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NUMERICAL SOLUTION OF A PARABOLIC EQUATION WITH A WEAKLY SINGULAR POSITIVE-TYPE MEMORY TERM

Marián Slodička

Abstract. We find a numerical solution of an initial and boundary value problem. This problem is a parabolic integro-differential equation whose integral is the convolution product of a positive-definite weakly singular kernel with the time derivative of the solution. The equation is discretized in space by linear finite elements, and in time by the backward-Euler method. We prove existence and uniqueness of the solution to the continuous problem, and demonstrate that some regularity is present. In addition, convergence of the discrete sequence of iterations is shown.

1. Introduction

Physical processes, such as heat conduction in materials with memory, population dynamics, and visco-elasticity can be described by one of the following parabolic integro-differential equations

$$\partial_t u + Au = \int_0^t K(t-s)Bu(s)ds + f(t) \quad \text{in } \Omega,\ t>0$$

or

$$\partial_t u + \int_0^t \beta(t-s)Au(s)ds = f(t) \quad \text{in } \Omega,\ t>0$$

with homogenous Dirichlet conditions. Here A is a second-order selfadjoint positive-definite differential operator; B is a general partial differential operator of second order with smooth coefficients; K is weakly singular and β is a positive-definite kernel (c.f. Chen-Thomée-Wahlbin [1], McLean-Thomée [6], Thomée [10], etc.).

Our aim is to describe a product integration method for the discretization of the Volterra term in the equation

$$\begin{aligned} \partial_t u(t) - \Delta u(t) + \int_0^t a(t-s)\partial_s u(s)ds &= f(t,u(t)) \quad \text{in } \Omega,\ t>0 \\ u &= 0 \quad \text{on } \partial\Omega,\ t>0 \\ u(0) &= v \quad \text{in } \Omega\,. \end{aligned} \tag{1.1}$$

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This problem arises in the application of homogenization techniques to diffusion models for fractured media (cf. Hornung [4] and its references).

A fully discretized method for solving (1.1) (with $f = f(t)$) was presented in Peszynska [7]. There the author establishes a rate of convergence using the strong regularity assumptions

$$u \in C^2((0,T) \times \Omega), \text{ and } u_{tt} \in L_1((0,T), H^2(\Omega) \cap \overset{\circ}{H}{}^1(\Omega)) .$$

Our main goal is to show a fully discretized numerical method for solving (1.1). We use the backward Euler method for the discretization in time (also called Rothe method; see, e.g., Kačur [5]), and finite elements for space-discretization. We use a right rectangular quadrature rule, and some results for weakly singular positive-definite kernels, for handling the Volterra term. The storage problem associated with this convolution integral has been discussed by Peszynska [7].

We prove existence and uniqueness of a solution, and the convergence of our approximation scheme to a solution u that satisfies

$$u \in C((0,T), L_2(\Omega)) \cap L_\infty((0,T), \overset{\circ}{H}{}^1(\Omega)) \text{ and } u_t \in L_2((0,T), L_2(\Omega)).$$

We extend the results of Hornung-Showalter [3] (where $f = f(t)$), and of Peszynska [7] (where $f = f(t,u)$).

Remark 1. The differential operator $-\Delta$ in (1.1) can be replaced by a general linear elliptic differential operator.

Remark 2. The values $C, \varepsilon, C_\varepsilon$ are generic and positive constants independent of the discretization parameter σ, to be introduced below. The value ε is small, and $C_\varepsilon = C(\varepsilon^{-1})$.

Remark 3. The right-hand side f can depend on Volterra terms containing u, linear terms depending on ∇u, and linear Volterra terms containing ∇u.

2. Assumptions

In this section we establish hypotheses on the data and state the continuous and the fully discretized problem.

