Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 50, pp. 1–3.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EIGENVALUES OF STURM-LIOUVILLE OPERATORS AND PRIME NUMBERS

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Abstract. We show that there is no function $q(x) \in L_2(0,1)$ which is the potential of a Sturm-Liouville problem with Dirichlet boundary condition whose spectrum is a set depending nonlinearly on the set of prime numbers as suggested by Mingarelli [7].

1. Introduction

We consider the Sturm-Liouville problem

$$\begin{gathered} -y'' + q(x)y = (\pi N(\lambda))^2 y \\ y(0) = y(1) = 0, \end{gathered} \tag{1.1}$$

with

$$N(\lambda) = \lambda, \quad N(\lambda) = \frac{\lambda}{\ln(\lambda)}, \quad \text{or} \quad N(\lambda) = li(\lambda) := \int_0^\lambda \frac{dt}{\ln(t)} \tag{1.2}$$

where $li(x)$ is defined as in [1, p. 228]. A real number λ is called an eigenvalue of (1.1) if it has a nontrivial solution. The set of all such eigenvalues is called the spectrum of (1.1) .

The purpose of this note is to prove the following results.

Theorem 1.1. *If $N(\lambda) = \lambda/\ln(\lambda)$ then there is no function $q \in L_2[0,1]$ such that the spectrum of (1.1) is the set of prime numbers.*

Theorem 1.2. *If $N(\lambda) = li(\lambda)$ then is no function $q \in L_2[0,1]$ such that the spectrum of (1.1) is the set of prime numbers.*

The case $N(\lambda) = \lambda$ was asked by Zettl [9, p.299] and answered by Mingarelli [7]. In turn, Mingarelli [7] asked the question answered by Theorems 1.1 and 1.2.

Our proofs are based on the asymptotic distribution of prime numbers and the asymptotic distribution of the eigenvalues for $N(\lambda) = \lambda$. In fact, letting $\pi(x)$ denote the number of prime number less than or equal to x, by the Prime Number Theorem, see [5], we have

$$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\ln x}} = 1 \quad \text{and} \quad \lim_{x\to\infty} \frac{\pi(x)}{li(x)} = 1. \tag{1.3}$$

2010 *Mathematics Subject Classification.* 34B05, 34B07, 11Z05.
Key words and phrases. Sturm-Liouville; spectrum; prime numbers.

Submitted June 1, 2016. Published February 20, 2017.

On the other hand for $N(\lambda)=\lambda$ we have

$$\pi\lambda_n = n\pi + \frac{\int_0^1 q(t)dt}{2n\pi} + O(n^{-2}), \tag{1.4}$$

see [2, (3.15), p. 81].

2. Main Results

Proof of Theorem 1.1. Suppose the exists $q \in L_2[0,1]$ such that the spectrum of (1.1) is the set of prime numbers. Let p_n denote the n-th prime number. By (1.4), see [2, 4, 8],

$$\Big(\frac{\pi p_n}{\ln(p_n)}\Big)^2 = n^2\pi^2 + \int_0^1 q(t)dt + c_n \tag{2.1}$$

where $c_n \in l_2$,

From the results by Dusart [3] we have

$$\pi(x) \geq \frac{x}{\ln x}(1 + \frac{1}{\ln x} + \frac{1.8}{\ln^2 x}) \tag{2.2}$$

for $x \geq 32299$. Hence

$$\begin{aligned} \lim_{n\to\infty}\Big(\big(\pi\frac{p_n}{\ln p_n}\big)^2 - n^2\pi^2\Big) &= \lim_{n\to\infty}\Big(\big(\pi\frac{p_n}{\ln p_n}\big)^2 - (\pi(p_n))^2\pi^2\Big) \\ &\leq -\lim_{n\to\infty}\frac{p_n^2}{\ln^4(p_n)} = -\infty. \end{aligned} \tag{2.3}$$

Since (2.3) contradicts (1.4), the proof is complete. □

Proof of Theorem 1.2. The classical Littlewood theorem, see [6, 5], proves that $\pi(x)-li(x)$ changes sign infinitely often. More precisely, it establishes the existence of increasing sequences $\{x_n\}_n$ and $\{y_n\}$ converging to $+\infty$ such that

$$\lim_{n\to+\infty}\pi(x_n) - li(x_n) = +\infty \quad \text{and} \quad \lim_{n\to+\infty}\pi(y_n) - li(y_n) = -\infty. \tag{2.4}$$

It is not difficult to see that if p_j denotes the largest prime number less than or equal to x_j then

$$\lim_{n\to+\infty}\pi(p_n) - li(p_n) = +\infty. \tag{2.5}$$

Similarly, if p_j denotes the smallest prime number greater than or equal to y_j then

$$\lim_{n\to+\infty}\pi(p_n) - li(p_n) = -\infty. \tag{2.6}$$

Assuming that the set of prime numbers is the spectrum for $N(\lambda) = li(\lambda)$ from (2.1) we have

$$\lim_{n\to\infty}((\pi li(\lambda_n))^2 - n^2\pi^2) = \int_0^1 q(t)dt,$$

which contradicts (2.5) and (2.6). This completes the proof. □

Acknowledgements. The authors would like to thank Alfonso Castro and the referees for their valuable comments and remarks which led to improvements of this article.

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