Electronic Journal of Differential Equations, Vol. **2000**(2000), No. 03, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

ON THE OPTIMAL GROWTH OF FUNCTIONS WITH BOUNDED LAPLACIAN

Lavi Karp & Henrik Shahgholian

Abstract. Using a compactness argument, we introduce a Phragmén Lindelöf type theorem for functions with bounded Laplacian. The technique is very useful in studying unbounded free boundary problems near the infinity point and also in approximating integrable harmonic functions by those that decrease rapidly at infinity. The method is flexible in the sense that it can be applied to any operator which admits the standard elliptic estimate.

1. Introduction and main result

Let u be a function with bounded Laplacian in $\mathbb{R}^N$. Then we ask for conditions that force u to have a quadratic growth at infinity. The main motivation for this problem comes from studying unbounded free boundary problems (see [8], [10]). Other applications are Phragmén Lindelöf principle for the Cauchy problem and approximation of harmonic functions, in the L^1-norm, by rapidly decreasing ones (see [17], [18], [19]).

As there are harmonic polynomials of arbitrarily large degree, it is clear that one has to impose certain types of conditions on the function u in order to get the desired growth. In this note we introduce a general method which gives the desired quadratic growth under the condition that u and its gradient vanish on a sufficiently large set.

To fix the idea, let u be a function with polynomial growth and satisfy (in the sense of distributions)

$$\|\Delta u\|_\infty \leq L < \infty\,. \tag{1.1}$$

Our main result asserts that if $\Lambda(u) := \{x \in \mathbb{R}^N : \; u = |\nabla u| = 0\}$ has positive "Capacity density" (see below) at infinity, then

$$|u(x)| \leq CL(1+|x|)^2 \qquad \forall x \in \mathbb{R}^N. \tag{1.2}$$

In many cases, one actually has the estimate (see [8])

$$|u(x)| \leq C\|f\|_\infty (1+|x|)^2 \log(2+|x|) \qquad \forall x \in \mathbb{R}^N, \tag{1.3}$$

1991 Mathematics Subject Classifications: 35J05, 35J60, 31C45.
Key words and phrases: Optimal growth, bounded Laplacian, linear and semi-linear operators, capacity density condition.

Submitted October 15, 1999. Published January 1, 2000.
The second author was supported by the Swedish Natural Science Research Council.

where u solves the Poisson equation $\Delta u = f$ in $\mathbb{R}^N$ and $f \in L^\infty(\mathbb{R}^N)$. Thus, the main problem is to *get rid of* the logarithmic term in (1.3), when $\Lambda(u)$ is not too thin.

In a special case the estimate (1.2) can be extracted (see [16]) from a Phragmén-Lindelöf type theorem due to Fuchs [2]. A weaker version of the Fuchs theorem asserts that if $s(z)$ $(z = x_1 + ix_2)$ is an analytic function in $\Omega \subset \mathbb{C}$, continuous on $\overline{\Omega}$, $\infty \in \partial\Omega$ and $|s(z)| \le 1$ on $\partial\Omega \setminus \{\infty\}$. Then either

$$\lim_{r\to\infty} \frac{\log\left(\sup_{|z|\le r} |s(z)|\right)}{\log r} = \infty\,,$$

or

$$|s(z)| \le 1 \qquad \text{for all } \; z \in \Omega. \tag{1.4}$$

Now, suppose u satisfies: $\Delta u = \chi_\Omega$ in $\mathbb{R}^2$, $|u| = |\nabla u| = 0$ on $\mathbb{R}^2 \setminus \Omega$. We also assume both Ω and its complement (Ω^c) are unbounded, and that the origin is in the interior of Ω^c. Let $v(x) = |x|^2 - 4u(x)$ and $s(z) = (\partial v/\partial z)/z$. Then $s(z)$ is analytic in Ω, $s(z) = \bar{z}/z$ for $z \in \partial\Omega \setminus \{\infty\}$. Therefore, if u has a polynomial growth, then by the Fuchs theorem the estimate (1.4) holds; hence (1.2) holds as well.

Our method will give the estimate (1.2) in any space dimension $N \ge 2$ and for any $f \in L^\infty(\mathbb{R}^N)$. The condition $\infty \in \partial\Omega$, is replaced by a capacity density condition on $\mathbb{R}^N \setminus \Omega$ at the infinity point. In [5], the authors prove (1.2) by a completely different method. Their method, however, works only for the Laplace operator, while ours is more flexible and applies also to nonlinear operators.