We assume that

$$\Omega \subset \mathbb{R}^d \quad \text{is a polyhedral with bounded domain and } d \geq 1. \tag{2.1}$$

Let $\{S_h\}_h$ be a family of decompositions $S_h = \{S_k\}_{k=1}^K$ of Ω into closed d-simplices such that $\overline{\Omega} = \bigcup_{k=1}^K S_k$ (h stands for the mesh size). We suppose that

$$\{S_h\}_h \text{ is regular (c.f. Ciarlet [2]).} \tag{2.2}$$

Let $V_h = \left\{\chi \in C\left(\overline{\Omega}\right);\ \chi \text{ is linear on } S_k\ \forall k = 1, \dots, K;\ \chi = 0 \text{ on } \partial\Omega\right\}$ be the discrete space with which we shall work. We denote the scalar product in

$L_2(\Omega)$ by $(\cdot,\cdot)$ and $\langle u,v\rangle = (\nabla u,\nabla v)$. The corresponding discrete inner product is defined by

$$(u,v)_h = \sum_{k=1}^{K}\int_{S_k}\Pi_h(u,v)dx = \sum_{k=1}^{K}\frac{\text{meas } S_k}{d+1}\sum_{l=1}^{d+1}u(A_l)v(A_l)$$

for any two piecewise continuous functions u,v. Π_h stands for the local linear interpolation operator and A_l $(l=1,\dots,d+1)$ are the vertices of S_k. It is known that $(\cdot,\cdot)_h$ is the inner product in V_h for which

$$C_1\|u\|^2 \le \|u\|_h^2 \le C_2\|u\|^2 \quad \forall u\in V_h, \tag{2.3}$$

where $\|u\|^2=(u,u)$, $\|u\|_h^2=(u,u)_h$.

The well-known estimate

$$|(u,v)-(u,v)_h| \le Ch^2\|u\|_1\|v\|_1 \quad \forall u,v\in V_h, \tag{2.4}$$

takes the effect of numerical integration into account, where $\|u\|_1^2 = \langle u,u\rangle = (\nabla u,\nabla u)$.

Furthermore, we suppose that the inverse inequality holds for our discretization, i.e.,

$$\|u\|_1 \le Ch^{-1}\|u\| \quad \forall u\in V_h. \tag{2.5}$$

Now we introduce the discrete H^1 projection operator P_h , i.e., for $z\in \overset{\circ}{H}{}^1(\Omega)$ we define P_hz as follows

$$\langle P_hz,\phi\rangle = \langle z,\phi\rangle \quad \forall\phi\in V_h.$$

Concerning the time discretization, let the time interval be denoted by $I=(0,T_0)$, and the time step by $\tau=\frac{T_0}{n}$. For short notation let

$$t_i = i\tau, \quad z_i = z(t_i), \quad \delta z_i = \frac{z_i - z_{i-1}}{\tau}$$

for $i=1,\dots,n$ (where n is a positive integer).

Assume that the right-hand side of (1.1) fulfills

$$|f(t,x)-f(s,y)| \le C[|t-s|(1+|x|+|y|)+|x-y|] \ \ \forall t,s,x,y\in\mathbb{R}, \tag{2.6}$$

and the initial data satisfies

$$v\in \overset{\circ}{H}{}^1(\Omega). \tag{2.7}$$

The integral kernel a satisfies

$$(-1)^j a^{(j)}(t) \ge 0 \quad \forall t>0;\ j=0,1,2;\ a'\ne 0. \tag{2.8}$$

These hypotheses are physical and imply the strong positiveness of the kernel a (c.f. Staffans [9]), i.e.,

$$\int_0^T\int_0^t a(t-s)\phi(s)\phi(t)\,ds\,dt \ge 0 \quad \forall T>0, \phi\in C(\langle 0,T\rangle). \tag{2.9}$$

We assume that all occurring functions are real-valued. Moreover we assume that

$$a(t) \le Ct^{-\alpha} \quad \alpha\in\langle 0,1),\ t>0. \tag{2.10}$$

Now we can state the variational formulation of (1.1):

Problem C. *Find $u \in C(I, L_2(\Omega)) \cap L_\infty(I, \overset{\circ}{H}{}^1(\Omega))$ with $\frac{du}{dt} \in L_2(I, L_2(\Omega))$, such that*

$$\left(\frac{du(t)}{dt}, \phi\right) + \langle u(t), \phi\rangle + \left(\int_0^t a(t-s)\frac{du(s)}{ds}ds, \phi\right) = (f(t, u(t)), \phi) \qquad (2.11)$$
$$u(0) = v$$

holds for any $\phi \in \overset{\circ}{H}{}^1(\Omega)$ and a.e. $t \in I$.

