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# SUB-SUPER SOLUTION METHOD FOR NONLOCAL SYSTEMS INVOLVING THE $p(x)$-LAPLACIAN OPERATOR

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Abstract. In this article we study the existence of solutions for nonlocal systems involving the $p(x)$-Laplacian operator. The approach is based on a new sub-super solution method.

## 1. Introduction

In this work we are interested in the nonlocal system

$$\begin{gathered} -\mathcal{A}(x,|v|_{L^{r_1(x)}})\Delta_{p_1(x)}u = f_1(x,u,v)|v|^{\alpha_1(x)}_{L^{q_1(x)}} + g_1(x,u,v)|v|^{\gamma_1(x)}_{L^{s_1(x)}} \quad \text{in } \Omega, \\ -\mathcal{A}(x,|u|_{L^{r_2(x)}})\Delta_{p_2(x)}v = f_2(x,u,v)|u|^{\alpha_2(x)}_{L^{q_2(x)}} + g_2(x,u,v)|u|^{\gamma_2(x)}_{L^{s_2(x)}} \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N (N > 1)$ with $C^2$ boundary, $|\cdot|_{L^{m(x)}}$ is the norm of the space $L^{m(x)}(\Omega)$, $-\Delta_{p(x)}u := -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplacian operator, $r_i, p_i, q_i, s_i, \alpha_i, \gamma_i : \Omega \to [0,\infty), i = 1, 2$ are measurable functions and $\mathcal{A}, f_1, f_2, g_1, g_2 : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying certain conditions.

In the previous decades there have been several works related to the $p$ and $p(x)$ Laplacian operator; see for example [1, 4, 9, 12, 25, 26, 27, 28, 29, 34, 35, 38, 39] and the references therein. Partial differential equations involving the $p(x)$-Laplacian arise in several areas of Science and Technology such as nonlinear elasticity, fluid mechanics, non-Newtonian fluids and image processing. Regarding the mentioned applications we point out [1, 14, 36, 41, 42].

The nonlocal term $|\cdot|_{L^{m(x)}}$ with the condition $p(x) = r(x) \equiv 2$ was considered in the well known Carrier's equation

$$\rho u_{tt} - a(x,t,|u|^2_{L^2})\Delta u = 0$$

which models the vibrations of a elastic string under certain contidions. See [11] for more details. We also quote the applicability of such nonlocal term in Population Dynamics, see [15, 17]. Several works related to (1.1) in the $p$-Laplacian case, that is, with $p(x) = p$ (a constant) can be found, see [10, 13, 19, 20, 23, 43] and the references provided in such manuscripts. For example Corrêa & Lopes [20] studied the system

$$-\Delta u^m = a|v|^{\alpha}_{L^p} \quad \text{in } \Omega,$$

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$$-\Delta v^n = b|u|_{L^q}^{\beta} \quad \text{in } \Omega, u = v = 0 \quad \text{on } \partial\Omega,$$

and in [13] a related system was considered using the Galerkin method.

In [19] the authors used a theorem due to Rabinowitz [40] to study the problem

$$\begin{gathered} -\Delta_{p_1} u = |v|_{L^{q_1}}^{\alpha_1} \quad \text{in } \Omega, \\ -\Delta_{p_2} v = |u|_{L^{q_2}}^{\alpha_2} \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

The system

$$\begin{gathered} -\mathcal{A}(x, |v|_{L^{r_1(x)}})\Delta u = f_1(x,u,v)|v|_{L^{q_1(x)}}^{\alpha_1(x)} + g_1(x,u,v)|v|_{L^{s_1(x)}}^{\gamma_1(x)} \quad \text{in } \Omega, \\ -\mathcal{A}(x, |u|_{L^{r_2(x)}})\Delta u = f_2(x,u,v)|u|_{L^{q_2(x)}}^{\alpha_2(x)} + g_2(x,u,v)|u|_{L^{s_2(x)}}^{\gamma_2(x)} \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where $\mathcal{A} : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a function satisfying some conditions, was considered in [43]. The approach in such paper consists in use an abstract result involving sub and supersolutions, whose proof is based on the Schaefer's fixed point theorem. Specifically, it was considered a sublinear system, a concave-convex problem and a system of logistic equations.

The scalar version of (1.1),

$$\begin{gathered} -\mathcal{A}(x, |u|_{L^{r(x)}})\Delta_{p(x)} u = f(x,u)|u|_{L^{q(x)}}^{\alpha(x)} + g(x,u)|u|_{L^{s(x)}}^{\gamma(x)} \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

was considered in [44]. The authors obtained an abstract result involving sub and super solutions for (1.1) that generalizes [43, Theorem 1]. As an application of such result the authors generalized for the $p(x)$-Laplacian operator the three applications of [43, Theorem 1].

The goal of this work is to prove [43, Theorem 2] for the $p(x)$-Laplacian operator and use it in three applications of the mentioned paper. Thus, we provide a generalization of [43] with respect to systems with variable exponents. Next we describe the main differences and difficulties of this work when compared with [43].

(i) The homogeneity of the Laplacian operator $(-\Delta, H_0^1(\Omega))$ and the eigenfunction associated to the first eigenvalue were used in [43] for constructing a subsolution. Differently from the $p$-Laplacian ($p(x) \equiv p$ constant) the $p(x)$-Lapalcian is not homogeneous. Besides that, it can occurs that the first eigenvalue and the first eigenfunction of the $p(x)$-Laplacian operator $(-\Delta_{p(x)}, W_0^{1,p(x)}(\Omega))$ do not exist. Even if the first eigenvalue and the associated eigenfunction exist the homogeneity, in general, does not allows to use the first eigenfunction to construct a subsolution. In order to avoid such difficulties we explore some arguments of [44].

(ii) Some arguments of [43] were improved and weaker conditions on $r_i, q_i, s_i, \alpha_i$, $\gamma_i, i = 1, 2$ are considered here.

(iii) We generalize [43, Theorem 2] and as an application it is considered some nonlocal problems that generalizes the three systems studied in [43].

(iv) As in [43, Theorem 2] and differently from several works that consider the nonlocal term $\mathcal{A}(x, |u|_{L^{r(x)}})$ satisfying $\mathcal{A}(x,t) \geq a_0 > 0$ (where $a_0$ is a constant), Theorem 1.1 permits us to study (1.1) in the mentioned case and in situations where $\mathcal{A}(x, 0) = 0$.

(v) The abstract result involving sub and super solutions is proved by using a different argument. It is used a theorem due to Rabinowitz that can be found in [40] and some arguments of [43] are improved.

In this work we assume that $r_i, p_i, q_i, s_i, \alpha_i, \gamma_i$ satisfy

(H1) $p_i \in C^1(\overline{\Omega}), r_i, q_i, s_i \in L^\infty_+(\Omega)$, where
$$L^\infty_+(\Omega) = \big\{m \in L^\infty(\Omega) \text{ with} \operatorname{ess\,inf} m(x) \geq 1\big\}$$
and for $i = 1, 2$, $\alpha_i, \gamma_i \in L^\infty(\Omega)$ and satisfy
$$1 < p_i^- := \inf_\Omega p_i(x) \leq p_i^+ := \sup_\Omega p_i(x) < N, \quad \alpha_i(x), \gamma_i(x) \geq 0 \quad \text{a.e in } \Omega\,.$$

Some definitions are needed to present the main results. We say that the pair $(u_1, u_2)$ is a weak solution of (1.1), if $u_i \in W_0^{1,p_i(x)}(\Omega) \cap L^\infty(\Omega)$ and
$$\int_\Omega |\nabla u_i|^{p_i(x)-2} \nabla u_i \nabla \varphi = \int_\Omega \Big( \frac{f_i(x, u_1, u_2)|u_j|^{\alpha_i(x)}_{L^{q_i(x)}}}{\mathcal{A}(x, |u_j|_{L^{r_i(x)}})} + \frac{g_i(x, u_1, u_2)|u_j|^{\gamma_i(x)}_{L^{s_i(x)}}}{\mathcal{A}(x, |u_j|_{L^{r_i(x)}})} \Big)\varphi,$$
for all $\varphi \in W_0^{1,p_i(x)}(\Omega)$ and $i \neq j$ with $i, j = 1, 2$. Given $u, v \in \mathcal{S}(\Omega)$ we write $u \leq v$ if $u(x) \leq v(x)$ a.e. in $\Omega$. If $u \leq v$ we define
$$[u, v] := \big\{w \in \mathcal{S}(\Omega) : u(x) \leq w(x) \leq v(x) \text{ a.e. in } \Omega\big\}.$$
To simplify the next definition we denote
$$\widetilde{f}_1(x, t, s) = f_1(x, t, s), \quad \widetilde{g}_1(x, t, s) = g_1(x, t, s),$$
$$\widetilde{f}_2(x, t, s) = f_2(x, s, t), \quad \widetilde{g}_2(x, t, s) = g_2(x, s, t).$$
We say that the pairs $(\underline{u}_i, \overline{u}_i), i = 1, 2$ are a sub-super solutions for (1.1) if $\underline{u}_i \in W_0^{1,p_i(x)}(\Omega) \cap L^\infty(\Omega)$, $\overline{u}_i \in W^{1,p_i(x)}(\Omega) \cap L^\infty(\Omega)$ with $\underline{u}_i \leq \overline{u}_i$, $\underline{u}_i = 0 \leq \overline{u}_i$ on $\partial\Omega$ and for all $\varphi \in W_0^{1,p_i(x)}(\Omega)$ with $\varphi \geq 0$ the following inequalities hold
$$\int_\Omega |\nabla \underline{u}_i|^{p_i(x)-2} \nabla \underline{u}_i \nabla \varphi \leq \int_\Omega \Big( \frac{\widetilde{f}_i(x, \underline{u}_i, w)|\underline{u}_j|^{\alpha_i(x)}_{L^{q_i(x)}}}{\mathcal{A}(x, |w|_{L^{r_i(x)}})} + \frac{\widetilde{g}_i(x, \underline{u}_i, w)|\underline{u}_j|^{\gamma_i(x)}_{L^{s_i(x)}}}{\mathcal{A}(x, |w|_{L^{r_i(x)}})} \Big)\varphi,$$
$$\int_\Omega |\nabla \overline{u}_i|^{p_i(x)-2} \nabla \overline{u}_i \nabla \varphi \geq \int_\Omega \Big( \frac{\widetilde{f}_i(x, \overline{u}_i, w)|\overline{u}_j|^{\alpha_i(x)}_{L^{q_i(x)}}}{\mathcal{A}(x, |w|_{L^{r_i(x)}})} + \frac{\widetilde{g}_i(x, \overline{u}_i, w)|\overline{u}_j|^{\gamma_i(x)}_{L^{s_i(x)}}}{\mathcal{A}(x, |w|_{L^{r_i(x)}})} \Big)\varphi, \tag{1.3}$$
for all $w \in [\underline{u}_j \overline{u}_j]$ where $i, j = 1, 2$ with $i \neq j$. Our main result reads as follows.

