta < 1$; $0 < N_1 \leq 1 \leq M_1$, $0 < N_2 \leq 1 \leq M_2$ such that for any $0 < t < 1$, $u > 0$, $v \leq 0$ satisfying

$$\begin{gathered} \bar{c}^{\mu} f(t,u,v) \leq f(t,\bar{c}u,v) \leq \bar{c}^{\lambda} f(t,u,v), \quad 0 < \bar{c} \leq N_1, \\ \bar{c}^{\lambda} f(t,u,v) \leq f(t,\bar{c}u,v) \leq \bar{c}^{\mu} f(t,u,v), \quad \bar{c} \geq M_1; \\ \bar{c}^{\beta} f(t,u,v) \leq f(t,u,\bar{c}v) \leq \bar{c}^{\alpha} f(t,u,v), \quad 0 < \bar{c} \leq N_2, \\ \bar{c}^{\alpha} f(t,u,v) \leq f(t,u,\bar{c}v) \leq \bar{c}^{\beta} f(t,u,v), \quad \bar{c} \geq M_2. \end{gathered} \tag{1.4}$$

A typical function satisfying the above hipothesis is

$$f(t,u,v) = \sum_{i=1}^{n} p_i(t) u^{\alpha_i} (-v)^{\beta_i},$$

where $p_i(t) \in C[(0,1), R^+]$, $\lambda = \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_k < 0 < \alpha_{k+1} \leq \dots \leq \alpha_n = \mu$, $0 \leq \beta_i < 1$, $k = 1, 2, \dots, n-1$, $i = 1, 2, \dots, n$.

To the best of our knowledge, there is no paper that considers (1.3) with general boundary-value conditions. As a result, the goal of present paper discusses and treats the extension of focal boundary- value problems to more general n-th order boundary value problems and hence fill the gap in this area. The main features here are as follows. Firstly, the nonlinear term f include $u^{(4n-2)}$. Secondly, the boundary- value conditions are more extensive. Thirdly, the singularity of f on u is arbitrary.

The main techniques used in this paper are some new constructed construct cones and cone expansion and compression fixed point theorems. Comparing with previous literature to study the singular problems, neither the approximation method nor upper-lower solution approach is applied. In this paper, we obtain some necessary and sufficient conditions for the existence of C^{4n-2} positive solutions.

We say $u \in C^{4n-2}[0,1] \cap C^{4n}(0,1)$ is a $C^{4n-2}[0,1]$ positive solution of (1.3) if $u(t)$ satisfies (1.3) and $u(t) > 0$ for $t \in (0,1)$.

Now we state the following lemma from the literature which will be used in section 2.

Lemma 1.3 ([6]). *Let K be a cone of real Banach space E, Ω_1, Ω_2 be bounded open sets of E, $0 \in \overline{\Omega}_1 \subset \Omega_2$. Suppose that $A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ is completely continuous such that one of the following two conditions is satisfied:*

(i) $\|Ax\| \le \|x\|$ *for* $x \in K \cap \partial\Omega_1$; $\|Ax\| \ge \|x\|$ *for* $x \in K \cap \partial\Omega_2$,

(ii) $\|Ax\| \le \|x\|$ *for* $x \in K \cap \partial\Omega_2$; $\|Ax\| \ge \|x\|$ *for* $x \in K \cap \partial\Omega_1$.

Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2. Main Results

Theorem 2.1. *Suppose* (1.4) *holds and $b = d = 0$. Then* (1.3) *has a $C^{4n-2}[0,1]$ positive solution if and only if*

$$0 < \int_0^1 t(1-t)f(t, t(1-t), -1)dt < +\infty; \tag{2.1}$$

$$\lim_{t\to 0+} t\int_t^1 (1-s)f(s, s(1-s), -1)ds = 0; \tag{2.2}$$

$$\lim_{t\to 1-} t\int_t^1 (1-s)f(s, s(1-s), -1)ds = 0. \tag{2.3}$$

Theorem 2.2. *Suppose* (1.4) *holds and $b = 0, d > 0$. Then* (1.3) *has a $C^{4n-2}[0,1] \cap C^{4n-1}(0,1]$ positive solution if and only if*

$$0 < \int_0^1 tf(t, t(1-t), -1)dt < +\infty, \tag{2.4}$$

$$\lim_{t\to 0+} t\int_t^1 f(s, s(1-s), -1)ds = 0. \tag{2.5}$$

Theorem 2.3. *Suppose* (1.4) *holds and $b > 0, d = 0$. Then* (1.3) *has a $C^{4n-2}[0,1] \cap C^{4n-1}[0,1)$ positive solution if and only if*

$$0 < \int_0^1 (1-t)f(t, t(1-t), -1)dt < +\infty, \tag{2.6}$$

$$\lim_{t\to 1^-} (1-t)\int_0^t f(s, s(1-s), -1)ds = 0. \tag{2.7}$$

It is well known that

$$G(t,s)=\frac{1}{\Delta}\begin{cases}(b+as)[d+c(1-t)], & s<t;\\ (b+at)[d+c(1-s)], & t\le s,\end{cases} \tag{2.8}$$

is the Green function of homogeneous boundary-value problem

$$\begin{gathered}-u''(t)=0 \quad 0\le t\le 1,\\ au(0)-bu'(0)=0,\\ cu(1)+du'(1)=0.\end{gathered} \tag{2.9}$$

It is easy to see that

$$\begin{gathered}G(t,s)\ge \frac{[c(1-s)(b+as)+ads][c(1-t)(b+at)+adt]}{\Delta^2},\\ G(t,s)\le G(t,t),\quad G(t,s)\le G(s,s).\end{gathered} \tag{2.10}$$

Since

$$\frac{G(t,s)}{G(\tau,s)}=\begin{cases}\frac{(b+as)[d+c(1-t)]}{(b+a\tau)[d+c(1-s)]}, & \tau<s<t;\\ \frac{d+c(1-t)}{d+c(1-\tau)}, & s\le t,\tau;\\ \frac{b+at}{b+a\tau}, & t,\tau\le s;\\ \frac{(b+at)[d+c(1-s)]}{(b+as)[d+c(1-\tau)]}, & t<s<\tau,\end{cases}$$

we know that

$$G(t,s)\ge e(t)G(\tau,s), \tag{2.11}$$

where

$$e(t)=\frac{(b+at)[d+c(1-t)]}{(b+a)(c+d)}. \tag{2.12}$$

It follows from (2.8) that some special Green function of different homogeneous boundary-value problems corresponding to (2.9) are

$$G_1(t,s)=\begin{cases}s(1-t),\ s<t;\\ t(1-s),\ t\le s,\end{cases}\qquad (b=0,\ d=0) \tag{2.13}$$

$$G_2(t,s)=\frac{1}{c+d}\begin{cases}s[d+c(1-t)],\ s<t;\\ t[d+c(1-s)],\ t\le s,\end{cases}\qquad (b=0,\ d>0) \tag{2.14}$$

$$G_3(t,s)=\frac{1}{a+b}\begin{cases}(b+as)(1-t),\ s<t;\\ (b+at)(1-s),\ t\le s.\end{cases}\qquad (b>0,\ d=0) \tag{2.15}$$

Let $E=\{u\in C^{4n-2}[0,1]: u(0)=u(1)=0\}$, and define the norm $\|u\|=\max\{\|u\|_0,\|u\|_{4n-2}\}$, for all $u\in E$, where

$$\|u\|_0=\sup_{0\le t\le 1}|u(t)|,\quad \|u\|_{4n-2}=\sup_{0\le t\le 1}|u^{(4n-2)}(t)|,\quad \forall u\in E.$$

