Electronic Journal of Differential Equations, Vol. 2006(2006), No. 51, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

A GLOBAL DESCRIPTION OF SOLUTIONS TO NONLINEAR PERTURBATIONS OF THE WIENER-HOPF INTEGRAL EQUATIONS

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Abstract. We establish the solvability, the number of solutions and the covering dimension of the solution set of nonlinear Wiener-Hopf equations. The induced linear mapping is assumed to be of nonnegative index, while the nonlinearities are such that projection like methods are applicable. Solvability of nonlinear integral equations on the real line has been also discussed.

1. Introduction

Consider a nonlinear perturbation of the Wiener-Hopf equation

$$\lambda x(s) - \int_0^\infty k(s-t)x(t)dt + (Nx)(s) = y(s) \tag{1.1}$$

where $k : \mathbb{R} \to \mathbb{C}$ is in $L_1(\mathbb{R}, \mathbb{C})$ and $y(s) \in L_1(\mathbb{R}^+, \mathbb{C})$ and N is a suitable nonlinear mapping. The corresponding linear Wiener-Hopf equations is

$$\lambda x(s) - \int_0^\infty k(s-t)x(t)dt = y(s). \tag{1.2}$$

Let $i(\lambda)$ be the index of the homogeneous equation corresponding to (1.2) with $y(s) = 0$. Then, if $i(\lambda) > 0$, the homogeneous equation has an $i(\lambda)$-dimensional space of solutions, and the nonhomogeneous linear equation (1.2) has infinitely many solutions for each y in a suitable space. If $i(\lambda) = 0$, then (1.2) has a unique solution for each y. These results have been proven in the seminal paper by Krein [9]. Detailed study of (1.2) can be found in Corduneanu [4]. Many problems in mathematical physics lead to (1.2). In particular, it appears in studying questions of transfer of radiant energy. We also note that the study of (1.2) with the integral over $\mathbb{R}$ is much simpler and is based on using integral Fourier transform (see Section 5).

In this paper, we shall extend these results to the nonlinear perturbed Wiener-Hopf equation (1.1). As mentioned above, if $x(s)$ is a solution of (1.2), then one also has the $i(\lambda)$-dimensional plane of solutions of (1.2) consisting of $\{x(s)+x_0(s) : x_0(s)$ in the null space of $A\}$, where A is the linear mapping defined by (1.2). The

2000 *Mathematics Subject Classification.* 47H15, 35L70, 35L75, 35J40.
Key words and phrases. Number of solutions; covering dimension; Wiener-Hopf equations; nonlinear; (pseudo) A-proper maps; surjectivity.

Submitted February 22, 2006. Published April 18, 2006.

results for (1.1) we will prove consist in describing the manner in which this $i(\lambda)$-dimensional aspect of the solutions of (1.2) persists in the global description of the set of solutions of the nonlinear (1.1).

Since N is nonlinear, one cannot expect the solutions of (1.1) to have any linear structure. We prove various dimension results for the solution set of (1.1) with $i(\lambda) > 0$, where by dimension we will mean the natural extension of the linear concept of dimension, namely, the Lebesgue covering dimension. Moreover, if $i(\lambda) = 0$, we prove that (1.1) has a constant number of solutions on certain connected components for almost all f.

Our method consists in reformulating (1.1) as an operator equation of the form

$$Ax + N(u, x) = f, \quad (u, x) \in D \subset \mathbb{R}^i \times X, \tag{1.3}$$

where X is a Banach space, D is an open subset in $\mathbb{R}^i \times X$, and $A + N : D \to X$ is an A-proper mapping. Then, under suitable assumptions on A and N, we prove some abstract results for (1.3) using some dimension results of [5, 6, 11, 12, 13], which are needed in the study of (1.1).

In Section 2, we give some definitions and state our multiplicity and dimension results for (1.1). Section 3 is devoted to proving some abstract results for (1.3) needed in Section 2. We prove the results stated in Section 2 in Section 4. Moreover, we look at special classes of nonlinearities that are of integral Hammerstein or Nemitskii type. Possible extensions of these results to integral equations on the real line are indicated briefly in Section 5.

2. Multiplicity and dimension results for semi-abstract equation (1.1)

Let us first recall some facts about the Wiener-Hopf equation (1.2), where $k : \mathbb{R} \to \mathbb{C}$ is in $L_1(\mathbb{R}, \mathbb{C})$ and $y(s) \in L_1(\mathbb{R}^+, \mathbb{C})$ (see [4, 9]). Let $Y = L_1(\mathbb{R}, \mathbb{C})$ and $x(s) = y(s) = 0$ for $s < 0$. Then (1.2) becomes

$$\lambda x(s) - \int_{-\infty}^{\infty} k(s-t)x(t)dt = z(s) \tag{2.1}$$

where $z(s) = y(s)$ for $s \geq 0$ and $z(s) = -\int_0^\infty k(s-t)x(t)dt$ for $s < 0$. Let $\hat{k}(\xi)$ be the Fourier transform of k, i.e.,

$$\hat{k}(\xi) = \int_{-\infty}^{+\infty} k(t)e^{i\xi t}dt.$$

Applying the Fourier transform to (2.1), we get $\lambda\hat{x}(\xi) - \hat{k}(\xi)\hat{x}(\xi) = \hat{z}(\xi)$ on $\mathbb{R}$. Let $X = L_p(\mathbb{R}^+, \mathbb{C})$ and $K : X \to X$ be given by

$$Kx(s) = \int_0^\infty k(s-t)x(t)dt, \quad s \in \mathbb{R}^+. \tag{2.2}$$

It turns out that $\lambda I - K : X \to X$ is a Fredholm mapping if and only if $\lambda - \hat{k}(\xi) \neq 0$ in $\mathbb{R} \cup \{-\infty, +\infty\}$ and the index $i(\lambda)$ =index$(\lambda I - K) = -w(\Gamma_\lambda, 0)$ for $\Gamma_\lambda = \{\lambda - \hat{k}(\xi) : -\infty \leq \xi \leq +\infty\}$. Γ_λ is a closed curve and $w(\Gamma_\lambda, 0)$ is the winding number. If $i(\lambda) \geq 0$, then $\dim N(\lambda I - K) = i(\lambda)$ and the range $\mathbb{R}(\lambda I - K) = X$. If $i(\lambda) < 0$, then $\dim N(\lambda I - K) = \{\emptyset\}$ and $\operatorname{codim} \mathbb{R}(\lambda I - K) = -i(\lambda)$. Suppose that $\lambda - \hat{k}(\xi) \neq 0$. Then $\lambda I - K$ is of index zero if, for example, $k(t-s)$ $(0 \leq t, s < \infty)$ is a symmetric kernel, that is, if $k(t)$ ($-\infty < t < \infty$) is an even function. In this case $\hat{k}(\xi)$ is also an even function. Another interesting case is when $k(t-s)$ is a hermitian kernel, i.e., $k(-t) = \overline{k(t)}$. Then $\hat{k}(\xi)$ is real, and $\lambda - \hat{k}(\xi) \neq 0$ if and only if it is positive. Hence $\lambda I - K$ is of index zero in this case if $\lambda - \hat{k}(\xi) > 0$. Throughout the paper A will stand for the operator $\lambda I - K$ for some $\lambda \in \mathbb{C}$.