**Theorem 1.1.** *Suppose that $r_i, p_i, q_i, s_i, \alpha_i$ and $\gamma_i$ satisfy* (H1), *that $(\underline{u}_i, \overline{u}_i)$ is a sub-super solution for* (1.1) *with $\underline{u}_i > 0$ a.e. in $\Omega$, that $f_i(x, t, s), g_i(x, t, s) \geq 0$ in $\overline{\Omega} \times [0, |\overline{u}_1|_{L^\infty}] \times [0, |\overline{u}_2|_{L^\infty}]$ and that $\mathcal{A} : \overline{\Omega} \times (0, \infty) \to \mathbb{R}$ is a continuous function with $\mathcal{A}(x, t) > 0$ in $\overline{\Omega} \times \big[\underline{\sigma}, \overline{\sigma}\big]$, where $\underline{\sigma} := \min\big\{|\underline{w}|_{L^{r_i(x)}}, i = 1, 2\big\}$, $\overline{\sigma} := \max\big\{|\overline{w}|_{L^{r_i(x)}}, i = 1, 2\big\}$, $\underline{w} := \min\{\underline{u}_i, i = 1, 2\}$ and $\overline{w} := \max\{\overline{u}_i, i = 1, 2\}$. Then* (1.1) *has a weak positive solution $(u_1, u_2)$ with $u_i \in [\underline{u}_i, \overline{u}_i]$, $i = 1, 2$.*

## 2. Preliminaries

In this section, we present some facts regarding the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ that will be often used in this work. For more details see Fan-Zhang [27] and the references therein.

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain. Given $p \in L^\infty_+(\Omega)$, we define the generalized Lebesgue space

$$L^{p(x)}(\Omega) = \{u \in \mathcal{S}(\Omega) : \int_\Omega |u(x)|^{p(x)} dx < \infty\},$$

where $\mathcal{S}(\Omega) := \{u : \Omega \to \mathbb{R} : u \text{ is measurable}\}$. Then $L^{p(x)}(\Omega)$ is a Banach space with the norm

$$|u|_{p(x)} := \inf \{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \leq 1\}.$$

Given $m \in L^\infty(\Omega)$, we define

$$m^+ := \operatorname{ess\,sup}_\Omega m(x), \quad m^- := \operatorname{ess\,inf}_\Omega m(x).$$

**Proposition 2.1.** *Let $\rho(u) := \int_\Omega |u|^{p(x)} dx$. Then for $u, u_n \in L^{p(x)}(\Omega)$, and $n \in \mathbb{N}$, the following assertions hold*

- (i) *Let $u \neq 0$ in $L^{p(x)}(\Omega)$, then $|u|_{L^{p(x)}} = \lambda \Leftrightarrow \rho(\frac{u}{\lambda}) = 1$.*
- (ii) *If $|u|_{L^{p(x)}} < 1$ $(= 1, > 1)$, then $\rho(u) < 1$ $(= 1, > 1)$.*
- (iii) *If $|u|_{L^{p(x)}} > 1$, then $|u|^{p^-}_{L^{p(x)}} \leq \rho(u) \leq |u|^{p^+}_{L^{p(x)}}$.*
- (iv) *If $|u|_{L^{p(x)}} < 1$, then $|u|^{p^+}_{L^{p(x)}} \leq \rho(u) \leq |u|^{p^-}_{L^{p(x)}}$.*
- (v) *$|u_n|_{L^{p(x)}} \to 0 \Leftrightarrow \rho(u_n) \to 0$, and $|u_n|_{L^{p(x)}} \to \infty \Leftrightarrow \rho(u_n) \to \infty$.*

**Theorem 2.2.** *Let $p, q \in L^\infty_+(\Omega)$. Then the following statements hold*

- (i) *If $p^- > 1$ and $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$ a.e. in $\Omega$, then*
$$\Big|\int_\Omega uv dx\Big| \leq \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u|_{L^{p(x)}}|v|_{L^{q(x)}}.$$
- (ii) *If $q(x) \leq p(x)$ a.e. in $\Omega$ and $|\Omega| < \infty$, then $L^{p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.*

We define the generalized Sobolev space as

$$W^{1,p(x)}(\Omega) := \{u \in L^{p(x)}(\Omega) : \frac{\partial u}{\partial x_j} \in L^{p(x)}(\Omega), j = 1, \dots, N\}$$

with the norm

$$\|u\|_* = |u|_{L^{p(x)}} + \sum_{j=1}^N \Big|\frac{\partial u}{\partial x_j}\Big|_{L^{p(x)}}.$$

The space $W^{1,p(x)}_0(\Omega)$ is defined as the closure of $C^\infty_0(\Omega)$ with respect to the norm $\|\cdot\|_*$.

**Theorem 2.3.** *If $p^- > 1$, then $W^{1,p(x)}(\Omega)$ is a Banach, separable and reflexive space.*

**Proposition 2.4.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $p, q \in C(\overline{\Omega})$. Define the function $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ and $p^*(x) = \infty$ if $N \geq p(x)$. Then the following statements hold.*

- (i) *(Poincaré inequality) If $p^- > 1$, then there is a constant $C > 0$ such that $|u|_{L^{p(x)}} \leq C|\nabla u|_{L^{p(x)}}$ for all $u \in W^{1,p(x)}_0(\Omega)$.*
- (ii) *If $p^-, q^- > 1$ and $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact.*

From (i) of Proposition 2.4, we have that $\|u\| := |\nabla u|_{L^{p(x)}}$ defines a norm in $W^{1,p(x)}_0(\Omega)$ which is equivalent to the norm $\|\cdot\|_*$.

**Definition 2.5.** For $u, v \in W^{1,p(x)}(\Omega)$, we say that $-\Delta_{p(x)} u \le -\Delta_{p(x)} v$, if
$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla\varphi \le \int_\Omega |\nabla v|^{p(x)-2}\nabla v\nabla\varphi,$$
for all $\varphi \in W_0^{1,p(x)}(\Omega)$ with $\varphi \ge 0$.

The following result appears in [29, Lemma 2.2] and [26, Proposition 2.3].

**Proposition 2.6.** *Let $u, v \in W^{1,p(x)}(\Omega)$. If $-\Delta_{p(x)} u \le -\Delta_{p(x)} v$ and $u \le v$ on $\partial\Omega$, (i.e., $(u-v)^+ \in W_0^{1,p(x)}(\Omega)$) then $u \le v$ in $\Omega$. If $u, v \in C(\overline{\Omega})$ and $S = \{x \in \Omega : u(x) = v(x)\}$ is a compact set of $\Omega$, then $S = \emptyset$.*

**Lemma 2.7** ([26, Lemma 2.1])**.** *Let $\lambda > 0$ be the unique solution of the problem*
$$\begin{gathered} -\Delta_{p(x)} z_\lambda = \lambda \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.1}$$
*Define $\rho_0 = \frac{p^-}{2|\Omega|^{\frac{1}{N}} C_0}$. If $\lambda \ge \rho_0$ then $|z_\lambda|_{L^\infty} \le C^* \lambda^{\frac{1}{p^- -1}}$, and $|z_\lambda|_{L^\infty} \le C_* \lambda^{\frac{1}{p^+ -1}}$ if $\lambda < \rho_0$. Here $C^*$ and $C_*$ are positive constants depending only on $p^+, p^-, N, |\Omega|$ and $C_0$, where $C_0$ is the best constant of the embedding $W_0^{1,1}(\Omega) \hookrightarrow L^{\frac{N}{N-1}}(\Omega)$.*

Regarding the function $z_\lambda$ of the previous result, it follows from [25, Theorem 1.2] and [29, Theorem 1] that $z_\lambda \in C^1(\overline{\Omega})$ with $z_\lambda > 0$ in $\Omega$. The proof of Theorem 1.1 is mainly based on the following result by Rabinowitz:

**Theorem 2.8** ([40])**.** *Let $E$ be a Banach space and $\Phi : \mathbb{R}^+ \times E \to E$ a compact map such that $\Phi(0, u) = 0$ for all $u \in E$. Then the equation*
$$u = \Phi(\lambda, u)$$
*possesses an unbounded continuum $\mathcal{C} \subset \mathbb{R}^+ \times E$ of solutions with $(0, 0) \in \mathcal{C}$.*

We point out that a mapping $\Phi : E \to E$ is compact if it is continuous and for each bounded subset $U \subset E$, the set $\overline{\Phi(U)}$ is compact.

## 3. Proof of main results

*Proof of Theorem 1.1.* For $i = 1, 2$ consider the operators $T_i : L^{p_i(x)}(\Omega) \to L^\infty(\Omega)$ defined by
$$T_i z(x) = \begin{cases} \underline{u}_i(x), & \text{if } z(x) \le \underline{u}_i(x), \\ z(x), & \text{if } \underline{u}_i(x) \le z(x) \le \overline{u}_i(x), \\ \overline{u}_i(x), & \text{if } z(x) \ge \overline{u}_i(x). \end{cases}$$
Since $T_i z \in [\underline{u}_i, \overline{u}_i]$ and $\underline{u}_i, \overline{u}_i \in L^\infty(\Omega)$ it follows that the operators $T_i$ are well-defined.

We define $p_i'(x) = p_i(x)/\big(p_i(x) - 1\big)$ and consider the operators $H_i : [\underline{u}_1, \overline{u}_1] \times [\underline{u}_2, \overline{u}_2] \to L^{p_i'(x)}(\Omega)$ given by
$$H_i(u_1, u_2)(x) = \frac{f_i(x, u_1(x), u_2(x))|u_j|^{\alpha_i(x)}_{L^{q_i(x)}}}{\mathcal{A}(x, |u_j|_{L^{r_i(x)}})} + \frac{g_i(x, u_1(x), u_2(x))|u_j|^{\gamma_i(x)}_{L^{s_i(x)}}}{\mathcal{A}(x, |u_j|_{L^{r_i(x)}})}$$
where $i \ne j$ with $i, j = 1, 2$, and $|\cdot|_{L^{m(x)}}$ denotes the norm of the space $L^{m(x)}(\Omega)$.