In order to solve our continuous problem we shall start with:

Problem D. *Find $u_i^h \in V_h \quad (i = 1, \dots, n)$, such that*

$$\left(\delta u_i^h, \phi\right)_h + \langle u_i^h, \phi\rangle + \left(\sum_{j=1}^{i} a_{i+1-j}\delta u_j^h \tau, \phi\right)_h = \left(f(t_i, u_{i-1}^h), \phi\right)_h \qquad (2.12)$$
$$u_0^h = P_h v$$

holds for any $\phi \in V_h$.

3. Stability

According to (2.10) we have $a \in L_1(I)$ and $\tau a(\tau) \to 0$ for $\tau \to 0$. Since the matrix of the linear system (corresponding to the Problem D) is symmetric and positive-definite, the solution u_i^h exists and is unique. Thus we can solve this system successively for $i = 1, \dots, n$.

We show that a similar inequality to (2.9) holds in a discrete form. Denoting $b_j = a_{j+1}\tau$ for $j \in \{0, \dots, n\}$ and $b_j = 0$ for $j \notin \{0, \dots, n\}$, one can easily check that $\{b_j\}_{j=0}^\infty \in l_\infty$ is positive, convex and then (c.f. Zygmund [11])

$$\frac{b_0}{2} + \sum_{j=1}^{\infty} b_j \cos(j\Theta) \ge 0 \quad \forall \Theta \in \mathbb{R}. \qquad (3.1)$$

Hence, applying McLean-Thomée [6, L4.1], we get

$$B_m(\phi) = \sum_{i=1}^{m}\sum_{j=1}^{i} b_{i-j}\phi^j\phi^i \ge 0 \quad \forall \phi = (\phi^1, \dots, \phi^m) \in \mathbb{R}^m,\ m \ge 1.$$

This can be rewritten as follows

$$\tau^2 \sum_{i=1}^{m}\sum_{j=1}^{i} a_{i+1-j}\phi^j\phi^i \ge 0 \quad \forall \phi = (\phi^1, \dots, \phi^m) \in \mathbb{R}^m,\ m \ge 1. \qquad (3.2)$$

Remark 4. The non negativity of the term $B_m(\phi)$ can be proved using Fourier transform. Let us denote

$$\hat{b}(\Theta) = \sum_{j=0}^{\infty} b_j e^{ij\Theta}.$$

A simple calculation with $\phi^j = 0$ for $j \notin \{1,\dots,m\}$ gives

$$B_m(\phi) = \frac{1}{2\pi}\int_0^{2\pi} \hat{b}(\Theta)|\hat{\phi}(\Theta)|^2 d\Theta = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\ \hat{b}(\Theta)|\hat{\phi}(\Theta)|^2 d\Theta,$$

since $B_m(\phi)$ is real-valued. Further we can write

$$\mathrm{Re}\ \hat{b}(\Theta) = \sum_{j=0}^{\infty} b_j \cos(j\Theta) \geq 0.$$

□

Now we establish a-priori estimates for energy norms.

Lemma 1. *Let* (2.1)-(2.8) *and* (2.10) *be satisfied. Then*

$$\sum_{i=1}^m \|\delta u_i^h\|_h^2 \tau + \|u_m\|_1^2 + \sum_{i=1}^m \|u_i^h - u_{i-1}^h\|_1^2 \leq C$$

for $m = 1,\dots,n$.

Proof. Setting $\phi = \delta u_i^h \tau$ into (2.12) and adding together the identities for $i = 1,\dots,m$, we can write

$$\sum_{i=1}^m \|\delta u_i^h\|_h^2 \tau + \sum_{i=1}^m \langle u_i^h, u_i^h - u_{i-1}^h\rangle + \sum_{i=1}^m \left(\sum_{j=1}^i a_{i+1-j}\delta u_j^h \tau, \delta u_i^h\right)_h \tau$$
$$= \sum_{i=1}^m \left(f(t_i, u_{i-1}^h), \delta u_i^h\right)_h \tau.$$

Using integration by parts in the second term, we have

$$2\sum_{i=1}^m \langle u_i^h, u_i^h - u_{i-1}^h\rangle = \|u_m^h\|_1^2 - \|u_0^h\|_1^2 + \sum_{i=1}^m \|u_i^h - u_{i-1}^h\|_1^2.$$