Notation. For a C^1-function u defined in the entire space $\mathbb{R}^N$, we set $\Lambda(u) := \{x : u(x) = |\nabla u(x)| = 0\}$; $\mathrm{CAP}(\cdot)$ denotes the Newtonian capacity for $N \ge 3$ and the logarithmic capacity in $\mathbb{R}^2$ (see e.g. [11]); $B_r = \{x :\ |x| < r\}$, $\|f\|_\infty$ denotes the supremum norm of f (in $\mathbb{R}^N$) and

$$S(r,u) = \sup_{x\in B_r} |u(x)|.$$

Definition 1.1. Let L, K, m and ε be positive numbers. A function u belongs to the class $\mathcal{G}(L,K,m,\varepsilon)$ if:

(a) $\|\Delta u\|_\infty \le L$;
(b) $|u(x)| \le K(1+|x|)^m \quad \forall x \in \mathbb{R}^N$;
(c) $\liminf_{r\to\infty} \frac{\mathrm{CAP}(\Lambda(u)\cap B_r)}{\mathrm{CAP}(B_r)} \ge \varepsilon$.

Our main result in this section is the following:

Theorem 1.2. *There is a positive constant $K' = K'(L,K,m,\varepsilon)$ such that*

$$\mathcal{G}(L,K,m,\varepsilon) \subset \mathcal{G}(L,K',m,\varepsilon),$$

i.e., for any $u \in \mathcal{G}(L,K,m,\varepsilon)$, there holds

$$|u(x)| \le K'(1+|x|)^2 \qquad \forall x \in \mathbb{R}^N.$$

Corollary 1.3. *Let u be a C^1-function with a bounded Laplacian in $\mathbb{R}^N$ and satisfying*

$$\liminf_{r\to\infty} \frac{CAP(\Lambda(u)\cap B_r)}{CAP(B_r)} > 0. \tag{1.5}$$

Then either

$$\limsup_{r\to\infty} \frac{\log(S(r,u))}{\log r} = \infty$$

or

$$|u(x)| \le K'(1+|x|)^2 \qquad \forall x \in \mathbb{R}^N.$$

Remarks.

(i) Uniformly fat sets. A set $E \subset \mathbb{R}^N$ is uniformly fat (or satisfies the capacity density condition) at infinity if

$$\liminf_{r\to\infty} \frac{\mathrm{CAP}(E\cap B_r)}{\mathrm{CAP}(B_r)} > 0.$$

This concept has previously been used in different contexts; see [1], [7], [6], [12], and [14]. One can verify, using explicit calculations of the Newtonian potential of ellipsoids (see [4] or [11]), that $\{x \in \mathbb{R}^3 : \ x_1^2 + x_2^2 \le 1\}$ is not uniformly fat, while $\{x \in \mathbb{R}^3 : \ |x_1| \le 1, \ x_2 \ge 0\}$ is uniformly fat.

(ii) Indispensable conditions. For a tempered distribution u satisfying (1.1), one may ask whether conditions (b) and (c), in Definition 1.1, are necessary for the conclusion of Theorem 1.2. To show the indispensability of these conditions, let $v(x_1,x_2) := \exp(x_2)\cos(x_1)$, and let $\varphi \in C^\infty(\mathbb{R})$ satisfy: $0 \le \varphi(t) \le 1$, $\varphi(t) = 0$ for $t \le 0$, and $\varphi(t) = 1$ for $t \ge 1$. Set $u(x_1,x_2) = \varphi(x_2)v(x_1,x_2)$. Then, $\Lambda(u)$ is the lower half plane. Also u has a bounded Laplacian in $\mathbb{R}^2$ and is of exponential growth. This shows that condition (b) cannot be removed.

As to condition (c), let $u(x_1,x_2,x_3) := x_1x_2x_3$, and note that $\Lambda(u)$ is the union of the three axes. Hence, $\mathrm{CAP}(\Lambda(u)\cap B_r) = 0$ for all $r > 0$. This example shows that condition (c) cannot be removed.