We consider in the space $L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega)$ with the norm
$$|(u, v)|_{1,2} = |u|_{L^{p_1(x)}} + |v|_{L^{p_2(x)}}.$$

Since $f_i, g_i, \mathcal{A}$ are continuous functions, $\mathcal{A}(x,t) > 0$ in the compact set $\overline{\Omega} \times [\underline{\sigma}, \overline{\sigma}]$, $T_i z_i \in [\underline{u}_i, \overline{u}_i]$ for all $z_i \in L^{p_i(x)}(\Omega)$, $\underline{u}_i, \overline{u}_i \in L^\infty(\Omega)$, and $|w|^{\theta(x)}_{L^{m(x)}} \leq |w|^{\theta^-}_{L^{m(x)}} + |w|^{\theta^+}_{L^{m(x)}}$ for all $w \in L^{m(x)}(\Omega)$ with $\theta \in L^\infty(\Omega)$, it follows that there are constants $K_i > 0$ such that

$$|H_i(T_1 z_1, T_2 z_2)| \leq K_i \tag{3.1}$$

for all $(z_1, z_2) \in L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega)$.

By the Lebesgue Dominated Convergence Theorem, the mappings $(z_1, z_2) \mapsto H_i(T_1 z_1, T_2 z_2)$ are continuous from $L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega)$ in $L^{p_i'(x)}(\Omega)$, $i = 1, 2$.

From [27, Theorem 4.1] the operator $\Phi : \mathbb{R}^+ \times L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega) \to L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega)$ given by

$$\Phi(\lambda, z_1, z_2) = (u_1, u_2),$$

where $(u_1, u_2) \in W_0^{1,p_1(x)}(\Omega) \times W_0^{1,p_2(x)}(\Omega)$ is the unique solution of

$$\begin{gathered} -\Delta_{p_1(x)} u_1 = \lambda H_1(T_1 z_1, T_2 z_2) \quad \text{in } \Omega, \\ -\Delta_{p_2(x)} u_2 = \lambda H_2(T_1 z_1, T_2 z_2) \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.2}$$

is well-defined.

**Claim 1: $\Phi$ is compact.** Let $(\lambda_n, z_n^1, z_n^2) \subset \mathbb{R}^+ \times L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega)$ be a bounded sequence and consider $(u_n^1, u_n^2) = \Phi(\lambda_n, z_n^1, z_n^2)$. The definition of $\Phi$ imply that

$$\int_\Omega |\nabla u_n^i|^{p_i(x)-2} \nabla u_n \nabla \varphi = \lambda_n \int_\Omega H_i(T_1 z_n^1, T_2 z_n^2)\varphi, \quad \forall \varphi \in \ W_0^{1,p_i(x)}(\Omega),$$

where $i, j = 1, 2$ blue with $i \neq j$.

Considering the test function $\varphi = u_n^i$, the boundness of $(\lambda_n)$ and inequality (3.1), we obtain

$$\int_\Omega |\nabla u_n^i|^{p_i(x)} \leq \overline{\lambda} K_i \int_\Omega |u_n^i|$$

for all $n \in \mathbb{N}$. Here $\overline{\lambda}$ is a constant that does not depend on $n \in \mathbb{N}$.

Since $p_i^- > 1$, the embedding $L^{p_i(x)}(\Omega) \hookrightarrow L^1(\Omega)$ holds. Combining such embedding with the Poincaré inequality we obtain

$$\int_\Omega |\nabla u_n^i|^{p_i(x)} \leq C K_i \|u_n^i\|,$$

for all $n \in \mathbb{N}$. Suppose that $|\nabla u_n^i|_{L^{p_i(x)}} > 1$. Thus by Proposition 2.1 we have $\|u_n^i\|^{p^- - 1} \leq C K_i$ for all $n \in \mathbb{N}$ where $C$ is a constant that does not depend on $n$. Then we conclude that $(u_n^i)$ is bounded in $W_0^{1,p_i(x)}(\Omega)$. The reflexivity of $W_0^{1,p_i(x)}(\Omega)$ and the compact embedding $W_0^{1,p_i(x)}(\Omega) \hookrightarrow L^{p_i(x)}(\Omega)$ provides the result.

**Claim 2: $\Phi$ is continuous.** Consider a sequence $(\lambda_n, z_n^1, z_n^2)$ in $\mathbb{R}^+ \times L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega)$ converging to $(\lambda, z^1, z^2)$ in $\mathbb{R}^+ \times L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega)$. Define $(u_n^1, u_n^2) = \Phi(\lambda_n, z_n^1, z_n^2)$ and $(u^1, u^2) = \Phi(\lambda, z^1, z^2)$. Using the definition of $\Phi$ we obtain

$$\int_\Omega |\nabla u_n^i|^{p_i(x)-2} \nabla u_n^i \nabla \varphi = \lambda_n \int_\Omega H_i(T_1 z_n^1, T_2 z_n^2)\varphi, \tag{3.3}$$

$$\int_\Omega |\nabla u^i|^{p_i(x)-2} \nabla u^i \nabla \varphi = \lambda \int_\Omega H_i(T_1 z^1, T_2 z^2)\varphi \tag{3.4}$$

for all $\varphi \in W_0^{1,p_i(x)}(\Omega)$ where $i,j=1,2$ and $i \neq j$.

Considering $\varphi = (u_n^i - u^i)$ in (3.3) and (3.4) and subtracting (3.4) from (3.3) we obtain

$$\begin{aligned}
&\int_\Omega \big\langle |\nabla u_n^i|^{p_i(x)-2}\nabla u_n^i - |\nabla u^i|^{p_i(x)-2}\nabla u^i, \nabla(u_n^i - u^i)\big\rangle \\
&= \int_\Omega \lambda_n H(T_1 z_n^1, T_2 z_n^2)(u_n^i - u^i) - \int_\Omega \lambda H(T_1 z^1, T_2 z^2)\big](u_n^i - u^i).
\end{aligned}$$

Using Hölder's inequality we have

$$\begin{aligned}
&\big|\int_\Omega \big\langle |\nabla u_n^i|^{p_i(x)-2}\nabla u_n^i - |\nabla u|^{p_i(x)-2}\nabla u^i, \nabla(u_n^i - u)\big\rangle\big| \\
&\leq |u_n^i - u^i|_{p_i(x)} |\lambda_n H_i(T_1 z_n^1, T_2 z_n^2) - \lambda H_i(T_1 z^1, T_2 z^2)|_{p_i'(x)}
\end{aligned}$$

The arguments above ensures that $(u_n^i)$ is bounded in $W_0^{1,p_i(x)}(\Omega)$. Since $\lambda_n \to \lambda$ and $H_i(T_1 z_n^1, T_2 z_n^2) \to H_i(T_1 z^1, T_2 z^2)$ in $L^{p_i'(x)}(\Omega)$ for $i=1,2$ we have

$$\big|\int_\Omega \big\langle |\nabla u_n^i|^{p_i(x)-2}\nabla u_n^i - |\nabla u|^{p_i(x)-2}\nabla u^i, \nabla(u_n^i - u)\big\rangle\big| \to 0.$$

Therefore $u_n^i \to u^i$ in $L^{p_i(x)}(\Omega)$ for $i=1,2$ which proves the continuity of $\Phi$.

Combining the fact that $\Phi(0,z_1,z_2) = (0,0,0)$ for all $(z_1,z_2) \in L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega)$ with the previous claims we have by Theorem 2.8 that the equation $\Phi(\lambda,u,v) = (u,v)$ possesses an unbounded continuum $\mathcal{C} \subset \mathbb{R}^+ \times L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega)$ of solutions with $(0,0,0) \in \mathcal{C}$.

**Claim 3: $\mathcal{C}$ is bounded with respect to the parameter $\lambda$.** Suppose that there exists $\lambda^* > 0$ such that $\lambda \leq \lambda^*$ for all $(\lambda,u^1,u^2) \in \mathcal{C}$. For $(\lambda,u^1,u^2) \in \mathcal{C}$ the definition of $\Phi$ imply that

$$\begin{gathered}
-\Delta_{p_1(x)} u_1 = \lambda H_1(T_1 u_1, T_2 u_2) \quad \text{in } \Omega, \\
-\Delta_{p_2(x)} u_2 = \lambda H_2(T_1 u_1, T_2 u_2) \quad \text{in } \Omega, \\
u_1 = u_2 = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{3.5}$$

Using the test function $u_i$ in (3.5) and considering (3.1) we obtain

$$\int_\Omega |\nabla u_i|^{p_i(x)} \leq \lambda^* C |u_i|_{L^{p(x)}}.$$

Suppose that $|\nabla u_i|_{L^{p(x)}} > 1$. Then using Proposition 2.1 and the Poincaré inequality we obtain that

$$|u_i|_{L^{p_i(x)}}^{p_i-1} \leq \lambda^* C.$$

Thus $\mathcal{C}$ is bounded in $\mathbb{R}^+ \times L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega)$, which is a contradiction.

Considering $\lambda = 1$, by (3.5) we have

$$\begin{aligned}
\int_\Omega |\nabla u_i|^{p_i(x)-2}\nabla u_i \nabla\varphi &= \int_\Omega \Big(\frac{f_i(x,T_1u_1,T_2u_2)|T_j u_j|_{L^{q_i(x)}}^{\alpha_i(x)}}{\mathcal{A}(x,|T_j u_j|_{L^{r_i(x)}})}\Big)\varphi \\
&\quad + \int_\Omega \Big(\frac{g_i(x,T_1u_1,T_2u_2)|T_j u_j|_{L^{s_i(x)}}^{\gamma_i(x)}}{\mathcal{A}(x,|T_j u_j|_{L^{r_i(x)}})}\Big)\varphi,
\end{aligned} \tag{3.6}$$

for all $\varphi \in W_0^{1,p_i(x)}(\Omega)$ where $i,j=1,2$ with $i \neq j$.

