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SUPERLINEAR EQUATIONS AND A UNIFORM ANTI-MAXIMUM PRINCIPLE FOR THE MULTI-LAPLACIAN OPERATOR

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Abstract. In the first part of this paper, we study a nonlinear equation with the multi-Laplacian operator, where the nonlinearity intersects all but the first eigenvalue. It is proved that under certain conditions, involving in particular a relation between the spatial dimension and the order of the problem, this equation is solvable for arbitrary forcing terms. The proof uses a generalized Mountain Pass theorem. In the second part, we analyze the relationship between the validity of the above result, the first nontrivial curve of the Fučík spectrum, and a uniform anti-maximum principle for the considered operator.

1. Introduction

The main theme of this paper are the following superlinear equations with the multi-Laplacian operator:

$$\begin{gathered} (-\Delta)^m u = \lambda u + g(x,u) + h(x) \quad \text{in } \Omega \\ \frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = \dots = \frac{\partial \Delta^{m-1} u}{\partial n} = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.1}$$

and

$$\begin{gathered} (-\Delta)^m u = \lambda u + g(x,u) + h(x) \quad \text{in } \Omega \\ u = \Delta u \dots = \Delta^{m-1} u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.2}$$

with $\Omega \subseteq \mathbb{R}^N$ a bounded smooth domain (say of class $\mathcal{C}^\infty$), $h \in L^2(\Omega)$ and

$$g \in \mathcal{C}^0(\overline{\Omega} \times \mathbb{R}), \quad \lim_{s\to-\infty} \frac{g(x,s)}{s} = 0, \quad \lim_{s\to+\infty} \frac{g(x,s)}{s} = +\infty \tag{1.3}$$

uniformly with respect to $x \in \overline{\Omega}$.

We will assume for these problems the following hypotheses on the order of the operator and the dimension of the set Ω:

$$N < 2m, \quad \text{so that} \quad H^m(\Omega) \subseteq \mathcal{C}^0(\overline{\Omega}); \tag{1.4}$$

$$N < 2(m-1), \quad \text{so that} \quad H^m(\Omega) \subseteq \mathcal{C}^1(\overline{\Omega}); \tag{1.5}$$

in particular (1.4) will be assumed for problem (1.1) and (1.5) for problem (1.2).

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Some hypotheses on the growth at infinity of the nonlinearity g will be needed in order to obtain the PS condition for the functional associated to the above problems: Defining $G(x,s)=\int_0^s g(x,\xi)d\xi$, we require

$$\exists\,\theta\in\big(0,\frac{1}{2}\big),\ s_0>0 \text{ such that } 0<G(x,s)\le\theta sg(x,s)\ \forall s>s_0\,. \tag{1.6}$$

Moreover, for λ equal to the first eigenvalue of the operator, we will assume the nonresonance condition

$$g(x,s)>0,\quad \lim_{s\to-\infty} g(x,s)=0 \tag{1.7}$$

uniformly with respect to $x\in\overline{\Omega}$.

We will refer to the boundary conditions in (1.1) as the case (N) and to those in (1.2) as the case (D); moreover, we will usually write the results for the case (N) and when needed remark in parentheses what is different for the case (D).

In the following we will denote by $0\le\lambda_1<\lambda_2\le\lambda_3\le\dots\le\lambda_k\le\dots$ the eigenvalues of $-\Delta$ in $H^1(\Omega)$ (resp. in $H^1_0(\Omega)$ when considering the case (D)) and with $\{\phi_k\}_{k=1,2,..}$ the corresponding eigenfunctions, which will be taken orthogonal and normalized with $\|\phi_k\|_{L^2}=1$ and $\phi_1>0$.

The main result of the paper is the following.

Theorem 1.1. *Under hypotheses* (1.4) *(resp.* (1.5)*),* (1.3) *and* (1.6)*, there exists* $\gamma>\lambda_1^m$ *such that if* $\lambda\in(\lambda_1^m,\gamma)$*, then there exists a solution of problem* (1.1) *(resp.* (1.2)*) for all* $h\in L^2(\Omega)$*.*

Moreover, if $\lambda=\lambda_1^m$ *and in addition hypothesis* (1.7) *holds, then there exists a solution for* $h\in L^2(\Omega)$ *if and only if* $\int_\Omega h\phi_1<0$*.*

Remark 1.2. The hypotheses (1.3) and (1.6) are satisfied for example by the function $g(x,s)=e^s$, which satisfies also (1.7).

Theorem (1.1) is proved in section (3), except for the technical PS condition, which is proved in section (5). The proof uses a generalized Mountain Pass theorem: we prove the existence of a linking structure for the functional associated to the problems (1.1) and (1.2), by building a suitable set on which it is simple to estimate both the nonlinearity g and the principal part of the functional and which divides the function space into two components containing respectively ϕ_1 and $-\phi_1$.

The value γ in theorem (1.1) is obtained by the following variational characterization

$$\gamma=\inf\Big\{\frac{\int_\Omega|\nabla^m u|^2}{\int_\Omega u^2}\text{ with } u\in H^m_*(\Omega)\backslash\{0\}\text{ and }\sup_{x\in\Omega}\frac{u(x)}{\phi_1(x)}=0\Big\} \tag{1.8}$$

(see section (2) for the definition of the space $H^m_*(\Omega)$).

In the second part of the paper (section (4)) we show, by variational techniques, the existence of a connection between the above value γ, the first nontrivial curve of the Fučík spectrum and a uniform anti-maximum principle (uAMP for short) for the operators in problem (1.1) and (1.2).

In brief, the uAMP is a reversing sign property of the operator $((-\Delta)^m-\lambda)$: it holds when, for λ in a certain interval, a positive forcing term produces a negative solution (see in section (4) for more details about the uAMP and the Fučík spectrum).

The result is given in theorem (4.2); in particular we show that, under hypothesis (1.4) (resp. (1.5)) the uAMP holds in $(\lambda_1^m,\gamma]$ and γ also coincides with the asymptote of the first nontrivial curve of the Fučík spectrum. On the other hand, when (1.4) (resp. (1.5)) is not satisfied, no uAMP holds and the asymptote of the first nontrivial curve of the Fučík spectrum coincides with λ_1^m.

Since γ is characterized variationally, the above connection also provides a characterization for the limit of validity of the uAMP.

1.1. **Related results.** Theorem (1.1) (and the techniques used in its proof) is in the same spirit as the results in [8] and in [13]: these results were obtained for the Laplacian operator in an interval and only allowed conditions of the type (N) (that is the Neumann problem); actually in the case (N) hypothesis (1.4) with $m = 1$ implies $N = 1$ and so theorem (1.1) corresponds to the result in [8], while in the case (D) hypothesis (1.5) may be satisfied only for $m \geq 2$.

In [8] and [13] the value γ was obtained both variationally and explicitly; here we obtain the characterization in (1.8) for the general case and we will be able to calculate it only for the one dimensional fourth order case (see the propositions (3.9) and (3.10)).

The anti-maximum and uniform anti-maximum principles are largely treated in [3, 7, 1] for Laplacian and p-Laplacian operators; in the latter two the authors proved, respectively for the Laplacian and the p-Laplacian, the strict relationship of these properties with the behavior of the first nontrivial curve of the Fučík spectrum; in particular they obtained, as we do here for the higher order problem, that in those cases in which the asymptote of this curve is bounded away from the first eigenvalue, the uAMP holds indeed between this eigenvalue and the asymptote, while it does not hold when the asymptote and the eigenvalue coincide. In fact, the techniques we use here are inspired from these two papers, and for $m = 1$ our result corresponds to that in [7].

Results concerning the AMP and uAMP for higher order operators with boundary conditions like in (1.2) (and more general ones) have recently been found in [4, 5, 10]. In particular it is obtained in [4, 5] that the uAMP holds under hypothesis (1.5), while in [10] it is proved that hypothesis (1.5) is indeed necessary for the uAMP to hold. In [4], boundary conditions of mixed type (Robin) are also considered, obtaining the uAMP under hypothesis (1.4).

We remark that in these papers about higher order operators, there is no estimate of the upper limit of validity of the uAMP, so that the variational characterization we obtain here looks to be an interesting result by itself.

2. Variational formulation of the problem

In this work we consider the differential operator $(-\Delta)^m$; in dealing with it, we will use the notation $\nabla^{2h}u = \Delta^h u$ and $\nabla^{2h+1}u = \nabla(\Delta^h u)$.

We will look for weak solutions in the space $H^m(\Omega)$: let B_N (resp. B_D) be the operator that maps u to the vector of the traces on $\partial\Omega$ of the derivatives of order strictly lower than m which are imposed in problem (1.1) (resp. (1.2)): then the problem in variational form reads

$$\begin{gathered} u \in H^m_*(\Omega) \quad \text{such that} \\ \int_\Omega \nabla^m u \nabla^m v - \lambda \int_\Omega uv - \int_\Omega g(x,u)v - \int_\Omega hv = 0 \quad \text{for all } v \in H^m_*(\Omega)\,, \end{gathered} \tag{2.1}$$

where

$$H^m_*(\Omega) = \{u \in H^m(\Omega) \text{ such that } B_* u = 0\}\,, \tag{2.2}$$

and with B_* we denoted B_N or B_D when considering respectively (1.1) or (1.2). Observe that for $m = 1$ the above spaces reduce to $H^1_N(\Omega) = H^1(\Omega)$ and $H^1_D(\Omega) = H^1_0(\Omega)$.