(iii) The local problem. Let $x_0 \in \Lambda(u)$, then the same method, with some minor modifications, gives

$$|u(x)| \le K'|x-x_0|^2, \tag{1.6}$$

provided the local analogue of the capacity density condition (1.5) holds. A consequence of (1.6) is that u is a $C^{1,1}$-function, if $f \in C^\lambda$; see [9; Theorem 2.1].

We prove Theorem 1.2 in Section 2. In Section 3, we will indicate applications of the method to other elliptic operators.

Proof of Theorem 1.2

To prove the theorem, it suffices to show $S(r,u) \le 4K'r^2$ for all $r \ge 1$, when $u \in \mathcal{G}(L,K,m,\varepsilon)$, and where $S(r,u) = \sup_{x\in B_r}|u(x)|$. We need the following lemma, which concerns a "doubling" condition for functions with polynomial growth.

Lemma 2.1. *Let $\{a_j\}$ be sequence of nondecreasing positive numbers satisfying*

$$a_j \le K2^{jm} \qquad \forall j \tag{2.1}$$

where K and m are positive constants. Then, there is an infinite maximal subset $\mathbb{M} = \mathbb{M}(\{a_j\})$, of natural numbers $\mathbb{N}$, such that

$$a_{j+1} \le 2^{(m+1)} a_j \qquad \forall\ j \in \mathbb{M}(\{a_j\}). \tag{2.2}$$

Proof. Define

$$g_j := \frac{a_j}{a_{j+1}}\,.$$

If the conclusion of the lemma fails, then there is a positive integer $j_0 = j_0(\{a_j\})$ such that

$$g_j < \frac{1}{2^{(m+1)}} \qquad \forall\ j \ge j_0. \tag{2.3}$$

Let $j \ge j_0$ and $k \ge 1$, then by (2.1) and (2.3),

$$\begin{aligned} a_j &= g_j a_{j+1} = \cdots \\ &= g_j g_{j+1} \cdots g_{j+k} a_{(j+k+1)} \\ &\le \left(\frac{1}{2^{(m+1)}}\right)^k K(2^{(j+k+1)m}). \end{aligned}$$

Letting $k \to \infty$, we obtain by the latter inequality that $a_j = 0$ for all $j \ge j_0$, which is a contradiction. $\square$

In our applications of this lemma we'll let

$$a_j = S(2^j, u),$$

for a given u in the class $\mathcal{G}$. In particular we have the following form of the above lemma.

Lemma 2.1'. *Suppose u satisfies*

$$|u(x)| \le K(1+|x|)^m \qquad \forall x \in \mathbb{R}^N, \tag{2.1'}$$

where K and m are positive constants. Then, there is an infinite maximal subset $\mathbb{M} = \mathbb{M}(u)$, of natural numbers $\mathbb{N}$, such that

$$S(2^{(j+1)}, u) \le 2^{(m+1)} S(2^j, u) \qquad \forall\ j \in \mathbb{M}(u). \tag{2.2'}$$

Lemma 2.2. *There is a positive constant $K' = K'(L, K, m, \varepsilon)$ such that for any $u \in \mathcal{G}(L, K, m, \varepsilon)$ and $j \in \mathbb{M}(u)$, there holds*

$$S(2^j, u) \le K' \left(2^j\right)^2. \tag{2.4}$$

In the proof of Lemma 2.2 we shall use two known results. The first one is the easy part of Choquet's theorem concerning the capacitablity of the Newtonian capacity. The second one, is a result of L. Robbiano and J. Salazar [15].

Lemma 2.3. *(see e.g. [11]) The compact sets are capacitable, i.e., if $A \subset \mathbb{R}^N$ is compact, then for any $\delta > 0$, there is an open set $O \supset A$ such that*

$$CAP(O) \leq CAP(A) + \delta.$$

Theorem 2.4. *(Robbiano and Salazar) Let h be a harmonic function in a domain Ω. Then for any $A \subset \Omega$, and compact, either $\Lambda(h) \cap A$ has zero Newtonian capacity or $h \equiv 0$ in Ω .*