The third term on the left is nonnegative because of (3.2). For the right-hand side we put

$$\sum_{i=1}^m \left(f(t_i, u_{i-1}^h), \delta u_i^h\right)_h \tau \leq \varepsilon \sum_{i=1}^m \|\delta u_i^h\|_h^2 \tau + C_\varepsilon \sum_{i=1}^m \|f(t_i, u_{i-1}^h)\|_h^2 \tau$$
$$\leq \varepsilon \sum_{i=1}^m \|\delta u_i^h\|_h^2 \tau + C_\varepsilon \left(1 + \sum_{i=1}^m \sum_{j=1}^i \|\delta u_j\|_h^2 \tau^2\right).$$

Thus setting ε sufficiently small, we get

$$\sum_{i=1}^m \|\delta u_i^h\|_h^2 \tau + \|u_m^h\|_1^2 + \sum_{i=1}^m \|u_i^h - u_{i-1}^h\|_1^2$$
$$\leq C\left(1 + \sum_{i=1}^m \sum_{j=1}^i \|\delta u_j^h\|_h^2 \tau^2\right).$$

The rest of the proof is a trivial consequence of the Gronwall lemma. □

It would be useful to have an a-priori estimate for the δu_i^h in the $H^{-1}(\Omega)$ norm. We are working in discrete spaces, thus we are only able to prove the following Lemma.

Lemma 2. *Let* (2.1)-(2.8) *and* (2.10) *be satisfied. Then*

$$\left|(\delta u_i^h, \phi)_h\right| \leq C\|\phi\|_1$$

for all $\phi \in V_h$ *and* $i = 1, \ldots, n$.

Proof. This is a simple consequence of Lemma 1. In fact one can write $(\forall \phi \in V_h)$

$$(\delta u_i^h, \phi)_h = -\langle u_i^h, \phi\rangle - \left(\sum_{j=1}^{i} a_{i+1-j}\delta u_j^h \tau, \phi\right)_h + (f(t_i, u_{i-1}^h), \phi)_h.$$

Hence

$$\begin{aligned}\left|(\delta u_i^h, \phi)_h\right| &\leq C\|\phi\|_1 + \sum_{j=1}^{i} a_{i+1-j}|(\delta u_j^h, \phi)_h|\tau + C\|\phi\|_h \\ &\leq C\|\phi\|_1 + \sum_{j=1}^{i} a_{i+1-j}|(\delta u_j^h, \phi)_h|\tau.\end{aligned}$$

The integral kernel a is weakly singular and $\tau a(\tau) \to 0$ for $\tau \to 0$. Thus

$$\left|(\delta u_i^h, \phi)_h\right| \leq C\left[\|\phi\|_1 + \sum_{j=1}^{i-1} (t_i - t_j)^{-\alpha} |(\delta u_j^h, \phi)_h|\tau\right].$$

Now we apply the following discrete analogue of the Gronwall lemma (c.f. Slodička [8]):
Let $\{A_n\}, \{w_n\}$ be sequences of nonnegative real numbers satisfying

$$w_n \leq A_n + C\sum_{k=1}^{n-1} (t_n - t_k)^{\beta-1} w_k \tau$$

for $0 < \tau < 1,\ 0 < \beta \leq 1,\ C > 0,\ t_n = n\tau \leq T$. Then

$$w_n \leq C\left[A_n + \sum_{k=1}^{n-1} A_k\tau + \sum_{k=1}^{n-1} (t_n - t_k)^{\beta-1} A_k\tau\right],$$

where $C = C(\beta, T)$.

