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A CHARACTERIZATION OF BALLS USING THE DOMAIN DERIVATIVE

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Abstract. In this note we give a characterization of balls in $\mathbb{R}^N$ using the domain derivative. As a byproduct we will show that an overdetermined Stekloff eigenvalue problem is solvable if and only if the domain of interest is a ball.

1. Introduction

In this note we give a characterization of balls in $\mathbb{R}^N$ using the domain derivative. As an application we prove that an overdetermined Stekloff eigenvalue problem is solvable if the domain of interest is a ball. This work is motivated by the following result.

Theorem 1.1. *A domain $D \subset \mathbb{R}^N$ is a ball if and only if there exists a constant c such that the following integral equality is valid*

$$\int_D h\,dx = c \int_{\partial D} h\,d\sigma, \tag{1.1}$$

for every harmonic function h.

For the proof of the above theorem, the reader is referred to [1, 3].

Our characterization replaces (1.1) by another integral equation which involves the domain derivative of the solution of the Saint-Venant equation in D. This result will enable us to show that an overdetermined Stekloff eigenvalue problem is solvable if and only if the domain of the problem is a ball.

2. Main result

To state the main result we need some preparation. Henceforth D is a smooth simply connected bounded domain in $\mathbb{R}^N$. By u we denote the unique solution of the Saint-Venant problem in D; i.e.,

$$\begin{gathered} -\Delta u = 1 \quad \text{in } D \\ u = 0 \quad \text{on } \partial D \end{gathered} \tag{2.1}$$

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Given a vector field $V \in C^2(\mathbb{R}^N;\mathbb{R}^N)$, we denote by u', the domain derivative of u at D in direction of V; the reader is referred to [5] for a thorough treatment of the concept of domain derivatives. Using [5, Theorems 3.1 and 3.2], it follows that

$$\begin{aligned} \Delta u' = 0 \quad \text{in } D \\ u' = -\frac{\partial u}{\partial \nu} V \cdot \nu \quad \text{on } \partial D, \end{aligned} \tag{2.2}$$

where ν stands for the unit outward normal vector on ∂D. Now we state our main result.

Theorem 2.1. *The domain D is a ball if and only if there exists a constant c such that the following integral equation is valid*

$$\int_D u'\,dx = c \int_{\partial D} u'\,d\sigma, \tag{2.3}$$

for every vector field $V \in C^2(\mathbb{R}^N;\mathbb{R}^N)$.

We need the following result.

Lemma 2.2. *Suppose $f \in C(\partial D)$ and the following equation holds*

$$\int_{\partial D} fV \cdot \nu\,d\sigma = 0, \tag{2.4}$$

for every $V \in C^2(\mathbb{R}^N, \mathbb{R}^N)$. Then f vanishes on ∂D.

Proof. To derive a contradiction suppose $f(x_0) \neq 0$, for some $x_0 \in \partial D$. Let us assume that in fact $f(x_0) > 0$; the case $f(x_0) < 0$ can be addressed similarly. Since f is continuous, we readily infer existence of an open component of ∂D, denoted γ, where

$$f(x) \geq \frac{1}{k}, \quad \forall x \in \gamma,$$

for some integer k. Thanks to smoothness of ∂D we can make the following observation; namely, ∂D is locally star-shaped. This means: For every $\xi \in \partial D$, there exists a ball B_ξ centered at ξ, and a point $x_\xi \in D$, such that

$$(x - x_\xi) \cdot \nu(x) > 0, \quad \forall x \in B_\xi \cap \partial D.$$

Without loss of generality we may assume there exists $x^* \in D$ such that

$$(x - x^*) \cdot \nu(x) > 0, \quad \forall x \in \gamma.$$

Let us now consider a non-negative test function $\phi \in C_0^\infty(\mathbb{R}^N)$, where the intersection of the support of ϕ with ∂D is a proper subset of γ and has positive measure. Now we choose $V = \phi(x)(x - x^*)$ in (2.4); note that V is admissible since it belongs to $C^2(\mathbb{R}^N, \mathbb{R}^N)$. Thus

$$\int_\gamma f(x)\phi(x)(x - x^*) \cdot \nu(x)\,d\sigma = 0. \tag{2.5}$$

However

$$\int_\gamma f(x)\phi(x)\;(x - x^*) \cdot \nu(x)\,d\sigma \geq \frac{1}{k} \int_{\text{support}(\phi)\cap\gamma} \phi(x)(x - x^*) \cdot \nu(x)\,d\sigma > 0,$$

which contradicts (2.5). Thus f must vanish on ∂D, as desired. $\square$

Proof of Theorem 2.1. Assume that (2.3) is satisfied. Let us fix $V \in C^2(\mathbb{R}^N;\mathbb{R}^N)$. We claim

$$\int_D u'\,dx = \int_{\partial D} \big(\frac{\partial u}{\partial \nu}\big)^2 V\cdot\nu\,d\sigma. \tag{2.6}$$

To prove (2.6) we observe that from the differential equation in (2.1) we have $\int_D u'\,dx = -\int_D u'\Delta u\,dx$. Since u' is harmonic in D it then follows that

$$\int_D u'\,dx = \int_D (u\Delta u' - u'\Delta u)\,dx.$$

Now an application of the Green identity to the right hand side of the above equation yields

$$\int_D u'\,dx = \int_{\partial D} \big(u\frac{\partial u'}{\partial \nu} - u'\frac{\partial u}{\partial \nu}\big)\,d\sigma.$$

