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SYMMETRY ANALYSIS, BIFURCATION AND EXACT SOLUTIONS OF NONLINEAR WAVE EQUATION IN SEMICONDUCTORS WITH STRONG SPATIAL DISPERSION

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Abstract. Based on Lie symmetry analysis and steady bifurcation method, we study the nonlinear wave equation in semiconductors with strong spatial dispersion. The similarity reductions and exact solutions are obtained based on the optimal system and power series method. Then, steady bifurcation and solitary waves are presented. Especially, the existence and solvability of solitary or period wave are discussed, and all kinds of the solitary and period wave solutions are given by direct integration.

1. Introduction

The celebrated nonlinear wave model called Korteweg de-Vries (KdV) equation [12] is

$$u_t + uu_x + u_{xxx} = 0. \tag{1.1}$$

This equation can be used in shallow water waves models, and a lot of other fields such as fluid mechanics, optical fibers, electromagnetic waves, acoustic waves in plasmas and so on [5]. Several noticeable attempts to improve the KdV model were taken over the years. In 2008, Al'shin et al [1] give the following nonlinear wave equation, known as the improved KdV equation, and is based on the theory of electromagnetism.

$$(u - u_{xx})_t + uu_x + u_{xxx} = 0, \tag{1.2}$$

which describes waves in semiconductors with strong spatial dispersion. This equation can be derived from the nonstationary processes in semiconductors that are described by systems consisting of stationary field equation, continuity equation and constitutive equation. 1-soliton solution and conservation laws of (1.2) are given by Anjan Biswas and Kara[3].

The main purpose of this work is to study the symmetry analysis, bifurcation and exact solutions of (1.2). The similarity reductions and exact solutions are obtained based on the optimal system and power series method. Steady bifurcation and solitary waves are presented based on the ideas in [6, 7, 8, 9, 10, 11, 13]. Our result may be of great interest for both mathematician and physicist.

The rest of this article is organized as follows. In Section 2, the vector fields and the optimal systems are obtained by employing Lie symmetry analysis method. In

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Section 3, the similarity reductions and exact solutions are obtained. In Section 4, steady bifurcation are analyzed by selecting the integration constant as the bifurcation control parameter, then the existence and solvability of solitary or period wave are discussed. Conclusions and remarks are presented at the end of the paper.

2. Lie symmetry analysis of (1.2)

In this section, we perform Lie symmetry analysis of (1.2). Lie symmetry analysis method is described in many books, e.g. [14, 4]. First of all, let us consider a one-parameter group of infinitesimal transformation:

$$\begin{aligned} \bar{t} &= t + \epsilon\tau(x,t,u) + O(\epsilon^2), \\ \overline{x} &= x + \epsilon\xi(x,t,u) + O(\epsilon^2), \\ \overline{u} &= u + \epsilon\eta(x,t,u) + O(\epsilon^2), \end{aligned} \tag{2.1}$$

where ϵ is a group parameter. The vector field associated with the above group of transformations can be written as

$$V = \tau(x,t,u)\frac{\partial}{\partial t} + \xi(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{2.2}$$

Applying the third prolongation $Pr^{(3)}V$ to (1.2), we find that the coefficient functions $\tau(x,t,u), \xi(x,t,u)$ and $\eta(x,t,u)$ must satisfy the invariant condition

$$\eta^t + u\eta^x + u_x\eta + \eta^{xxx} - \eta^{xxt} = 0, \tag{2.3}$$

where

$$\begin{aligned} \eta^t &= D_t(\eta) - u_tD_t(\tau) - u_xD_t(\xi), \\ \eta^x &= D_x(\eta) - u_tD_x(\tau) - u_xD_x(\xi), \\ \eta^{xxx} &= D_x(\eta^{xx}) - u_{xxt}D_x(\tau) - u_{xxx}D_x(\xi), \\ \eta^{xxt} &= D_t(\eta^{xx}) - u_{xtt}D_t(\tau) - u_{xxt}D_t(\xi). \end{aligned} \tag{2.4}$$

Here, D_x, D_t denote the total derivative operators with respect to x and t, respectively.