Now we claim that $u_i \in [\underline{u}_i, \overline{u}_i]$ for $i = 1, 2$. To prove the claim we define

$$L_1(\underline{u}_1 - u_1)_+ := \int_{\{\underline{u}_1 \geq u_1\}} \big\langle |\nabla \underline{u}_1|^{p_1(x)-2}\nabla \underline{u}_1 - |\nabla u_1|^{p_1(x)-2}\nabla u_1, \nabla(\underline{u}_1 - u_1)\big\rangle.$$

Using the facts that $T_2u_2 \in [\underline{u}_2, \overline{u}_2]$, $\underline{u}_i(x) > 0$ a.e. in $\Omega$, $i = 1, j = 2$, considering $w = T_2u_2$ and $\varphi = (\underline{u}_1 - u_1)_+$ in the first inequality of (1.3) and combining with equation (3.6) we obtain

$$\begin{aligned} L_1(\underline{u}_1 - u_1)_+ \leq \int_{\{\underline{u}_1 \geq u_1\}} &\frac{f_1(x, \underline{u}_1, T_2u_2)(|\underline{u}_2|^{\alpha_1(x)}_{L^{q_1(x)}} - |T_2u_2|^{\alpha_1(x)}_{L^{q_1(x)}})}{\mathcal{A}(x, |T_2u_2|_{L^{r_1(x)}})}(\underline{u}_1 - u_1) \\ &+ \int_{\{\underline{u}_1 \geq u_1\}} \frac{g_1(x, \underline{u}_1, T_2u_2)(|\underline{u}_2|^{\gamma_1(x)}_{L^{s_1(x)}} - |T_2u_2|^{\gamma_1(x)}_{L^{s_1(x)}})}{\mathcal{A}(x, |T_2u_2|_{L^{r_1(x)}})}(\underline{u}_1 - u_1), \end{aligned}$$

which implies that

$$\int_{\{\underline{u}_1 \geq u_1\}} \big\langle |\nabla \underline{u}_1|^{p_1(x)-2}\nabla \underline{u}_1 - |\nabla u_1|^{p_1(x)-2}\nabla u_1, \nabla(\underline{u}_1 - u_1)\big\rangle \leq 0.$$

Therefore $\underline{u}_1 \leq u_1$. The same reasoning imply the other inequalities. Since $u_i \in [\underline{u}_i, \overline{u}_i]$, we have $T_iu_i = u_i$. Therefore the pair $(u_1, u_2)$ is a weak positive solution of $(S)$. □

## 4. Applications

In this section we apply Theorem 1.1 to some nonlocal problems.

4.1. **A sublinear problem:** In this section, we use Theorem 1.1 to study the nonlocal problem

$$\begin{gathered} -\mathcal{A}(x, |v|_{L^{r_1(x)}})\Delta_{p_1(x)}u = (u^{\beta_1(x)} + v^{\gamma_1(x)})|v|^{\alpha_1(x)}_{L^{q_1(x)}} \quad \text{in } \Omega, \\ -\mathcal{A}(x, |u|_{L^{r_2(x)}})\Delta_{p_2(x)}v = (u^{\beta_2(x)} + v^{\gamma_2(x)})|u|^{\alpha_2(x)}_{L^{q_2(x)}} \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.1}$$

This problem with $p_1(x) \equiv p_1(x) \equiv 2$, was considered in [43]. The result in this section generalizes [43, Theorem 6].

**Theorem 4.1.** *Suppose that $p_i, q_i, r_i, s_i, i = 1, 2$ satisfy* (H1) *and $\alpha_i, \beta_i \in L^\infty(\Omega)$, $i = 1, 2$. Assume also that*

$$\begin{gathered} 0 < \alpha_1^+ + \gamma_1^+ < p_i^- - 1, \quad 0 < \frac{\alpha_1^+}{p_2^- - 1} + \frac{\beta_1^+}{p_1^- - 1} < 1, \\ 0 < \alpha_2^+ + \gamma_2^+ < p_i^- - 1, \quad 0 < \frac{\alpha_2^+}{p_1^- - 1} + \frac{\beta_2^+}{p_2^- - 1} < 1 \end{gathered}$$

*for $i = 1, 2$. Let $a_0 > 0$ be a positive constant. Suppose that one of the following two sets of conditions holds*

$$\mathcal{A}(x, t) \geq a_0 \quad \textit{in } \overline{\Omega} \times [0, \infty), \tag{4.2}$$

*or*

$$\begin{gathered} 0 < \mathcal{A}(x, t) \leq a_0 \quad \textit{in } \overline{\Omega} \times (0, \infty) \quad \textit{and} \\ \lim_{t \to +\infty} \mathcal{A}(x, t) = a_\infty > 0 \quad \textit{uniformly in } \Omega. \end{gathered} \tag{4.3}$$

*Then* (4.1) *has a positive solution.*

*Proof.* Suppose that (4.2) holds. We will start by constructing $(\overline{u},\overline{v})$. Let $\lambda>0$ be a positive number, which will be chosen later and denote by $z_\lambda\in W_0^{1,p_1(x)}(\Omega)\cap L^\infty(\Omega)$ and $y_\lambda\in W_0^{1,p_2(x)}(\Omega)\cap L^\infty(\Omega)$ the unique solutions of (2.1) respectively.

For $\lambda>0$ sufficiently large it follows from Lemma 2.7 that there is a constant $K>1$ that does not depend on $\lambda$ such that

$$0<z_\lambda(x)\le K\lambda^{\frac{1}{p_1^- -1}}\quad\text{in }\Omega, \tag{4.4}$$

$$0<y_\lambda(x)\le K\lambda^{\frac{1}{p_2^- -1}}\quad\text{in }\Omega. \tag{4.5}$$

Since $\alpha_1^+ + \gamma_1^+ < p_2^- - 1$ and $\frac{\alpha_1^+}{p_2^- -1}+\frac{\beta_1^+}{p_1^- -1}<1$, it is possible to choose $\lambda>1$ such that (4.4), (4.5) and

$$\frac{1}{a_0}\big(K^{\beta_1^+}\lambda^{\frac{\beta_1^+}{p_1^- -1}+\frac{\alpha_1^+}{p_2^- -1}}+K^{\gamma_1^+}\lambda^{\frac{\alpha_1^+ +\gamma_1^+}{p_2^- -1}}\big)\max\{|K|_{L^{q_1(x)}}^{\alpha_1^-},|K|_{L^{q_1(x)}}^{\alpha_1^+}\}\le\lambda \tag{4.6}$$

hold. By (4.4), (4.5) and (4.6), we obtain

$$\frac{1}{a_0}(z_\lambda^{\beta_1(x)}+w^{\gamma_1(x)})|y_\lambda|_{L^{q_1(x)}}^{\alpha_1(x)}\le\lambda, w\in[0,y_\lambda].$$

Thus for $w\in[0,y_\lambda]$ we obtain

$$-\Delta_{p_1(x)}z_\lambda\ge\frac{1}{\mathcal{A}(x,|w|_{L^{r_1(x)}})}(z_\lambda^{\beta_1(x)}+w^{\gamma_1(x)})|y_\lambda|_{L^{q_1(x)}}^{\alpha_1(x)}\quad\text{in }\Omega,$$
$$z_\lambda=0\quad\text{on }\partial\Omega.$$

Considering, if necessary, a larger $\lambda>0$, the previous reasoning imply that

$$-\Delta_{p_2(x)}y_\lambda\ge\frac{1}{\mathcal{A}(x,|w|_{L^{r_2(x)}})}(w^{\beta_2(x)}+y_\lambda{}^{\gamma_2(x)})|z_\lambda|_{L^{q_2(x)}}^{\alpha_2(x)}\quad\text{in }\Omega,$$
$$y_\lambda=0\quad\text{on }\partial\Omega,$$

for all $w\in[0,z_\lambda]$.

Now we construct $(\underline{u}_i,\underline{v}_i), i=1,2$. Since $\partial\Omega$ is $C^2$, there is a constant $\delta>0$ such that $d\in C^2(\overline{\Omega_{3\delta}})$ and $|\nabla d(x)|\equiv 1$, where $d(x):=dist(x,\partial\Omega)$ and $\overline{\Omega_{3\delta}}:=\{x\in\overline{\Omega}; d(x)\le 3\delta\}$. From [34, Page 12], we have that, for $\sigma\in(0,\delta)$ sufficiently small, the function $\phi_i=\phi_i(k,\sigma)$, $i=1,2$ defined by

$$\phi_i(x)=\begin{cases}e^{kd(x)}-1 & \text{if } d(x)<\sigma,\\ e^{k\sigma}-1+\int_\sigma^{d(x)}ke^{k\sigma}\big(\frac{2\delta-t}{2\delta-\sigma}\big)^{\frac{2}{p_i^- -1}}dt & \text{if } \sigma\le d(x)<2\delta,\\ e^{k\sigma}-1+\int_\sigma^{2\delta}ke^{k\sigma}\big(\frac{2\delta-t}{2\delta-\sigma}\big)^{\frac{2}{p_i^- -1}}dt & \text{if } 2\delta\le d(x),\end{cases}$$

belongs to $C_0^1(\overline{\Omega})$, where $k>0$ is an arbitrary number and that

$$-\Delta_{p_i(x)}(\mu\phi_i)$$

$$= \begin{cases} -k(k\mu e^{kd(x)})^{p_i(x)-1}\big[(p_i(x)-1)+(d(x)+\frac{\ln k\mu}{k})\nabla p_i(x)\nabla d(x)+\frac{\Delta d(x)}{k}\big] \\ \quad \text{if } d(x)<\sigma, \\ \Big\{\frac{1}{2\delta-\sigma}\frac{2(p_i(x)-1)}{p_i^- -1}-\big(\frac{2\delta-d(x)}{2\delta-\sigma}\big)\Big[\ln k\mu e^{k\sigma}\big(\frac{2\delta-d(x)}{2\delta-\sigma}\big)^{\frac{2}{p_i^- -1}}\nabla p_i(x)\nabla d(x) \\ +\Delta d(x)\Big]\Big\}(k\mu e^{k\sigma})^{p_i(x)-1}\big(\frac{2\delta-d(x)}{2\delta-\sigma}\big)^{\frac{2(p_i(x)-1)}{p_i^- -1}-1} \\ \quad \text{if } \sigma<d(x)<2\delta, \\ 0 \quad \text{if } \ 2\delta<d(x), \end{cases}$$

for all $\mu>0$ and $i=1,2$.

Define $\mathcal{A}_\lambda := \max\big\{\mathcal{A}(x,t) : (x,t)\in\overline{\Omega}\times\big[0,\max\{|y_\lambda|_{L^{r_1(x)}}|z_\lambda|_{L^{r_2(x)}}\}\big]\big\}$. Then we have

$$a_0 \le \mathcal{A}(x,|w|_{L^{r_1(x)}}) \le \mathcal{A}_\lambda \quad \text{in } \Omega$$

for all $w\in[0,y_\lambda]$. Let $\sigma=\frac{1}{k}\ln 2$ and $\mu=e^{-ak}$ where

$$a=\frac{\min\{p_1^- -1,p_2^- -1\}}{\max\{\max_{\overline{\Omega}}|\nabla p_1|+1,\max_{\overline{\Omega}}|\nabla p_2|+1\}}.$$

Then $e^{k\sigma}=2$ and $k\mu\le 1$ if $k>0$ is sufficiently large.