Then $(E,\|\cdot\|)$ is a Banach space. Define

$$\begin{aligned}P=\big\{u\in E: R_1(u)=R_2(u)=0,\ u(t)\ge 0,\ u^{(4n-2)}(t)\le e(t)u^{(4n-2)}(s)\le 0,\\ u(t)\ge -kt(1-t)u^{(4n-2)}(s),\forall t,s\in[0,1]\big\}.\end{aligned} \tag{2.16}$$

where $e(t)$ is given by (2.12), $R_1(u)$ and $R_2(u)$ are defined by (1.3), and

$$\begin{aligned} k =&\Big(2ac + 5bc + 5ad)(15abcd + 15b^2cd + 15abd^2 + 10b^2c^2 + 5abc^2 \\ &+ 5a^2cd + a^2c^2 + 10a^2d^2\Big)/\big(1800(a+b)(c+d)\big) \\ &\times \Big(\frac{5abc^2 + 10b^2c^2 + 10abcd + a^2c^2 + 10a^2d^2 + 5a^2cd}{30}\Big)^{2n-3} \frac{1}{\Delta^{4n-4}}. \end{aligned} \tag{2.17}$$

It is easy to see that P is a cone of E. From

$$\begin{aligned} u(t) &= \int_0^1 \dots \int_0^1 G_1(t, s_{2n-1}) G(s_{2n-1}, s_{2n-2}) \dots G(s_2, s_1)(-u^{(4n-2)}(s_1)) \\ &\quad \times ds_1 \dots ds_{2n-1} \\ &\le \int_0^1 G_1(t, s_{2n-1}) ds_{2n-1} \int_0^1 \frac{(b + as_{2n-2})[d + c(1 - s_{2n-2})]}{\Delta} ds_{2n-2} \\ &\quad \times \dots \int_0^1 \frac{(b + as_1)[d + c(1 - s_1)]}{\Delta} ds_1 \cdot \|u\|_{4n-2} \\ &= \frac{l^{2n-2}}{2} t(1-t) \|u\|_{4n-2}, \end{aligned} \tag{2.18}$$

where

$$l = \frac{ac + 3ad + 3bc + 6bd}{6\Delta}, \tag{2.19}$$

for fixed $u \in P$, we have

$$kt(1-t)\|u\|_{4n-2} \le u(t) \le \frac{l^{2n-2}}{2} t(1-t)\|u\|_{4n-2}. \tag{2.20}$$

Moreover, for $u \in P$, $t \in J_0 = [\tau, \gamma]$, $0 < \tau < \gamma < 1$, we get $\tau(1-\gamma) \le t(1-t) \le 1/4$, $(t, s) \in J_0 \times J_0$. The inequality (2.20) together with (2.16) yields

$$k\tau(1-\gamma)\|u\|_{4n-2} \le \|u\|_0 \le \frac{l^{2n-2}}{8} \|u\|_{4n-2}, \tag{2.21}$$

where k and l are defined by (2.17) and (2.19), respectively.

Also, for $e(t), l, k$ corresponding to different settings of boundary-value problem (1.3), we have: (1) For $b = d = 0$,

$$e_1(t) = t(1-t), \quad l_1 = \frac{1}{6}, \quad k_1 = \frac{1}{30^{2n-1}}. \tag{2.22}$$

(2) For $b = 0, d > 0$,

$$\begin{gathered} e_2(t) = \frac{t[d + c(1-t)]}{c+d}, \quad l_2 = \frac{c + 3d}{6(c+d)}, \\ k_2 = \frac{(2c + 5d)(5a^2cd + a^2c^2 + 10a^2d^2)}{1800(c+d)} \Big(\frac{a^2c^2 + 10a^2d^2 + 5a^2cd}{30}\Big)^{2n-3} \\ \times \frac{1}{(ac + ad)^{4n-4}}. \end{gathered} \tag{2.23}$$

(3) For $b > 0, d = 0$,

$$e_3(t) = \frac{(b+at)(1-t)}{b+a}, \quad l_3 = \frac{a+3b}{6(a+b)},$$
$$k_3 = \frac{(2a+5b)(10b^2c^2+5abc^2+a^2c^2)}{1800(a+b)}\Big(\frac{5abc^2+10b^2c^2+a^2c^2}{30}\Big)^{2n-3} \times \frac{1}{(ac+bc)^{4n-4}}. \tag{2.24}$$

In the following, we give the proof of Theorems 2.1, 2.2, and 2.3.

Proof of Theorem 2.1. **Sufficiency.** In this theorem, the cone P is

$$\begin{aligned} P_1 = \big\{&u \in E : R_1(u) = R_2(u) = 0,\ u(t) \geq 0,\ u^{(4n-2)}(t) \leq e_1(t)u^{(4n-2)}(s) \leq 0,\\ &u(t) \geq -k_1 t(1-t)u^{(4n-2)}(s), \forall t, s \in [0,1]\big\}. \end{aligned} \tag{2.25}$$

where $e_1(t), k_1$ are given by (2.23), $R_1(u) = u^{(2k)}(0)$, $R_2(u) = u^{(2k)}(1)$, $k = 1, 2, \dots, 2n-1$. By (2.21) and (2.22), we get

$$\|u\| = \|u\|_{4n-2}, \quad \forall u \in P_1. \tag{2.26}$$

Furthermore, from (2.16),(2.20) and (2.26), we have

$$\begin{gathered} \frac{1}{30^{2n-1}}t(1-t)\|u\| \leq u(t) \leq \frac{1}{2\times 6^{2n-2}}t(1-t)\|u\|,\\ t(1-t)\|u\| \leq -u^{(4n-2)}(t) \leq \|u\|. \end{gathered} \tag{2.27}$$

Define an operator A on $P_1 \setminus \{0\}$ by

$$(Au)(t) = \int_0^1 h_1(t,s)f(s, u(s), u^{(4n-2)}(s))ds, \quad \forall\ u \in P_1 \setminus \{0\}, \tag{2.28}$$

where

$$h_1(t,s) = \int_0^1 \dots \int_0^1 G_1(t, s_{2n-1})G_1(s_{2n-1}), s_{2n-2}) \dots G_1(s_1, s)ds_1 \dots ds_{2n-1};$$

and $G_1(t,s)$ is defined by (2.13). Clearly,

$$G_1(t,s) \leq G_1(s,s), \quad G_1(t,s) \leq G_1(t,t), \quad G_1(t,s) \geq t(1-t)s(1-s),$$

for all $t,\ s \in [0,1]$. Then

$$\begin{aligned} &h_1(t,s)\\ &\leq \int_0^1 t(1-t)ds_{2n-1}\int_0^1 s_{2n-1}(1-s_{2n-1})ds_{2n-2}\dots\int_0^1 s_2(1-s_2)s(1-s)ds_1\\ &\leq t(1-t)\int_0^1\dots\int_0^1 s_{2n-1}(1-s_{2n-1})\dots s_2(1-s_2)s(1-s)ds_{2n-1}\dots ds_2\\ &\leq t(1-t)s(1-s)\\ &\leq s(1-s), \quad \forall\ t,s \in [0,1]. \end{aligned} \tag{2.29}$$

Now we claim that Au is well defined on $P_1\setminus\{0\}$. First, for $\forall u \in P_1\setminus\{0\}$, we can see that $\|u\| \neq 0$. At the same time, notice that $G_1(t,s) \leq G_1(s,s),\ \forall\ t,\ s \in [0,1]$. This together with (2.1) yields that $\int_0^1 G_1(s_1,s)f(s,u(s),u^{(4n-2)}(s))ds$ is convergent.