This discrete version of the Gronwall lemma implies

$$\left|(\delta u_i^h, \phi)_h\right| \leq C\|\phi\|_1$$

which concludes the proof. □

4. Main results

Let us first introduce some notation. We denote for $t \in (t_{i-1}, t_i\rangle, \sigma = (\tau, h)$

$$f_\tau(t,\xi) = f(t_i,\xi), \quad a_\tau(t_k - t) = a(t_k - t_i) \quad \text{for } k > i,$$
$$\overline{u}_\sigma(t) = u_i^h, \quad u_\sigma(0) = u_0^h = P_h v, \quad u_\sigma(t) = u_{i-1}^h + (t - t_{i-1})\delta u_i^h .$$

Hence we rewrite (2.12) as follows

$$\begin{aligned}\left(\frac{du_\sigma(t)}{dt}, \phi\right)_h + \langle \overline{u}_\sigma(t), \phi\rangle + \left(\int_0^{t_i} a_\tau(t_i + \tau - s)\frac{du_\sigma(s)}{ds}ds, \phi\right)_h \\ = (f_\tau(t, \overline{u}_\sigma(t-\tau)), \phi)_h\end{aligned} \tag{4.1}$$

for all $\phi \in V_h$ and $t \in (t_{i-1}, t_i\rangle$.

First of all, we show the uniqueness of a solution of the Problem C.

Theorem 1. *Let u_1 and u_2 be two solutions of the Problem C. Then $u_1 = u_2$.*

Proof. Using (2.11), we can write

$$\begin{aligned}\left(\frac{d(u_1(t) - u_2(t))}{dt}, \phi\right) + \langle u_1(t) - u_2(t), \phi\rangle + \left(\int_0^t a(t-s)\frac{d(u_1(s) - u_2(s))}{ds}ds, \phi\right) \\ = (f(t, u_1(t)) - f(t, u_2(t)), \phi) .\end{aligned}$$

Now, setting $\phi = u_1(t) - u_2(t)$ and integrating the whole equation over $(0, T)$ for any $T \in I$, we obtain

$$\begin{aligned}\int_0^T \left(\frac{d(u_1(t) - u_2(t))}{dt}, u_1(t) - u_2(t)\right) dt + \int_0^T \langle u_1(t) - u_2(t), u_1(t) - u_2(t)\rangle dt \\ + \int_0^T \left(\int_0^t a(t-s)\frac{d(u_1(s) - u_2(s))}{ds}ds, u_1(t) - u_2(t)\right) dt \\ = \int_0^T (f(t, u_1(t)) - f(t, u_2(t)), u_1(t) - u_2(t))\, dt.\end{aligned}$$

Due to (2.9) and (2.6) we arrive at

$$||u_1(T) - u_2(T)||^2 + \int_0^T ||u_1(t) - u_2(t)||_1^2 dt \le C \int_0^T ||u_1(t) - u_2(t)||^2 dt.$$

The Gronwall lemma implies $||u_1(T) - u_2(T)||^2 \le 0$. This is valid for an arbitrary $T \in I$, thus $u_1 = u_2$. □

Now, we are in the position to prove our main result.

Theorem 2. *Let* (2.1)-(2.8) *and* (2.10) *be satisfied. Then there exists a solution* u *of the Problem* C *such that as* $\sigma \to 0$*,*

$$\begin{aligned} u_\sigma &\to u \quad \textit{in } C(I, L_2(\Omega))\,, \\ u_\sigma &\rightharpoonup u \quad \textit{in } L_2(I, \overset{\circ}{H}{}^1(\Omega)) \\ \frac{du_\sigma}{dt} &\rightharpoonup \frac{du}{dt} \quad \textit{in } L_2(I, L_2(\Omega))\,. \end{aligned}$$

Proof. Lemma 1 and the reflexivity of $L_2(I, \overset{\circ}{H}{}^1(\Omega))$ imply the existence of a subsequence of $\overline{u}_\sigma$ (we denote it by $\overline{u}_\sigma$ again) for which

$$\overline{u}_\sigma \rightharpoonup u \quad \text{in } L_2(I, \overset{\circ}{H}{}^1(\Omega)),$$

and

$$\int_I \|\overline{u}_\sigma - u_\sigma\|_1^2 \le C\tau.$$

This implies (for a subsequence of u_σ)

$$u_\sigma \rightharpoonup u \quad \text{in } L_2(I, \overset{\circ}{H}{}^1(\Omega)),$$

and

$$u_\sigma \to u \quad \text{in } L_2(I, L_2(\Omega)),$$

because of $L_2(I, \overset{\circ}{H}{}^1(\Omega)) \circlearrowleft\circlearrowleft L_2(I, L_2(\Omega))$. Lemma 1 yields