Let $x\in\Omega$ with $d(x)<\sigma$. If $k>0$ is large enough we have $|\nabla d(x)|=1$ and then

$$\begin{aligned} \big|d(x)+\frac{\ln(k\mu)}{k}\big||\nabla p_1(x)||\nabla d(x)| &\le \big(|d(x)|+\frac{|\ln(k\mu)|}{k}\big)|\nabla p_1(x)| \\ &\le \Big(\sigma-\frac{\ln(k\mu)}{k}\Big)|\nabla p_1(x)| \\ &= \Big(\frac{\ln 2}{k}-\frac{\ln k}{k}\Big)|\nabla p_1(x)|+a|\nabla p_1(x)| \\ &< p_1^- -1. \end{aligned} \tag{4.7}$$

Note also that there exists a constant $A>0$, that does not depend on $k$, such that $|\Delta d(x)|<A$ for all $x\in\overline{\partial\Omega_{3\delta}}$. Using the last inequality and the expression of $-\Delta_{p_1(x)}(\mu\phi)$, we obtain $-\Delta_{p_1(x)}(\mu\phi_1)\le 0$ for $x\in\Omega$ with $d(x)<\sigma$ or $d(x)>2\delta$ for $k>0$ large enough. Therefore

$$\begin{aligned} -\Delta_{p_1(x)}(\mu\phi_1) \le 0 &\le \frac{1}{\mathcal{A}_\lambda}(\mu\phi_1)^{\beta_1(x)}|\mu\phi_2|^{\alpha_1(x)}_{L^{q_1(x)}} \\ &\le \frac{1}{\mathcal{A}_\lambda}((\mu\phi_1)^{\beta_1(x)}+w^{\gamma_1(x)})|\mu\phi_2|^{\alpha_1(x)}_{L^{q_1(x)}} \end{aligned}$$

for all $w\in L^\infty(\Omega)$ with $w\ge\mu\phi_2$ and $d(x)<\sigma$ or $2\delta<d(x)$. Using the idea in the proof of [34, estimate (3.10)] we obtain

$$\begin{aligned} -\Delta_{p_1(x)}(\mu\phi_1) &\le \tilde{C}(k\mu)^{p_1^- -1}|\ln k\mu| \\ &= \tilde{C}(k\mu)^{p_1^- -1}\big|\ln\frac{k}{e^{ak}}\big| \quad \text{if } \sigma<d(x)<2\delta. \end{aligned} \tag{4.8}$$

From the proof of [44, Theorem 2] and the fact that $\alpha_1^+ +\gamma_1^+ < p_1^- -1$ we obtain

$$\lim_{k\to+\infty}\frac{\tilde{C}k^{p_1^- -1}}{e^{ak(p_1^- -1-(\alpha_1^+ +\gamma_1^+))}}\big|\ln\frac{k}{e^{ak}}\big|=0. \tag{4.9}$$

Note that $\phi_1(x)\ge 1$ if $\sigma\le d(x)<2\delta$ because $\phi_1(x)\ge e^{k\sigma}-1$ and $e^{k\sigma}=2$ for all $k>0$. Thus, there is a constant $C_0>0$ that does not depend on $k$ such that

$|\phi_2|^{\alpha_1(x)}_{L^{q_1(x)}(\Omega)} \geq C_0$ if $\sigma < d(x) < 2\delta$. By (4.9), we can choose $k > 0$ large enough such that

$$\frac{\tilde{C}k^{p_1^- -1}}{e^{ak[(p_1^- -1)-(\alpha_1^+ + \beta_1^+)]}}\Big|\ln\frac{k}{e^{ak}}\Big| \leq \frac{C_0}{\mathcal{A}_\lambda}. \tag{4.10}$$

Therefore from (4.8) and (4.10) we have

$$-\Delta_{p_1(x)}(\mu\phi_1) \leq \frac{1}{\mathcal{A}_\lambda}((\mu\phi_1)^{\beta_1(x)} + w^{\gamma_1(x)})|\mu\phi_2|^{\alpha_1(x)}_{L^{q_1(x)}},$$

for all $w \in L^\infty(\Omega)$ with $w \geq \mu\phi_2$ and $\sigma < d(x) < 2\delta$ for $k > 0$ large enough. Thus it is possible to conclude that

$$-\Delta_{p_1(x)}(\mu\phi_1) \leq \frac{1}{\mathcal{A}_\lambda}((\mu\phi_1)^{\beta_1(x)} + w^{\gamma_1(x)})|\mu\phi_2|^{\alpha_1(x)}_{L^{q_1(x)}} \quad \text{in } \Omega.$$

Fix $k > 0$ satisfying the above property and $-\Delta_{p_1(x)}(\mu\phi_1) \leq 1$. For $\lambda > 1$ we have $-\Delta_{p_1(x)}(\mu\phi_1) \leq -\Delta_{p_1(x)}z_\lambda$. Therefore $\mu\phi_1 \leq z_\lambda$. Since $\alpha_2^+ + \gamma_2^+ < p_2^- - 1$, a similar reasoning imply that there is $\mu > 0$ small enough such that

$$-\Delta_{p_2(x)}(\mu\phi_2) \leq \frac{1}{\mathcal{A}(x, |w|_{L^{r_2(x)}})}(w^{\beta_2} + (\mu\phi_2)^{\gamma_2})|\mu\phi_1|^{\alpha_2(x)}_{L^{q_2(x)}(\Omega)} \quad \text{in } \Omega$$

for all $w \in L^\infty(\Omega)$ with $w \geq \mu\phi_1$ and that $\mu_2\phi \leq y_\lambda$. The first part of the result is proved.

Now suppose that $0 < \mathcal{A}(x,t) \leq a_0$ in $\overline{\Omega} \times (0,\infty)$. Let $\delta, \sigma, \mu, a, \lambda, z_\lambda, y_\lambda$ and $\phi_i$ for $i = 1,2$ as before. From the previous arguments there exist $k > 0$ large enough and $\mu > 0$ small such that

$$-\Delta_{p_1(x)}(\mu\phi_1) \leq 1, \quad -\Delta_{p_1(x)}(\mu\phi) \leq \frac{1}{a_0}((\mu\phi_1)^{\beta_1(x)} + w^{\gamma_1(x)})|\mu\phi_2|^{\alpha_1(x)}_{L^{q_1(x)}} \tag{4.11}$$

in $\Omega$ for all $w \in [\mu\phi_2, y_\lambda]$, and

$$-\Delta_{p_2(x)}(\mu\phi_2) \leq 1, \quad -\Delta_{p_2(x)}(\mu\phi_2) \leq \frac{1}{a_0}(w^{\beta_2(x)} + (\mu\phi_2)^{\gamma_2(x)})|\mu\phi_1|^{\alpha_2(x)}_{L^{q_2(x)}} \tag{4.12}$$

in $\Omega$ for all $w \in [\mu\phi_1, z_\lambda]$.

Since $\lim_{t\to\infty}\mathcal{A}(x,t) = a_\infty > 0$ uniformly in $\Omega$ there is a large constant $a_1 > 0$ such that $\mathcal{A}(x,t) \geq \frac{a_\infty}{2}$ on $\overline{\Omega} \times (a_1,\infty)$. Let

$$m_k := \min\big\{\mathcal{A}(x,t) : (x,t) \in \overline{\Omega} \times [\min\{|\mu\phi_1|_{L^{r_1(x)}}, |\mu\phi_2|_{L^{r_2(x)}}\}, a_1]\big\} > 0$$

and $\mathcal{A}_k := \min\big\{m_k, \frac{a_\infty}{2}\big\}$. Then we have

$$\mathcal{A}(x,t) \geq \mathcal{A}_k \quad \text{in } \overline{\Omega} \times [\min\{|\mu\phi_1|_{L^{r_1(x)}}, |\mu\phi_2|_{L^{r_2(x)}}\}, \infty).$$

Fix $k > 0$ satisfying (4.11) and (4.12). Consider $\lambda > 1$ such that (4.4), (4.5) and

$$\frac{1}{\mathcal{A}_k}\Big(K^{\beta_1^+}\lambda^{\frac{\beta_1^+}{p_1^- -1} + \frac{\alpha_1^+}{p_2^- -1}} + K^{\gamma_1^+}\lambda^{\frac{\alpha_1^+ + \gamma_1^+}{p_2^- -1}}\Big)\max\{|K|^{\alpha_1^-}_{L^{q_1(x)}}, |K|^{\alpha_1^+}_{L^{q_1(x)}}\} \leq \lambda,$$

$$\frac{1}{\mathcal{A}_k}\Big(K^{\beta_2^+}\lambda^{\frac{\beta_2^+ + \alpha_2^+}{p_1^- -1}} + K^{\gamma_2^+}\lambda^{\frac{\gamma_2^+}{p_2^- -1} + \frac{\alpha_2^+}{p_1^- -1}}\Big)\max\{|K|^{\alpha_2^+}_{L^{q_2(x)}}, |K|^{\alpha_2^-}_{L^{q_2(x)}}\} \leq \lambda,$$

where $K > 1$ is a constant that does not depend on $k$ or $\lambda$ (see Lemma 2.7). Therefore,

$$-\Delta_{p_1(x)}z_\lambda \leq \frac{1}{\mathcal{A}(x, |w|_{L^{r_1(x)}})}(z_\lambda^{\beta_1(x)} + w^{\gamma_1(x)})|y_\lambda|^{\alpha_1(x)}_{L^{q_1(x)}} \quad \text{in } \Omega, \quad w \in [\mu\phi_2, y_\lambda].$$

Arguing as before and considering a suitable choice for $\lambda$ and $k$ we obtain

$$-\Delta_{p_2(x)}y_\lambda \leq \frac{1}{\mathcal{A}(x,|w|_{L^{r_2(x)}})}(w^{\beta_2(x)}+y_\lambda^{\beta_2(x)})|z_\lambda|_{L^{q_2(x)}}^{\alpha_2(x)} \quad \text{in } \Omega, \quad w\in[\mu\phi_1,z_\lambda].$$

The comparison principle implies that $\mu\phi_1 \leq z_\lambda$ and $\mu\phi_2 \leq y_\lambda$ if $\mu$ is small. The proof is complete. ☐

4.2. **A concave-convex problem.** In this section we consider the following nonlocal problem with concave-convex nonlinearities

$$\begin{gathered}
-\mathcal{A}(x,|v|_{L^{r_1(x)}})\Delta_{p_1(x)}u = \lambda|u|^{\beta_1(x)-1}u|v|_{L^{q_1(x)}}^{\alpha_1(x)} + \theta|v|^{\eta_1(x)-1}v|v|_{L^{s_1(x)}}^{\gamma_1(x)} \quad \text{in } \Omega,\\
-\mathcal{A}(x,|u|_{L^{r_2(x)}})\Delta_{p_2(x)}v = \lambda|v|^{\beta_2(x)-1}v|u|_{L^{q_2(x)}}^{\alpha_2(x)} + \theta|u|^{\eta_2(x)-1}u|u|_{L^{s_2(x)}}^{\gamma_2(x)} \quad \text{in } \Omega,\\
u=v=0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{4.13}$$

The scalar and local version of (4.13) with $p(x)\equiv 2$ and constant exponents was considered in the famous paper by Ambrosetti-Brezis-Cerami [5] in which a sub-supersolution argument is used. In [43], problem (4.13) was studied with $p(x)\equiv 2$. The following result generalizes [43, Theorem 7].