Since u vanishes on ∂D, the above equation implies

$$\int_D u'\,dx = -\int_{\partial D} u'\,\frac{\partial u}{\partial \nu}\,d\sigma. \tag{2.7}$$

From (2.7) and the boundary condition in (2.2) we derive (2.6). From the hypothesis and (2.6) we obtain $c\int_{\partial D} u'\,d\sigma = \int_{\partial D}\big(\frac{\partial u}{\partial \nu}\big)^2 V\cdot\nu\,d\sigma$. So again using the boundary condition in (2.2) we derive

$$-c\int_{\partial D} \partial u/\partial\nu V\cdot\nu\,d\sigma = \int_{\partial D}\big(\frac{\partial u}{\partial \nu}\big)^2 V\cdot\nu\,d\sigma.$$

So

$$\int_{\partial D}\Big(\big(\frac{\partial u}{\partial \nu}\big)^2 + c\frac{\partial u}{\partial \nu}\Big)V\cdot\nu\,d\sigma = 0.$$

Since $V \in C^2(\mathbb{R}^N;\mathbb{R}^N)$ is arbitrary Lemma 2.2, applied to the above equation, guarantees that

$$\frac{\partial u}{\partial \nu}\big(\frac{\partial u}{\partial \nu} + c\big) = 0 \quad \text{on } \partial D.$$

By the Hopf boundary point lemma applied to (2.1) we infer that $\partial u/\partial\nu$ is negative on ∂D. So the last equation implies $\partial u/\partial\nu = -c$ on ∂D. This result added to (2.1) yields the following overdetermined boundary value problem

$$\begin{gathered} -\Delta u = 1 \quad \text{in } D\\ u = 0 \quad \text{on } \partial D\\ \frac{\partial u}{\partial \nu} = -c \quad \text{on } \partial D \end{gathered} \tag{2.8}$$

It is classical, see [4, 6], that (2.8) is solvable if and only if D is a ball.

Conversely, let us assume that D is a ball. Without loss of generality we may assume that D is the ball with radius R centered at the origin. Note that in this case the solution of (2.1) is

$$u(x) = \frac{1}{2N}(R^2 - |x|^2).$$

Therefore $\partial u/\partial\nu$ will be equal to $-R/N$ on ∂D. So if we apply (2.7) we find that

$$\int_D u'\,dx = -\frac{R}{N}\int_{\partial D} u'\,d\sigma,$$

which coincides with the integral equation (2.3), with $c = -R/N$. This completes the proof. ∎

Note that $c = -R/N$, as in the above argument, could also be written as $c = -\frac{\omega_N R^N}{N\omega_N R^{N-1}} = -\frac{V(D)}{S(D)}$, where ω_N stands for the volume of the unit N-dimensional ball, and $V(D)$, $S(D)$ denote the volume and the surface area of D, respectively.

In the remaining of this section we focus on the Stekloff eigenvalue problem; i.e.,

$$\begin{gathered} \Delta w = 0 \quad \text{in } D. \\ \frac{\partial w}{\partial \nu} = pw \quad \text{on } \partial D \end{gathered} \tag{2.9}$$

In (2.9), p denotes the eigenvalue. It is well known that there are infinitely many eigenvalues $0 = p_1 < p_2 \leq p_3 \leq \dots$ for which (2.9) has non trivial solutions. These solutions are the corresponding eigenfunctions denoted by $w_1, w_2, \dots$, where w_1 is clearly constant. We now prove the following result.

Theorem 2.3. *The overdetermined boundary-value problem*

$$\begin{gathered} \Delta w = 0 \quad \text{in } D \\ \frac{\partial w}{\partial \nu} = pw \quad \text{on } \partial D \\ \int_D w_k \, dx = 0 \quad \forall k \geq 2. \end{gathered} \tag{2.10}$$

is solvable if and only if D is a ball.

Proof. Let us assume D is a ball. Let w_k be an eigenfunction corresponding to p_k, $k = 2, 3, \dots$. Since w_k is harmonic it follows from the mean value property that

$$\int_D w_k \, dx = d \int_{\partial D} w_k \, d\sigma,$$

for some constant d. Thus using the boundary condition in (2.9) in conjunction with the Divergence Theorem we infer

$$\int_D w_k \, dx = \frac{d}{p_k} \int_D \Delta w_k \, dx.$$

Since w_k is harmonic in D we obtain $\int_D w_k \, dx = 0$, as desired.

To prove the converse we proceed along the same lines as in [2, Theorem 2] to prove the converse. To this end, let u be the solution of the Saint-Venant problem in D, $V \in C^2(\mathbb{R}^N; \mathbb{R}^N)$, and u' the domain derivative of u in direction of V. Since D is smooth it follows from (2.2) that $u' \in C^2(\overline{D})$. Hence u' can be represented in terms of the eigenfunctions w_k as follows

$$u'(x) = \sum_{i=1}^{\infty} \gamma_i \, w_i(x),$$

where

$$\gamma_i = \int_{\partial D} w_i u' \, d\sigma.$$

Integrating the equation before the last, over D, and taking into account that $\int_D w_i \, dx = 0$, for $i = 2, 3, \dots$ yields

$$\int_D u' \, dx = \gamma_1 \int_D w_1 \, dx = k \int_{\partial D} u' \, d\sigma,$$

where k is a constant independent of the vector field V. Since V is arbitrary we can apply Theorem 2.1 to conclude that D must be a ball, as desired. □

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