Next, let us recall the idea of covering dimension. If D is a topological space, and k is a positive integer, then D is said to have covering dimension equal to k provided that k is the smallest integer with the property that whenever U is a family of open subsets of D whose union covers D, there exists a refinement, U', of U, whose union also covers D, and no subfamily of U' consisting of more than $k+1$ members has nonempty intersection. If D fails to have this refinement property for each positive integer, then D is said to have infinite dimension. When $a \in D$, we say that D has a dimension k at a if each neighborhood, in D, of a has dimension at least k. In the absence of a manifold structure on D, the concept of dimension is the natural way in which to describe its size.

For a mapping $T : X \to Y$, let Σ be the set of all points $x \in X$ where T is not locally invertible, and let $\operatorname{card} T^{-1}(\{f\})$ be the cardinal number of the set $T^{-1}(\{f\})$. Let $X = L_p(\mathbb{R}^+, \mathbb{C})$, $1 \leq p < \infty$ and $P : X \to N(A)$ be the projection onto $N(A)$. Throughout the paper we assume that a nonlinear mapping N is quasibounded with the quasinorm

$$|N| = \limsup_{\|x\| \to \infty} \|Nx\|/\|x\| < \infty$$

Theorem 2.1 (Nonlinear Fredholm Alternative)**.** *Let $A = \lambda I - K : X \to X$ be a Fredholm mapping of index $i(A) \geq 0$ induced by (1.2) and $N : X \to X$ be a k-ball contractive mapping with k and $|N|$ sufficiently small. Then, either*

- (i) *the equation $Ax = 0$ has a unique zero solution, i.e, $i(A) = 0$, in which case (1.1) is approximation solvable for each $y \in X$ and $(A+N)^{-1}(\{y\})$ is compact for each $y \in X$ and the cardinal number $\operatorname{card}(A+N)^{-1}(\{y\})$ is constant, finite and positive on each connected component of $X \setminus (A+N)(\Sigma)$, or*
- (ii) *$N(A) \neq \{0\}$, i.e., $i(A) = \dim N(A) > 0$, in which case, for each $y \in X$, there is a connected closed subset C of $(A+N)^{-1}(\{y\})$ whose dimension at each point is at least $m = i(A)$ and the projection P maps C onto $N(A)$.*

The uniqueness result in Theorem 2.1 with $i(A) = 0$ and N a k-contraction has been proven by Corduneanu [4] using the contraction mapping principle. Next, we shall look at monotone like perturbations.

Theorem 2.2. *Let $A : H = L_2(\mathbb{R}^+, \mathbb{R}) \to H$ be a Fredholm mapping of index $i(A) = 0$ induced by (1.2) and $N : H \to H$ be such that either*

- (a) *A is monotone and N is bounded continuous and of type (S_+) with $|N|$ sufficiently small, or*
- (b) *A is c-strongly monotone for some $c > 0$ and $N = N_1 + N_2$ with N_1 monotone and N_2 is k-contractive for some $k < c$ with $|N_2|$ sufficiently small.*

Then, the equation $Ax = 0$ has a unique zero solution and (1.1) is approximation solvable for each $y \in H$ and $(A + N)^{-1}(\{y\})$ is compact for each $y \in H$ and the cardinal number card$(A + N)^{-1}(\{y\})$ is constant, finite and positive on each connected component of $H \setminus (A + N)(\Sigma)$.

Under just the monotonicity assumption on N, we can only prove the solvability of (1.1).

Theorem 2.3. *Let A be a monotone Fredholm mapping of index $i(A) = 0$ induced by (1.2) and $N : H \to H$ be continuous, bounded and monotone such that $|N|$ is sufficiently small. Then (1.1) has a solution for each $y \in H$.*

For odd nonlinearities, we have the following result.

Theorem 2.4. *Let A be a Fredholm mapping of index $i(A) > 0$ induced by (1.2) and $N : X \to X$ be a continuous odd k-ball contractive mapping with k sufficiently small. Let S_0 be the solution set of (1.1) with $y = 0$. Then, for any positive real number r and $B(0,r) = \{x \in X : \|x\| < r\}$, the dimension of $S_0 \cap \partial B(0,r)$ is at least $i(A) - 1$, when $i(A) > 1$, and $S_0 \cap \partial B(0,r)\}$ contains at least two points when $i(A) = 1$.*

Remark 2.5. If $i(A) = 0$, then (1.1) is equivalent to the Hammerstein operator equation $x + A^{-1}Nx = A^{-1}y$. Then the recent results of the author [15] apply to (1.1) with F of pseudo monotone type and A selfadjoint or P-(quasi) positive.

3. Nonlinear Fredholm alternative and the number of solutions

We begin by proving some generalizations of the first Fredholm theorem and the Fredholm alternative to general (pseudo) A-proper mappings. In the case of A-proper mappings, we also establish the number of solutions of these equations and the dimension of the solution set. These results are needed for the proofs of the theorems in Section 2. There is a large literature on nonlinear Fredholm theory which has dealt only with the existence results [2]. The index and the covering dimension results for various operator equations involving A-proper and epi-maps can be found in [5, 6, 11, 13] and the literature in there.

Let us recall some definitions first. Let $\{X_n\}$ and $\{Y_n\}$ be finite dimensional subspaces of Banach spaces X and Y respectively such that $\dim X_n - \dim Y_n = i \geq 0$ for each n and $\operatorname{dist}(x, X_n) \to 0$ as $n \to \infty$ for each $x \in X$. Let $P_n : X \to Y_n$ and $Q_n : Y \to Y_n$ be linear projections onto X_n and Y_n respectively such that $P_n x \to x$ for each $x \in X$ and $\delta = \max \|Q_n\| < \infty$. Then $\Gamma_i = \{X_n, P_n; Y_n, Q_n\}$ is a projection scheme for (X, Y). Such schemes are needed for studying multiparameter problems and nonlinear perturbations of Fredholm mappings of index $i \geq 0$.

A mapping $T : D \subset X \to Y$ is said to be *approximation-proper* (*A-proper* for short) with respect to Γ_i if (i) $Q_n T : D \cap X_n \to Y_n$ is continuous for each n and (ii) whenever $\{x_{n_k} \in D \cap X_{n_k}\}$ is bounded and $\|Q_{n_k} T x_{n_k} - Q_{n_k} f\| \to 0$ for some $f \in Y$, then a subsequence $x_{n_{k(i)}} \to x$ and $Tx = f$. T is said to be *pseudo A-proper* with respect to Γ_i if in (ii) above we do not require that a subsequence of $\{x_{n_k}\}$ converges to x for which $Tx = f$. If f is given in advance, we say that T is (pseudo) A-proper at f.