To find a solution of problem (1.1) (resp. (1.2)) we will look for critical points of the $\mathcal{C}^1$ functional

$$F : H^m_*(\Omega) \to \mathbb{R} : u \mapsto F(u) = \frac{1}{2}\int_\Omega |\nabla^m u|^2 - \frac{\lambda}{2}\int_\Omega u^2 - \int_\Omega G(x,u) - \int_\Omega hu \ . \quad (2.3)$$

Some useful lemmas. We give here some results about the properties of the spaces we will work with. The proofs will not be reported here, but can be found in [12].

Remark that these results are consequence of the particular sets of the chosen boundary conditions, in particular of the fact that one may see the differential operator as the m^{th} power of the Laplacian with Neumann or Dirichlet boundary conditions.

Lemma 2.1. *The norm $\|u\| = (\|\nabla^m u\|^2_{L^2} + \|u\|^2_{L^2})^{1/2}$ is an equivalent norm for $H^m_*(\Omega)$.*

Lemma 2.2. *The eigenvalues of the operators in* (1.1) *(resp* (1.2)*) are the m^{th} power of those of the Laplacian with Neumann (resp. Dirichlet) boundary conditions, while the eigenfunctions are the same of those cases.*

Moreover, the eigenfunctions are orthogonal also in the H^m scalar product and they form a basis for it.

Finally, we still have a variational characterization of the first eigenvalue:

$$\lambda_1^m = \inf\Big\{\int_\Omega |\nabla^m u|^2 : u \in H^m_*(\Omega);\ \|u\|_{L^2} = 1\Big\} . \quad (2.4)$$

3. The main result

In this section we will show the existence of a mountain pass structure for the functional (2.3) in order to prove the existence of a solution for problems (1.1) and (1.2), for suitable values of the parameter λ. The technique we use is inspired from [8] and [13], where the case $m = N = 1$ was considered.

Given $u \in H^m_*(\Omega)$ with m satisfying hypothesis (1.4) (resp. (1.5)), we define:

$$\sigma(u) = \sup_{x\in\Omega} \frac{u(x)}{\phi_1(x)} . \quad (3.1)$$

Remark 3.1. In the case (N), ϕ_1 is a constant function and so $\sigma(u) = \sup_{x\in\Omega}[u(x)]$, which is finite by the inclusion $H^m(\Omega) \subseteq \mathcal{C}^0(\overline{\Omega})$.

In the case (D), ϕ_1 is the first eigenfunction of the Laplacian, which is known to have the property that $\inf_{x\in\partial\Omega} \frac{\partial\phi_1}{\partial n_{int}}(x) > 0$; this property and the inclusion $H^m(\Omega) \subseteq \mathcal{C}^1(\overline{\Omega})$ implies that $\sigma(u)$ is finite also in this case.

Then we define

$$E = \Big\{u \in H^m_*(\Omega) : \int_\Omega u\phi_1 = 0\Big\} , \quad (3.2)$$

$$S_0 = \{u \in H^m_*(\Omega) : \sigma(u) = 0\} , \quad (3.3)$$

$$\gamma = \inf\Big\{\frac{\int_\Omega |\nabla^m u|^2}{\int_\Omega u^2} \text{ with } u \in S_0\backslash\{0\}\Big\} . \quad (3.4)$$

First we will prove some properties of the above definitions:

Lemma 3.2. *The function $\sigma : H^m_*(\Omega) \to \mathbb{R} : u \mapsto \sigma(u)$ is continuous.*

Proof. In the case (N) we have, by the hypothesis (1.4),
$$|\sigma(u) - \sigma(v)| \le \|u - v\|_{L^\infty(\Omega)} \le C\|u - v\|_{H^m_N(\Omega)} \ . \tag{3.5}$$
In the case (D), we have
$$|\sigma(u) - \sigma(v)| \le \Big\|\frac{u - v}{\phi_1}\Big\|_{L^\infty(\Omega)} . \tag{3.6}$$

To estimate the last norm, we may exploit the fact that $\phi_1 \in \mathcal{C}^1(\overline{\Omega})$, vanishes (just) on the smooth boundary $\partial\Omega$, and $\eta = \inf_{\xi\in\partial\Omega} \frac{\partial\phi_1}{\partial n_{int}}(\xi) > 0$: this allows to compute the ratio in (3.6) near to the boundary using l'Hôpital's rule, and so to estimate with the $\mathcal{C}^1$ norm of $u - v$ and then, by hypothesis (1.5), with the H^m norm:
$$|\sigma(u) - \sigma(v)| \le \Big\|\frac{u - v}{\phi_1}\Big\|_{L^\infty(\Omega)} \le C_1\|u - v\|_{\mathcal{C}^1(\overline{\Omega})} \le C_2\|u - v\|_{H^m_D(\Omega)} . \tag{3.7}$$
□

Lemma 3.3. *The set S_0 is homeomorphic to E, moreover S_0 divides $H^m_*(\Omega)$ into two components containing respectively $\{t\phi_1 : \quad t > 0\}$ and $\{t\phi_1 : \quad t < 0\}$.*

Proof. The map $M : E \to S_0 : u \mapsto u - \sigma(u)\phi_1$ is continuous by the previous lemma and has the orthogonal projection on E as its inverse, so it is a homeomorphism. Moreover, it is clear by the definitions that $H^m_*(\Omega)$ is divided into the two components $\{u \in H^m_*(\Omega) : \sigma(u) > 0\}$ and $\{u \in H^m_*(\Omega) : \sigma(u) < 0\}$. □

Lemma 3.4. *Let γ be given by (3.4). Then $\gamma > \lambda^m_1$ and it is attained, that is there exists $u \in S_0\backslash\{0\}$ such that $\gamma = \frac{\int_\Omega |\nabla^m u|^2}{\int_\Omega u^2}$.*

Proof. Let us take a minimizing sequence $\{u_n\} \subseteq S_0\backslash\{0\}$: by the homogeneity of the definition of γ and S_0 we may assume $\|u_n\|_{L^2} = 1$; since $\int_\Omega |\nabla^m u_n|^2 \to \gamma$, u_n is bounded in H^m_* and we can extract a subsequence such that $u_n \to u$ weakly in H^m_* and strongly in L^2 and in $\mathcal{C}^0(\overline{\Omega})$ (resp. in $\mathcal{C}^1(\overline{\Omega})$) by hypothesis (1.4) (resp. (1.5)).

The strong convergences implies that $\sigma(u) = 0$ and $\|u\|_{L^2} = 1$ and so $u \in S_0\backslash\{0\}$. Then $\int_\Omega |\nabla^m u|^2 \ge \gamma$ by the definition of γ, but by the weak convergence this implies $\int_\Omega |\nabla^m u|^2 = \gamma$ and so u realizes the value γ.

Finally $\gamma \ge \lambda^m_1$ by the variational characterization of λ^m_1 and if, by contradiction, $\gamma = \lambda^m_1$, then the minimizer would be a multiple of ϕ_1, which is a contradiction since $\operatorname{span}\{\phi_1\} \cap S_0 = \{0\}$. □

Now, we proceed to prove the existence of the linking structure for the functional (2.3). Since $h \in L^2$ and using hypothesis (1.3) we can find, for $\delta, M > 0$, constants $C_1(\delta, h)$, $C_2(\delta, g)$ and $C_3(M, g)$ as follows:
$$\Big|\int_\Omega hu\Big| \le \frac{\delta}{4}\|u\|^2_{L^2} + C_1(\delta, h) , \tag{3.8}$$
$$\Big|\int_\Omega G(x, -u^-)\Big| \le \frac{\delta}{4}\|u\|^2_{L^2} + C_2(\delta, g) , \tag{3.9}$$
$$\int_\Omega G(x, u^+) \ge \frac{M}{2}\|u^+\|^2_{L^2} - C_3(M, g) . \tag{3.10}$$

Lemma 3.5. $\lim_{\rho\to+\infty} F(\rho\phi_1) = -\infty$.