$$\int_I \left\| \frac{du_\sigma}{dt} \right\|^2 \le C.$$

$L_2(I, L_2(\Omega))$ is a reflexive Banach space, thus

$$\frac{du_\sigma}{dt} \rightharpoonup w \quad \text{in } L_2(I, L_2(\Omega)).$$

Now for arbitrary $t \in I$ and $\psi \in H^{-1}(\Omega)$ (dual space to $\overset{\circ}{H}{}^1(\Omega)$), as $\sigma \to 0$ we get

$$\begin{aligned} (u_\sigma(t) - u(0), \psi) &= \left(\int_0^t \frac{du_\sigma}{ds}, \psi \right) \\ \downarrow \qquad & \qquad\quad \downarrow \\ (u(t) - u(0), \psi) &= \left(\int_0^t w, \psi \right)\,, \end{aligned}$$

where the differentiation with respect to t gives $w = \dfrac{du}{dt}$.

Due to Arzela-Ascoli theorem, the convergence

$$u_\sigma \to u \quad \text{in } L_2(I, L_2(\Omega)),$$

and the estimate

$$\int_I \left\| \frac{du_\sigma}{dt} \right\|^2 + \int_I \left\| \frac{du}{dt} \right\|^2 \le C$$

imply that there is a subsequence for which

$$u_\sigma \to u \quad \text{in } C(I, L_2(\Omega)).$$

Collecting all considerations above, we have proved that there exist a function u and a subsequence of u_σ (denote again by u_σ) for which we have (as $\sigma \to 0$)

$$\begin{aligned} u_\sigma &\to u \quad \text{in } C(I, L_2(\Omega))\,, \\ u_\sigma &\rightharpoonup u \quad \text{in } L_2(I, \overset{\circ}{H}{}^1(\Omega)) \\ \frac{du_\sigma}{dt} &\rightharpoonup \frac{du}{dt} \quad \text{in } L_2(I, L_2(\Omega))\,.. \end{aligned} \tag{4.2}$$

Now, we have to prove that u is the solution of the Problem C. To do this, we integrate (4.1) on $(0,T)$ and then we pass to the limit as $\sigma \to 0$. We will demonstrate this on each term of (4.1) separately. Let us fix such a $\mu > 0$ for which $V_\mu \subset V_h \quad \forall h$. Now we set $\phi = \phi_\mu = P_\mu \psi \in V_\mu$ for any $\psi \in \overset{\circ}{H}{}^1(\Omega)$. For such a ϕ_μ (4.1) holds true. Hence we can write $(t \in (t_{i-1}, t_i\rangle, T \in I)$

$$\begin{aligned} &\int_0^T \left(\frac{du_\sigma(t)}{dt}, \phi_\mu \right)_h dt + \int_0^T \langle \overline{u}_\sigma(t), \phi_\mu \rangle \, dt + \int_0^T \int_0^{t_i} a_\tau(t_i + \tau - s) \left(\frac{du_\sigma(s)}{ds}, \phi_\mu \right)_h ds\, dt \\ &\qquad = \int_0^T \left(f_\tau(t, \overline{u}_\sigma(t-\tau)), \phi_\mu \right)_h \, dt. \end{aligned} \tag{4.3}$$

Now, one can easily see that

$$\int_0^T \left(\frac{du_\sigma(t)}{dt}, \phi_\mu \right)_h dt = \left(u_\sigma(T) - u_\sigma(0), \phi_\mu \right)_h .$$

According to (2.4) we get

$$|(u_\sigma(t), \phi_\mu)_h - (u_\sigma(t), \phi_\mu)| \le C \|\phi_\mu\|_1 h^2$$

and (4.2) yields

$$(u_\sigma(t), \phi_\mu) \to (u(t), \phi_\mu) \text{ for } t \in \langle 0, T\rangle \qquad \text{as } \sigma \to 0\,.$$

Thus, we have shown

$$\int_0^T \left(\frac{du_\sigma(t)}{dt}, \phi_\mu \right) dt \to (u(T) - v, \phi_\mu) \qquad \text{as } \sigma \to 0\,. \tag{4.4}$$