Substituting (2.4) in the invariant condition (2.3), one can get

$$\xi = -c_1t + c_3, \quad \tau = c_1t + c_2, \quad \eta = -c_1 - c_1u, \tag{2.5}$$

where c_1, c_2 and c_3 are arbitrary constants.

Hence the Lie algebra of infinitesimal symmetries of (1.2) is spanned by the vector fields

$$V_1 = -t\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - (1+u)\frac{\partial}{\partial u}, \quad V_2 = \frac{\partial}{\partial t}, \quad V_3 = \frac{\partial}{\partial x}. \tag{2.6}$$

Then, all of the infinitesimal generators of (1.2) can be expressed as

$$V = c_1V_1 + c_2V_2 + c_3V_3. \tag{2.7}$$

The commutation relations of Lie algebra determined by V_1, V_2, V_3, are shown in Table 1. It is obvious that $\{V_1, V_2, V_3\}$ is commute under the Lie bracket.

To compute the adjoint representation, we use the commutation Table 1 and following Lie series

$$Ad(\exp(\varepsilon V_i))V_j = V_j - \varepsilon[V_i, V_j] + \frac{1}{2}\varepsilon^2[V_i, [V_i, V_j]] + \cdots, \tag{2.8}$$

TABLE 1. Commutation table of Lie algebra

$[V_i, V_j]$	V_1	V_2	V_3
V_1	0	$V_3 - V_2$	0
V_2	$V_2 - V_3$	0	0
V_3	0	0	0

we have the following results

$$\begin{gathered} Ad(\exp(\varepsilon V_i))V_i = V_i, \quad i = 1, 2, 3 \\ Ad(\exp(\varepsilon V_1))V_2 = (1+\varepsilon)V_2 - \varepsilon V_3, \quad Ad(\exp(\varepsilon V_1))V_3 = V_3, \\ Ad(\exp(\varepsilon V_2))V_1 = V_1 - \varepsilon(V_2 - V_3), \quad Ad(\exp(\varepsilon V_2))V_3 = V_3, \\ Ad(\exp(\varepsilon V_3))V_1 = V_1, \quad Ad(\exp(\varepsilon V_3))V_2 = V_2. \end{gathered} \tag{2.9}$$

Based on the adjoint representations of the vector fields, we obtain the optimal systems of the (1.2) as follows:

$$\{V_1 + V_2,\ V_1 + V_3\} \tag{2.10}$$

3. Symmetry reductions and power series solutions

In the previous section, we obtained the vector fields and the optimal system of (1.2). In this section, we will deal with the symmetry reductions and exact solutions based on the optimal system and power series method.

3.1. **Generator** $V_1 + V_2$**.** The similarity variables are $\xi = (t+1)e^{-t-x}, f(\xi) = (t+1)u + t$, and the group-invariant solution is $u = \frac{f(\xi)-t}{t+1}$. Substituting the group-invariant solution in (1.2), we obtain the following reduction equation

$$1 - f - \xi f f' - \xi^3 f''' - 2\xi^2 f'' = 0, \tag{3.1}$$

where $f' = \frac{df}{d\xi}$. we can not get exact solutions of reduction equation (3.1) by using the elementary functions or some already well known mathematical functions, but we know that the power series can be used to deal with differential equations, including many complicated nonlinear differential equations with non-constant coefficients[2]. Next, we will consider the exact analytic solution of the reduction equation (3.1) by using the power series method.