Proof. Remembering that $\phi_1 > 0$ in Ω we estimate

$$\begin{aligned}\frac{F(\rho\phi_1)}{\rho^2} &= \frac{1}{2}\int_\Omega |\nabla^m\phi_1|^2 - \frac{\lambda}{2}\int_\Omega \phi_1^2 - \int_\Omega \frac{G(x,\rho\phi_1)}{\rho^2} - \int_\Omega \frac{h\phi_1}{\rho}\\ &\le \frac{\lambda_1^m-\lambda}{2}\int_\Omega \phi_1^2 - \Big(\frac{M}{2}\int_\Omega\phi_1^2 - \frac{C_3(M,g)}{\rho^2}\Big) + \Big(\frac{\delta}{2}\int_\Omega\phi_1^2 + \frac{C_1(\delta,h)}{\rho^2}\Big)\\ &\le \frac{\lambda_1^m-\lambda-M+\delta}{2} + \frac{C_1(\delta,h)+C_3(M,g)}{\rho^2}\,;\end{aligned}$$

then by choosing $M > \lambda_1^m - \lambda + \delta$ the lemma is proved. □

Lemma 3.6. $\lim_{\rho\to+\infty} F(-\rho\phi_1) = -\infty$, *provided (i) or (ii) holds:*

(i) $\lambda > \lambda_1^m$
(ii) $\lambda = \lambda_1^m$, $\int_\Omega h\phi_1 < 0$ *and hypothesis* (1.7) *holds.*

Proof. Estimating as before we now get for $\lambda > \lambda_1^m$

$$\begin{aligned}\frac{F(-\rho\phi_1)}{\rho^2} &= \frac{1}{2}\int_\Omega |\nabla^m\phi_1|^2 - \frac{\lambda}{2}\int_\Omega \phi_1^2 - \int_\Omega \frac{G(x,-\rho\phi_1)}{\rho^2} - \int_\Omega \frac{-h\rho\phi_1}{\rho^2}\\ &\le \frac{\lambda_1^m-\lambda}{2}\int_\Omega \phi_1^2 + \Big(\frac{\delta}{4}\int_\Omega\phi_1^2 + \frac{C_2(\delta,g)}{\rho^2}\Big) + \Big(\frac{\delta}{4}\int_\Omega\phi_1^2 + \frac{C_1(\delta,h)}{\rho^2}\Big)\\ &\le \frac{\lambda_1^m-\lambda+\delta}{2} + \frac{C_1(\delta,h)+C_2(\delta,g)}{\rho^2}\,;\end{aligned}$$

then by choosing $\delta < \lambda - \lambda_1^m$ the first part of the lemma is proved.

For $\lambda = \lambda_1^m$ we need a finer estimate: since $\lim_{s\to-\infty} g(x,s) = 0$ we may estimate:

for any $\varepsilon > 0$ there exists C_ε such that
$|g(x,s)| \le \varepsilon + \frac{C_\varepsilon}{|s-1|^2}$ and $|G(x,s)| \le \varepsilon|s| + \frac{C_\varepsilon}{|s-1|}$, $\qquad \forall s \le 0$.

Then

$$\Big|\int_\Omega \frac{G(x,-\rho\phi_1)}{\rho}\Big| \le \int_\Omega \varepsilon\phi_1 + \frac{C_\varepsilon}{\rho(1+\rho\phi_1)} \le \big(\varepsilon + \frac{C_\varepsilon}{\rho}\big)C_\Omega \tag{3.11}$$

and so

$$\limsup_{\rho\to+\infty}\Big|\int_\Omega \frac{G(x,-\rho\phi_1)}{\rho}\Big| \le \varepsilon\, C_\Omega \tag{3.12}$$

for any choice of ε and hence the limit is zero. Then we conclude

$$\lim_{\rho\to+\infty}\frac{F(-\rho\phi_1)}{\rho} = \rho\frac{\lambda_1^m-\lambda}{2} + \int_\Omega h\phi_1 \tag{3.13}$$

that for $\lambda = \lambda_1^m$ and $\int_\Omega h\phi_1 < 0$ implies that this last limit is negative and so the second part of the lemma is proved too. □

Lemma 3.7. *For $\lambda < \gamma$, $F|_{S_0}$ is bounded from below.*

Proof. For $u \in S_0$ we have $u(x) \le 0$ and $\int_\Omega |\nabla^m u|^2 \ge \gamma\|u\|_{L^2}^2$, then we may estimate:

$$\begin{aligned}F(u) &= \frac{1}{2}\int_\Omega |\nabla^m u|^2 - \frac{\lambda}{2}\int_\Omega u^2 - \int_\Omega G(x,u) - \int_\Omega hu\\ &\geq \frac{\gamma-\lambda}{2}\|u\|^2_{L^2} - \Big(\frac{\delta}{4}\int_\Omega u^2 + C_2(\delta,g)\Big) - \Big(\frac{\delta}{4}\int_\Omega u^2 + C_1(\delta,h)\Big)\\ &\geq \frac{\gamma-\lambda-\delta}{2}\int_\Omega u^2 - C_2(\delta,g) - C_1(\delta,h)\end{aligned}$$

and so it is enough to choose $\delta < \gamma - \lambda$ to obtain $F(u) \geq -C_2(\delta,g) - C_1(\delta,h)$. $\square$

In section (5) we will prove the following lemma.

Lemma 3.8. *Under hypotheses* (1.4) *(resp.* (1.5)*),* (1.3) *and* (1.6)*, with* $h \in L^2(\Omega)$*, the functional* (2.3) *defined in* $H^m_N(\Omega)$ *(resp. in* $H^m_D(\Omega)$*) satisfies the PS condition for* $\lambda \in (\lambda_1^m, \gamma)$*. Moreover under hypothesis* (1.7) *and* $\int_\Omega h\phi_1 < 0$ *it satisfies the PS condition also for* $\lambda = \lambda_1^m$*.*

Then we may conclude the following.

Proof of theorem (1.1)*.* The previous lemmas allow us to apply the generalized mountain pass theorem to get a solution of problem (1.1) and (1.2).

In fact, define

$$f = \inf_{\gamma\in\Gamma_R} \sup_{u\in\gamma([0,1])} F(u) \tag{3.14}$$

where

$$\Gamma_R = \{\gamma \in \mathcal{C}^0([0,1], H^m_*(\Omega)) \quad s.t. \quad \gamma(0) = -R\phi_1 \quad and \quad \gamma(1) = R\phi_1\}: \tag{3.15}$$

provided R is large enough to have $F(\pm R\phi_1) < -C_2(\delta,g) - C_1(\delta,h)$ where δ is the value fixed in the proof of lemma (3.7), one may apply the deformation lemma and then prove that f is a free critical value for F.

In particular, the condition $\int_\Omega h\phi_1 < 0$ for $\lambda = \lambda_1^m$ is necessary: considering the variational equation with the test function ϕ_1 one gets

$$\int_\Omega \nabla^m u \nabla^m \phi_1 - \lambda_1^m \int_\Omega u\phi_1 - \int_\Omega g(x,u)\phi_1 - \int_\Omega h\phi_1 = 0, \tag{3.16}$$

that is $-\int_\Omega g(x,u)\phi_1 - \int_\Omega h\phi_1 = 0$ which, by hypothesis (1.7), implies $\int_\Omega h\phi_1 < 0$. $\square$

The fourth order one dimensional case. In dimension 1 and with $m = 2$ we can find the minimizing functions of (3.4), and then the value of γ; we will proceed in a way similar to [13]. Let $\Omega = (0,1)$: we start by considering the case (N):

• Claim: the minimizer of (3.4) satisfies $u(x) < 0 \quad \forall x \in (0,1)$.

Proof of the claim. In dimension 1 we have that $H^2_N(0,1) \subseteq \mathcal{C}^1([0,1])$, so if $u(x_0) = 0$ with $x_0 \in (0,1)$, since $u \in S_0$, then x_0 is a maximum and so $u'(x_0) = 0$; this implies that $u_l(x) = u(x_0 x)$ and $u_r(x) = u(1-(1-x_0)(1-x))$ with $x \in (0,1)$ are both in H^2_N and also in S_0; we claim that one of the two realizes a lower value than

$\frac{\int_0^1 |u''|^2}{\int_0^1 u^2}$. The claim follows from

$$\int_0^1 u_l^2 = \frac{1}{x_0}\int_0^{x_0} u^2\,, \quad \int_0^1 u_r^2 = \frac{1}{1-x_0}\int_{x_0}^1 u^2\,,$$
$$\int_0^1 |u_l''|^2 = x_0^3\int_0^{x_0} |u''|^2\,, \quad \int_0^1 |u_r''|^2 = (1-x_0)^3\int_{x_0}^1 |u''|^2$$

and the inequality

$$\frac{a+b}{c+d} \geq \min\big\{\frac{a}{c}, \frac{b}{d}\big\}\,, \tag{3.17}$$

valid for reals $a, b, c, d > 0$. $\square$

• The previous claim implies that the minimizer needs to reach zero in the boundary of $(0,1)$; by symmetry, we may look for a minimizer with $u(1) = 0$. In particular we consider the problem

$$\delta = \inf\big\{\frac{\int_0^1 |u''|^2}{\int_0^1 u^2} \text{ with } u \in H^2_N(0,1)\backslash\{0\} \text{ and } u(1) = 0\big\}\ : \tag{3.18}$$

if we show that the minimizer of (3.18) is in $S_0\backslash\{0\}$ then it is also the minimizer we are looking for and so $\delta = \gamma$.
• By standard calculations the minimizer of (3.18) needs to satisfy the eigenvalue problem

$$\begin{gathered} u'''' = \delta u \quad \text{in } (0,1) \\ u'(0) = u'''(0) = 0 \\ u(1) = u'(1) = 0\,; \end{gathered} \tag{3.19}$$

setting $q^4 = \delta$ with $q > 0$, the solutions of (3.19) are of the form

$$A\cos(qx) + B\sin(qx) + C\sinh(qx) + D\cosh(qx)\,; \tag{3.20}$$

from $u'(0) = u'''(0) = 0$ we get $B = C = 0$ and forcing the remaining conditions we get

$$\frac{A}{D} = -\frac{\cosh(q)}{\cos(q)} = \frac{\sinh(q)}{\sin(q)}\,. \tag{3.21}$$

To have the minimal value of δ we get the first positive solution of $\tanh(q) = -\tan(q)$: this will be in $(\frac{\pi}{2}, \pi)$, so $\sin(q) > 0$ and the resulting minimizer is

$$\tilde{u} = A\big(\cos(qx) + \cosh(qx)\frac{\sin(q)}{\sinh(q)}\big)\ : \quad A < 0\,. \tag{3.22}$$

We observe that, by the choice of q, $\tilde{u}$ does not change sign and then $\tilde{u} \in S_0$, as required.