Proof of Lemma 2.2. We argue by contradiction. Thus, we may assume, as we do, that for any $j \geq K$, there is $u_j \in \mathcal{G}(L, K, m, \varepsilon)$ and $k_j \in \mathbb{M}(u_j)$ such that

$$S(2^{k_j}, u_j) \geq j \left(2^{k_j}\right)^2; \tag{2.5}$$

note that by condition (b), $k_j \to \infty$ which is an important fact in using condition (c) later on. Define now

$$\tilde{u}_j(x) := \frac{u_j(2^{(k_j+1)}x)}{S(2^{k_j}, u_j)}. \tag{2.6}$$

The function $\tilde{u}_j$ enjoys the following properties:

$$\sup_{B_1} |\tilde{u}_j| = \frac{S(2^{(k_j+1)}, u_j)}{S(2^{k_j}, u_j)} \leq 2^{(m+1)}, \qquad \text{(by Lemma 2.1')} \tag{2.7}$$

$$\sup_{B_{(1/2)}} |\tilde{u}_j| = 1, \qquad \text{(by the definition)} \tag{2.8}$$

and

$$|\Delta \tilde{u}_j(x)| \leq \frac{4L}{j} \qquad \text{by (2.5)}. \tag{2.9}$$

Let $W^{2,p}(\Omega)$ be the standard Sobolev space. The elliptic estimate (see e.g. [3; Theorem 9.11])

$$\|v\|_{W^{2,p}(B_{(3/4)})} \leq C \left(\|v\|_{L^p(B_1)} + \|\Delta v\|_{L^p(B_1)}\right), \qquad (1 < p < \infty)$$

combined with (2.7) and (2.9) implies that $\{\tilde{u}_j\}$ is bounded in $W^{2,p}(B_{(3/4)})$. Taking p sufficiently large and $0 < \sigma < 1 - N/p$, we may use the compactness of the embedding $W^{2,p}(B_{(3/4)}) \hookrightarrow C^{1,\sigma}(\bar{B}_{(3/4)})$, to conclude the convergence (for a subsequence) of $\tilde{u}_j$ to a function $\tilde{u}_0$ in the norm of $C^{1,\sigma}(\bar{B}_{(3/4)})$. By (2.8)–(2.9), $\tilde{u}_0 \not\equiv 0$, and is harmonic in $B_{(3/4)}$. Now Theorem 2.4, in conjunction with Assertion (see below) gives a contradiction. Hence the proof will be completed once we prove the following assertion.

Assertion.

$$CAP\left(\Lambda(\tilde{u}_0) \cap \overline{B_{(1/2)}}\right) > 0. \tag{2.10}$$

Proof of Assertion. Define $\Lambda_\infty = \{x : \ x$ is a limit point of a sequence $\{x_j\}$, where $x_j \in \Lambda(\tilde{u}_j)\}$. Since $\tilde{u}_j \to \tilde{u}_0$ and $\nabla \tilde{u}_j \to \nabla \tilde{u}_0$ uniformly on $\overline{B_{(1/2)}}$,

$$\Lambda_\infty \cap \overline{B_{(1/2)}} \subset \Lambda(\tilde{u}_0) \cap \overline{B_{(1/2)}}. \tag{2.11}$$

Let $\delta > 0$. Then by Lemma 2.3 there is an open set O, $O \supset (\Lambda_\infty \cap \overline{B_{(1/2)}})$ and satisfies

$$\mathrm{CAP}(O) \leq \mathrm{CAP}\left(\Lambda_\infty \cap \overline{B_{(1/2)}}\right) + \delta. \tag{2.12}$$

We claim there is j_0 such that

$$\left(\Lambda(\tilde{u}_j) \cap \overline{B_{(1/2)}}\right) \subset O \qquad \text{for all} \quad j \geq j_0. \tag{2.13}$$

Since, otherwise, there is a sequence $\{x_j\} \subset \left(\Lambda(\tilde{u}_j) \cap \overline{B_{(1/2)}}\right) \setminus O$, with $x_j \to x \in \left(\Lambda_\infty \cap \overline{B_{(1/2)}}\right) \subset O$. Since O is open, $x_j \in O$ for all j large enough. This is a contradiction.