We state now a number of examples of A-proper and pseudo A-proper mappings. To look at ϕ-condensing mappings, we recall that the *set measure of noncompactness* of a bounded set $D \subset X$ is defined as $\gamma(D) = \inf\{d > 0 : D$ has a finite covering by sets of diameter less than $d\}$. The *ball-measure of noncompactness* of D is defined as $\chi(D) = \inf\{r > 0 | D \subset \cup_{i=1}^n B(x_i, r),\ x \in X,\ n \in N\}$. Let ϕ denote either the set or the ball-measure of noncompactness. Then a mapping $N : D \subset X \to X$ is said to be $k - \phi$ *contractive* (ϕ-*condensing*) if $\phi(N(Q)) \leq k\phi(Q)$ (respectively $\phi(N(Q)) < \phi(Q)$) whenever $Q \subset D$ (with $\phi(Q) \neq 0$). A k- ball contractive vector field with $k < 1$ is A-proper [16].

Recall that a function $N : X \to X^*$ is said to be of type (S_+) if $x_n \rightharpoonup x$ and $\limsup(Nx_n, x_n - x) \leq 0$ imply that $x_n \to x$. A ball-condensing perturbation of a c-strongly monotone mapping, i.e., $(Tx - Ty, x - y) \geq c\|x - y\|^2$ for all $x, y \in X$, is an A-proper mapping.

We need the following classes of A-proper mappings.

Example 3.1. *(*[11]*)* Let $A : X \to Y$ be a linear Fredholm mapping of index $i(A) \geq 0$, $X = N(A) \oplus \tilde{X}$, $Y = Y_0 \oplus R(A)$, $\|Ax\| \geq c\|x\|$ for some $c > 0$ and all $x \in \tilde{X}$, and $N : X \to Y$ be a bounded and continuous k-ball contraction with $k < c$. Then $A, A + N : X \to Y$ are A-proper with respect to $\Gamma_i = \{X_0 \oplus X_n, Y_0 \oplus Y_n, \tilde{Q}_n\}$ where $\cup_{n\geq 1} X_n$ is dense in $\tilde{X}$, $Y_n = A(X_n)$, $\tilde{Q}_n(y_0 + y_1) = y_0 + Q_n y_1$ and $Q_n : \tilde{Y} \to Y_n$ are projections onto Y_n for each n. Moreover, if $Ax_n + Nx_n = f$ with $\{x_n = x_{0n} + x_{1n}\} \in X = N(A) \oplus \tilde{X}$ and $\{x_{0n}\}$ both bounded, then $\{x_n\}$ has a convergent subsequence.

Example 3.2. Let $A : X \to X^*$ be monotone and $N : X \to X^*$ be of type (S_+). Then $A + N : X \to X^*$ is A-proper with respect to $\Gamma_0 = \{X_n, R(P_n^*), P_n^*\}$, where $P_n : X \to X_n$ are projections onto X_n.

Example 3.3 ([10, 17])**.** Let $A : X \to X^*$ be continuous and c-strongly monotone and $N : X \to X^*$ be continuous and k-ball contractive. Then $A + N : X \to X^*$ is A-proper with respect to $\Gamma_0 = \{X_n, R(P_n^*), P_n^*\}$, where $P_n : X \to X_n$ are projections onto X_n.

We say that a mapping $T : X \to Y$ satisfies condition (+) if whenever $Tx_n \to f$ in Y then $\{x_n\}$ is bounded in X. T is locally injective at $x_0 \in X$ if there is a neighborhood $U(x_0)$ of x_0 such that T is injective on $U(x_0)$. T is locally injective on X if it is locally injective at each point $x_0 \in X$. A continuous mapping $T : X \to Y$ is said to be locally invertible at $x_0 \in X$ if there are a neighborhood $U(x_0)$ and a neighborhood $U(T(x_0))$ of $T(x_0)$ such that T is a homeomorphism of $U(x_0)$ onto $U(T(x_0))$. It is locally invertible on X if it is locally invertible at each point $x_0 \in X$.

We need the following basic theorem on the number of solutions of nonlinear equations for A-proper mappings [14].

Theorem 3.4. *Let $T : X \to Y$ be a continuous A-proper mapping w.r.t. Γ_0 that satisfies condition (+). Then*

(a) *The set $T^{-1}(\{f\})$ is compact (possibly empty) for each $f \in Y$.*
(b) *The range $R(T)$ of T is closed and connected.*
(c) *Σ and $T(\Sigma)$ are closed subsets of X and Y, respectively, and $T(X \setminus \Sigma)$ is open in Y.*
(d) *$\operatorname{card} T^{-1}(\{f\})$ is constant and finite (it may be 0) on each connected component of the open set $Y \setminus T(\Sigma)$.*

Next, we shall prove a nonlinear Fredholm alternative for nonlinear perturbations of linear Fredholm mappings

$$Ax + Nx = f \tag{3.1}$$

where $A : X \to Y$ is a linear Fredholm mapping of index $i(A) \geq 0$ with $R(A) = Y$, and N is a nonlinear quasibounded mapping with $|N|$ sufficiently small. If X_0 is the null space of A, then $X = X_0 \oplus \tilde{X}$ and $i(A) = \dim X_0$. Let $P : X \to X_0$ be a linear projection onto X_0. Let $\Gamma_0 = \{X_n, Y_n, Q_n\}$ be an approximation scheme for $(\tilde{X}, Y)$ with $\sup\|Q_n\|$ finite and define a new approximation scheme $\Gamma_i = \{X_0 \oplus X_n, Y_n, Q_n\}$

for (X, Y). Then $\dim X_0 \oplus X_n - \dim Y_n = \dim X_0 = i(A)$ for each n. Let $\Sigma = \{x \in X : A + N \text{ is not invertible at } x\}$

Theorem 3.5 (Nonlinear Fredholm Alternative)**.** *Let $A : X \to Y$ be a linear Fredholm mapping of index $i(A) \geq 0$ with $R(A) = Y$, and $N : X \to Y$ be a continuous nonlinear mapping.*
(a) If $A, A + N : X \to Y$ are A-proper with respect to $\Gamma_i = \{X_0 \oplus X_n, Y_n, Q_n\}$ and $|N| < c$ with c sufficiently small, then either

- (i) *$N(A) = \{0\}$, in which case* (3.1) *is approximation solvable for each $f \in Y$ and $(A + N)^{-1}(\{f\})$ is compact for each $f \in Y$ and the cardinal number card$(A + N)^{-1}(\{f\})$ is constant, finite and positive on each connected component of the set $Y \setminus (A + N)(\Sigma)$, or*
- (ii) *$N(A) \neq \{0\}$ and then for each $f \in Y$ $(= N(A^*)^{\perp})$ there is a connected closed subset C of $(A + N)^{-1}(\{f\})$ whose dimension at each point is at least $i(A)$ and the projection P maps C onto X_0.*

(b) If $N(A) = \{0\}$ and $A + N$ is pseudo A-proper with respect to Γ_0 and $\delta|N| < c$, where $\delta = max\|Q_n\|$, then $(A + N)(X) = Y$.