**Theorem 4.2.** *Suppose that $r_i,p_i,q_i,s_i,\alpha_i,\eta_i$ satisfy* (H1) *for $i=1,2$ and that $\beta_i\in L^\infty(\Omega)$, $i=1,2$ are nonnegative functions with $0<\alpha_i^-+\beta_i^-\leq\alpha_i^++\beta_i^+<p_i^--1$, $i=1,2$. Let $a_0,b_0>0$ be positive numbers. Then the following assertions hold*

1. *If $p_2^+-1<\eta_1^-+\gamma_1^-$, $p_1^+-1<\eta_2^-+\gamma_2^-$ and $\mathcal{A}(x,t)\geq a_0$ in $\overline{\Omega}\times[0,b_0]$, then for each $\theta>0$ there exists $\lambda_0>0$ such that for each $\lambda\in(0,\lambda_0)$, problem* (4.13) *has a positive solution $u_{\lambda,\theta}$.*
2. *$p_2^+-1<\eta_1^-+\gamma_1^-$, $p_1^+-1<\eta_2^-+\gamma_2^-$ and*
$$\frac{\beta_1^+}{p_1^--1}+\frac{\alpha_1^+}{p_2^--1}<1, \quad \frac{\beta_2^+}{p_2^--1}+\frac{\alpha_2^+}{p_1^--1}<1\,.$$

*Suppose that $0<\mathcal{A}(x,t)\leq a_0$ in $\overline{\Omega}\times(0,\infty)$ and $\lim_{t\to\infty}\mathcal{A}(x,t)=b_0$ uniformly in $\overline{\Omega}$. Then given a $\lambda>0$, there exists $\theta_0>0$ such that for each $\theta\in(0,\theta_0)$, problem* (4.13) *has a positive solution $u_{\lambda,\theta}$.*

*Proof.* Suppose that (1) occurs. Consider $z_\lambda\in W_0^{1,p_1(x)}(\Omega)\cap L^\infty(\Omega)$ and $y_\lambda\in W_0^{1,p_2(x)}(\Omega)\cap L^\infty(\Omega)$ the unique solutions of (2.1) respectively, where $\lambda\in(0,1)$ will be chosen later.

Lemma 2.7 imply that for $\lambda>0$ small enough there exists a constant $K>1$ that does not depend on $\lambda$ such that

$$0<z_\lambda(x)\leq K\lambda^{\frac{1}{p_1^+-1}} \quad \text{in } \Omega, \tag{4.14}$$

$$0<y_\lambda(x)\leq K\lambda^{\frac{1}{p_2^+-1}} \quad \text{in } \Omega. \tag{4.15}$$

To construct $\overline{u}_i$ we will prove, for each $\theta>0$, that there exists $\lambda_0>0$ such that

$$\frac{1}{a_0}\Big(\lambda|z_\lambda|^{\beta_1(x)-1}z_\lambda|y_\lambda|_{L^{q_1(x)}}^{\alpha_1(x)}+\theta|w|^{\eta_1(x)-1}w|y_\lambda|_{L^{s_1(x)}}^{\gamma_1(x)}\Big)\leq\lambda, \quad \forall w\in[0,y_\lambda], \tag{4.16}$$

$$\frac{1}{a_0}\Big(\lambda|y_\lambda|^{\beta_2(x)-1}y_\lambda|z_\lambda|_{L^{q_2(x)}}^{\alpha_2(x)}+\theta|w|^{\eta_2(x)-1}w|z_\lambda|_{L^{s_2(x)}}^{\gamma_2(x)}\Big)\leq\lambda, \quad \forall w\in[0,z_\lambda]. \tag{4.17}$$

Let

$$\overline{K} := \max_{i=1,2}\left\{K^{\beta_i^+}|K|^{\alpha_i^+}_{L^{q_i(x)}}, K^{\beta_i^+}|K|^{\alpha_i^-}_{L^{q_i(x)}}, K^{\eta_i^+}|K|^{\gamma_i^+}_{L^{s_i(x)}}, K^{\eta_i^+}|K|^{\gamma_i^-}_{L^{s_i(x)}}\right\}. \tag{4.18}$$

Since $0 < \alpha_1^- + \beta_1^-$ and $p_2^+ - 1 < \eta_1^- + \gamma_1^-$, there exists $\lambda_0 > 0$ such that

$$\frac{1}{a_0}\Big(\lambda^{\frac{p_1^+-1+\beta_1^-}{p_1^+-1}+\frac{\alpha_1^-}{p_2^+-1}}\overline{K} + \theta\lambda^{\frac{\eta_1^-+\gamma_1^-}{p_2^+-1}}\overline{K}\Big) \le \lambda, \tag{4.19}$$

for all $\lambda \in (0, \lambda_0)$.

If necessary, we consider small $\lambda_0 > 0$ such that $|y_\lambda|_{L^{r_1(x)}} \le |K|_{L^{r_1(x)}}\lambda^{\frac{1}{p_2^+-1}} \le b_0$ for all $\lambda \in (0, \lambda_0)$. Therefore $\mathcal{A}(x, |w|_{L^{r_1(x)}}) \ge a_0, w \in [0, y_\lambda]$. It follows from (4.14), (4.15) and (4.19) that (4.16) holds. Then we can conclude that

$$-\Delta_{p_1(x)} z_\lambda \ge \frac{1}{\mathcal{A}(x, |w|_{L^{r_1(x)}})}\Big(\lambda {z_\lambda}^{\beta_1(x)}|y_\lambda|^{\alpha_1(x)}_{L^{q_1(x)}} + \theta w^{\eta_1(x)}|y_\lambda|^{\gamma_1(x)}_{L^{s_1(x)}}\Big), \tag{4.20}$$

for all $w \in [0, y_\lambda]$. Assume also that $\lambda_0$ satisfies

$$\frac{1}{a_0}\Big(\lambda^{\frac{p_2^+-1+\beta_2^-}{p_2^+-1}+\frac{\alpha_2^-}{p_1^+-1}}\overline{K} + \theta\lambda^{\frac{\eta_2^-+\gamma_2^-}{p_1^+-1}}\overline{K}\Big) \le \lambda \tag{4.21}$$

and $|z_\lambda|_{L^{r_2(x)}} \le |K|_{L^{r_2(x)}}\lambda^{\frac{1}{p_1^+-1}} \le b_0$ for all $\lambda \in (0, \lambda_0)$. Therefore $\mathcal{A}(x, |w|_{L^{r_2(x)}}) \ge a_0, w \in [0, z_\lambda]$. Thus from (4.14), (4.15) and (4.21) we have that (4.17) holds. Then we can conclude that

$$-\Delta_{p_2(x)} y_\lambda \ge \frac{1}{\mathcal{A}(x, |w|_{L^{r_2(x)}})}\Big(\lambda {z_\lambda}^{\beta_2(x)}|z_\lambda|^{\alpha_2(x)}_{L^{q_2(x)}} + \theta w^{\eta_2(x)}|z_\lambda|^{\gamma_2(x)}_{L^{s_2(x)}}\Big) \tag{4.22}$$

for all $w \in [0, z_\lambda]$.

To construct $\underline{u}_i$ consider $\phi_i, \delta, \sigma, \mu$ as in the proof of Theorem 4.1. Using the inequalities $\alpha_i^+ + \beta_i^+ < p_i^- - 1, i = 1, 2$ and repeating the arguments of Theorem 4.1, we have that exists a number $\mu > 0$ such that

$$\mu\phi_1 \le z_\lambda, \quad \mu\phi_2 \le y_\lambda, \quad -\Delta_{p_1(x)}(\mu\phi_1) \le \lambda,$$

$$-\Delta_{p_1(x)}(\mu\phi_1) \le \frac{1}{\mathcal{A}(x, |w|_{L^{r_1(x)}})}\Big(\lambda(\mu\phi_1)^{\beta_1(x)}|\mu\phi_1|^{\alpha_1(x)}_{L^{q_1(x)}} + \theta w^{\eta_1(x)}|\mu\phi_2|^{\gamma_1(x)}_{L^{s_1(x)}}\Big),$$

for all $w \in [\mu\phi_2, y_\lambda]$ and

$$-\Delta_{p_2(x)}(\mu\phi_2) \le \lambda,$$

$$-\Delta_{p_2(x)}(\mu\phi_2) \le \frac{1}{\mathcal{A}(x, |w|_{L^{r_2(x)}})}\Big(\lambda(\mu\phi_2)^{\beta_2(x)}|\mu\phi_1|^{\alpha_2(x)}_{L^{q_2(x)}} + \theta w^{\eta_2(x)}|\mu\phi_1|^{\gamma_2(x)}_{L^{s_2(x)}}\Big),$$

for all $w \in [\mu\phi_2, z_\lambda]$. Then by Theorem 1.1 we have the desired result.

Now we consider the condition (2). Let $\phi_i, \delta$ and $\sigma_i, i = 1, 2$ as in the first part of the result and let $\lambda > 0$ fixed. Since $\alpha_i^+ + \beta_i^+ < p_i^- - 1$, $i = 1, 2$ there exists $\mu > 0$ depending only on $\lambda$ such that

$$-\Delta_{p_i(x)}(\mu\phi_i) \le 1, \quad -\Delta_{p_i(x)}(\mu\phi) \le \frac{1}{a_0}\lambda(\mu\phi_i)^{\beta_i(x)}|\mu\phi_j|^{\alpha_i(x)},$$

for $w \in L^\infty(\Omega)$ with $w \ge \mu\phi_j$, $i \ne j$ and $i, j = 1, 2$.