In fact, for $\forall\ u \in P_1 \setminus \{0\}$, choose positive numbers $c_1 \le \min\{N_1,\ \frac{\|u\|}{30^{2n-1}M_1}\}$ and $c_2 \ge \max\{M_2,\ \frac{\|u\|}{N_2}\}$. By (1.4) and (2.27), we obtain

$$\begin{aligned}
&\int_0^1 G_1(s_1,s)f(s,u(s),u^{(4n-2)}(s))ds \le \int_0^1 s(1-s)f(s,u(_s),u^{(4n-2)}(s))ds \\
&\le \int_0^1 s(1-s)f(s,c_1\frac{u(s)}{c_1 s(1-s)}s(1-s),(-1)c_2\frac{-u^{(4n-2)}(s)}{c_2})ds \\
&\le \int_0^1 s(1-s)c_1^{\lambda}(\frac{u(s)}{c_1 s(1-s)})^{\mu}c_2^{\beta}(\frac{-u^{(4n-2)}(s)}{c_2})^{\alpha}f(s,s(1-s),-1)ds \\
&\le \int_0^1 s(1-s)c_1^{\lambda-\mu}(\frac{\|u\|}{2\times 6^{2n-2}})^{\mu}c_2^{\beta}(\frac{\|u\|}{c_2})^{\alpha}f(s,s(1-s),-1)ds \\
&\le (\frac{1}{2\times 6^{2n-2}})^{\mu}c_1^{\lambda-\mu}c_2^{\beta-\alpha}\|u\|^{\mu+\alpha}\int_0^1 s(1-s)f(s,s(1-s),-1)ds \\
&= c_3\|u\|^{\mu+\alpha}\int_0^1 s(1-s)f(s,s(1-s),-1)ds < \infty,
\end{aligned} \tag{2.30}$$

where

$$c_3 = (\frac{1}{2\times 6^{2n-2}})^{\mu}c_1^{\lambda-\mu}c_2^{\beta-\alpha}. \tag{2.31}$$

Also, by (2.29) and the process similar to the proof of (2.30), for $\forall\ u \in P_1 \setminus \{0\}$, there exist positive constants c_1 and c_2 such that

$$\begin{aligned}
Au(t) &= \int_0^1 h_1(t,s)f(s,u(s),u^{(4n-2)}(s))ds \\
&\le \int_0^1 s(1-s)f(s,u(s),u^{(4n-2)}(s))ds \\
&\le c_3\|u\|^{\mu+\alpha}\int_0^1 s(1-s)f(s,s(1-s),-1)ds < \infty,
\end{aligned} \tag{2.32}$$

where c_3 is the same as (2.31). This together with (2.1) yields that A is well defined on $P_1 \setminus \{0\}$. Obviously, if (2.1)-(2.3) hold, then (1.3)($b = d = 0$) has a positive solution u if and only if A has a fixed point in $P_1 \setminus \{0\}$. So we need to prove only that A has a fixed point in $P_1 \setminus \{0\}$.

Now we show that $A:\ P_1 \setminus \{0\} \to P_1$ is completely continuous. Firstly, we show that $A(P_1 \setminus \{0\}) \subset P_1$. To see this, for all $u \in P_1 \setminus \{0\}$, notice that

$$\begin{aligned}
(Au)^{(4n-2)}(t) &= -\int_0^1 G_1(t,\tau)f(\tau,u(\tau),u^{(4n-2)}(\tau))d\tau \\
&\le -t(1-t)\int_0^1 G_1(s,\tau)f(\tau,u(\tau),u^{(4n-2)}(\tau))d\tau \\
&= t(1-t)(Au)^{(4n-2)}(s) \le 0, \quad \forall\ t,\ s \in [0,1],
\end{aligned}$$

and

$$\begin{aligned}
(Au)(t) = \int_0^1 \dots \int_0^1 & G_1(t,s_{2n-1})G_1(s_{2n-1},s_{2n-2})\dots G_1(s_2,s_1) \\
&\times (-(Au)^{(4n-2)}(s_1))ds_1 ds_2 \dots ds_{2n-1}
\end{aligned}$$

$$\begin{aligned}
&\geq -t(1-t)\int_0^1 \dots \int_0^1 s_{2n-1}^2(1-s_{2n-1}))^2 \dots s_2^2(1-s_2)^2 \\
&\quad \times s_1^2(1-s_1)^2(Au)^{(4n-2)}(s)ds_1 \dots ds_{2n-1} \\
&\geq -\frac{t(1-t)}{30^{2n-1}}(Au)^{(4n-2)}(s).
\end{aligned}$$

Then we have $A(P_1 \setminus \{0\}) \subset P_1$.

Secondly, we show that A is bounded. In fact, let $V \subset P_1 \setminus \{0\}$ be a bounded set. There exists a positive constant L satisfying $\|u\| \leq L$, for all $u \in V$. Choose $c_1 \leq \min\{N_1, \frac{L}{30^{2n-1}M_1}\}$ and $c_2 \geq \max\{M_2, \frac{L}{N_2}\}$. By (2.1), (2.30), and (2.31), we get

$$\begin{aligned}
|(Au)^{(4n-2)}(t)| &= \int_0^1 G_1(t,s)f(s,u(s),u^{(4n-2)}(s))ds \\
&\leq \int_0^1 s(1-s)f(s, c_1\frac{u(s)}{c_1 s(1-s)}s(1-s), (-1)c_2\frac{-u^{(4n-2)}(s)}{c_2})ds \\
&\leq c_3 L^{\mu+\alpha}\int_0^1 s(1-s)f(s,s(1-s),-1)ds \\
&< +\infty, \quad \forall t \in [0,1], \quad \forall u \in P_1 \setminus \{0\}.
\end{aligned}$$

Therefore, this together with (2.26) implies

$$\|Au\| \leq c_3 L^{\mu+\alpha}\int_0^1 s(1-s)f(s,s(1-s),-1)ds < +\infty, \tag{2.33}$$

where c_3 is defined by (2.31). Namely, AV is uniformly bounded.

Thirdly, by (2.33) and the Ascoli-Arzela theorem, we need to show only that AV is equicontinuous on $[0,1]$. Therefore, we need to prove only that $(Au)^{(4n-2)}(t) \to 0$ as $t \to 0^+$ and $t \to 1^-$ uniformly with respect to $u \in V$ and AV are equicontinuous on any closed subinterval of (0,1). In fact, notice that

$$\begin{aligned}
&-(Au)^{(4n-2)}(t) \\
&= \int_0^1 G_1(t,s)f(s,u(s),u^{(4n-2)}(s))ds \\
&= (1-t)\int_0^t sf(s,u(s),u^{(4n-2)}(s))ds + t\int_t^1 (1-s)f(s,u(s),u^{(4n-2)}(s))ds,
\end{aligned}$$

Then this together with (2.1) and (2.2) guarantees $(Au)^{(4n-2)}(t) \to 0$, as $t \to 0^+$ or $t \to 1^-$, uniformly with respect to $u \in V$.