We conclude following statement.

Proposition 3.9. *In the case (N), with $m = 2$ and $\Omega = (0,1)$, we have $\gamma = q^4$ where q is the first positive solution of* $\tanh(q) = -\tan(q)$*; moreover $\tilde{u}$ in* (3.22) *is a minimizer for* (3.4)*. An approximate value for γ is* $0.32\pi^4$ *($q = 0.753\pi$).*

In the case (D) one may repeat the same argument: one obtains $u(x) < 0 \quad \forall x \in (0,1)$, from this fact deduces that $u' = 0$ in 0 or in 1 since $u \in S_0$, and then one considers the problem (by symmetry we suppose $u'(1) = 0$)

$$\delta = \inf \Big\{ \frac{\int_0^1 |u''|^2}{\int_0^1 u^2} \text{ with } u \in H^2_D(0,1)\backslash\{0\} \text{ and } u'(1) = 0 \Big\}, \tag{3.23}$$

which corresponds to the eigenvalue problem

$$\begin{gathered} u'''' = \delta u \quad \text{in } (0,1) \\ u(0) = u''(0) = 0 \\ u(1) = u'(1) = 0\,; \end{gathered} \tag{3.24}$$

imposing the boundary conditions to (3.20) we obtain $A = D = 0$ and

$$\frac{B}{C} = -\frac{\sinh(q)}{\sin(q)} = -\frac{\cosh(q)}{\cos(q)}\,, \tag{3.25}$$

so we get the first positive solution of $\tanh(q) = \tan(q)$, which will be in $(\pi, \frac{3\pi}{2})$, so $\cos(q) < 0$ and the resulting minimizer is

$$\tilde{u} = B\big(\sin(qx) - \sinh(qx)\frac{\cos(q)}{\cosh(q)}\big) \;:\quad B < 0\,. \tag{3.26}$$

Again $\tilde{u}$ does not change sign and then $\tilde{u} \in S_0\backslash\{0\}$.

Then we conclude the following statement.

Proposition 3.10. *In the case (D), with $m = 2$ and $\Omega = (0,1)$, we have $\gamma = q^4$ where q is the first positive solution of $\tanh(q) = \tan(q)$; moreover $\tilde{u}$ in (3.26) is a minimizer for (3.4). An approximate value for γ is $2.44\pi^4$ ($q = 1.2499\pi$).*

In figure (1), we plot the shape of the minimizers $\tilde{u}$ for the case (N) (on the left) and the case (D) (on the right). Remark that in both cases $\gamma \in (\lambda_1^2, \lambda_2^2)$, that is $(0, \pi^4)$ in the case (N) and $(\pi^4, 16\pi^4)$ in the case (D).

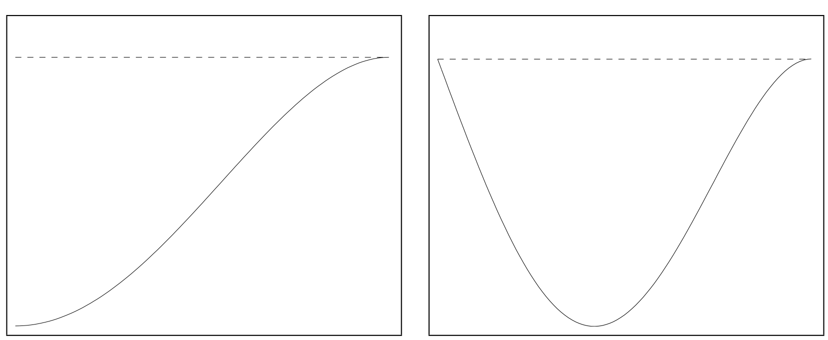

Figure 1. Minimizers of (3.4) in the fourth order one dimensional case (case (N) and case (D)).

4. Uniform anti-maximum principle, Fučík spectrum and the value γ

In this section we will show that the hypothesis (1.4) (resp. (1.5)), which by the way guarantees the wellposedness of the definition of γ in (3.4) and the fact that it is larger than λ_1^m, also guarantees (and is in fact necessary for) a uniform anti-maximum principle as well as the existence of a gap between the first eigenvalue (λ_1^m) and the first nontrivial curve of the Fučík spectrum, for the operator $(-\Delta)^m$ in $H_*^m(\Omega)$.

Introduction to the matter. The anti-maximum and uniform anti-maximum principles are largely treated in [3, 7, 1] for Laplacian and p-Laplacian operators; going to our case, let $\tilde{B}_*$ represent the boundary conditions in problem (1.1) (resp. (1.2)) and consider the problem

$$\begin{gathered}(-\Delta)^m u = \lambda u + h \quad \text{in } \Omega\\ \tilde{B}_* u = 0 \quad \text{on } \partial\Omega\ :\end{gathered} \tag{4.1}$$

the uniform anti-maximum principle (uAMP) is said to hold in $(\lambda_1^m, \delta]$ if for $\lambda \in (\lambda_1^m, \delta]$, $h \in L^2(\Omega)$ and $h \geq 0$ a.e. one gets $u \leq 0$ a.e; a less demanding property is the (non-uniform) AMP, that is when the above property holds for a $\delta_h > \lambda_1^m$ which is no more independent of the given h.

Observe that conversely, for $\lambda < \lambda_1^m$, one has the usual maximum principle, that is $h \geq 0$ a.e. implies $u \geq 0$ a.e.

The notion of Fučík spectrum was introduced in [9] and [6] for the Laplacian operator; for the operators we are considering it may be defined as the set $\Sigma \subseteq \mathbb{R}^2$ of points (λ^+, λ^-) for which there exists a non trivial solution of the problem

$$\begin{gathered}(-\Delta)^m u = \lambda^+ u^+ - \lambda^- u^- \quad \text{in } \Omega\\ \tilde{B}_* u = 0 \quad \text{on } \partial\Omega\,,\end{gathered} \tag{4.2}$$

where $u^+(x) = \max\{0, u(x)\}$ and $u^-(x) = \max\{0, -u(x)\}$.

It is simple to prove that the lines $\{\lambda^+ = \lambda_1^m\}$ and $\{\lambda^- = \lambda_1^m\}$ are in Σ and that any other point in Σ lies in the quadrant $\{\lambda^\pm > \lambda_1^m\}$ but not in the squares $\{\lambda^\pm \in (\lambda_k^m, \lambda_{k+1}^m)\}_{k=1,2,\dots}$.

In [7, 2], the first nontrivial curve was characterized variationally respectively for the Laplacian and p-Laplacian operator; in [11], the author gave a variational characterization of additional parts of the Fučík spectrum for the Laplacian operator which, for the first nontrivial curve, is very similar to the one in [2].

Since the space $H_*^m(\Omega)$ equipped with the norm $\left(\|\nabla^m u\|_{L^2}^2 + \|u\|_{L^2}^2\right)^{1/2}$ has the same functional properties as $H_*^1(\Omega)$ with the norm $\left(\|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2\right)^{1/2}$, the characterization in [11] may be applied also to problem (4.2).

We recall here briefly an adaption to the multi-Laplacian case of that result. The idea is to find, for $\varepsilon > 0$ small enough, a critical point of the functional

$$J_\varepsilon(u) = \int_\Omega |\nabla^m u|^2 - (\lambda_1^m + \varepsilon)\int_\Omega u^2 \tag{4.3}$$

constrained to the set

$$Q_r = \big\{u \in H_*^m(\Omega) \text{ such that } \int_\Omega (u^+)^2 + r(u^-)^2 = 1\big\} \tag{4.4}$$

with $r \in (0,1]$. In order to do this one considers the class of maps

$$\Gamma_r = \{\gamma : [-1,1] \to Q_r \text{ continuous such that } \gamma(-1) = \frac{-\phi_1}{\sqrt{r}} \text{ and } \gamma(1) = \phi_1\} \quad (4.5)$$

and defines

$$d_{\varepsilon,r} = \inf_{\gamma \in \Gamma_r} \sup_{u \in \gamma([-1,1])} J_\varepsilon(u)\,; \quad (4.6)$$

proceeding as in [11] one gets, by a standard "Linking Theorem", that

Proposition 4.1. *Provided $\lambda_1^m + \varepsilon < \lambda_2^m$, the level $d_{\varepsilon,r} > 0$ is a critical value for $J_\varepsilon(u)$ constrained to Q_r. Moreover the critical points associated to this critical value are non trivial solutions of the Fučík problem* (4.2) *with coefficients* (λ^+, λ^-), *where $\lambda^+ = \lambda_1^m + \varepsilon + d_{\varepsilon,r}$ and $\lambda^- = \lambda_1^m + \varepsilon + r d_{\varepsilon,r}$.*