Let $M > 0$ that will be chosen later and assume $z_M \in W_0^{1,p_1(x)}(\Omega) \cap L^\infty(\Omega)$ is a solution of

$$-\Delta_{p_1(x)} z_M = M \quad \text{in } \Omega,$$

$$z_M = 0 \quad \text{on } \partial\Omega,$$

and $y_M \in W_0^{1,p_2(x)}(\Omega) \cap L^\infty(\Omega)$ is a solutions of

$$\begin{gathered} -\Delta_{p_2(x)} y_M = M \quad \text{in } \Omega, \\ y_M = 0 \quad \text{on } \partial\Omega\,. \end{gathered}$$

For $M$ large enough from Lemma 2.7, there exists a constant $K > 1$ that does not depend on $M$ such that

$$0 < z_M(x) \le KM^{\frac{1}{p_1^- - 1}} \quad \text{in } \Omega, \tag{4.23}$$

$$0 < y_M(x) \le KM^{\frac{1}{p_2^- - 1}} \quad \text{in } \Omega. \tag{4.24}$$

To construct $\overline{u}_i$ we will show that exist $\theta_0 > 0$ depending on $\lambda$ with the following property: if we assume $\theta \in (0, \theta_0)$ then there is a constant $M$ depending only on $\lambda$ and $\theta$ satisfying

$$M \ge \frac{1}{\mathcal{A}(x, |w|_{L^{r_1(x)}})} \Big( \lambda {z_M}^{\beta_1(x)} |y_M|_{L^{q_1(x)}}^{\alpha_1(x)} + \theta w^{\eta_1(x)} |y_M|_{L^{s_1(x)}}^{\gamma_1(x)} \Big), \tag{4.25}$$

for $w \in [\mu\phi_2, y_M]$, and

$$M \ge \frac{1}{\mathcal{A}(x, |w|_{L^{r_2(x)}})} \Big( \lambda {y_M}^{\beta_2(x)} |z_M|_{L^{q_2(x)}}^{\alpha_2(x)} + \theta w^{\eta_2(x)} |z_M|_{L^{s_2(x)}}^{\gamma_2(x)} \Big), \tag{4.26}$$

for $w \in [\mu\phi_1, z_M]$.

Since $\mathcal{A}$ is continuous and $\lim_{t\to+\infty} \mathcal{A}(x,t) = b_0 > 0$ uniformly in $\Omega$, there exists $a_1 > 0$ large enough such that $\mathcal{A}(x,t) \ge \frac{b_0}{2}$ in $\overline{\Omega} \times (a_1, +\infty)$. Define

$$m_\lambda := \{\mathcal{A}(x,t) : (x,t) \in \overline{\Omega} \times [\min\{|\mu\phi_1|_{L^{r_1(x)}}, |\mu\phi_2|_{L^{r_2(x)}}\}, a_1]\}$$

and $\mathcal{A}_\lambda := \min\{m_\lambda, \frac{b_0}{2}\}$. Then $\mathcal{A}(x,t) \ge \mathcal{A}_\lambda$ in $\overline{\Omega} \times [\min\{|\mu\phi_1|_{L^{r_1(x)}}, |\mu\phi_2|_{L^{r_2(x)}}\}, \infty)$. Thus $\mathcal{A}_\lambda \le \mathcal{A}(x, |w|_{L^{r_1(x)}}) \le a_0$ for all $w \in L^\infty(\Omega)$ with $\mu\phi_1 \le w$ or $\mu\phi_2 \le w$. Note that from (4.23) and (4.24) the inequalities (4.25) and (4.26) hold if we have simultaneously the inequalities

$$\begin{gathered} \frac{1}{\mathcal{A}_\lambda} \Big( \lambda \overline{K} M^{\frac{\beta_1^+}{p_1^- - 1} + \frac{\alpha_1^+}{p_2^- - 1}} + \theta \overline{K} M^{\frac{\eta_1^+ + \gamma_1^+}{p_2^- - 1}} \Big) \le M, \\ \frac{1}{\mathcal{A}_\lambda} \Big( \lambda \overline{K} M^{\frac{\beta_2^+}{p_2^- - 1} + \frac{\alpha_2^+}{p_1^- - 1}} + \theta \overline{K} M^{\frac{\eta_2^+ + \gamma_2^+}{p_1^- - 1}} \Big) \le M, \end{gathered}$$

where $\overline{K}$ is given by (4.18). To obtain such inequalities we will study the inequality

$$\frac{1}{\mathcal{A}_\lambda} \left( \lambda \overline{K} M^{\rho - 1} + \theta \overline{K} M^{\tau - 1} \right) \le 1 \tag{4.27}$$

where

$$\begin{gathered} \rho := \max\Big\{ \frac{\beta_1^+}{p_1^- - 1} + \frac{\alpha_1^+}{p_2^- - 1}, \frac{\beta_2^+}{p_2^- - 1} + \frac{\alpha_2^+}{p_1^- - 1} \Big\}, \\ \tau := \max\Big\{ \frac{\eta_1^+ + \gamma_1^+}{p_2^- - 1}, \frac{\eta_2^+ + \gamma_2^+}{p_1^- - 1} \Big\}. \end{gathered}$$

Define

$$\Psi_{\lambda,\theta}(M) := \frac{\lambda \overline{K}}{\mathcal{A}_\lambda} M^{\rho - 1} + \frac{\theta \overline{K}}{\mathcal{A}_\lambda} M^{\tau - 1}, \quad M > 0.$$

Since $0 < \rho < 1$ and $\tau > 1$ we have $\lim_{M\to 0^+} \Psi_{\lambda,\theta}(M) = \lim_{M\to+\infty} \Psi_{\lambda,\theta}(M) = +\infty$. Note that ${\Psi_{\lambda,\theta}}'(M) = 0$ if, and only if

$$M = M_{\lambda,\theta} := \big(\frac{\lambda}{\theta}\big)^{\frac{1}{\tau-\rho}} c, \quad c := \Big(\frac{1-\rho}{\tau-1}\Big)^{\frac{1}{\tau-\rho}}. \tag{4.28}$$

From the above properties of $\Psi_{\lambda,\mu}$ we have that the global minimum of $\Psi_{\lambda,\theta}$ is attained at $M_{\lambda,\theta}$. The inequality (4.27) is equivalent to finding $M_{\lambda,\theta} > 0$ such that $\Psi_{\lambda,\theta}(M_{\lambda,\theta}) \le 1$. By (4.28), we have that $\Psi_{\lambda,\theta}(M_{\lambda,\theta}) \le 1$, if and only if

$$\frac{\lambda\overline{K}}{\mathcal{A}_\lambda}\big(\frac{\lambda}{\theta}\big)^{\frac{\rho-1}{\tau-\rho}} c^{\rho-1} + \theta^{1-(\frac{\tau-1}{\tau-\rho})}\frac{\overline{K}}{\mathcal{A}_\lambda}\lambda^{\frac{\tau-1}{\tau-\rho}} c^{\tau-1} \le 1. \tag{4.29}$$

Thus from (4.28) and (4.29), we have that given $\lambda > 0$ there exists $\theta_0 > 0$ such that for each $\theta \in (0,\theta_0)$ there exists $M_{\lambda,\theta}$ satisfying

$$M_{\lambda,\theta} \ge 1 \quad \text{and} \quad \frac{1}{\mathcal{A}_\lambda}\big(\lambda\overline{K}{M_{\lambda,\theta}}^{\rho-1} + \theta\overline{K}{M_{\lambda,\theta}}^{\tau-1}\big) \le 1.$$

Therefore,

$$-\Delta_{p_1(x)} z_M \ge \frac{1}{\mathcal{A}_\lambda}\Big(\lambda {z_M}^{\beta_1(x)}|y_M|^{\alpha_1(x)}_{L^{q_1(x)}} + \theta w^{\eta_1(x)}|y_M|^{\gamma_1(x)}_{L^{s_1(x)}}\Big) \quad \text{in } \Omega,$$

for all $w \in [\mu\phi_2, y_M]$, and

$$-\Delta_{p_2(x)} y_M \ge \frac{1}{\mathcal{A}_\lambda}\Big(\lambda {y_M}^{\beta_2(x)}|z_M|^{\alpha_2(x)}_{L^{q_2(x)}} + \mu w^{\eta_2(x)}|z_M|^{\gamma_w(x)}_{L^{s_w(x)}}\Big) \quad \text{in } \Omega,$$

for all $w \in [\mu\phi_1, z_M]$.

Since $M_{\lambda,\theta} \to +\infty$ as $\theta \to 0^+$ and the map $\theta \longmapsto M_{\lambda,\theta}$ is decreasing we have

$$-\Delta_{p_1(x)}(\mu\phi_1) \le 1 \le M_{\lambda,\theta_0} \le M_{\lambda,\theta}, \quad \theta \in (0,\theta_0)$$

for $\theta_0$ small enough. Similarly, we have $-\Delta_{p_2(x)}(\mu\phi_2) \le M_{\lambda,\theta_0} \le M_{\lambda,\theta}$ for all $\theta \in (0,\theta_0)$, for $\theta_0$ small. The weak maximum principle imply that $\mu\phi_1 \le z_M$ and $\mu\phi_2 \le y_M$. The proof is complete. □

4.3. **A generalization of the logistic equation.** In the previous sections, we considered at least one of the conditions $\mathcal{A}(x,t) \ge a_0 > 0$ or $0 < \mathcal{A}(x,t) \le a_\infty, t > 0$. In this section we study a generalization of the classic logistic equation where the function $\mathcal{A}(x,t)$ satisfies

$$\mathcal{A}(x,0) \ge 0, \quad \lim_{t\to 0^+} \mathcal{A}(x,t) = \infty, \quad \text{and} \quad \lim_{t\to+\infty} \mathcal{A}(x,t) = \pm\infty.$$

We consider the problem

$$\begin{gathered} -\mathcal{A}(x,|v|_{L^{r_1(x)}})\Delta_{p_1(x)} u = \lambda f_1(u)|v|^{\alpha_1(x)}_{L^{q_1(x)}} \quad \text{in } \Omega, \\ -\mathcal{A}(x,|u|_{L^{r_2(x)}})\Delta_{p_2(x)} v = \lambda f_2(v)|u|^{\alpha_2(x)}_{L^{q_2(x)}} \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.30}$$

We suppose that there are numbers $\theta_i > 0$, $i = 1,2$ such that the functions $f_i : [0,\infty) \to \mathbb{R}$ satisfy the following conditions:

- (H2) $f_i \in C^0([0,\theta_i],\mathbb{R})$, $i = 1,2$;
- (H3) $f_i(0) = f_i(\theta_i) = 0$, $f_i(t) > 0$ in $(0,\theta_i)$ for $i = 1,2$.