Proof. (a) First, assume that $N(A) = \{0\}$. We claim that the homotopy $H(t, x) = Ax + tNx$ satisfies condition (+), i.e., if $H(t_n, x_n) \to f$ then $\{x_n\}$ is bounded in X. Since A is a continuous bijection, it follows that for some $c > 0$

$$\|Ax\| \geq c\|x\|, \quad x \in X.$$

Let $\epsilon > 0$ be such that $|N| + \epsilon < c$ and $R = R(\epsilon) > 0$ such that

$$\|Nx\| \leq (|N| + \epsilon)\|x\| \quad \text{for all } \|x\| \geq R.$$

Then, for $x \in X \setminus B(0, R)$, we get that

$$\|Ax + tNx\| \geq (c - |N| - \epsilon)\|x\|$$

and therefore, $\|H(t, x)\| = \|Ax + tNx\| \to \infty$ as $\|x\| \to \infty$ independently of t. Hence, condition (+) holds. Arguing by contradiction, we see that for each $f \in Y$ there is an $r > R$ and $\gamma > 0$ such that

$$\|H(t, x) - tf\| \geq \gamma \text{ for all } t \in [0, 1],\ x \in \partial B(0, r).$$

Since H_t is an A-proper homotopy, this implies that there is an $n_0 \geq 1$ such that

$$Q_nH(t, x) \neq tQ_nf \text{ for all } t \in [0, 1],\ x \in \partial B(0, r) \cap X_n,\ n \geq n_0.$$

By the Brouwer degree properties and the A-properness of H_1, there is an $x_n \in B(0, r) \cap X_n$ such that $Q_nAx_n + Q_nNx_n = Q_nf$ and a subsequence $x_{n_k} \to x$ with $Ax - Nx = f$. Hence, part (i) follows from Theorem 3.4.

Next, let $N(A) \neq \{0\}$. For a given $f \in Y$, let $Bx = Nx - f$. We need to show that $A + N : X_0 \oplus \tilde{X} \to Y$ is complemented by the projection P of X onto X_0. To that end, it suffices to show (see [5]) that $\deg(Q_n(A+B)|X_n, X_n, 0) \neq 0$ for all large n. Define the homotopy $H_n : [0, 1] \times X_n \to Y_n$ by $H_n(t, x_1) = Q_nAx_1 + tQ_nBx_1$. Consider the restriction of $A + B$ to $\tilde{X}$. Since A restricted to $\tilde{X}$ is a bijection from $\tilde{X}$ onto Y, as in the first case we get that there are $n_0 \geq 1$ and $r \geq R$ such that $H_n(t, x_1) \neq 0$ for $x_1 \in X_n$ with $n \geq n_0$ and $t \in [0, 1]$. Thus, for each $n \geq n_0$, $\deg(Q_n(A + B)|X_n, X_n, 0) = \deg(Q_nA|X_n, X_n, 0) \neq 0$.

Next, we need to show that the projection $P : X = X_0 \oplus \tilde{X} \to X_0$ is proper on $(A + B)^{-1}(f)$. To see this, it suffices to show that if $\{x_n\} \subset X_0 \times \tilde{X}$ is such that

$Ax_n + Bx_n = f$ and $\{Px_n\}$ is bounded, then $\{x_n\}$ is bounded since the A-proper mapping $A+B$ with respect to Γ_i is proper when restricted to bounded sets. We have that $x_n = x_{0n} + x_{1n}$ with $x_{0n} \in X_0$ and $x_{1n} \in \tilde{X}$ and $c\|x_{1n}\| \leq \|Ax_{1n}\| \leq (|N|+\epsilon)(\|x_{0n}\| + \|x_{1n}\|) + \|f\|$ for some $\epsilon > 0$ with $|N| + \epsilon < c$ if $\|x_n\| \geq R$. This implies that $\{x_{1n}\}$ is bounded by the boundedness of $\{x_{0n}\}$. Since $\{x_{0n}\} = \{Px_n\}$ is bounded, it follows that $\{x_n\}$ is also bounded. Hence, the conclusions of (a)-(ii) follow from [5, Theorem 1.2].

(b) Since A is an A-proper injection, it is easy to see that there is an $n_0 \geq 1$ and $c > 0$ such that for each $n \geq n_0$

$$\|Q_n Ax\| \geq c\|x_n\| \quad \text{for all } x \in X_n.$$

Let $\epsilon > 0$ be such that $|N| + \epsilon < c/\delta$ and $R = R(\epsilon) > 0$ such that

$$\|Nx\| \leq (|N| + \epsilon)\|x\| \quad \text{for all } \|x\| \geq R.$$

Define the homotopy $Q_n H(t,x) = Q_n Ax + (1-t)Q_n Nx + (1-t)Q_n f$ on $[0,1] \times X_n$ for $n \geq n_0$. It follows easily from the above remarks that $Q_n H(t,x) \neq 0$ on $\partial B(0,R) \cap X_n$ for each $n \geq n_0$. Thus, the there is an $x_n \in B(0,R) \cap X_n$ such that $Q_n Ax_n + Q_n Nx_n = Q_n f$ for each $n \geq n_0$. Hence, $Ax + Nx = f$ for some $x \in X$ by the pseudo A-properness of $A + N$. □

Remark 3.6. Theorem 3.5 is valid also for Fredholm mappings of index zero under the additional assumption that the range $R(N) \subset R(A)$ [12]. Just the solvability of $x - Kx + Nx = y$ with K and N compact under this condition was first established by Kachurovskii [8].

Next, we shall look at odd mappings. Recall that a nonempty and symmetric subset A of $X \setminus \{0\}$ has genus p, $\gamma(A) = p$, if there is an odd continuous mapping $\phi : A \to \mathbb{R}^p \setminus \{0\}$ and p is the smallest integer having this property; $\gamma(\emptyset) = 0$. We also have that $\dim A \geq \gamma(A) - 1$. The following dimension result from [11] for odd mappings will be needed in the sequal.

Theorem 3.7. *Let $T : X \to Y$ be an odd A-proper mapping at zero with respect to Γ_i with $i \geq 1$. For each $r > 0$, let $S_r = \{x \in \partial B(0,r) : Tx = 0\}$. Then the genus $\gamma(S_r) \geq i$ and the dimension $\dim(S_r) \geq \gamma(S_r) - 1$ if $i > 1$, and S_r contains at least two points when $i = 1$.*

4. Proofs of Theorems 2.1-2.4 and special cases

In this section, we provide proofs to the results in Section 2 and look at some special classes of N. To that end we need to have suitable approximation schemes. It is known (R. Beals) that for $1 \leq p < \infty$ and a finite dimensional Banach space M, $L_p(\mathbb{R}, M)$ is a π_1-space and thus has a monotone Schauder basis, i.e. there is an increasing sequence of finite dimensional subspaces $\{X_n\}$ of $L_p(\mathbb{R}, M)$ and linear projections of L_p onto X_n with $\|P_n\| = 1$ and $\cup_{n\geq 1} X_n$ dense in L_p. If S is any closed interval or $(0,\infty)$, then $L_p(S,M)$ is also a π_1-space as a subspace of $L_p(\mathbb{R}, M)$, since each $f \in L_p(S,M)$ is in $L_p(\mathbb{R}, M)$ if we set $f = 0$ outside of S.

Equation (1.1) is equivalent to the operator equation $Ax + Nx = y$ in X.