We seek a solution of (3.1) in the power series of the form

$$f(\xi) = \sum_{n=0}^{\infty} c_n \xi^n. \tag{3.2}$$

Substituting (3.2) in (3.1), we have

$$\begin{aligned} &1 - c_0 - (c_1 + c_0 c_1)\xi - (c_2 + 2c_0 + c_1^2 + 4c_1)\xi^2 - \sum_{n=0}^{\infty} c_{n+3}\xi^{n+3} \\ &- \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n+2}(n+3-k)c_k c_{n-k+3}\Big)\xi^{n+3} - \sum_{n=0}^{\infty}(n+1)(n+2)(n+3)c_{n+3}\xi^{n+3} \\ &- 2\sum_{n=0}^{\infty}(n+2)(n+3)c_{n+3}\xi^{n+3} = 0. \end{aligned} \tag{3.3}$$

Through comparing coefficients of ξ, ξ^2 and a constant, we have

$$c_0 = 1, \quad c_1 = 0, \quad c_2 = -2. \tag{3.4}$$

When $n \geq 0$, we have

$$c_{n+3} = \frac{-1}{1 + 2(n+2)(n+3) + (n+1)(n+2)(n+3)} \Big[\sum_{k=0}^{n+2} (n+3-k) c_k c_{n-k+3} \Big].$$

Therefore, the power series solution for (3.1) can be written as

$$\begin{aligned} f(\xi) = 1 - 2\xi^2 - \sum_{n=0}^{\infty} & \frac{1}{1 + 2(n+2)(n+3) + (n+1)(n+2)(n+3)} \\ & \times \Big(\sum_{k=0}^{n+2} (n+3-k) c_k c_{n-k+3} \Big) \xi^{n+3}. \end{aligned} \tag{3.5}$$

3.2. **Generator $V_1 + V_3$.** The similarity variables are $\xi = \ln(t) - x - t, f(\xi) = (1+u)t$, and the group-invariant solution is $u = \frac{f(\xi)}{t} - 1$. Substituting the group-invariant solution in (1.2), we obtain the reduction equation

$$f - f' + ff' - f'' + f''' = 0. \tag{3.6}$$

where $f' = \frac{df}{d\xi}$. Similarly, we will consider the exact analytic solutions of the reduction equation (3.6) by using the power series method.

We seek a solution of (3.6) in the power series of the form

$$f(\xi) = \sum_{n=0}^{\infty} c_n \xi^n. \tag{3.7}$$

Substituting (3.7) into (3.6), we have

$$\begin{aligned} & \sum_{n=0}^{\infty} c_n \xi^n - \sum_{n=0}^{\infty} (n+1) c_{n+1} \xi^n + \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} (n-k+1) c_k c_{n-k+1} \Big) \xi^n \\ & - \sum_{n=0}^{\infty} (n+1)(n+2) c_{n+2} \xi^n + \sum_{n=0}^{\infty} (n+1)(n+2)(n+3) c_{n+3} \xi^n = 0. \end{aligned} \tag{3.8}$$

Through comparing the coefficients of $\xi^i (i = 0, 1, 2, \dots)$, we have

$$\begin{aligned} c_{n+3} = \frac{1}{(n+1)(n+2)(n+3)} \Big(& (n+1) c_{n+1} + (n+1)(n+2) c_{n+2} - c_n \\ & - \sum_{k=0}^{n} (n-k+1) c_k c_{n-k+1} \Big). \end{aligned} \tag{3.9}$$

Therefore, the power series solution for Eq.(3.6) can be written as follows

$$\begin{aligned} f(\xi) = \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)(n+3)} \Big(& (n+1) c_{n+1} + (n+1)(n+2) c_{n+2} - c_n \\ & - \sum_{k=0}^{n} (n-k+1) c_k c_{n-k+1} \Big) \xi^n. \end{aligned} \tag{3.10}$$

4. Bifurcation and solitary waves of (1.2)

In this section, we consider the linear combination of generators $V_2 + cV_3$, gives rise to the traveling wave solutions, and study the bifurcation of the two-dimensional dynamic system which satisfied the traveling wave transformation, then existence and solvability of solitary or period wave of Eq.(1.2) by analyzing the homoclinic orbits or period orbits of two-dimensional dynamic system based on the bifurcation diagram.