For the second term we put ($T \in (t_{m-1}, t_m\rangle$)

$$\begin{aligned}\int_0^T \langle \overline{u}_\sigma(t), \phi_\mu\rangle\, dt =& \int_0^T \langle u_\sigma(t), \phi_\mu\rangle dt + \int_{t_m}^T \langle \overline{u}_\sigma(t) - u_\sigma(t), \phi_\mu\rangle dt \\ &+ \int_0^{t_m} \langle \overline{u}_\sigma(t) - u_\sigma(t), \phi_\mu\rangle dt \\ =& I_1 + I_2 + I_3 .\end{aligned}$$

Lemma 1 yields

$$\begin{aligned}|I_3| \le& C \sum_{i=1}^m \|\phi_\mu\|_1 \|u_i^h - u_{i-1}^h\|_1 \tau \le C\|\phi_\mu\|_1 \sqrt{\tau} \\ |I_2| \le& C \int_T^{t_m} \left(\|\overline{u}_\sigma(t)\|_1 + \|u_\sigma(t)\|_1 \right) \|\phi_\mu\|_1 dt \le C\|\phi_\mu\|_1 \tau \, .\end{aligned}$$

Thus, these estimates together with (4.2) give

$$\int_0^T \langle \overline{u}_\sigma(t), \phi_\mu\rangle dt \to \int_0^T \langle u(t), \phi_\mu\rangle dt \quad \text{as } \sigma \to 0. \tag{4.5}$$

The situation with the third term is more delicate. Let $t \in (t_{i-1}, t_i\rangle$. Then Lemma 2 implies

$$\left| \int_t^{t_i} a_\tau(t_i + \tau - s) \left(\frac{du_\sigma(s)}{ds}, \phi_\mu \right)_h ds \right| \le C\|\phi_\mu\|_1 \tau a(\tau) \to 0 \quad \text{as } \sigma \to 0.$$

Further

$$a_\tau(t_i + \tau - s) \to a(t-s) \quad \text{as } \tau \to 0$$

and Lemma 2 together with the Lebesgue theorem give

$$\begin{aligned}&\left| \int_0^t \left(a_\tau(t_i + \tau - s) - a(t-s) \right) \left(\frac{du_\sigma(s)}{ds}, \phi_\mu \right)_h ds \right| \\ \le& C\|\phi_\mu\|_1 \int_0^t |a_\tau(t_i + \tau - s) - a(t-s)| ds \ \to 0 \quad \text{as } \sigma \to 0 \, .\end{aligned}$$

According to these facts it is sufficient to pass to the limit as $\sigma \to 0$ in the term

$$\int_0^T \int_0^t a(t-s) \left(\frac{du_\sigma(s)}{ds}, \phi_\mu \right)_h ds\, dt$$

instead of the third term of (4.3).

One can write

$$\begin{aligned}&\int_0^T \int_0^t a(t-s) \left(\frac{du_\sigma(s)}{ds}, \phi_\mu \right)_h ds\, dt \\ &= \int_0^T \int_0^t a(t-s) \left(\frac{du_\sigma(s)}{ds}, \phi_\mu \right) ds\, dt \\ &\quad + \int_0^T \int_0^t a(t-s) \left\{ \left(\frac{du_\sigma(s)}{ds}, \phi_\mu \right)_h - \left(\frac{du_\sigma(s)}{ds}, \phi_\mu \right) \right\} ds\, dt \\ &= R_1 + R_2 \, .\end{aligned}$$

Using a change of order of integration, (2.4) and (2.5), we estimate

$$
\begin{aligned}
|R_2| = &\left| \int_0^T \left\{ \left(\frac{du_\sigma(s)}{ds}, \phi_\mu \right)_h - \left(\frac{du_\sigma(s)}{ds}, \phi_\mu \right) \right\} \int_0^{T-s} a(t)\, dt\, ds \right| \\
\le & Ch \int_0^T \left\| \frac{du_\sigma(s)}{ds} \right\| \ \|\phi_\mu\|_1 \, ds \\
\le & Ch \|\phi_\mu\|_1 \quad \to 0 \quad \text{as } \sigma \to 0.
\end{aligned}
$$