Proof of Theorem 2.1. By our assumptions A and $A + N$ are A-proper mappings with respect to $\Gamma_i = \{X_0 \oplus X_n, A(X_n), P_n\}$ by Example 3.1, where $P_n : X \to A(X_n)$ are the projections onto $A(X_n)$ with $\sup\|P_n\|$ finite. Thus, the conclusions of the theorem follow from Theorem 3.5. since $R(A) = X$. □

Proof of Theorem 2.2. By our assumptions $A + N : H \to H$ is A-proper with respect to $\Gamma_0 = \{A(H_n), P_n\}$ by Example 3.2, where $\cup_{h\geq 1} H_n$ is dense in H. Moreover, A is also A-proper with respect to Γ_0 by Example 3.1. If (b) holds, then $A+N_1$ c-strongly monotone and N_2 is k-ball contractive. Hence, A and $A+N$ are A-proper with respect to Γ_0 by Example 3.3. Since $R(A) = H$ in either case, the conclusions of the theorem follow from Theorem 3.5(a). □

Proof of Theorem 2.3. Note that the mapping A is A-proper with respect to $\Gamma_0 = \{A(H_n), P_n\}$ by Example 3.1, and $A + N$ is pseudo A-proper with respect to Γ_0, where $\cup_{n\geq 1} H_n$ is dense in H. Moreover, $|N|$ is sufficiently small. Hence, the conclusion of the theorem follows from Theorem 3.5(b). □

Proof of Theorem 2.4. The mapping $A + N : X \to X$ is odd and A-proper with respect to Γ_i by Example 3.1. Hence, the conclusions follow from Theorem 3.7. □

Next, we shall look at various classes of k-ball contractive mappings N. Let $X = L_p(\mathbb{R}^+, \mathbb{C})$ with $1 < p < \infty$. First, we assume that the operator N is formally given by the formula

$$(Nx)(s) = \int_0^\infty k_0(s,t)F(t,x(t))dt, \quad s \in \mathbb{R}^+, \tag{4.1}$$

with $k_0(s,t)$ and F satisfying appropriate conditions. Let

$$K_0x(s) = \int_0^\infty k_0(s,t)x(t)dt, \quad s \in \mathbb{R}^+.$$

Then $N = K_0F$ and (1.1) becomes

$$\lambda x(s) - \int_0^\infty k(s-t)x(t)dt + \int_0^\infty k_0(s,t)F(t,x(t))dt = y(s), \quad s \in \mathbb{R}^+. \tag{4.2}$$

In the operator form, (4.2) is $Ax + K_0Fx = y$. For a given $k_0(s)$, define

$$\tilde{K}_0x(s) = \int_0^\infty k_0(s-t)x(t)dt, \quad s \in \mathbb{R}^+.$$

Corollary 4.1 (Nonlinear Fredholm Alternative)**.** *Let $A = \lambda I - K : X \to X$ be a Fredholm mapping of index $i(A) \geq 0$ induced by* (1.2) *and $k_0(s,t)$ be a measurable complex-valued function of (s,t) for $0 \leq s,t < \infty$ and*

$$|\int_0^\infty k_0(s,t)x(t)dt| \leq |\int_0^\infty k_0(s-t)x(t)dt|, \quad s \in \mathbb{R}^+$$

for $x \in X^* = L_q$ and some $k_0(s) \in L_1(\mathbb{R},\mathbb{C}) \cap L_\infty(\mathbb{R},\mathbb{C})$ with $(\tilde{K}_0x, x) \geq 0$ for all $x \in X^*$. Assume that for $p \geq 2$, $F : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{C}$ is a Caratheodory function such that $F(s,0) \in X^*$ and

$$|F(s,u) - F(s,v)| \leq k|u-v|^{p-1} \quad \text{for all } s \in \mathbb{R}^+,\ u,v \in \mathbb{R}$$

and some k such that $k\|K_0\|$ is sufficiently small. Then, either

- (i) *the equation $Ax = 0$ has a unique zero solution, i.e., $i(A) = 0$, in which case* (4.2) *is uniquely approximation solvable for each $y \in X$ with respect to Γ_0 for X, or*
- (ii) *$N(A) \neq \{0\}$, i.e., $i(A) = \dim N(A) > 0$, in which case, for each $y \in X$, there is a connected closed subset C of $(A+N)^{-1}(y)$ whose dimension at each point is at least $m = i(A)$ and the projection P maps C onto of $N(A)$, where N is given by* (4.1), *or*

(iii) *$N(A) \neq \{0\}$ and if $F(s,-u) = -F(s,u)$ for $(s,u) \in \mathbb{R}^+ \times \mathbb{R}$, $k\|K_0\| < 1$, and if S_0 is the solution set of (4.2) with $y = 0$, then, for any positive real number r and $B(0,r) = \{x \in H : \|x\| < r\}$, the dimension of $S_0 \cap \partial B(0,r)$ is at least $i(A) - 1$, when $i(A) > 1$ and $S_0 \cap \partial B(0,r)$ contains at least two points when $i(A) = 1$.*

Proof. We claim that $K_0 : X^* \to X$, where K_0 is given above. Indeed, by Holder's inequality, for each $x \in X^*$,

$$\begin{aligned}
\Big|\int_0^\infty k_0(s-t)x(t)dt\Big| &\leq \int_0^\infty |k_0(s-t)|^{1/p}(|k_0(s-t)|^{1/q}|x(t)|)dt \\
&\leq \Big(\int_0^\infty |k_0(s-t)|dt\Big)^{1/p}\Big(\int_0^\infty |k_0(s-t)||x(t)|^q dt\Big)^{1/q},
\end{aligned}$$

and therefore

$$|\int_0^\infty k_0(s-t)x(t)dt|^p \leq \|k_0\|_{L_1}(\int_0^\infty |k_0(s-t)||x(t)|^q dt)^{p/q}.$$

The integral on the right hand side is in $L_\infty \cap L_1$ and therefore it is in $L_{p/q}(\mathbb{R}^+,\mathbb{C})$ since $p/q = p-1 \geq 1$. Hence, $\tilde{K}_0 : X^* \to X$ and is continuous since the linear mapping $\tilde{K}_0$ is monotone. Moreover, $\|K_0 x\|_{L_p} \leq \|\tilde{K}_0 x\|_{L_p} \leq \|\tilde{K}_0\|\ \|x\|_{L_q}$. Hence, $K_0 : X^* \to X$ is continuous.