Now, we are in position to show that for $\forall\, a \in (0, \frac{1}{2})$, AV are equicontinuous on $[a, 1-a]$. For all $t_1,\ t_2 \in [a, 1-a]$, $t_1 < t_2$, for all $u \in V$, by (2.31), we get

$$\begin{aligned}
&|(Au)^{(4n-2)}(t_2) - (Au)^{(4n-2)}(t_1)| \\
&= \Big|\int_0^{t_1}(t_1-t_2)sf(s,u(s),u^{(4n-2)}(s))ds + \int_{t_1}^{t_2}(1-t_2)sf(s,u(s),u^{(4n-2)}(s))ds \\
&\quad + \int_{t_2}^1 (t_2-t_1)(1-s)f(s,u(s),u^{(4n-2)}(s))ds \\
&\quad - \int_{t_1}^{t_2} t_1(1-s)f(s,u(s),u^{(4n-2)}(s))ds\Big|
\end{aligned}$$

$$\leq c_3 L^{\mu+\alpha}[(t_2-t_1)\int_0^{1-(t_2-t_1)} sf(s,s(1-s),-1)ds$$
$$+ (t_2-t_1)\int_{t_2-t_1}^1 (1-s)f(s,s(1-s),-1)ds + 2\int_{t_1}^{t_2} s(1-s)f(s,s(1-s),-1)ds].$$

Also, as $|t_1-t_2|\to 0$, (2.1)-(2.3) imply

$$(t_2-t_1)\int_0^{1-(t_2-t_1)} sf(s,s(1-s),-1)ds \to 0,$$
$$(t_2-t_1)\int_{t_2-t_1}^1 (1-s)f(s,s(1-s),-1)ds \to 0,$$
$$\int_{t_1}^{t_2} s(1-s)f(s,s(1-s),-1)ds \to 0.$$

This guarantees $|(Au)^{(4n-2)}(t_2)-(Au)^{(4n-2)}(t_1)| \to 0(|t_1-t_2|\to 0)$.

Similar to the above proof, we can get $(Au)(t)\to 0$, as $t\to 0^+$ or $t\to 1^-$ uniformly with respect to $u\in V$ and for all $t_1,t_2\in[a,1-a]$, $t_1<t_2$, for all $u\in V$, we have $|(Au)(t_2)-(Au)(t_1)| \to 0(|t_1-t_2|\to 0)$. Therefore, AV is relatively compact.

Finally, it remains to show A is continuous. Suppose $u_n, u_0\in P$, and $\|u_n-u_0\|\to 0$ $(n\to\infty)$. Then $\{u_n\}$ is a bounded set and

$$\|u_n-u_0\|_0\to 0,\quad \|u_n-u_0\|_{4n-2}\to 0\quad (n\to\infty).$$

Let $M=\sup\{\|u_n\|,\ n=0,\ 1,\ 2,\ \dots\}$. Then we may still choose positive constants $c_1\leq \min\{N_1,\frac{M}{30^{2n-1}M_1}\}$ and $c_2\geq \max\{M_2,\frac{M}{N_2}\}$. Similar to the proof of (2.30), we get

$$f(t,u_n(t),u_n^{(4n-2)}(t)) \leq c_3 M^{\mu+\alpha} f(t,t(1-t),-1),\quad t\in(0,1), \tag{2.34}$$

$$\begin{aligned}&|(Au_n)^{(4n-2)}(t)-(Au_0)^{(4n-2)}(t)|\\ &\leq \int_0^1 s(1-s)|f(s,u_n(s),u_n^{(4n-2)}(s))-f(s,u_0(s),{u_0}^{(4n-2)}(s))|ds,\end{aligned} \tag{2.35}$$

and

$$\begin{aligned}&|(Au_n)(t)-(Au_0)(t)|\\ &\leq \int_0^1 s(1-s)|f(s,u_n(s),u_n^{(4n-2)}(s))-f(s,u_0(s),{u_0}^{(4n-2)}(s))|ds.\end{aligned}$$

The above inequality, (2.1), (2.34), (2.35), the Lebesgue dominated convergence theorem, and Ascoli-Arzela theorem guarantee that

$$\|Au_n-Au_0\|\to 0\quad (n\to\infty),$$

that is, A is continuous. Summing up, $A: P_1\setminus 0\to P_1$ is completely continuous.

For $0<r<1<R$, let

$$P_{1,r}=\{u\in P_1:\|u\|\leq r\},\quad P_{1,R}=\{u\in P_1:\|u\|\leq R\}.$$

Choose r such that

$$0<r\leq \min\{(2^{-(2+(10n-5)\mu+4\beta)}3^\beta\int_{\frac14}^{\frac34} s(1-s)f(s,s(1-s),-1)ds)^{\frac{1}{1-(\mu+\beta)}},$$

$$2\times 6^{2n-2}N_1, N_2\big\}.$$

Then for $u\in\partial P_{1,r}$, we have

$$\begin{gathered}\frac{r}{30^{2n-1}}t(1-t)\le u(t)\le \frac{r}{2\times 6^{2n-2}}t(1-t)\le N_1 t(1-t),\\ \frac{3r}{16}\le rt(1-t)\le -u^{(4n-2)}(t)\le r\le N_2,\quad \forall\, t\in[\frac14,\frac34].\end{gathered}$$

By the properties of $G_1(t,s)$, (2.1), and (1.4), we get

$$\begin{aligned}-(Au)^{(4n-2)}(t) &= \int_0^1 G_1(t,s)f(s,u(s),u^{(4n-2)}(s))ds\\ &\ge \frac14\int_{\frac14}^{\frac34} s(1-s)f(s,\frac{u(s)}{s(1-s)}s(1-s),(-1)(-u^{(4n-2)}(s)))ds\\ &\ge \frac14\int_{\frac14}^{\frac34} s(1-s)(\frac{u(s)}{s(1-s)})^{\mu}(-u^{(4n-2)}(s))^{\beta}f(s,s(1-s),-1)ds\\ &\ge \frac14\int_{\frac14}^{\frac34} s(1-s)(\frac{r}{30^{2n-1}})^{\mu}(\frac{3r}{16})^{\beta}f(s,s(1-s),-1)ds\\ &\ge \frac{1}{2^2}\int_{\frac14}^{\frac34}(\frac{r}{2^{(10n-5)}})^{\mu}(\frac{3r}{2^4})^{\beta}s(1-s)f(s,s(1-s),-1)ds\\ &= 2^{-(2+(10n-5)\mu+4\beta)}3^{\beta}r^{\mu+\beta}\int_{\frac14}^{\frac34} s(1-s)f(s,s(1-s),-1)ds\\ &\ge r=\|u\|_{4n-2}=\|u\|,\quad \forall\, u\in\partial P_{1,r}.\end{aligned}$$

This guarantees

$$\|Au\|\ge\|u\|,\quad \forall u\in\partial P_{1,r}. \tag{2.36}$$

On the other hand, Choose R such that

$$\begin{aligned}&R\ge \max\big\{30^{2n-1}M_1, M_2N_2,\\ &\qquad [(2\times 6^{2n-2})^{-\mu}N_2^{\alpha-\beta}\int_0^1 s(1-s)f(s,s(1-s),-1)ds)]^{\frac{1}{1-(\mu+\beta)}}\big\}.\end{aligned}$$

and let $c=\frac{N_2}{R}$. Then for $u\in\partial P_{1,R}$, we have

$$\begin{gathered}M_1t(1-t)\le \frac{R}{30^{2n-1}}t(1-t)\le u(t)\le \frac{R}{2\times 6^{2n-2}}t(1-t),\\ -cu^{(4n-2)}(t)\le c\|u\|=cR=N_2.\end{gathered}$$