According to (4.2) we have

$$
\begin{aligned}
R_1 &= \int_0^T \left(\frac{du_\sigma(s)}{ds}, \phi_\mu \right) \int_0^{T-s} a(t)\, dt\, ds \\
&\to \int_0^T \left(\frac{du(s)}{ds}, \phi_\mu \right) \int_0^{T-s} a(t)\, dt\, ds \\
&= \int_0^T \int_0^t a(t-s) \left(\frac{du(s)}{ds}, \phi_\mu \right) ds\, dt \quad \text{as } \sigma \to 0\,.
\end{aligned}
$$

Summarizing the previous facts, we arrive at ($t \in (t_{i-1}, t_i)$)

$$
\begin{aligned}
&\int_0^T \int_0^{t_i} a_\tau(t_i + \tau - s) \left(\frac{du_\sigma(s)}{ds}, \phi_\mu \right)_h ds\, dt \\
&\to \int_0^T \int_0^t a(t-s) \left(\frac{du(s)}{ds}, \phi_\mu \right) ds\, dt \quad \text{as } \sigma \to 0\,.
\end{aligned} \tag{4.6}
$$

For the right-hand side we write

$$
\begin{aligned}
&\int_0^T (f_\tau(t, \overline{u}_\sigma(t-\tau)), \phi_\mu)_h dt \\
&= \int_0^T [(f_\tau(t, \overline{u}_\sigma(t-\tau)), \phi_\mu)_h - (f_\tau(t, \overline{u}_\sigma(t-\tau)), \phi_\mu)]\, dt \\
&\quad + \int_0^T [(f_\tau(t, \overline{u}_\sigma(t-\tau)), \phi_\mu) - (f(t, \overline{u}_\sigma(t-\tau)), \phi_\mu)]\, dt \\
&\quad + \int_0^T [(f(t, \overline{u}_\sigma(t-\tau)), \phi_\mu) - (f(t, u_\sigma(t)), \phi_\mu)]\, dt \\
&\quad + \int_0^T (f(t, u_\sigma(t)), \phi_\mu)\, dt = F_1 + F_2 + F_3 + F_4\,.
\end{aligned}
$$

Now, we proceed in a standard way

$$
\begin{aligned}
|F_1| \le & Ch \int_0^T \|f_\tau(t, \overline{u}_\sigma(t-\tau))\| \ \|\phi_\mu\|_1 dt \le Ch \|\phi_\mu\|_1, \\
|F_2| \le & C\tau \|\phi_\mu\|, \\
|F_3| \le & C \int_0^T \|\overline{u}_\sigma(t-\tau) - u_\sigma(t)\| \ \|\phi_\mu\| \ dt \le C\tau \|\phi_\mu\|,
\end{aligned}
$$

and according to (4.2) we obtain

$$F_4 \quad \to \quad \int_0^T (f(t,u(t)),\phi_\mu)dt \qquad \text{as } \sigma \to 0\,.$$

Thus we have proved

$$\int_0^T (f_\tau(t,\overline{u}_\sigma(t-\tau)),\phi_\mu)_h dt \quad \to \quad \int_0^T (f(t,u(t),\phi_\mu)\,dt \qquad \text{as } \sigma \to 0\,. \tag{4.7}$$

Finally, (4.3)-(4.7) imply

$$\int_o^T \left(\frac{du(t)}{dt},\phi_\mu\right)dt + \int_0^T \langle u(t),\phi_\mu\rangle\,dt + \int_0^T\int_0^t a(t-s)\left(\frac{du(s)}{ds},\phi_\mu\right)ds\,dt$$
$$= \int_0^T f(t,u(t)),\phi_\mu)\,dt\,.$$

This is true for any $\phi_\mu \in V_\mu$ and for any T from our time interval.

By virtue of the fact that $\phi_\mu \to \psi$ in $L_2(\Omega)$ and $\phi_\mu \rightharpoonup \psi$ in $\overset{\circ}{H}{}^1(\Omega)$, passing to the limit as $\mu \to 0$, and then differentiating the identity with respect to T, we see that u is a solution of Problem C. Due to Lemma 1, Lemma 2 and Theorem 1, we see that the whole sequence u_σ converges to u. □

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M. Slodička, Department of Computer Science, University of the Federal Armed Forces Munich, 85577 Neubiberg, Germany

E-mail address: marian@informatik.unibw-muenchen.de