Next, (a) implies that $(Fx)(s) = F(s,x(s))$ maps X into X^* and therefore, the nonlinear mapping given by $N = K_0 F$, i.e., by (4.1), maps X into itself. Moreover, for each $x, y \in L_p$

$$\|Fx - Fy\|^q = \int_0^\infty |F(t,x(t)) - F(t,y(t))|^q dt \leq k^q \|x-y\|^p.$$

Hence, F is a k-contraction and $\|Nx - Ny\| \leq k\|K_0\|\ \|x-y\|$ with $k_1 = k\|K_0\| < 1$. Thus, N is a k_1-ball contraction. If $i(A) = 0$, then the equation $Ax + Nx = y$ is equivalent to $x + A^{-1}Nx = A^{-1}y$, where $A^{-1}N$ is a $k_2 = k_1/c$-contraction with $k_2 < 1$, since $\|x - Kx\| \geq c\|x\|$ and $c > k_1$. Hence, it is uniquely solvable by the contraction principle. This proves (i), while (ii) follows from Theorem 2.1 and (iii) follows from Theorem 2.4. ☐

Corollary 4.2 (Nonlinear Fredholm Alternative). *Let $A = \lambda I - K : X \to X$ be a Fredholm mapping of index $i(A) \geq 0$ induced by (1.2) and $k_0(s,t)$ be a measurable complex valued function of (s,t) for $0 \leq s,t < \infty$ and either*

$$\sup_{s\in\mathbb{R}^+} \int_0^\infty |k_0(s,t)|dt < \infty \tag{4.3}$$

or

$$|k_0(s,t)| \leq |k_0(s-t)|, \quad s,t \in \mathbb{R}^+. \tag{4.4}$$

for some $k_0 \in L_1(\mathbb{R}^+,\mathbb{C})$. Assume that $F : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{C}$ is a Caratheodory function such that $F(s,0) \in L_p(\mathbb{R}^+,\mathbb{R})$ and

$$|F(s,u) - F(s,v)| \leq k|u-v| \quad \text{for all } s \in \mathbb{R}^+,\ u,v \in \mathbb{R}$$

and some k sufficiently small. Then, either

(i) *the equation $Ax = 0$ has a unique zero solution, i.e., $i(A) = 0$, in which case (4.2) is uniquely approximation solvable for each $y \in X$ with respect to Γ_0 for X, or*

(ii) *$N(A) \neq \{0\}$, i.e., $i(A)$ =dim $N(A) > 0$, in which case, for each $y \in X$, there is a connected closed subset C of $(A+N)^{-1}(y)$ whose dimension at each point is at least $m = i(A)$ and the projection P maps C onto of $N(A)$, where N is given by (4.1), or*

(iii) *$N(A) \neq \{0\}$ and if $F(s,-u) = -F(s,u)$ for $(s,u) \in \mathbb{R}^+ \times \mathbb{R}$, and if S_0 is the solution set of (4.2) with $y = 0$, then, for any positive real number r and $B(0,r) = \{x \in H : \|x\| < r\}$, the dimension of $S_0 \cap \partial B(0,r)$ is at least $i(A)-1$, when $i(A) > 1$ and $S_0 \cap \partial B(0,r)$ contains at least two points when $i(A) = 1$.*

Proof. Let $N = K_0F$ be given by (4.1). If (4.3) holds, then K_0 maps $L_p(\mathbb{R}^+, \mathbb{C})$ into itself, $1 \leq p \leq \infty$, and is a $\|K_0\|$-contraction by the Riesz convexity theorem (see [7, Chapter 11]). If (4.4) holds, then $K_0 : L_p(\mathbb{R}^+, \mathbb{C}) \to L_p(\mathbb{R}^+, \mathbb{C})$ is continuous by Young's inequality and $\|K_0\| \leq \|k_0\|_1$ for each $1 \leq p \leq \infty$. If $(Fx)(s) = F(s, x(s))$, then $F : L_p \to L_p$ is a k-contraction. Thus, $N = K_0F$ is a $k_1 = k\|K_0\|$-contraction in L_p, and is therefore k_1-ball contractive. The conclusions now follow from Theorems 2.1 and 3.7 since (i) follows as in Corollary 4.1. □

Corollary 4.3 (Nonlinear Fredholm Alternative)**.** *Let $A = \lambda I - K : X \to X$ be a Fredholm mapping of index $i(A) \geq 0$ induced by (1.2) and $k_0(s,t)$ be a measurable complex valued function of (s,t) for $0 \leq s, t < \infty$ and*

$$\int_0^\infty [\int_0^\infty |k_0(s,t)|^p ds]^{q/p} dt \leq c^q,$$

for $1 < p < \infty$, $q = p/(p-1)$, $c > 0$. Let $F : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{C}$ be a Caratheodory function such that

$$|F(s,u)| \leq c(s) + c_0(s)|u|, \quad s \in \mathbb{R}^+,\ u \in \mathbb{R}$$

where $c(s) \in L_p(\mathbb{R}^+, \mathbb{R})$ and $c_0(s) \in L_\infty(\mathbb{R}^+, \mathbb{C})$. Then, either

(i) *the equation $Ax = 0$ has a unique zero solution, i.e., $i(A) = 0$, in which case (4.2) is approximation solvable for each $f \in X$, $(A+N)^{-1}(\{f\})$ is compact for each $f \in X$ and the cardinal number* card$(A+N)^{-1}(\{f\})$ *is constant, finite and positive on each connected component of $X \setminus (A+N)(\Sigma)$, or*

(ii) *$N(A) \neq \{0\}$, i.e., $i(A) = \dim N(A) > 0$, in which case, for each $f \in X$, there is a connected closed subset C of $(A+N)^{-1}(f)$ whose dimension at each point is at least $m = i(A)$ and the projection P maps C onto of $N(A)$, where N is given by (4.1), or*

(iii) *$N(A) \neq \{0\}$ and if $F(s,-u) = -F(s,u)$ for $(s,u) \in \mathbb{R}^+ \times \mathbb{R}$, and if S_0 is the solution set of (4.2) with $y = 0$, then, for any positive real number r and $B(0,r) = \{x \in H : \|x\| < r\}$, the dimension of $S_0 \cap \partial B(0,r)$ is at least $i(A)-1$, when $i(A) > 1$ and $S_0 \cap \partial B(0,r)$ contains at least two points when $i(A) = 1$.*

Proof. The mapping K_0 defined above is a completely continuous linear operator in L_p with $\|K_0\| \leq c$ [4]. Moreover, $(Fx)(s) = F(s, x(s))$ is a continuous bounded mapping from $L_p(\mathbb{R}^+, \mathbb{C})$ into itself. Hence, $N = K_0F : L_p(\mathbb{R}^+, \mathbb{C}) \to L_p(\mathbb{R}^+, \mathbb{C})$ is a compact mapping and the conclusions follow from Theorems 2.1, 3.4 and 3.7. □

Theorem 2.1 also applies when N is the Nemitskii operator, i.e., $(Nx)(s) = F(s,x(s))$ for some function F. Then (1.1) becomes

$$\lambda x(s) - \int_0^\infty k(s-t)x(t)dt + F(t,x(t)) = y(s), \quad s \in \mathbb{R}^+ \tag{4.5}$$

We have the following result.