The linear combination of generators $V_2 + cV_3$ leads to the group-invariant solution $u(x, t) = u(\xi)$, where $\xi = x - ct$ is the similarity variables of $V_2 + cV_3$. Substituting the group-invariant solution $u(x, t) = u(\xi)$ into (1.2), then (1.2) reduces to an ordinary differential equation as follows

$$-cU' + UU' + (1 + c)U''' = 0, \tag{4.1}$$

where $U' = dU/d\xi$. Integrating (4.1) once with respect to ξ, and taking integration constant equal to h, one obtains

$$-cU + \frac{U^2}{2} + (1 + c)U'' = h, \tag{4.2}$$

or equivalent to the following two-dimensional dynamic system

$$\begin{aligned} \frac{dU}{d\xi} &= \frac{1}{1+c}V \\ \frac{dV}{d\xi} &= h + cU - \frac{U^2}{2}, \end{aligned} \tag{4.3}$$

which has the energy integral

$$E = \frac{1}{2(1+c)}V^2 + P(U), \tag{4.4}$$

where $P(U) = \frac{U^3}{6} - \frac{cU^2}{2} - hU$.

We select the integration constant h as the bifurcation control parameter. System (4.3) has equilibrium points $(U_1, 0)$ and $(U_2, 0)$, and the bifurcation point is $(U_b, -\frac{c^2}{2})$, where $U_1 = c - \sqrt{c^2 + 2h}$, $U_2 = c + \sqrt{c^2 + 2h}$ and $U_b = c$. Since the characteristic equation for the equilibrium points $(U_i, 0)(i = 1, 2)$ is $\lambda^2 - \frac{1}{1+c}(c - U_i) = 0$, we know that the point $(U_1, 0)$ is unstable saddle , and the points $(U_2, 0)$ is stable center. According to the above analysis, we see that the bifurcation diagram of system (4.3) is the parabolic shape curve in Figure 1.

Next, we discuss the existence and solvability of solitary waves of (1.2). The solitary wave of (1.2) fact correspond to the homoclinic orbits of two-dimensional dynamic system (4.3) passing through the saddle $(U_1, 0)$. The period wave of (1.2) are in fact corresponding to the period orbits of two-dimensional dynamic system (4.3) passing through the center $(U_2, 0)$. We should discuss two cases:

Case 1. $h > -c^2/2$**.** When $E = P(U_1)$, from (4.3) and (4.4), we obtain

$$\frac{dU}{d\xi} = \pm\sqrt{\frac{2}{1+c}G_1(U)}, \tag{4.5}$$

where $G_1(U) = P(U_1) - P(U) \geq 0$ in some range of U, which depends on the values of the bifurcation control parameter, and we have

$$G_1'(U_1) = -P'(U) = -\frac{1}{2}(U - U_1)(U - U_2), \tag{4.6}$$

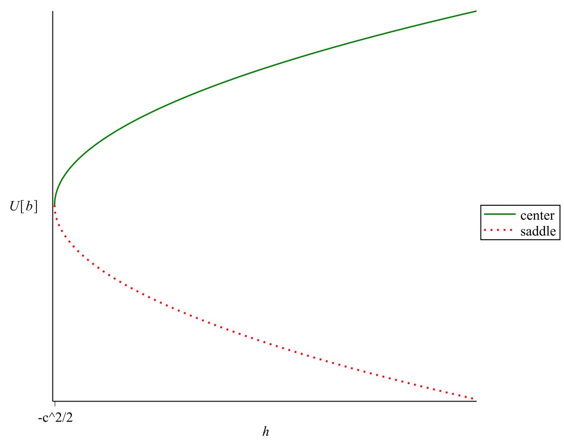

FIGURE 1. Steady bifurcation diagram of system (4.3)

According to the existence theorem of zero points of the elementary continuous function on the closed interval, it is obvious that the following results:

(i) $G_1(U_1) = 0$, $G_1'(U_1) = 0$, i.e., $G_1(U)$ has the double root U_1;
(ii) $\lim_{U\to-\infty} G_1(U) = +\infty$ and $\lim_{U\to+\infty} G_1(U) = -\infty$ i.e., $G_1(U)$ has a single root $U_1^* > U_2$.