Now using (2.11)-(2.13), and letting $j \geq j_0$, we obtain

$$\begin{aligned}\mathrm{CAP}\left(\Lambda(\tilde{u}_0) \cap \overline{B_{(1/2)}}\right) &\geq \mathrm{CAP}\left(\Lambda_\infty \cap \overline{B_{(1/2)}}\right) \geq \mathrm{CAP}\left(\Lambda(\tilde{u}_j) \cap \overline{B_{(1/2)}}\right) - \delta \\ &= c(N)\frac{\mathrm{CAP}\left(\Lambda(u_j) \cap \overline{B_{2^{k_j}}}\right)}{\mathrm{CAP}(B_{2^{k_j}})} - \delta,\end{aligned}$$

where $c(N) = \mathrm{CAP}(B_1)/2^{(N-2)}$, for $N \geq 3$, and $c(2) = \mathrm{CAP}(B_1)/2$ (for the last equality see e.g. [11; pp. 158-160]). Since $k_j \to \infty$ (by (b)), condition (c) implies (2.10), if we chose δ sufficiently small. This completes the proof of the Assertion and hence that of Lemma 2.2. $\square$

Proof of Theorem 1.2. Obviously we may assume $m > 2$. For $u \in \mathcal{G}(L, K, m, \varepsilon)$ let $\mathbb{M}'(u)$ be the maximal subset of $\mathbb{N}$ such that (2.4) holds. We first show that

$$\mathbb{M}'(u) = \mathbb{N}. \tag{2.14}$$

By Lemma 2.1', $\mathbb{M}(u)$ is infinite and by Lemma 2.2, $\mathbb{M}(u) \subset \mathbb{M}'(u)$. Hence, $\mathbb{M}'(u)$ is an infinite subset of $\mathbb{N}$. Therefore in order to show (2.14), it suffices to show that if $j+1 \in \mathbb{M}'(u)$, then $j \in \mathbb{M}'rime(u)$. Suppose not, i.e., there is $j \notin \mathbb{M}'(u)$ such that $j+1 \in \mathbb{M}'(u)$. By the maximality of both $\mathbb{M}(u)$ and $\mathbb{M}'(u)$, neither (2.2') nor (2.4) holds for elements outside $\mathbb{M}'(u)$. Hence

$$\begin{gathered} S(2^j, u) > K'\left(2^j\right)^2 \\ S(2^{(j+1)}, u) > 2^{(m+1)} S(2^j, u). \end{gathered}$$

Since $j+1 \in \mathbb{M}'(u)$, we'll have by the latter inequalities,

$$K'\left(2^{(j+1)}\right)^2 \geq S(2^{(j+1)}, u) > 2^{(m+1)} S(2^j, u) > 2^{(m+1)} K'\left(2^j\right)^2,$$

which implies $2^2 > 2^m$, contradicting $m > 2$. Hence (2.4) holds for all $j \in \mathbb{N}$. For $2^j \leq r \leq 2^{(j+1)}$, we have

$$S(r, u) \leq S(2^{(j+1)}, u) \leq K'\left(2^{(j+1)}\right)^2 \leq 4K'r^2$$

and the proof is complete. $\square$

3. Application to other elliptic operators

The method presented in the previous section can be applied to other elliptic operators. In this section we shall obtain the optimal growth of solutions to certain type of elliptic operators. In order to avoid complicated notations, we present two examples. The corresponding local estimate (1.6) will be treated elsewhere.

Recall that Lemma 2.1' deals only with the doubling property of functions with polynomial growth. Therefore, to adopt the method of the previous section, we only need to deduce the corresponding "compactness" property in Lemma 2.2.

(i) A second order linear operator. Let

$$\mathcal{L}u = \Delta u + a_{ij}(x)\partial_i\partial_j u + b_i(x)\partial_i u + c(x)u,$$

where $\partial_i = \partial/\partial x_i$, $(\delta_{ij} + a_{ij}(x))\xi_i\xi_j \geq \lambda|\xi|^2$ in $\mathbb{R}^N$ $(\lambda > 0)$, $a_{ij}, b_i, c \in C(\mathbb{R}^N)$ and satisfy

$$\lim_{x\to\infty} |a_{ij}(x)| = \lim_{x\to\infty} |x|\ |b(x)| = \lim_{x\to\infty} |x|^2\ |c(x)| = 0. \tag{3.1}$$

Theorem 3.1. *If we replace the Laplace operator, in Definition 1.1, by the operator $\mathcal{L}$ above. Then the conclusions of Theorem 1.2, as well as Corollary 1.3 remain true.*