4.1. **Main result.** First observe that we may extend the definition of γ in (3.4) to the case in which the hypothesis (1.4) (resp. (1.5)) does not hold, by simply replacing the supremum with essential supremum, that is

$$\gamma = \inf\Big\{\frac{\int_\Omega |\nabla^m u|^2}{\int_\Omega u^2} \text{ with } u \in H_*^m(\Omega)\backslash\{0\} \text{ and } \operatorname*{ess\,sup}_{x\in\Omega} \frac{u(x)}{\phi_1(x)} = 0\Big\}\,. \quad (4.7)$$

In this case we may assert that $\gamma \geq \lambda_1^m$ by the variational characterization of the first eigenvalue in (2.4), but observe that maybe the inf is not attained. Then we define the following values:

$$\delta = \begin{cases} \sup\{t \in \mathbb{R} : \text{uAMP holds in } (\lambda_1^m, t]\} & \text{if uAMP holds for some } t > \lambda_1^m \\ \lambda_1^m & \text{otherwise,} \end{cases} \quad (4.8)$$

$$\bar\lambda = \inf\{\lambda^- > \lambda_1^m \text{such that} (\lambda^+, \lambda^-) \in \Sigma \text{ and } \lambda^\pm > \lambda_1^m\}\,. \quad (4.9)$$

We will obtain the following statement.

Theorem 4.2. *If hypothesis* (1.4) *(resp.* (1.5)*) holds, then*

$$\delta = \bar\lambda = \gamma > \lambda_1^m\,. \quad (4.10)$$

If hypothesis (1.4) *(resp.* (1.5)*) does not hold, then*

$$\delta = \bar\lambda = \gamma = \lambda_1^m\,. \quad (4.11)$$

Proof of theorem (4.2)**.** First we will need the following lemma.

Lemma 4.3. *Under hypothesis* (1.4) *(resp.* (1.5)*), if u is a minimizer for γ, let $\tilde u : \overline{\Omega} \to \mathbb{R}$ be the function $\frac{u}{\phi_1}$ extended up to the boundary, then $\tilde u$ vanishes in one single point in $\overline{\Omega}$.*

Proof. First observe that hypothesis (1.4) (resp. (1.5)) indeed guarantees that $\frac{u}{\phi_1}$ may be extended up to the boundary; we will denote in the proof by T the operator that maps a function $v \in H_*^m$ in the extension to $\overline{\Omega}$ of v/ϕ_1.

Let now Tu vanish in $x_0 \in \overline{\Omega}$ and $V_{x_0} = \{v \in S_0 : (Tv)(x_0) = 0\}$ (recall the definition of S_0 in (3.3)). Then u is a minimizer also in this class for the same value γ, and so we claim that $\int_\Omega \nabla^m u \nabla^m v - \gamma \int_\Omega uv \geq 0$ for any $v \in V_{x_0}$; indeed $u + tv \in V_{x_0}$ for any $t \geq 0$ and so $\int_\Omega |\nabla^m(u+tv)|^2 \geq \gamma \int_\Omega (u+tv)^2$, from which one gets by standard calculations the claim.

Given any $x_1 \in \overline{\Omega}\backslash\{x_0\}$ and V_{x_1} defined in an analogous way, one can always find functions $v_1 \in V_{x_1}$ and $v_0 \in V_{x_0}$, such that $v_0 + v_1 = -\phi_1$; if, for the sake of contradiction, $(Tu)(x_1) = 0$, one would have

$$\int_\Omega \nabla^m u \nabla^m (v_1 + v_2) - \gamma \int_\Omega u(v_1 + v_2) \geq 0\,, \tag{4.12}$$

that is

$$\int_\Omega \nabla^m u \nabla^m (-\phi_1) - \gamma \int_\Omega u(-\phi_1) = (\lambda_1^m - \gamma) \int_\Omega u(-\phi_1) \geq 0\,, \tag{4.13}$$

which gives rise to a contradiction since $\lambda_1^m < \gamma$ and $u \leq 0$ but not identically zero. ◻

Now we prove the following three lemmas, which will imply (4.10).

Lemma 4.4. *Under hypothesis* (1.4) *(resp.* (1.5)*), the uniform anti-maximum principle holds for $\lambda \in (\lambda_1^m, \gamma]$. This implies*

$$\gamma \leq \delta\,. \tag{4.14}$$

Moreover if $h \geq 0$ a.e and $h \not\equiv 0$, then the solution u of (4.1) *is such that $u \notin S_0$, that is $\sigma(u) < 0$.*

Proof. Suppose $h \geq 0$ a.e, $\lambda \in (\lambda_1^m, \gamma)$ and assume for sake of contradiction that u is a solution of (4.1) with $u > 0$ in some set of positive measure: we have

$$\int_\Omega \nabla^m u \nabla^m v = \lambda \int_\Omega uv + \int_\Omega hv \quad \text{for any } v \in H_*^m\,. \tag{4.15}$$

With $v = \phi_1$ we get

$$0 = (\lambda - \lambda_1^m) \int_\Omega u\phi_1 + \int_\Omega h\phi_1 \tag{4.16}$$

which, since the second term is not negative, implies $u < 0$ in some set of positive measure, that is u changes sign. Moreover we have

$$\int_\Omega \nabla^m \phi_1 \nabla^m v = \lambda_1^m \int_\Omega \phi_1 v = \lambda \int_\Omega \phi_1 v - (\lambda - \lambda_1^m) \int_\Omega \phi_1 v\,. \tag{4.17}$$

Subtracting (4.17) multiplied by a constant c from (4.15) and rearranging the terms we get

$$\int_\Omega \nabla^m (u - c\phi_1) \nabla^m v = \lambda \int_\Omega (u - c\phi_1)v + (\lambda - \lambda_1^m) \int_\Omega c\phi_1 v + \int_\Omega hv \quad \forall v \in H_*^m\,, \tag{4.18}$$

which, choosing $v = (u - c\phi_1)$, gives

$$\int_\Omega (\nabla^m (u - c\phi_1))^2 - \lambda \int_\Omega (u - c\phi_1)^2 - (\lambda - \lambda_1^m) \int_\Omega c\phi_1 (u - c\phi_1) - \int_\Omega h(u - c\phi_1) = 0\,. \tag{4.19}$$

Now let $c = \sigma(u)$: this gives, since $(u - c\phi_1) \in S_0$,

$$0 \geq (\gamma - \lambda) \int_\Omega (u - c\phi_1)^2 - (\lambda - \lambda_1^m) \int_\Omega c\phi_1 (u - c\phi_1) - \int_\Omega h(u - c\phi_1)\,; \tag{4.20}$$

however $u - c\phi_1 \leq 0$, $h \geq 0$, $\phi_1 \geq 0$ and since $u > 0$ in some set of positive measure $\sigma(u) > 0$; then $\lambda \in (\lambda_1^m, \gamma)$ implies that each term in the right hand side is not negative and so $u - c\phi_1 \equiv 0$, which implies $u \in span\{\phi_1\}$: contradiction since we proved that u changes sign.

Now that we proved $u \le 0$, assume that $h \ge 0$, $h \not\equiv 0$ and suppose for the sake of contradiction that $\sigma(u) = 0$: then equation (4.20) becomes

$$\int_\Omega hu \ge (\gamma - \lambda) \int_\Omega u^2 \,, \tag{4.21}$$

giving a contradiction since equation (4.16) implies $u \not\equiv 0$ and then the right hand side is strictly positive while the left hand side is non positive.

For the case $\lambda = \gamma$ we may do the same and observe that the inequality in (4.20) and (4.21) is strict unless $(u - c\phi_1)$ is a minimizer for γ: if this is the case, then the contradiction comes since $\int_\Omega h(u - c\phi_1) < 0$ by lemma (4.3). □

Lemma 4.5. *If $(\lambda^+, \lambda^-) \in \Sigma$ with $\lambda^\pm > \lambda_1^m$, then uAMP does not hold for $\lambda = \lambda^-$. This implies $\delta \le \bar\lambda$.*

Proof. Let $(\lambda^+, \lambda^-) \in \Sigma$ with $\lambda^\pm > \lambda_1^m$ and u be the corresponding non trivial solution of problem (4.2). It is known that u needs to change sign, actually testing the Fučík equation against ϕ_1 one gets $(\lambda_1^m - \lambda^+) \int_\Omega u^+ \phi_1 = (\lambda_1^m - \lambda^-) \int_\Omega u^- \phi_1$.