Therefore,

$$\begin{aligned}-(Au)^{(4n-2)}(t) &= \int_0^1 G_1(t,s)f(s,u(s),u^{(4n-2)}(s))ds\\ &\le \int_0^1 s(1-s)f(s,\frac{u(s)}{s(1-s)}s(1-s),(-1)\frac1c(-cu^{(4n-2)}(s))ds\\ &\le \int_0^1 s(1-s)(\frac{u(s)}{s(1-s)})^{\mu}(\frac1c)^{\beta}(-cu^{(4n-2)}(s))^{\alpha}f(s,s(1-s),-1)ds\\ &\le \int_0^1 s(1-s)(\frac{R}{2\times 6^{2n-2}})^{\mu}c^{\alpha-\beta}R^{\alpha}f(s,s(1-s),-1)ds\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 s(1-s)(\frac{R}{2\times 6^{2n-2}})^{\mu}(\frac{N_2}{R})^{\alpha-\beta}R^{\alpha}f(s,s(1-s),-1)ds \\
&= (2\times 6^{2n-2})^{-\mu}N_2^{\alpha-\beta}R^{\mu+\beta}\int_0^1 s(1-s)f(s,s(1-s),-1)ds \\
&\le R = \|u\|_{4n-2} = \|u\|, \quad \forall u \in \partial P_{1,R},
\end{aligned}
$$

This implies

$$
\|Au\| \le \|u\|, \quad \forall u \in \partial P_{1,R}. \tag{2.37}
$$

By the complete continuity of A, (2.36), and (2.37), and Lemma 1.3, we obtain that A has a fixed point $u_*(t)$ in $\overline{P_{1,R}} \setminus P_{1,r}$. Consequently, (1.3) has a $C^{4n-2}[0,1]$ positive solution $u_*(t)$ in $\overline{P_{1,R}} \setminus P_{1,r}$.

Necessity. Let $u(t)$ be a $C^{(4n-2)}[0,1]$ positive solution of (1.3). It follows from the boundary value conditions that there exists $t_0 \in (0,1)$ such that $u^{(4n-1)}(t_0) = 0$. Obviously, by virtue of $u^{(4n)}(t) \ge 0,\ t \in (0,1)$, we get

$$
u^{(4n-1)}(t) \le 0, \quad t \in (0,t_0); \quad u^{(4n-1)}(t) \ge 0, \quad t \in (t_0,1).
$$

Hence, $u^{(4n-2)}(t) \le 0,\ t \in [0,1]$. Similarly, when $t \in [0,1]$, by induction we know

$$
u^{(4n-4)}(t) \ge 0, \quad u^{(4n-6)}(t) \le 0, \quad \dots, \quad u^{(4)}(t) \ge 0, \quad u''(t) \le 0.
$$

Therefore, there exists $0 < m_1 < 1 < m_2$ such that for all $t \in [0,1]$

$$
m_1 t(1-t) \le u(t) \le m_2 t(1-t). \tag{2.38}
$$

Choose positive numbers $c_1 \le \min\{N_1, \frac{1}{M_1 m_2}\}$ and $c_2 \ge \max\{\frac{1}{N_2}, M_2\|u\|\}$. Then we can get

$$
\begin{aligned}
f(t,t(1-t),-1) &= f(t, c_1\frac{t(1-t)}{c_1 u(t)}u(t), \frac{1}{c_2}\frac{c_2}{-u^{(4n-2)}(t)}u^{(4n-2)}(t)) \\
&\le c_1^{\lambda}(\frac{t(1-t)}{c_1 u(t)})^{\mu}(\frac{1}{c_2})^{\alpha}(\frac{c_2}{-u^{(4n-2)}(t)})^{\beta}f(t,u(t),u^{(4n-2)}(t)) \\
&\le c_1^{\lambda}(\frac{1}{c_1 m_1})^{\mu}(\frac{1}{c_2})^{\alpha}(-\frac{c_2}{u^{(4n-2)}(t)})^{\beta}f(t,u(t),u^{(4n-2)}(t)) \\
&= c_1^{\lambda-\mu}c_2^{\beta-\alpha}m_1^{-\mu}(-u^{(4n-2)}(t))^{-\beta}f(t,u(t),u^{(4n-2)}(t)).
\end{aligned} \tag{2.39}
$$

By (2.39), and $-u^{(4n-2)}(t)$ being nondecreasing on $(0,t_0)$, integrate the first equality of (1.3) from t_0 to t to obtain

$$
-u^{(4n-1)}(t) = \int_t^{t_0} f(s,u(s),u^{(4n-2)}(s))ds,\ t \in (0,t_0), \tag{2.40}
$$

and

$$
\begin{aligned}
0 &< \int_0^{t_0} t f(t, t(1-t), -1) dt = \int_0^{t_0} dt \int_t^{t_0} f(s, s(1-s), -1) ds \\
&\le \int_0^{t_0} dt \int_t^{t_0} c_1^{\lambda-\mu} c_2^{\beta-\alpha} m_1^{-\mu} (-u^{(4n-2)}(s))^{-\beta} f(s, u(s), u^{(4n-2)}(s)) ds \\
&\le c_1^{\lambda-\mu} c_2^{\beta-\alpha} m_1^{-\mu} \int_0^{t_0} (-u^{(4n-2)}(t))^{-\beta} dt \int_t^{t_0} f(s, u(s), u^{(4n-2)}(s)) ds \\
&= c_1^{\lambda-\mu} c_2^{\beta-\alpha} m_1^{-\mu} \int_0^{t_0} \frac{-u^{(4n-1)}(t)}{(-u^{(4n-2)}(t))^{\beta}} dt \\
&= c_1^{\lambda-\mu} c_2^{\beta-\alpha} m_1^{-\mu} (1-\beta)^{-1} (-u^{(4n-2)}(t_0))^{1-\beta} < \infty;
\end{aligned} \tag{2.41}
$$

Similar to (2.40) and (2.41), we can also prove that

$$
u^{(4n-1)}(t) = \int_{t_0}^{t} f(s, u(s), u^{(4n-2)}(s)) ds, \ t \in (t_0, 1), \tag{2.42}
$$

$$
\begin{aligned}
0 &< \int_{t_0}^{1} (1-t) f(t, t(1-t), -1) dt = \int_{t_0}^{1} dt \int_{t_0}^{t} f(s, s(1-s), -1) ds \\
&\le c_1^{\lambda-\mu} c_2^{\beta-\alpha} m_1^{-\mu} (1-\beta)^{-1} (-u^{(4n-2)}(t_0))^{1-\beta} < \infty.
\end{aligned} \tag{2.43}
$$

Consequently, inequalities (2.41) and (2.43) yield (2.1). For $t \in (0, t_0)$, by (2.39), and integrating (2.40), we have

$$
\begin{aligned}
& t \int_t^{t_0} f(s, s(1-s), -1) ds \le \int_0^t d\xi \int_\xi^{t_0} f(s, s(1-s), -1) ds \\
&\le \int_0^t d\xi \int_\xi^{t_0} c_1^{\lambda-\mu} c_2^{\beta-\alpha} m_1^{-\mu} (-u^{(4n-2)}(s))^{-\beta} f(s, u(s), u^{(4n-2)}(s)) ds \\
&= c_1^{\lambda-\mu} c_2^{\beta-\alpha} m_1^{-\mu} (1-\beta)^{-1} (-u^{(4n-2)}(t)).
\end{aligned} \tag{2.44}
$$

Noticing $u^{(4n-2)}(0) = 0$ and letting $t \to 0^+$ in (2.44), we obtain

$$
\lim_{t \to 0+} t \int_t^{t_0} f(s, s(1-s), -1) ds = 0.
$$

This implies (2.2). For $t \in (t_0, 1)$, noticing (2.39) and integrating (2.42), we get

$$
\begin{aligned}
& (1-t) \int_{t_0}^{t} f(s, s(1-s), -1) ds \\
&\le \int_t^1 d\xi \int_{t_0}^{\xi} f(s, s(1-s), -1) ds \\
&\le \int_t^1 d\xi \int_{t_0}^{\xi} c_1^{\lambda-\mu} c_2^{\beta-\alpha} m_1^{-\mu} (-u^{(4n-2)}(s))^{-\beta} f(s, u(s), u^{(4n-2)}(s)) ds \\
&= c_1^{\lambda-\mu} c_2^{\beta-\alpha} m_1^{-\mu} (1-\beta)^{-1} (-u^{(4n-2)}(t))
\end{aligned} \tag{2.45}
$$