Proof. Set

$$\mathcal{L}_r u = \Delta u + a_{ij}(rx)\partial_i\partial_j u + rb_i(rx)\partial_i u + r^2 c(rx)u$$

and

$$C(r) = \sup_{B_1}\left(|a_{ij}(rx)| + |rb_i(rx)| + |r^2c(rx)|\right).$$

Then by (3.1)

$$C(r) \to 0 \qquad \text{as} \ \ r \to \infty\,.$$

Hence the constant C, of the elliptic estimate

$$\|v\|_{W^{2,p}(B_{(3/4)})} \leq C\left(\|v\|_{L^p(B_1)} + \|\mathcal{L}_r v\|_{L^p(B_1)}\right), \qquad (1 < p < \infty) \tag{3.2}$$

is independent of r (see e.g [3; Theorem 9.11]). Furthermore, $\mathcal{L}_r \to \Delta$ in the sense of $W^{2,p}(B_1)$, i.e.,

$$\lim_{r\to\infty}\int_{B_1} \mathcal{L}_r v\varphi\,dx = \int_{B_1}\Delta v\varphi\,dx, \tag{3.3}$$

for all $v \in W^{2,p}(B_1)$ and for all $\varphi \in C_0(B_1)$. Now define $\tilde{u}_j$ as in (2.6), then (2.7) and (2.8) remain unchanged, while (2.9) becomes

$$|\mathcal{L}_{(2^{(k_j+1)})}\tilde{u}_j(x)| \leq \frac{4L}{j}. \tag{3.4}$$

Therefore, by (3.2), (3.3) and (3.4), we conclude, as we did in the proof of Lemma 2.2, that $\tilde{u}_j$ converges to a harmonic function $\tilde{u}_0$ in the norm of $C^{1,\sigma}(\bar{B}_{(3/4)})$. The rest of the proof follows precisely as in the proof of Theorem 1.2. □

(ii) A semi-linear operator. We consider the growth of any solution u to

$$\Delta u = f(x,u) \qquad \text{in } \ \mathbb{R}^N,$$

where

$$|f(x,z)| \le L_0 |z|^{\gamma-1} + L_1 \qquad \text{in } \ \mathbb{R}^N,$$

for positive constants $L_0,\ L_1$, and $1 \le \gamma < 2$. For the existence, of infinitely many solutions, see e.g. [13].

Definition 3.2. Let L_0, L_1, K, m and ε be positive numbers and $1 \le \gamma < 2$. A function u belongs to the class $\mathcal{G}(L_0, L_1, K, m, \varepsilon, \gamma)$ if:

(a) $|\Delta u(x)| \le L_0 |u(x)|^{\gamma-1} + L_1 \quad \forall x \in \mathbb{R}^N$;
(b) $|u(x)| \le K(1+|x|)^m \quad \forall x \in \mathbb{R}^N$;
(c) $\liminf_{r\to\infty} \frac{\text{CAP}(\Lambda(u)\cap B_r)}{\text{CAP}(B_r)} \ge \varepsilon$.

Theorem 3.3. *There is a positive constant $K' = K'(L_0, L_1, K, m, \varepsilon, \gamma)$ such that for any $u \in \mathcal{G}(L_0, L_1, K, m, \varepsilon, \gamma)$, there holds*

$$|u(x)| \le K'(1+|x|)^{(\frac{2}{2-\gamma})} \qquad \forall x \in \mathbb{R}^N.$$

Proof. Arguing as in the proof of Lemma 2.2, we may assume

$$S(2^{k_j}, u_j) \ge j \left(2^{k_j}\right)^{(\frac{2}{2-\gamma})},$$

where $k_j \to \infty$. Now define $\tilde{u}_j$ as in (2.6), then, since $2/(2-\gamma) \ge 2$, we have by (2.2') and (a) (in Definition 3.2),

$$\begin{aligned}
\sup_{x\in B_1} |\Delta \tilde{u}_j(x)| &\le \frac{\left(2^{(k_j+1)}\right)^2}{S(2^{k_j}, u_j)} \left(L_0 \left(S(2^{(k_j+1)}, u_j)\right)^{(\gamma-1)} + L_1 \right) \\
&\le \frac{\left(2^{(k_j+1)}\right)^2}{S(2^{k_j}, u_j)} \left(L_0 \left(2^{(m+1)}\right)^{\gamma-1} \left(S(2^{k_j}, u_j)\right)^{(\gamma-1)} + L_1 \right) \\
&\le L_0 \left(2^{(m\gamma - m + \gamma)}\right) \left(\frac{\left(2^{k_j}\right)^{(\frac{2}{2-\gamma})}}{S(2^{k_j}, u_j)} \right)^{(2-\gamma)} + L_1 4 \left(\frac{\left(2^{k_j}\right)^{(\frac{2}{2-\gamma})}}{S(2^{k_j}, u_j)} \right).
\end{aligned}$$