Corollary 4.4 (Nonlinear Fredholm Alternative)**.** *Let $A = \lambda I - K$, $A : X = L_p(\mathbb{R}^+,\mathbb{R}) \to X$, be a Fredholm mapping of index $i(A) \geq 0$ induced by (1.2) and $F : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that $F(s,0) \in X$ and*

$$|F(s,u) - F(u,v)| \leq k|u-v| \quad \text{for all } s \in \mathbb{R}^+,\ u,v \in \mathbb{R}$$

for some k sufficiently small. Then, either

(i) *the equation $Ax = 0$ has a unique zero solution, i.e., $i(A) = 0$, in which case (4.5) is uniquely approximation solvable for each $y \in X$ with respect to Γ_0 for X, or*

(ii) *$N(A) \neq \{0\}$, i.e., $i(A) = \dim N(A) > 0$, in which case, for each $f \in H$, there is a connected closed subset C of $(A+N)^{-1}(f)$ whose dimension at each point is at least $m = i(A)$ and the projection P maps C onto of $N(A)$, where $Nx = F(s,x(s))$, or*

(iii) *$N(A) \neq \{0\}$ and if $F(s,-u) = -F(s,u)$ for $(s,u) \in \mathbb{R}^+ \times \mathbb{R}$, and if S_0 is the solution set of (4.2) with $y = 0$, then, for any positive real number r and $B(0,r) = \{x \in H : \|x\| < r\}$, the dimension of $S_0 \cap \partial B(0,r)$ is at least $i(A)-1$, when $i(A) > 1$ and $S_0 \cap \partial B(0,r)$ contains at least two points when $i(A) = 1$.*

Proof. Condition (a) implies that $(Nx)(s) = F(s,x(s))$ is a k-contraction from X into itself. Hence, N is a k-ball contraction. Part (i) follows as in Corollary 4.1, while (ii)-(iii) follow from Theorem 3.5-3.7 since $R(A) = H$. □

Next, we shall look at some special cases of Theorems 2.2-2.4 with nonlinearities of the form $(Nx)(s) = F(s,x(s))$.

Corollary 4.5. *Let $A : H = L_2(\mathbb{R}^+,\mathbb{R}) \to H$ be a Fredholm mapping of index $i(A) = 0$ induced by (1.2) and $F : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that*

(a) *$|F(s,u)| \leq c(s) + c_0(s)|u|$, $s \in \mathbb{R}^+$, $u \in \mathbb{R}$, where $c(s) \in L_2(\mathbb{R}^+)$ and $c_0(s) \in L_\infty(\mathbb{R}^+)$ with $\|c_0\|_2$ sufficiently small, and either*

(b) *A is monotone, $F(s,u)$ is strictly monotone increasing in u for each fixed s and*

$$F(s,u)u \geq c_2|u|^2 - c_1(s), \quad \text{for all } s \in \mathbb{R}^+, u \in \mathbb{R}$$

for some constant $c_2 > 0$ and a function $c_1(s) \in L_1(\mathbb{R}^+)$, or

(c) *A is c-strongly monotone for some $c > 0$ and $F = F_1 + F_2$ with $F_1(s,u)$ monotone increasing in u for each fixed s and*

$$|F_2(s,u) - F_2(s,v)| \leq k|u-v| \quad \text{for all } s \in \mathbb{R}^+,\ u,v \in \mathbb{R}$$

and some $k < c$.

Then, the equation $Ax = 0$ has a unique zero solution and (4.5) is approximation solvable for each $y \in H$, $(A+N)^{-1}(\{y\})$ is compact for each $y \in H$ and the cardinal number card$(A+N)^{-1}(\{y\})$ is constant, finite and positive on each connected component of $H \setminus (A+N)(\Sigma)$, where $(Nx)(s) = F(s,x(s))$.

Proof. Again, $N : H \to H$ is bounded and continuous. Condition (b) implies that $N : H \to H$ is of type (S_+) (cf., e.g., [2]). Since A is monotone, $A+N$ is A-proper with respect to $\Gamma_0 = \{A(H_n), P_n\}$ by Example 3.2, where $\cup_{h\geq 1} H_n$ is dense in H. Moreover, A is also A-proper with respect to Γ_0 by Example 3.1. If (c) holds, then $A+N_1$ is c-strongly monotone and N_2 is k-ball contractive. Hence, A and $A+N$ are A-proper with respect to Γ_0. by Example 3.3. Since $R(A) = H$ in either case, the conclusions of the theorem follow from Theorems 3.4-3.5. □

Corollary 4.6. *Let A be a Fredholm mapping of index $i(A) = 0$ induced by* (1.2) *and $F : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that*

(a) $|F(s,u)| \leq c(s) + c_0|u|$, $s \in \mathbb{R}^+$, $u \in \mathbb{R}$, *where $c(s) \in L_2(\mathbb{R}^+, \mathbb{R})$, $c_0 > 0$ is sufficiently small and*
(b) *A is monotone and $F(s,u)$ is monotone increasing in u for each fixed s.*

Then (4.5) *has a solution for each $y \in H$.*

Proof. The mapping $A+N : H \to H$ is bounded, continuous and monotone since such are A and N. Hence, A is A-proper with respect to $\Gamma_0 = \{A(H_n), P_n\}$ by Example 3.1, and $A+N$ is pseudo A-proper with respect to Γ_0, where $\cup_{n\geq 1} H_n$ is dense in H. Moreover, $|N|$ is sufficiently small. Hence, the conclusion of the theorem follows from Theorem 3.5(b). □

Let us look at some examples of monotone A. The monotonicity of A implies that $\lambda\|x\|^2 \geq (Kx, x)$ in L_2. Hence, let us look at some positive definite K, i.e., $(Kx, x) \geq 0$ on L_2. This means that

$$\int_0^\infty \int_0^\infty k(s-t)x(t)\overline{x(s)}\,ds\,dt \geq 0 \tag{4.6}$$

for all $x \in L_2(\mathbb{R}^+, \mathbb{C})$. The positive definitness of K implies that its kernel $k(s-t)$ is hermitian symmetric, i.e., $k(-t) = \overline{k(t)}$. If k is real, that means that it is an even function. Hence, in either case K is selfadjoint and $\lambda I - K$ is of index zero if $\lambda - \hat{k}(\xi) \neq 0$. If $k \in L_2(\mathbb{R}^+)$ is continuous, then condition (4.6) is equivalent to (see [3])

$$\Sigma_{i,j=1}^n k(x_i - x_j)\bar{\xi}_i\xi_j \geq 0 \tag{4.7}$$

for all positive integers n, $(x_1, \dots, x_n) \in \mathbb{R}^n$ and $(\xi_1, \dots, \xi_n) \in \mathbb{C}^n$. A complex valued function $k(t)$ satisfying (4.7) is called a positive definite function. The functions $e^{-c|t|}$ for $c > 0$, e^{-t^2}, $(1+t^2)^{-1}$ are positive definite by a theorem of Mathias since their Fourier transforms are positive and integrable. Moreover, any real, even, continuous function $k(t)$ which is convex on $(0, \infty)$, i.e., $k((t_1+t_2)/2) \leq (k(t_1)+k(t_2))/2$, and such that $\lim_{t\to\infty} k(t) = 0$ is also positive definite (Pólya). If, for example, $k(t) = e^{-c|t|}$ with $c > 0$, then $\hat{k}(\xi) = 2c/(\xi^2+c^2)$ and the corresponding linear map $\lambda I - K$ given by (1.2) is Fredholm of index zero in L_2 for each $\lambda \notin [0, 2/c]$ since $\lambda - \hat{k}(\xi) \neq 0$. Note that the spectrum of K is $[0, 2/c]$ and so K is not compact.