The function u may be seen as the solution of $(-\Delta)^m u = \lambda^- u + h$ where $h = (\lambda^+ - \lambda^-) u^+ \in L^2(\Omega)$, but since we have seen that u changes sign while h does not, the uAMP cannot hold for $\lambda = \lambda^-$. □

Lemma 4.6. *For any $\varepsilon > 0$ there exists a point $(\lambda^+, \lambda^-) \in \Sigma$ with $\lambda^\pm > \lambda_1^m$ and $\lambda^- \le \gamma + 2\varepsilon$. This implies $\bar\lambda \le \gamma$.*

Proof. Let $u \in H_*^m(\Omega) \backslash \{0\}$ be such that $\frac{\int_\Omega |\nabla^m u|^2}{\int_\Omega u^2} < \gamma + \varepsilon$ and $\operatorname{ess\,sup}_{x \in \Omega} \frac{u(x)}{\phi_1(x)} = 0$, then consider

$$\tilde\gamma : [-1, 1] \to H_*^m(\Omega) : t \mapsto v_t = t\phi_1 + (1 - |t|) u \ : \tag{4.22}$$

this is a path going from $-\phi_1$ to ϕ_1 which does not pass through the origin: projecting this path radially onto Q_r for some $r \in (0, 1]$ (see (4.4)), and considering the definition in (4.6) we get

$$r d_{\varepsilon, r} \le \max_{t \in [-1,1]} \frac{r J_\varepsilon(v_t)}{\|v_t^+\|_{L^2}^2 + r \|v_t^-\|_{L^2}^2} \,; \tag{4.23}$$

observe also that, by the choice made for u, $v_t \le 0$ if and only if $t \le 0$.

Let now $t(r)$ be such that the maximum in (4.23) is assumed in $v_{t(r)}$, consider any sequence $r_n \to 0^+$ and let $t_n = t(r_n)$: up to a subsequence we have

$$\begin{gathered} t_n \to t_0 \in [-1, 1] \,, \\ v_{t_n} \to v_{t_0} \quad \text{strongly in } H^m(\Omega), \\ r_n d_{\varepsilon, r_n} \to D \in [0, +\infty] \,; \end{gathered}$$

we assume for the moment $D > 0$. From (4.23) we get

$$J_\varepsilon(v_{t_n}) \ge \big(\|v_{t_n}^+\|_{L^2}^2 / r_n + \|v_{t_n}^-\|_{L^2}^2 \big) \, r_n d_{\varepsilon, r_n} \,, \tag{4.24}$$

that is

$$\|\nabla^m v_{t_n}\|_{L^2}^2 - (\lambda_1^m + \varepsilon) \|v_{t_n}\|_{L^2}^2 \ge \big(\|v_{t_n}^+\|_{L^2}^2 / r_n + \|v_{t_n}^-\|_{L^2}^2 \big) \, r_n d_{\varepsilon, r_n} \,; \tag{4.25}$$

since $v_{t_n} \to v_{t_0}$, the left hand side is bounded and then we obtain $\|v_{t_n}^+\|_{L^2} \to 0$ which implies $t_0 \le 0$ and $D < \infty$. Then taking the limit in (4.25) gives

$$\|\nabla^m v_{t_0}\|_{L^2}^2 \ge \|v_{t_0}\|_{L^2}^2 \, (D + \lambda_1^m + \varepsilon) \,; \tag{4.26}$$

using (4.22) and exploiting the properties of ϕ_1 this reads

$$\begin{aligned} &t_0^2\lambda_1^m + (1-|t_0|)^2\int_\Omega |\nabla^m u|^2 + 2t_0(1-|t_0|)\lambda_1^m\int_\Omega u\phi_1 \\ &\geq \Big(t_0^2 + (1-|t_0|)^2\int_\Omega u^2 + 2t_0(1-|t_0|)\int_\Omega u\phi_1\Big)(D+\lambda_1^m+\varepsilon)\ . \end{aligned} \tag{4.27}$$

If $t_0=-1$, (4.27) gives $\lambda_1^m \geq D+\lambda_1^m+\varepsilon$, contradiction; otherwise collect as

$$(1-|t_0|)^2\big(\int_\Omega |\nabla^m u|^2 - (D+\lambda_1^m+\varepsilon)\int_\Omega u^2\big) \geq (D+\varepsilon)\big(t_0^2 + 2t_0(1-|t_0|)\int_\Omega u\phi_1\big)$$

and observe that the right hand side is not negative (actually $t_0(1-|t_0|)\int_\Omega u\phi_1 \geq 0$) and then we get $\frac{\int_\Omega |\nabla^m u|^2}{\int_\Omega u^2} \geq D+\lambda_1^m+\varepsilon$ which implies, by the choice of u, that $\lambda_1^m+\varepsilon+D<\gamma+\varepsilon$. Then we conclude, since $D=\lim_{n\to\infty} r_n d_{\varepsilon,r_n}$, that there exists a point $(\lambda^+,\lambda^-)\in\Sigma$ with $\lambda^\pm>\lambda_1^m$ and $\lambda^-<\lambda_1^m+2\varepsilon+D<\gamma+2\varepsilon$.

Finally recall that we assumed during the proof $D>0$; but in the case $D=0$ one still gets the result since this implies the existence of a point $(\lambda^+,\lambda^-)\in\Sigma$ with $\lambda^\pm>\lambda_1^m$ and $\lambda^-<\lambda_1^m+2\varepsilon\leq\gamma+2\varepsilon$ (use (2.4) for the last inequality). □

At this point we have, by the lemmas (3.4), (4.4), (4.5) and (4.6), that under hypothesis (1.4) (resp. (1.5)) the chain of inequalities $\lambda_1^m<\gamma\leq\delta\leq\bar\lambda\leq\gamma$ holds, implying (4.10) of theorem (4.2).

The following two lemmas will complete the proof of theorem (4.2), giving (4.11).

Lemma 4.7. *In the case (D) with $N=2m-2$, one has $\gamma\leq\delta$.*

Proof. Let u be the solution of (4.1): since $h\in L^2$, by standard regularity theory $u\in H^{2m}(\Omega)\subseteq\mathcal{C}^1(\overline\Omega)$ and then $\sigma(u)=\operatorname{ess\,sup}_{x\in\Omega}\frac{u(x)}{\phi_1(x)}<\infty$.

This allows us to repeat the proof of lemma (4.4). In fact, the same result may be achieved whenever $N<4m-2$ for case (D) and $N<4m$ for case (N), but this will be proved in more generality in lemma (4.9). □

Corollary 4.8. *In the hypotheses of lemma* (4.7)*, in fact $\gamma=\lambda_1^m$.*

Proof. Since we know from [10] that the uAMP does not hold in the hypotheses of lemma (4.7), then $\lambda_1^m=\delta\geq\gamma$; but by the variational characterization of λ_1^m in (2.4) we have also $\lambda_1^m\leq\gamma$. □

Lemma 4.9. *If hypothesis* (1.4) *(resp.* (1.5)*) does not hold, then for any $\varepsilon>0$ there exists $u\in H_*^m(\Omega)$ such that $\frac{\int_\Omega|\nabla^m u|^2}{\int_\Omega u^2}<\lambda_1^m+\varepsilon$ and $\operatorname{ess\,sup}_{x\in\Omega}\frac{u(x)}{\phi_1(x)}=0$. This implies*

$$\gamma=\lambda_1^m\,. \tag{4.28}$$

Proof. In the case (D) with $N=2m-2$, the result follows from the definition of γ and the corollary (4.8). For $N\geq 2m$ (the argument works both for the case (N) and (D)), let the domain ω be such that $\overline\Omega\subseteq\Omega$ and $\widetilde u\in H_0^m(\omega)$ such that $\|\widetilde u\|_{H^m}=1$ and $\operatorname{ess\,sup}\widetilde u=+\infty$ (this is indeed possible since $H_0^m(\omega)\not\subseteq L^\infty(\omega)$ for $N\geq 2m$).

Having fixed $\varepsilon^*>0$, since $\mathcal{C}_0^\infty$ is dense in H_0^m, we may find $\widehat u\in\mathcal{C}_0^\infty(\omega)$ such that $\|\widehat u\|_{H^m}<1+\varepsilon^*$ and $\operatorname{ess\,sup}_{x\in\omega}\frac{\widehat u(x)}{\phi_1(x)}>2(\varepsilon^*)^{-1}$; by rescaling and extending by 0 on $\Omega\backslash\omega$ the function $\widehat u$, we redefine it such that $\|\widehat u\|_{H^m}<\varepsilon^*$ and $\operatorname{ess\,sup}\frac{\widehat u(x)}{\phi_1(x)}=1$.

Now consider the function $u=-\phi_1+\widehat{u}$:

$$\operatorname*{ess\,sup}_{x\in\Omega}\frac{u(x)}{\phi_1(x)}=\operatorname*{ess\,sup}_{x\in\Omega}\Big(-1+\frac{\widehat{u}(x)}{\phi_1(x)}\Big)=0\,,$$

$$\int_\Omega|\nabla^m u|^2=\lambda_1^m\int_\Omega|\phi_1|^2+\int_\Omega|\nabla^m\widehat{u}|^2-2\lambda_1^m\int_\Omega\phi_1\widehat{u}\le\lambda_1^m+(\varepsilon^*)^2+2\lambda_1^m\varepsilon^*\,,$$

$$\int_\Omega u^2=\int_\Omega\phi_1^2+\int_\Omega\widehat{u}^2-2\int_\Omega\phi_1\widehat{u}\ge 1-(\varepsilon^*)^2-2\varepsilon^*\,;$$

then by choosing ε^* small enough we may make the ratio $\frac{\int_\Omega|\nabla^m u|^2}{\int_\Omega u^2}<\lambda_1^m+\varepsilon$ for any $\varepsilon>0$ as small as desired, with $\operatorname{ess\,sup}_{x\in\Omega}\frac{u(x)}{\phi_1(x)}=0$. This gives $\gamma\le\lambda_1^m$, and again the variational characterization of λ_1^m in (2.4) gives $\gamma\ge\lambda_1^m$.