Hence we'll have

$$\sup_{x\in B_1} |\Delta \tilde{u}_j(x)| \le \frac{L_0 \left(2^{(m\gamma-m+\gamma)}\right) + L_1 4}{j},$$

and the rest of the proof follows now as in the proof of Theorem 1.2. $\square$

Acknowledgments. We would like to thank H.S. Shapiro for discussion on the capacity density condition. The first author would like to thank the department of mathematics at the University of Stockholm for a very fruitful and pleasant period while he was working there.

References

[1] A. Ancona, *On strong barriers and an inequality of Hardy for domaims in* $\mathbb{R}^n$, J. London Math. Soc. (2) **34** (1986), 274-290.

[2] W.H.J. Fuchs, *A Phragmén-Lindelöf theorem conjectured by D.J. Newman*, Trans. Amer. Math. Soc. **267** (1981), 285-293.

[3] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag 2^{nd} edition, Berlin Heidelberg New York Toko, 1983.

[4] W.K. Hayman, *Subharmonic Functions*, Volume 2, Academic Press, London, 1989.

[5] W.K. Hayman, L. Karp and H.S. Shapiro, *Newtonian capacity and quasi balayage*, In preparation.

[6] J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Clarendon Press, Oxfrd New York Tokyo, 1993.

[7] P.W. Jones and T.H. Wolff, *Hausdorff dimension of harmonic measures in the plane*, Acta Math. **161** (1988), 131-144.

[8] L. Karp and A.S. Margulis, *On the Newtonian potential theory for unbounded sources and its application to free boundary problems*, J. Anal. Math. 70 (1996), 1–63.

[9] L. Karp and H. Shahgholian, *Regularity of a free boundary problem*, J. of Geometric Analysis, (to appear).

[10] L. Karp and H. Shahgholian, *The regularity of a free boundary problem at the infinity point.*, Communication in partial diff. equation (to appear).

[11] N.S. Landkof, *Foundation of Modern Potential Theory*, Springer-Verlag, Berlin Heidelberg New York, 1972.

[12] J. L. Lewis, *Uniformly fat sets*, Trans. Amer. Math. Soc. **308** (1988), 177-196.

[13] J.M. Morel, *Runge property and multiplicity of solutions for elliptic equations in* $\mathbb{R}^N$*, with a monotone nonlinearity*, Manuscripta Math. **54** (1985), 165-185.

[14] Ch. Pommerenke, *Uniformly perfect sets*, Arch. Math. **32** (1979), 191-199.

[15] L. Robbiano and J. Salazar, *Dimension de Hausdorff et capapcité des points singuliers d'une solution d'un opérateur elliptique*, Bull Sci. Math. (2) **114** (1990), 329-336.

[16] M. Sakai, *Regularity of a boundary having a Schwarz function*, Acta Math. **166** (1991), 263-297.

[17] H. Shahgholian, *On quadrature domains and the Schwarz potential*, J. Math. Anal. Appl. **171** (1992), 61-78.

[18] H.S. Shapiro, *Global geometric aspects of the Cauchy's problem for the Laplace operator*, Geometrical and algebraical aspects in several complex variables (Cetraro, 1989), 309–324, Sem. Conf., 8, EditEl, Rende, 1991.

[19] H.S. Shapiro, *Quasi-balayage and á priori estimates for the Laplace operator*, I, II. Multivariate approximation (Witten-Bommerholz, 1996), 203–230, 231–254, Math. Res., 101, Akademie Verlag, Berlin, 1997. Stockholm.

Lavi Karp
Department of Applied Mathematics, Ort Braude College,
P.O. Box 78, Karmiel 21982, Israel,
and Department of Mathematics, Technion, 32000 Haifa, Israel.
E-mail address: `karp@techunix.technion.ac.il`

Henrik Shahgholian
Department of Mathematics, Royal Institute of Technology,
100 44 Stockholm, Sweden
E-mail address: `henriks@math.kth.se`