By $u^{(4n-2)}(1) = 0$, and letting $t \to 1^-$ in (2.45), we have

$$
\lim_{t \to 1^-} (1-t) \int_{t_0}^{t} f(s, s(1-s), -1) ds = 0.
$$

This yields (2.3). Summing up, the necessity follows. □

Proof of Theorem 2.2. **Sufficiency.** In this theorem, the cone P is

$$\begin{aligned} P_2 =&\big\{u \in E : R_1(u) = R_2(u) = 0,\ u(t) \geq 0,\ u^{(4n-2)}(t) \leq e_2(t)u^{4n-2}(s) \leq 0,\\ &u(t) \geq -k_2 t(1-t)u^{(4n-2)}(s), \forall t, s \in [0,\ 1]\big\}, \end{aligned}$$

where $e_2(t), k_2$ are given by (2.24), and $R_1(u) = u^{(2k)}(0)$, $R_2(u) = cu^{(2k)}(1) + du^{(2k+1)}(1)$, $k = 1, 2, \dots, 2n-1$. By (2.16), (2.20), (2.21), and (2.23), we have

$$\|u\| = \|u\|_{4n-2}, \quad \forall u \in P_2. \tag{2.46}$$

$$k_2 t(1-t)\|u\| \leq u(t) \leq \frac{l_2^{2n-2}}{2} t(1-t)\|u\|, \quad e_2(t)\|u\| \leq -u^{(4n-2)}(t) \leq \|u\|. \tag{2.47}$$

where l_2 is given by (2.24). Clearly, for $t \in [\frac{1}{4}, \frac{3}{4}]$, we have

$$e_2(t) \geq \frac{4d+c}{16(c+d)}. \tag{2.48}$$

Suppose (2.4) and (2.5) hold. Then (1.3) has a $C^{4n-2}[0,1] \cap C^{4n-1}(0,1]$ positive solution u if and only if u is a positive solution of the following integral equation

$$u(t) = (Au)(t) = \int_0^1 h_2(t,s) f(s, u(s), u^{(4n-2)}(s))ds, \quad \forall u \in P_2 \setminus \{0\}, \tag{2.49}$$

where

$$h_2(t,s) = \int_0^1 \dots \int_0^1 G_1(t, s_{2n-1})G_2(s_{2n-1}, s_{2n-2}) \dots G_2(s_1, s)ds_1 \dots ds_{2n-1};$$

and $G_1(t,s)$ and $G_2(t,s)$ are given by (2.13) and (2.14), respectively. For rall $u \in P_2 \setminus \{0\}$, it follows from (2.11), (2.14), and (2.24) that

$$\begin{aligned} (Au)^{(4n-2)}(t) &= -\int_0^1 G_2(t,\tau) f(\tau, u(\tau), u^{(4n-2)}(\tau))d\tau \\ &\leq -\frac{t[d + c(1-t)]}{c+d} \int_0^1 G_2(s,\tau) f(\tau, u(\tau), u^{(4n-2)}(\tau))d\tau \\ &= e_2(t)(Au)^{(4n-2)}(s) \leq 0, \quad \forall\ t,\ s \in [0,1]. \end{aligned}$$

Obviously, this together with (2.10) implies

$$\begin{aligned} &(Au)(t) \\ &= \int_0^1 \dots \int_0^1 G_1(t, s_{2n-1})G_2(s_{2n-1}, s_{2n-2}) \dots G_2(s_2, s_1)(-(Au)^{(4n-2)}(s_1)) \\ &\quad \times ds_1 ds_2 \dots ds_{2n-1} \\ &\geq \int_0^1 t(1-t)s_{2n-1}(1-s_{2n-1})ds_{2n-1} \\ &\quad \times \int_0^1 \frac{[ac(1-s_{2n-1})s_{2n-1} + ads_{2n-1}][ac(1-s_{2n-2})s_{2n-2} + ads_{2n-2}]}{a^2(c+d)^2} ds_{2n-2} \\ &\quad \times \dots \int_0^1 \frac{[ac(1-s_3)s_3 + ads_3][ac(1-s_2)s_2 + ads_2]}{a^2(c+d)^2} ds_2 \\ &\quad \times \int_0^1 \frac{[ac(1-s_2)s_2 + ads_2][ac(1-s_1)s_1 + ads_1]}{a^2(c+d)^2} \cdot (Au)^{(4n-2)}(s_1)ds_1 \end{aligned}$$

$$\begin{aligned}&\geq -t(1-t)\frac{2ac+5ad}{60}\cdot(\frac{a^2c^2+10a^2d^2+5a^2cd}{30})^{2n-3}\cdot\frac{(5a^2cd+a^2c^2+10a^2d^2)}{30a(c+d)}\\&\quad\times\frac{1}{(ac+ad)^{4n-4}}(Au)^{(4n-2)}(s),\\&=-k_2t(1-t)(Au)^{(4n-2)}(s).\end{aligned}$$

Thus, $A(P_2\setminus\{0\})\subset P_2$.

Similar to the proof of Theorem 2.1, we can show that $A: \quad P_2\setminus\{0\}\to P_2$ is completely continuous. For $0<r<1<R$, let

$$P_{2,r}=\{u\in P_2:\|u\|\leq r\},\quad P_{2,R}=\{u\in P_2:\|u\|\leq R\},$$

On the one hand, choose

$$r\leq\min\{[\frac{(c+4d)^2)}{256(c+d)^2}k_2^{\mu}(\frac{(c+4d)}{16(c+d)})^{\beta}\int_{\frac14}^{\frac34}sf(s,s(1-s),-1)ds]^{\frac{1}{1-(\mu+\beta)}},\frac{2N_1}{l_2^{2n-2}},N_2\}.$$

For $u\in\partial P_{2,r}$, combining (2.47) and (2.48), then we have

$$\begin{gathered}k_2rt(1-t)\leq u(t)\leq\frac{l_2^{2n-2}r}{2}t(1-t)\leq N_1t(1-t),\\\frac{c+4d}{16(c+d)}r\leq e_2(t)r\leq -u^{(4n-2)}(t)\leq r\leq N_2,\quad\forall\ t\in[\frac14,\frac34].\end{gathered}$$

In addition, by (2.4), (2.49), (1.4), and the properties of $G_2(t,s)$, we get

$$\begin{aligned}-(Au)^{(4n-2)}(t)&=\int_0^1G_2(t,s)f(s,u(s),u^{(4n-2)}(s))ds\\&\geq\int_{\frac14}^{\frac34}\frac{[c(1-s)s+ds][c(1-t)t+dt]}{(c+d)^2}f(s,\frac{u(s)}{s(1-s)}s(1-s),(-1)\\&\quad\times(-u^{(4n-2)}(s)))ds\\&\geq\frac{(c+4d)^2)}{256(c+d)^2}\int_{\frac14}^{\frac34}s(\frac{u(s)}{s(1-s)})^{\mu}(-u^{(4n-2)}(s))^{\beta}f(s,s(1-s),-1)ds\\&\geq\frac{(c+4d)^2)}{256(c+d)^2}\int_{\frac14}^{\frac34}(k_2r)^{\mu}(\frac{(c+4d)r}{16(c+d)})^{\beta}sf(s,s(1-s),-1)ds\\&=\frac{(c+4d)^2)}{256(c+d)^2}k_2^{\mu}(\frac{(c+4d)}{16(c+d)})^{\beta}r^{\mu+\beta}\int_{\frac14}^{\frac34}sf(s,s(1-s),-1)ds\\&\geq r=\|u\|_{4n-2}=\|u\|,\quad\forall\ u\in\partial P_{2,r}.\end{aligned}$$