5. Nonlinear perturbations of integral equations on the real line

The study of (1.2) with the integral over $\mathbb{R}$, i.e., of

$$\lambda x(s) - \int_{-\infty}^\infty k(s-t)x(t)dt = y(s) \tag{5.1}$$

where $k : \mathbb{R} \to \mathbb{C}$ is in $L_1(\mathbb{R}, \mathbb{C})$ and $y(s) \in L_1(\mathbb{R}, \mathbb{C})$, is much simpler and is based on using integral Fourier transform. Namely, if $k(t) \in L_1(-\infty, \infty)$ and $\lambda - \hat{k}(\xi) \neq 0$ in $(-\infty, +\infty)$, then (4.1) has a unique solution in $L_1(-\infty, \infty)$ for each $y \in L_1(-\infty, \infty)$ (Wiener). Hence, one can study nonlinear perturbations of such linear equations

$$\lambda x(s) - \int_{-\infty}^{\infty} k(s-t)x(t)dt + (Nx)(s) = y(s) \tag{5.2}$$

with a suitable nonlinear mapping N, as above, or using the theory of Hammerstein equations. Let K be a linear map defined by the integral in (5.1) and $A = \lambda I - K$. Then (5.2) is equivalent to $Ax + Nx = y$ and the results of Sections 2 and 4 (with $i(A) = 0$) are valid for (5.2) under the corresponding assumption on k and N. The problem is much more difficult if we assume that the kernel is general, i.e., if we look at

$$\lambda x(s) - \int_{-\infty}^{\infty} k(s,t)x(t)dt + (Nx)(s) = y(s). \tag{5.3}$$

Its operator form is $(\lambda I - K)x + Nx = y$ in $X = L_p(\mathbb{R}, \mathbb{C})$. Under some general conditions on K, it has been shown in [1] that $A = \lambda I - K$ is continuously invertible in X if $N(A) = \{0\}$. Hence, again the results of Sections 2 and 4 (with $i(A) = 0$) are valid for (5.3) with the corresponding assumptions imposed on $k(s,t)$ and N. For the sake of illustration, we just state explicitely the following result when N is the Nemitskii mapping. Denote by $BC(\mathbb{R})$ the space of complex valued continuous and bounded functions on $\mathbb{R}$.

Theorem 5.1. *Let $k \in L_1(\mathbb{R}, \mathbb{C})$, $X = L_p(\mathbb{R}, \mathbb{C})$ for $1 < p < \infty$, $Q \subset \mathbb{C}$ be compact and convex and $\lambda \neq 0$. Let, for every $z \in L_Q$, the equation*

$$\lambda x(s) - \int_{-\infty}^{\infty} k(s-t)z(t)x(t)dt = 0, \quad s \in \mathbb{R} \tag{5.4}$$

have only the trivial solution in $BC(\mathbb{R})$ and $F : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that $F(s, 0) \in X$ and

$$|F(s,u) - F(u,v)| \leq k|u - v| \text{ for all } s \in \mathbb{R}^+, \quad u, v \in \mathbb{R}$$

for some k sufficiently small. Then, for each $z \in L_Q$, the equation

$$\lambda x(s) - \int_{-\infty}^{\infty} k(s-t)z(t)x(t)dt + F(s, x(s)) = y(s), \quad s \in \mathbb{R} \tag{5.5}$$

is uniquely approximation solvable for each $y \in X$ with respect to Γ_0 for X.

Proof. Let $A : X \to X$ be the linear map defined by (5.4). Then A is continuously invertible on X [1]. If $(Nx)(s) = F(s, x(s))$, then $N : X \to X$ and (5.5) is equivalent to $x + A^{-1}Nx = A^{-1}y$, where $A^{-1}N$ is a $k_1 = k/c$-contraction with $k_1 < 1$, since $\|x - Kx\| \geq c\|x\|$ and $c > k$ with c depending only on $\lambda, k(s)$ and Q. Hence, it is uniquely solvable by the contraction principle. Since $A + N$ is A-proper with respect to Γ_0, the assertion follows from Theorem 2.1(a). $\square$

Acknowledgements. This work was done while the author was on the sabbatical leave at the Mathematics Department at Rutgers University, New Brunswick, NJ.

References

[1] T. Arens, S.N. Chandler-Wilde, K.O. Haseloh; (2003) Solvability and spectral properties of integral equations on the real line: II. L_p-spaces and applications, J. Integral Eq. Appl., 15(1), 1-35.

[2] F. E. Browder; (1971) Nonlinear functional analysis and nonlinear integral equations of Hammerstein and Urisohn type. In : Zarantonello, E. [ed.], pp. 425-500.

[3] J. Buescu, A.C. Paixão; (2006), Positive definite matrices and integral equations on unbounded domains, Diff. Integral Eq., 19(2), 189-210.

[4] C. Corduneanu; (1973) Integral equtions and stability of feedback systems, Acad. Press.

[5] P. M. Fitzpatrick, I. Massabo and J. Pejsachowicz, (1986) On the covering dimension of the set of solutions of some nonlinear equations, Trans. AMS, 296(2), 777-798.

[6] J. Ize, (1993) Topological bifurcation, Nonlinear Analysis, (M. Matzeu, A. Vignoli edts.), Birkhauser, Boston.

[7] K. Jörgens, (1982) Linear integral equations, Pitman, London.

[8] R. I. Kachurovski, (1971) On nonlinear operators whose ranges are subspaces, Dokl. Akad. Nauk SSSR, 192, 168-172.

[9] M. G. Krein, (1962) Integral equations on the half line, Transl. AMS 22, 163-288.

[10] P. S. Milojević, (1977) A generalization of the Leray-Schauder theorem and surjectivity results for multivalued A-proper and pseudo A-proper mappings, J. Nonlinear Anal., TMA, 1(3), 263-276.

[11] P. S. Milojević, (1982) Solvability of operator equations involving nonlinear perturbations of Fredholm mappings of nonnegative index and applications, Lecture Notes in Mathematics, vol. 957, 212-228, Springer-Verlag, NY.

[12] P. S. Milojević, (1987) Fredholm theory and semilinear equations without resonance involving noncompact perturbations, I, II, Applications, Publications de l'Institut Math. 42, 71-82 and 83-95.

[13] P. S. Milojević, (1995), On the dimension and the index of the solution set of nonlinear equations, Transactions Amer. Math. Soc., 347(3), 835-856.

[14] P. S. Milojević, (2000) Existence and the number of solutions of nonresonant semilinear equations and applications to boundary value problems, Mathematical and Computer Modelling, 32, 1395-1416.

[15] P. S. Milojević, (2004) Solvability and the number of solutions of Hammerstein equations, Electronic J. Diff. Equations, 2004, No 54, 1-25.

[16] R. D. Nussbaum, (1972) Degree theory for local condensing maps, J. Math. Anal. Appl. 37, 741-766.

[17] J. F. Toland, (1977) Global bifurcation theory via Galerkin method, J. Nonlinear Anal.,TMA, 1(3), 305-317.

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