In the more difficult case $N=2m-1$ (for the case (D)) we have $H^m(\Omega)\subseteq\mathcal{C}^0(\overline{\Omega})$ but $H^m(\Omega)\nsubseteq\mathcal{C}^{0,\alpha}(\overline{\Omega})$ for $\alpha\in(1/2,1]$: we will exploit this and the behavior of ϕ_1 at the boundary to obtain the function $\widehat{u}$ which will prove the claim as above.

First observe that by the regularity of $\partial\Omega$ we may always fix a point $x_c\in\partial\Omega$, denote by ν the internal normal at x_c and build a family $\{\beta_t\}_{t\in(0,T]}$ of balls $\beta_t=B(x_c+2t\nu,t)$ such that $\beta_t\subseteq\Omega$ for any $t\in(0,T]$; without loss of generality suppose T=1.

Having fixed $\varepsilon^*>0$ small enough, let B_1 be the unit ball and $u\in\mathcal{C}_0^\infty(B_1)$ be such that $\|u\|_{H^m}\le\varepsilon^*$ and $\sup_{x\in B_1}\frac{u(x)}{\phi_1(x)}=k\in(0,1)$, then denote by $u_t\in H_0^m(\Omega)$ the function

$$u_t=\begin{cases}t^\alpha u(\frac{x-(x_c+2t\nu)}{t}) & \text{for } x\in\beta_t\\ 0 & \text{for } x\notin\beta_t\end{cases}\tag{4.29}$$

with arbitrary $\alpha\in(1/2,1)$.

By the choice of u we have $(-\phi_1+u_1)<0$ in Ω; however since $\sup_{x\in\Omega}u_t(x)=t^\alpha\sup_{x\in B_1}u(x)$ and $\phi_1|_{\beta_t}<Ct$ for a suitable constant C, there exists a $\delta\in(0,1)$ such that $\sup_{x\in\Omega}[-\phi_1(x)+u_\delta(x)]>0$ and then by continuity there exists $\tau\in(\delta,1)$ such that

$$\sup_{x\in\Omega}[-\phi_1(x)+u_\tau(x)]=0,\quad\text{that is}\quad\sup_{x\in\Omega}\frac{u_\tau(x)}{\phi_1(x)}=1\,.\tag{4.30}$$

Now let us estimate the norms of the u_τ: by standard computation $\|u_\tau\|_{L^2}=\tau^{\alpha+N/2}\|u\|_{L^2}$ and $\|\nabla^m u_\tau\|_{L^2}=\tau^{\alpha+N/2-m}\|\nabla^m u\|_{L^2}$; then

$$\|u_\tau\|_{H^m}\le\tau^{\alpha+N/2-m}\|u\|_{H^m}\tag{4.31}$$

and since $\alpha+N/2-m>0$ we get $\|u_\tau\|_{H^m}\le\varepsilon^*$ and so the function u_τ is indeed the $\widehat{u}$ we were looking for. $\square$

Proof of theorem (4.2). As anticipated above, the lemmas (3.4), (4.4), (4.5) and (4.6) give, under hypothesis (1.4) (resp. (1.5)), the chain of inequalities $\lambda_1^m<\gamma\le\delta\le\bar\lambda\le\gamma$ which implies (4.10). When hypothesis (1.4) (resp. (1.5)) does not hold, from the lemmas (4.5), (4.6) one gets $\delta\le\bar\lambda\le\gamma$; then lemma (4.9) gives $\gamma=\lambda_1^m$ and so implies (4.11) (observe that $\delta,\bar\lambda\ge\lambda_1^m$ by their definition). $\square$

5. Proof of the PS condition for functional (2.3)

In this section we prove lemma (3.8); the proof will be adapted from that in [8].

First note that from hypothesis (1.3) one can always make the estimates: for any $\varepsilon > 0$, $\bar{s} \in \mathbb{R}$ and $M \in R$, there exist C_M, $C_\varepsilon \in \mathbb{R}$ (of course depending also on $\bar{s}$) such that

$$g(x,s) \geq Ms - C_M \quad \text{for } s > \bar{s}\,, \tag{5.1}$$

$$|g(x,s)| \leq \varepsilon(-s) + C_\varepsilon \quad \text{for } s \leq \bar{s}\ . \tag{5.2}$$

Let now $\{u_n\} \subseteq H^m_*(\Omega)$ be a PS sequence, i.e. there exist $T > 0$ and $\varepsilon_n \to 0^+$ such that

$$|F(u_n)| = \Big|\frac{1}{2}\int_\Omega |\nabla^m u_n|^2 - \frac{\lambda}{2}\int_\Omega |u_n|^2 - \int_\Omega G(x,u_n) - \int_\Omega h u_n\Big| \leq T \ , \tag{5.3}$$

$$\begin{aligned}|\langle F'(u_n), v\rangle| &= \Big|\int_\Omega \nabla^m u_n \nabla^m v - \lambda\int_\Omega u_n v - \int_\Omega g(x,u_n)v - \int_\Omega hv\Big| \\ &\leq \varepsilon_n \|v\|_{H^m}, \quad \forall v \in H^m_*\,.\end{aligned}$$

1. Suppose u_n is not bounded, then we can assume $\|u_n\|_{H^m} \geq 1$, $\|u_n\|_{H^m} \to +\infty$ and define $z_n = \frac{u_n}{\|u_n\|_{H^m}}$, so that z_n is a bounded sequence in H^m_* and we can select a subsequence such that $z_n \to z_0$ weakly in H^m_* and strongly in $L^2(\Omega)$ and $\mathcal{C}^0(\overline{\Omega})$ (resp. $\mathcal{C}^1(\overline{\Omega})$).
2. Claim: $z_0 \leq 0$.

Proof of the claim. Let $\Omega^+ = \{x \in \Omega : z_0(x) > 0\}$ and $v \in \mathcal{C}^\infty_0(\Omega^+)$ with $v \geq 0$ so that $z_p = z_0^+ v = z_0 v \in H^m_*(\Omega)$. By considering, for arbitrary v as above, $\Big|\frac{\langle F'(u_n), z_p\rangle}{\|u_n\|_{H^m}}\Big|$ we get

$$\int_\Omega \frac{g(x,u_n)z_p}{\|u_n\|_{H^m}} \leq \Big|\int_\Omega \nabla^m z_n \nabla^m z_p\Big| + \lambda\Big|\int_\Omega z_n z_p\Big| + \Big|\int_\Omega \frac{hz_p}{\|u_n\|_{H^m}}\Big| + \frac{\varepsilon_n \|z_p\|_{H^m}}{\|u_n\|_{H^m}}\,. \tag{5.4}$$

Now for any $\bar{x}$ such that $z_p(\bar{x}) > 0$, we have that $u_n(\bar{x}) > 0$ for n big enough and then we can use the estimate (5.1) to obtain

$$\frac{g(\bar{x},u_n)}{\|u_n\|_{H^m}} \geq M z_n(\bar{x}) - \frac{C_M}{\|u_n\|_{H^m}}\,; \tag{5.5}$$

by first taking $\liminf$ and then exploiting the arbitrariness of M we get

$$\lim_{n\to+\infty} \frac{g(\bar{x},u_n)}{\|u_n\|_{H^m}} = +\infty\ . \tag{5.6}$$

Joining equations (5.1) and (5.2) with $\bar{s} = 0$ and divided by $\|u_n\|_{H^m}$ we get

$$\frac{g(x,u_n)}{\|u_n\|_{H^m}} \geq -\varepsilon|z_n| - \frac{\max\{C_M, C_\varepsilon\}}{\|u_n\|_{H^m}}\,; \tag{5.7}$$

since z_n is uniformly bounded by its $\mathcal{C}^0$ convergence and $\|u_n\|_{H^m} \geq 1$, this implies that the functions $\frac{g(x,u_n)}{\|u_n\|_{H^m}}$ are bounded below uniformly so that we can use Fatou's Lemma and get from (5.4) and supposing $z_p \not\equiv 0$

$$\begin{aligned}+\infty &= \int_\Omega \lim_{n\to+\infty} \frac{g(x,u_n)z_p}{\|u_n\|_{H^m}} \leq \liminf_{n\to+\infty}\int_\Omega \frac{g(x,u_n)z_p}{\|u_n\|_{H^m}} \\ &\leq \liminf_{n\to+\infty}\Big(\Big|\int_\Omega \nabla^m z_n\nabla^m z_p\Big| + \lambda\Big|\int_\Omega z_n z_p\Big| + \Big|\int_\Omega \frac{hz_p}{\|u_n\|_{H^m}}\Big| + \frac{\varepsilon_n\|z_p\|_{H^m}}{\|u_n\|_{H^m}}\Big)\,,\end{aligned} \tag{5.8}$$

but the right hand side can be estimated since the first two terms are bounded by $(1+\lambda)\|z_n\|_{H^m}\|z_p\|_{H^m} \le (1+\lambda)\|z_p\|_{H^m}$ and the last two clearly go to zero; then equation (5.8) gives rise to a contradiction unless $z_0 \le 0$. □