Therefore, by (2.46), and the above inequality, we have

$$\|Au\|\geq\|u\|,\quad\forall u\in\partial P_{2,r}. \tag{2.50}$$

On the other hand, choose

$$R\geq\max\{[(\frac{l_2^{2n-2}}{2})^{\mu}N_2^{\alpha-\beta}\int_0^1sf(s,s(1-s),-1)ds)]^{\frac{1}{1-(\mu+\beta)}}],M_2N_2,\frac{M_1}{k_2}\}.$$

and Let $c=\frac{N_2}{R}$. Then for $u\in\partial P_{2,R}$, we obtain

$$M_1t(1-t)\leq k_2t(1-t)R\leq u(t)\leq\frac{Rl_2^{2n-2}}{2}t(1-t),$$

$$-cu^{(4n-2)}(t) \le c\|u\| = cR = N_2.$$

Consequently,

$$\begin{aligned}
-(Au)^{(4n-2)}(t) &= \int_0^1 G_2(t,s) f(s, u(s), u^{(4n-2)}(s))ds \\
&\le \int_0^1 \frac{s[d+c(1-s)]}{c+d} f(s, \frac{u(s)}{s(1-s)} s(1-s), (-1)\frac{1}{c}(-cu^{(4n-2)}(s))ds \\
&\le \int_0^1 s(\frac{u(s)}{s(1-s)})^{\mu} (\frac{1}{c})^{\beta} (-cu^{(4n-2)}(s))^{\alpha} f(s, s(1-s), -1)ds \\
&\le \int_0^1 s(\frac{Rl_2^{2n-2}}{2})^{\mu} c^{\alpha-\beta} R^{\alpha} f(s, s(1-s), -1)ds \\
&= (\frac{l_2^{2n-2}}{2})^{\mu} N_2^{\alpha-\beta} R^{\mu+\beta} \int_0^1 sf(s, s(1-s), -1)ds \\
&\le R = |u|_{4n-2} = \|u\|, \quad \forall u \in \partial P_{2,R},
\end{aligned}$$

which implies

$$\|Au\| \le \|u\|, \quad \forall\ u \in \partial P_{2,R}. \tag{2.51}$$

By the complete continuity of A, (2.50), and (2.51), we know that A has a fixed point $u_*(t)$ in $\overline{P_{2,R}} \backslash P_{2,r}$. Consequently, (1.3) has a $C^{4n-2}[0,1] \cap C^{4n-1}(0,1]$ positive solution $u_*(t)$ in $\overline{P_{2,R}} \setminus P_{2,r}$.

Necessity. Let $u(t)$ be a $C^{4n-2}[0,1] \cap C^{4n-1}(0,1] \cap C^{4n}(0,1)$ positive solution of (1.3). Then we get

$$\begin{gathered}
u^{(4n-2)}(t) = -\int_0^1 G_2(t,s) f(s, u(s), u^{(4n-2)}(s))ds \le 0, \\
u^{(4n-1)}(1) = -\frac{c}{d} u^{(4n-2)}(1) \ge 0.
\end{gathered}$$

Clearly, it follows from boundary-value conditions that there exists $t_0 \in (0,1]$ such that $u^{(4n-1)}(t_0) = 0$.

The following argument is broken into two cases: $t_0 < 1$ and $t_0 = 1$.

Case (1): Suppose that $t_0 < 1$. Then $u^{(4n-1)}(1) > 0$. Since $u^{(4n)}(t) \ge 0, t \in (0,1)$, we find

$$u^{(4n-1)}(t) \le 0, \quad t \in (0, t_0); \quad u^{(4n-1)}(t) \ge 0, \quad t \in (t_0, 1),$$

and hence $u^{(4n-2)}(t) \le 0$, $t \in [0,1]$. By the same way, we know $u''(t) \le 0$ for $t \in [0,1]$. Therefore, this implies (2.38)-(2.41), (2.44), (2.45), and

$$\begin{aligned}
\int_{t_0}^1 f(t, t(1-t), -1)dt &\le \int_{t_0}^1 c_1^{\lambda-\mu} c_2^{\beta-\alpha} m_1^{-\mu} (-u^{(4n-2)}(t))^{-\beta} f(t, u(t), u^{(4n-2)}(t))dt \\
&\le c_1^{\lambda-\mu} c_2^{\beta-\alpha} m_1^{-\mu} (-u^{(4n-2)}(1))^{-\beta} u^{(4n-1)}(1) < \infty.
\end{aligned}$$

Clearly, it follows from the above inequality and (2.41) that (2.4) holds. Moreover, by virtue of (2.44), (2.5) is satisfied.

Case (2). If $t_0 = 1$. Then $u^{(4n-1)}(1) = 0, u^{(4n-2)}(1) < 0$, and (2.38)- (2.40) hold. Also, by (2.39), we have

$$0 < \int_0^1 tf(t, t(1-t), -1)dt = \int_0^1 dt \int_t^1 f(s, s(1-s), -1)ds$$

$$\leq \int_0^1 dt \int_t^1 c_1^{\lambda-\mu} c_2^{\beta-\alpha} m_1^{-\mu} (-u^{(4n-2)}(s))^{-\beta} f(s, u(s), u^{(4n-2)}(s)) ds.$$

Notice that $-u^{(4n-2)}(s)$ is nondecreasing in s on $(0,1)$. Then we have

$$\begin{aligned} 0 &< \int_0^1 t f(t, t(1-t), -1) dt \\ &\leq c_1^{\lambda-\mu} c_2^{\beta-\alpha} m_1^{-\mu} \int_0^1 (-u^{(4n-2)}(t))^{-\beta} (-u^{(4n-1)}(t)) dt \\ &= c_1^{\lambda-\mu} c_2^{\beta-\alpha} m_1^{-\mu} \frac{(-u^{(4n-2)}(1))^{1-\beta}}{1-\beta} < \infty, \ t \in (0,1). \end{aligned}$$

Namely, (2.4) holds. By (2.40), and integrating (2.39), we obtain

$$\begin{aligned} & t \int_t^1 f(s, s(1-s), -1) ds \\ & \leq \int_0^t d\xi \int_\xi^1 f(s, s(1-s), -1) ds \\ & \leq c_1^{\lambda-\mu} c_2^{\beta-\alpha} m_1^{-\mu} \int_0^t d\xi \int_\xi^1 (-u^{(4n-2)}(s))^{-\beta} f(s, u(s), u^{(4n-2)}(s)) ds \\ & \leq c_1^{\lambda-\mu} c_2^{\beta-\alpha} m_1^{-\mu} \frac{(-u^{(4n-2)}(t))^{1-\beta}}{1-\beta}. \end{aligned} \tag{2.52}$$

Noting $u^{(4n-2)}(0) = 0, \ \beta < 1$, and letting $t \to 0^+$ in (2.52), we have

$$\lim_{t \to 0+} t \int_t^1 f(s, s(1-s), -1) ds = 0.$$

This implies (2.5). □

Proof of Theorem 2.3. **Sufficiency**. In this theorem, the cone P is

$$\begin{aligned} P_3 = \big\{ & u \in E : R_1(u) = R_2(u) = 0, \ u(t) \geq 0, \ u^{(4n-2)}(t) \leq e_3(t) u^{(4n-2)}(s) \leq 0, \\ & u(t) \geq -k_3 t(1-t) u^{(4n-2)}(s), \forall t, s \in [0, \ 1] \big\}, \end{aligned}$$

where $e_3(t), k_3$ are given by (2.24), $R_1(u) = au^{(2k)}(0) - bu^{(2k+1)}(0)$, $R_2(u) = u^{(2k)}(1)$, $k = 1, 2, \dots, 2n-1$. According to (2.21) and (2.24), we show

$$\|u\| = \|u\|_{4n-2}, \quad \forall \ u \in P_3,$$

$$k_3 t(1-t) \|u\| \leq u(t) \leq \frac{l_3^{2n-2}}{2} t(1-t) \|u\|, \quad e_3(t) \|u\| \leq -u^{(4n-2)}(t) \leq \|u\|,$$

where l_3 is defined by (2.24).