Therefore, $G_1(U) = \frac{1}{6}(U - U_1)^2(U_1^* - U)$, and substituting it into (4.5), we have

$$\frac{dU}{d\xi} = \pm|U - U_1|\sqrt{\frac{1}{3(1+c)}(U_1^* - U)}. \tag{4.7}$$

Integrating (4.7), we obtain the solitary wave solution

$$u(x,t) = (U_1 - U_1^*)\tanh\Big(\frac{1}{6}\sqrt{\frac{3(U_1^* - U_1)}{1+c}}(x - ct - \xi_0)\Big)^2 + U_1^* \tag{4.8}$$

where ξ_0 is determined by the initial value $u_0 = u(x_0, 0)$. It holds that $u \to U_1$ as $x \to \pm\infty$.

When $E = P(U_2)$, from (4.3) and (4.4), we obtain

$$\frac{dU}{d\xi} = \pm\sqrt{\frac{2}{1+c}G_2(U)}, \tag{4.9}$$

where $G_2(U) = P(U_2) - P(U) \geq 0$ in some range of U, which depends on the values of the bifurcation control parameter, and we have

$$G_2'(U_2) = -P'(U) = -\frac{1}{2}(U - U_1)(U - U_2), \tag{4.10}$$

According to the existence theorem of zero points of the elementary continuous function on the closed interval, it is obvious that the following results:

(i) $G_2(U_2) = 0$, $G_2'(U_2) = 0$, i.e., $G_2(U)$ has the double root U_2;
(ii) $\lim_{U\to-\infty} G_2(U) = +\infty$ and $\lim_{U\to+\infty} G_2(U) = -\infty$, i.e., $G_2(U)$ has a single root $U_2^* < U_1$.

Therefore, $G_2(U) = -\frac{1}{6}(U-U_2)^2(U-U_2^*)$, and substituting it into (4.9), we have

$$\frac{dU}{d\xi} = \pm|U-U_2|\sqrt{\frac{1}{3(1+c)}(U_2^*-U)}. \tag{4.11}$$

Integrating (4.11), we obtain the periodic wave solution

$$u(x,t) = (U_2^*-U_2)\tan\Big(\frac{1}{6}\sqrt{\frac{3(U_2-U_2^*)}{1+c}}(x-ct-\xi_0)\Big)^2 + U_2^* \tag{4.12}$$

Case 2. $h = -c^2/2$**.** In this case, the energy $E = P(U_b) = c^3/6$, from (4.3) and (4.4), we obtain

$$\frac{dU}{d\xi} = \pm\sqrt{\frac{2}{1+c}G_b(U)}, \tag{4.13}$$

where $G_b(U) = P(U_b) - P(U) \geq 0$ in some range of U, and we have

$$G_b(U) = G_b(U) = \frac{1}{6}(c-U)^3 \tag{4.14}$$

and substituting it in (4.13), we have

$$\frac{dU}{d\xi} = \pm\sqrt{\frac{1}{3(1+c)}(c-U)^3}. \tag{4.15}$$

Integrating (4.15), we obtain the breaking wave solution

$$u(x,t) = \frac{c(x-ct)^2 - 12c - 12}{(x-ct)^2}. \tag{4.16}$$

Conclusions. In this article, we studied the symmetry analysis, bifurcation and exact solutions of nonlinear wave equation (1.2) based on Lie symmetry analysis and steady bifurcation method. First, Lie symmetry and optimal systems of (1.2) are well presented. Then, exact analytic solutions of reduction equation are obtained by employing the power series method. Finally, we consider the influence of integration constant on the solitary or period wave, and selected the integration constant as the bifurcation control parameter. The existence and solvability of solitary or period wave are discussed based on bifurcation diagram. The result is shown that the existence condition of solitary or period wave is $h > -\frac{c^2}{2}$, and there exist a breaking wave solution if $h = -c^2/2$. It means that there is no solitary or period wave if $h < -c^2/2$.

We remark that the convergence of the power series solution (3.5) and (3.10) can be easily proved[2], thus power series solution is an exact analytic solution of reduction equation.

Also we remark that integration constant influence the existence of solitary or period wave, so we can control the phenomenon of solitary or period wave by controlling the value of the integral constant.

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