3. Claim:

$$\limsup_{n\to+\infty} \int_\Omega \frac{g(x,u_n)z_n}{\|u_n\|_{H^m}} \le 0 \,. \tag{5.9}$$

Proof of the claim. By considering $|2F(u_n) - \langle F'(u_n), u_n\rangle|$ we get:

$$\begin{aligned} &\int_{u_n>s_0} g(x,u_n)u_n - 2G(x,u_n) \\ &\le \int_{u_n\le s_0} 2G(x,u_n) - g(x,u_n)u_n + \big|\int_\Omega hu_n\big| + 2T + \varepsilon_n\|u_n\|_{H^m} \,. \end{aligned} \tag{5.10}$$

The right hand side may be estimated as follows:

- we use estimate (5.2) (once integrated and once multiplied by u_n) to get a constant D_ε such that

$$\int_{u_n\le s_0} 2G(x,u_n) - g(x,u_n)u_n \le \int_\Omega (\varepsilon u_n^2 + \tilde{D}_\varepsilon|u_n|) \le \varepsilon\|u_n\|_{L^2}^2 + D_\varepsilon\|u_n\|_{L^2} \,; \tag{5.11}$$

- $\big|\int_0^1 hu_n\big| \le \|h\|_{L^2}\|u_n\|_{L^2} \le \|h\|_{L^2}\|u_n\|_{H^m}$.

For the left hand side we use hypothesis (1.6) to obtain

$$(1-2\theta)\int_{u_n>s_0} g(x,u_n)u_n \le \int_{u_n>s_0} g(x,u_n)u_n - 2G(x,u_n) \tag{5.12}$$

and then, since $(1-2\theta)>0$, joining all estimates from (5.10) to (5.12), we get

$$\int_{u_n>s_0} g(x,u_n)u_n \le \frac{1}{1-2\theta}(A_\varepsilon\|u_n\|_{H^m} + \varepsilon\|u_n\|_{L^2}^2 + 2T) \,. \tag{5.13}$$

Finally, dividing by $\|u_n\|_{H^m}^2$ and estimating for $u_n \le s_0$ as in (5.11), we get (redefining the constants)

$$\int_\Omega \frac{g(x,u_n)z_n}{\|u_n\|_{H^m}} \le C\big(\varepsilon\frac{\|u_n\|_{L^2}^2}{\|u_n\|_{H^m}^2} + \frac{A_\varepsilon}{\|u_n\|_{H^m}} + \frac{T}{\|u_n\|_{H^m}^2}\big) \,: \tag{5.14}$$

by first taking lim sup and then exploiting the arbitrariness of ε one obtains the claim. □

4. Claim:

$$\limsup_{n\to+\infty}\int_\Omega |\nabla^m z_n|^2 \le \lim_{n\to+\infty}\lambda\int_\Omega z_n^2 = \lambda\int_\Omega z_0^2 \,. \tag{5.15}$$

Proof of the claim. By considering $\Big|\frac{\langle F'(u_n), z_n\rangle}{\|u_n\|_{H^m}}\Big|$ we get:

$$\int_\Omega |\nabla^m z_n|^2 \le \lambda\int_\Omega z_n^2 + \int_\Omega \frac{g(x,u_n)z_n}{\|u_n\|_{H^m}} + \big|\int_\Omega \frac{hz_n}{\|u_n\|_{H^m}}\big| + \frac{\varepsilon_n\|z_n\|_{H^m}}{\|u_n\|_{H^m}} \,,$$

where, taking lim sup and using equation (5.9), all the terms in the right hand side go to zero except the first that converges to $\lambda\|z_0\|_{L^2}^2$. □

5. Claim: if $\lambda \in (\lambda_1^m, \gamma)$ then $z_0 = 0$.

Proof of the claim. We will first prove that $z_0 \in S_0$ (see the definition in (3.3)): suppose for the sake of contradiction that $\sup_{x\in\Omega} \frac{z_0(x)}{\phi_1(x)} < 0$. Since $z_n \to z_0$ in $\mathcal{C}^0(\overline{\Omega})$ in the case (N) and in $\mathcal{C}^1(\overline{\Omega})$ in the case (D), we have that $\frac{z_n}{\phi_1} < \frac{1}{2}\frac{z_0}{\phi_1} < 0$ for $n > \bar{n}$ and then $u_n < 0$ in Ω for $n > \bar{n}$. This allows us to use the estimate (5.2) to obtain that

$$\Big|\frac{g(x,u_n)}{\|u_n\|_{H^m}}\Big| \le \varepsilon|z_n(x)| + \frac{C_\varepsilon}{\|u_n\|_{H^m}}\,; \tag{5.16}$$

by first taking lim sup and then exploiting the arbitrariness of ε we get

$$\lim_{n\to+\infty}\Big|\frac{g(x,u_n)}{\|u_n\|_{H^m}}\Big| = 0\,. \tag{5.17}$$

By considering $\big|\frac{\langle F'(u_n),\phi_1\rangle}{\|u_n\|_{H^m}}\big|$ we get

$$\big|(\lambda_1^m-\lambda)\int_\Omega z_n\phi_1\big| \le \Big|\int_\Omega \frac{g(x,u_n)\phi_1}{\|u_n\|_{H^m}}\Big| + \Big|\int_\Omega \frac{h\phi_1}{\|u_n\|_{H^m}}\Big| + \frac{\varepsilon_n\|\phi_1\|_{H^m}}{\|u_n\|_{H^m}}\,. \tag{5.18}$$

Since equation (5.16) also tells that the functions in the sequence are dominated (for $n > \bar{n}$) by $\max_{x\in\overline{\Omega}}|z_0| + 1 + C_{\varepsilon=1}$, we can use dominated convergence to assert that

$$\lim_{n\to+\infty}\Big|\int_\Omega \frac{g(x,u_n)\phi_1}{\|u_n\|_{H^m}}\Big| \le \lim_{n\to+\infty}\int_\Omega\Big|\frac{g(x,u_n)}{\|u_n\|_{H^m}}\Big|\phi_1 = \int_\Omega \lim_{n\to+\infty}\Big|\frac{g(x,u_n)}{\|u_n\|_{H^m}}\Big|\phi_1 = 0\,. \tag{5.19}$$

Now we may take limit in equation (5.18), to get

$$(\lambda_1^m-\lambda)\int_\Omega z_0\phi_1 = 0\,. \tag{5.20}$$

This, with $\lambda \ne \lambda_1^m$, gives $\int_\Omega z_0\phi_1 = 0$ which, since $z_0 \le 0$, would imply $z_0 \equiv 0$: we conclude that $z_0 \in S_0$ as claimed. Finally, this implies $\int_\Omega |\nabla^m z_0|^2 \ge \gamma\int_\Omega z_0^2$ by the definition of γ, which contradicts equation (5.15) unless $z_0 = 0$ since otherwise, by the weak convergence,

$$\int_\Omega |\nabla^m z_0|^2 \le \liminf_{n\to+\infty}\int_\Omega |\nabla^m z_n|^2 \le \lambda\int_\Omega z_0^2 < \gamma\int_\Omega z_0^2\,. \tag{5.21}$$

□

6. Claim: if $\lambda = \lambda_1^m$, $\int_\Omega h\phi_1 < 0$ and hypothesis (1.7) holds, then $z_0 = 0$.

Proof of the claim. Equation (5.15) and the weak convergence of z_n to z_0 implies

$$\int_\Omega |\nabla^m z_0|^2 \le \liminf_{n\to+\infty}\int_\Omega |\nabla^m z_n|^2 \le \lambda_1^m\int_\Omega z_0^2\,, \tag{5.22}$$

which implies that $z_0 \in \mathit{span}\{\phi_1\}$, that is $z_0 = -\rho\phi_1$ for some $\rho \ge 0$. By considering $|\langle F'(u_n),\phi_1\rangle|$ with $\lambda = \lambda_1^m$ we get

$$\Big|\int_\Omega g(x,u_n)\phi_1 + \int_\Omega h\phi_1\Big| \le \varepsilon_n\|\phi_1\|_{H^m}\,. \tag{5.23}$$

Taking lim sup gives

$$\limsup_{n\to+\infty}\int_\Omega g(x,u_n)\phi_1 = -\int_\Omega h\phi_1 > 0\,, \tag{5.24}$$

but this implies $z_0 = -\rho\phi_1 \equiv 0$ since otherwise

$$u_n(x) = z_n(x)\|u_n\|_{H^m} \leq -\frac{\rho}{2}\phi_1(x)\|u_n\|_{H^m} \to -\infty \quad \forall x \in \Omega \tag{5.25}$$

and so, by hypothesis (1.7), the limit in the left hand side of (5.24) would be zero. □

7. Claim: u_n is bounded.

Proof of the claim. The result follows since equation (5.15) now gives the contradiction $1 = \|\nabla^m z_n\|_{L^2}^2 + \|z_n\|_{L^2}^2 \to 0$. □

The PS condition follows now with standard calculations from the boundedness of u_n.

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