Assume (2.6) and (2.7) hold. Then (1.3) has a $C^{4n-2}[0,1] \cap C^{4n-1}[0,1)$ positive solution u if and only if u is a positive solution of the following integral equation

$$u(t) = (Au)(t) = \int_0^1 h_3(t,s) f(s, u(s), u^{(4n-2)}(s)) ds, \quad \forall \ u \in P_3 \setminus \{0\},$$

where

$$\begin{aligned} & h_3(t,s) \\ & = \int_0^1 \dots \int_0^1 G_1(t, s_{2n-1}) G_3(s_{2n-1}, s_{2n-2}) \dots G_3(s_2, s_1) G_3(s_1, s) ds_1 \dots ds_{2n-1}; \end{aligned}$$

and $G_1(t,s)$, $G_3(t,s)$ are defined by (2.13) and (2.15), respectively. The rest of the proof is very similar to Theorem 2.1 and Theorem 2.2. So it is omitted.

Necessity. Let $u(t)$ be a $C^{4n-2}[0,1]\cap C^{4n-1}[0,1)\cap C^{4n}(0,1)$ positive solution of (1.3). Then we claim that there is a constant $t_0\in[0,1)$ satisfying

$$u^{(4n-1)}(t_0)=0,\quad u^{(4n-1)}(0)=-\frac{a}{b}u^{(4n-2)}(0)\le 0.$$

Similar to the proof of necessity of Theorem 2.2, the argument can be broken into two cases: $t_0<0$ and $t_0=0$.
Case (1): Assume $t_0<0$. Then $u^{(4n-1)}(0)<0$. This implies (2.38)-(2.39), (2.42), (2.43), (2.45), and

$$\int_0^{t_0} f(t,t(1-t),-1)dt\le c_1^{\lambda-\mu}c_2^{\beta-\alpha}m_1^{-\mu}(-u^{(4n-2)}(0))^{-\beta}(-u^{(4n-1)}(0))<\infty.$$

Therefore, the above inequality and (2.43) guarantee (2.6). Also, by (2.45), we can deduce (2.7).
Case (2). If $t_0=0$, then $u^{(4n-1)}(0)=0$, $u^{(4n-2)}(0)<0$, and (2.38)-(2.39), (2.42) hold. Notice that $-u^{(4n-2)}(s)$ is decreasing in s on (0,1). Similar to the case (2) of Theorem 2.2, by (2.39), we have

$$\begin{aligned}0&<\int_0^1(1-t)f(t,t(1-t),-1)dt=\int_0^1 dt\int_0^t f(s,s(1-s),-1)ds\\&\le c_1^{\lambda-\mu}c_2^{\beta-\alpha}m_1^{-\mu}\frac{(-u^{(4n-2)}(0))^{1-\beta}}{1-\beta}<\infty,\ t\in(0,1).\end{aligned}$$

Namely, (2.6) holds. By (2.42), integrating (2.39), we get

$$\begin{aligned}&(1-t)\int_0^t f(s,s(1-s),-1)ds\\&\le\int_t^1 d\xi\int_0^\xi f(s,s(1-s),-1)ds\\&\le c_1^{\lambda-\mu}c_2^{\beta-\alpha}m_1^{-\mu}\int_0^t d\xi\int_\xi^1(-u^{(4n-2)}(s))^{-\beta}f(s,u(s),u^{(4n-2)}(s))ds\\&\le c_1^{\lambda-\mu}c_2^{\beta-\alpha}m_1^{-\mu}\frac{(-u^{(4n-2)}(t))^{1-\beta}}{1-\beta}.\end{aligned}\tag{2.53}$$

By $u^{(4n-2)}(1)=0$, and letting $t\to1^-$ in (2.53), we obtain

$$\lim_{t\to1^-}(1-t)\int_0^t f(s,s(1-s),-1)ds=0.$$

This implies (2.7). □

3. Examples

Example 3.1. Consider (1.3) with ($b=d=0$) and

$$f(t,u,v)=p_1(t)u^{-20}(-v)^{1/6}+p_2(t)u^{1/5}(-v)^{\frac{1}{5}},$$

where $p_i\in C[(0,1),R^+]$ ($i=1,2$).

It is easy to see, by Theorem 2.1, that (1.3) with ($b = d = 0$) has a C^{4n-2} positive solution if and only if

$$0 < \int_0^1 [p_1(t)(t(1-t))^{-19} + p_2(t)(t(1-t))^{6/5}]dt < +\infty,$$

$$\lim_{t\to 0+} t \int_t^1 [p_1(s)s^{-20}(1-s)^{-19} + p_2(s)s^{\frac{1}{5}}(1-s)^{6/5}]ds = 0,$$

$$\lim_{t\to 1-} t \int_t^1 [p_1(s)s^{-20}(1-s)^{-19} + p_2(s)s^{\frac{1}{5}}(1-s)^{6/5}]ds = 0.$$

Example 3.2. Consider (1.3) with ($b = 0, d > 0$) and

$$f(t, u, v) = q_1(t)u^{-18}(-v)^{\frac{1}{3}} + q_2(t)u^{\frac{1}{17}}(-v)^{\frac{1}{13}},$$

where $q_i \in C[(0, 1), R^+](i = 1, 2)$.

Obviously, by Theorem 2.2 , one can see that (1.3) with ($b = 0, d > 0$) has a $C^{4n-2}[0, 1] \cap C^{4n-1}(0, 1]$ positive solution if and only if

$$0 < \int_0^1 [q_1(t)t^{-17}(1-t)^{-18} + q_2(t)t^{\frac{18}{17}}(1-t)^{1/17}]dt < +\infty.$$

$$\lim_{t\to 0+} t \int_t^1 [q_1(s)(s(1-s))^{-18} + q_2(s)(s(1-s))^{1/17}]ds = 0.$$

Example 3.3. Consider (1.3) with ($b > 0, d = 0$) and

$$f(t, u, v) = m_1(t)u^{-\frac{1}{2}}(-v)^{\frac{1}{21}} + m_2(t)u^{\frac{1}{81}}(-v)^{\frac{18}{73}},$$

where $m_i \in C[(0, 1), R^+](i = 1, 2)$.

Clearly, according to Theorem 2.3, (1.3) with ($b > 0, d = 0$) has a $C^{4n-2}[0, 1] \cap C^{4n-1}[0, 1)$ positive solution if and only if

$$0 < \int_0^1 [m_1(t)t^{-\frac{1}{2}}(1-t)^{\frac{1}{2}} + m_2(t)t^{\frac{1}{81}}(1-t)^{\frac{82}{81}}]dt < +\infty,$$

$$\lim_{t\to 1^-} (1-t) \int_0^t [m_1(s)(s(1-s))^{-\frac{1}{2}} + m_2(s)(s(1-s))^{\frac{1}{81}}]ds = 0.$$

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