Problem (4.30) is a generalization of the problemes studied in [16, 18, 43]. The next result generalizes [43, Theorem 8].

**Theorem 4.3.** *Suppose that $r_i, p_i, q_i, \alpha_i$ satisfy* (H1)*. Also that $f_i$, $i = 1, 2$ satisfies* (H2), (H3) *and that $\mathcal{A}(x,t) > 0$ in $\overline{\Omega} \times \big(0, \max\{|\theta_1|_{L^{r_2(x)}}, |\theta_2|_{L^{r_1(x)}}\}\big]$. Then there exists $\lambda_0 > 0$ such that* (4.30) *has a positive solution for $\lambda \geq \lambda_0$.*

*Proof.* Consider the functions $\widetilde{f}_i(t) = f_i(t)$ for $t \in [0, \theta_i]$, and $\widetilde{f}_i(t) = 0$ for $t \in \mathbb{R} \setminus [0, \theta_i]$, $i = 1, 2$. The functional

$$
\begin{aligned}
&J_\lambda(u,v)\\
&= \int_\Omega \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} dx - \lambda \int_\Omega \widetilde{F}_1(u) dx + \int_\Omega \frac{1}{p_2(x)} |\nabla v|^{p_2(x)} dx - \lambda \int_\Omega \widetilde{F}_2(v) dx\\
&:= J_{1,\lambda}(u) + J_{2,\lambda}(v),
\end{aligned}
$$

where $\widetilde{F}_i(t) = \int_0^t \widetilde{f}_i(s) ds$ is of class $C^1(W_0^{1,p_1(x)} \times W_0^{1,p_2(x)}(\Omega), \mathbb{R})$ and $W_0^{1,p_1(x)}(\Omega) \times W_0^{1,p_2(x)}(\Omega)$ is a Banach space endowed with the norm

$$
|(u,v)| := \max\{|\nabla u|_{p_1(x)}, |\nabla v|_{p_2(x)}\}.
$$

Since $|\widetilde{f}_i(t)| \leq C$, $t \in \mathbb{R}$ for some constant which does not depends on $i = 1, 2$ we have that $J$ is coercive. Thus $J$ has a minimum $(z_\lambda, w_\lambda) \in W_0^{1,p_1(x)}(\Omega) \times W_0^{1,p_2(x)}(\Omega)$ with

$$
\begin{gathered}
-\Delta_{p_1(x)} z_\lambda = \lambda \widetilde{f}_1(z_\lambda) \quad \text{in } \Omega,\\
z_\lambda = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{4.31}
$$

and

$$
\begin{gathered}
-\Delta_{p_2(x)} w_\lambda = \lambda \widetilde{f}_2(w_\lambda) \quad \text{in } \Omega,\\
w_\lambda = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{4.32}
$$

Note that the unique solutions of (4.31) and (4.32) are given by the minimizers of functionals $J_{1,\lambda}$ and $J_{2,\lambda}$ respectively.

Consider a function $\varphi_0 \in W_0^{1,p_i(x)}(\Omega)$, $i = 1, 2$ with $\tilde{F}_i(\varphi_0) > 0$, $i = 1, 2$. Define $(z_0, w_0) := (z_{\tilde{\lambda}_0}, w_{\tilde{\lambda}_0})$, where $\widetilde{\lambda}_0$ satisfies

$$
\int_\Omega \frac{1}{p_i(x)} |\nabla \varphi_0|^{p_i(x)} dx < \widetilde{\lambda}_0 \int_\Omega \widetilde{F}_i(\varphi_0) dx, \quad i = 1, 2.
$$

We have $J_{1,\tilde{\lambda}_0}(z_0) \leq J_{1,\tilde{\lambda}_0}(\varphi_0) < 0$ and that $J_{2,\tilde{\lambda}_0}(z_0) < 0$. Therefore $z_0 \neq 0$ and $w_0 \neq 0$. Since $-\Delta_{p_1(x)} z_0$ and $-\Delta_{p_2(x)} w_0$ are nonnegative, we have $z_0, w_0 > 0$ in $\Omega$. Note that by [28, Theorem 4.1] and [25, Theorem 1.2], we obtain that $z_0, w_0 \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1]$.

Using the test function $\varphi = (z_0 - \theta_1)^+ \in W_0^{1,p_1(x)}(\Omega)$ in (4.31) we obtain

$$
\int_\Omega |\nabla z_0|^{p_1(x)-2} \nabla z_0 \nabla (z_0 - \theta_1)^+ dx = \widetilde{\lambda}_0 \int_{\{z_0 > \theta\}} \widetilde{f}_1(z_0)(z_0 - \theta_1) dx = 0.
$$

Therefore,

$$
\int_{\{z_0 > \theta\}} \big\langle |\nabla z_0|^{p(x)-2} \nabla z_0 - |\nabla \theta_1|^{p_1(x)-2} \nabla \theta_1, \nabla (z_0 - \theta_1) \big\rangle dx = 0,
$$

which imply $(z_0 - \theta_1)_+ = 0$ in $\Omega$. Thus $0 < z_0 \leq \theta_1$. A similar reasoning provides $0 < w_0 \leq \theta_2$.

Note that there is a constant $C > 0$ such that $|z_0|_{L^{q_1(x)}}^{\alpha_1(x)}, |w_0|_{L^{q_2(x)}}^{\alpha_2(x)} \geq C$. We define

$$
\mathcal{A}_0 = \max \Big\{ \mathcal{A}(x,t) : (x,t) \in \overline{\Omega} \times [\min\{|z_0|_{L^{r_2(x)}}, |w_0|_{L^{r_1(x)}}\},
$$

$$\max\{|\theta_1|_{L^{r_2(x)}}, |\theta_2|_{L^{r_1(x)}}\}\Big\}$$

and $\mu_0 = \frac{\mathcal{A}_0}{C}$. Then, we have

$$\begin{aligned}
-\Delta_{p_1(x)} z_0 &= \widetilde{\lambda}_0 f_1(z_0) \\
&= \frac{1}{\mathcal{A}_0}\widetilde{\lambda}_0 \mu_0 f_1(z_0)|w_0|^{\alpha_1(x)}_{L^{q_1(x)}} \frac{\mathcal{A}_0}{\mu_0 |z_0|^{\alpha_1(x)}_{L^{q_1(x)}}} \\
&\le \frac{1}{\mathcal{A}_0}\widetilde{\lambda}_0 \mu_0 f_1(z_0)|w_0|^{\alpha_1(x)}_{L^{q_1(x)}}.
\end{aligned}$$

Thus for each $\lambda \ge \lambda_0 := \widetilde{\lambda}_0\mu_0$ and $w \in [w_0, \theta_2]$, we obtain

$$-\Delta_{p_1(x)} z_0 \le \frac{1}{\mathcal{A}(x, |w|_{L^{r_1(x)}})} \lambda f_1(z_0)|w_0|^{\alpha_1(x)}_{L^{q_1(x)}}.$$

If necessary, we can consider a larger $\lambda_0 > 0$ such that

$$-\Delta_{p_2(x)} w_0 \le \frac{1}{\mathcal{A}(x, |w|_{L^{r_2(x)}})} \lambda f_2(w_0)|z_0|^{\alpha_2(x)}_{L^{q_2(x)}},$$

for all $\lambda \ge \lambda_0$ and $w \in [z_0, \theta_1]$.

Since $f_i(\theta_i) = 0$, $i = 1, 2$, we have that $(z_0, \theta_1)$ and $(w_0, \theta_2)$ are sub-super solutions pairs for (4.30). The proof is complete. □

We remark that is possible to use the functions $\phi_i$ from the proof of Theorem 4.1 for problem (4.30). However, more restrictions on the functions $p_i, f_i, i = 1, 2$ are needed.

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## References

[1] E. Acerbi, G. Mingione; *Regularity results for stationary electro-rheological fluids*, Arch. Ration. Mech. Anal. 164 (2002), no. 3, 213–259.
[2] W. Allegretto, Y. X. Huang; *A Picone's identity for the p-Laplacian an applications*, Nonlinear Anal., 32 (7) (1998) 819–830. doi: 10.1016/s0362-54X(97)00530-0.
[3] C. O. Alves, A. Moussaoui; *Existence and regularity of solutions for a class of singular $(p(x), q(x))$-Laplacian systems,* 2016, https: //doi.org/10.1080/17476933.2017.1298589
[4] C. O. Alves, A. Moussaoui, L. S. Tavares; *An elliptic system with logarithmic nonlinearity.* 2017, (to appear).
[5] A. Ambrosetti, H. Brezis, G. Cerami; *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal., 122 (1994), no. 2, 519–543.
[6] A. Anane, O. Chakrone, N. Moradi; *Regularity of the solutions to a nonlinear boundary problem with indefinite weight*, Bol. Soc. Parana. Mat., (3) 29 (1) (2011) 17–23. doi: 10.5269/bspm.v29i1.11402.
[7] A. Anane, O. Chakrone, N. Moradi; *Maximum and anti-maximum principle for the p-Laplacian with a nonlinear boundary condition*, Proceedings of the 2005 Oujda International Conference on Nonlinear Analysis, Electron. J. Differ. Equ., Conf. 14 (2006), pp. 95–107.
[8] M. Arias, J. Campos, J. P. Gossez; *On the antimaximum principle and the Fučik spectrum for the Neumann p-Laplacian*, Differential Integral Equations, 13 (1-3) (2000), 217–226.
[9] S. Baraket, S., G Molica Bisci; *Multiplicity results for elliptic Kirchhoff-type problems*, Adv. Nonlinear Anal., 6(2017), no. 1, 85–93.
[10] A. Cabadam F. J .S. A. Corrêa; *Existence of Solutions of a Nonlocal Elliptic System via Galerkin Method*, Abstr. and Appl. Anal., 2012, Art. ID 137379, 16 pp.

[11] G. F. Carrier; *On the Nonlinear Vibration Problem of the Elastic String*, Q. Appl. Math. 3 (1945), 157–165.
[12] M. Cencelj, D. Repovs, Z. Virk; *Multiple perturbations of a singular eigenvalue problem*, Nonlinear Anal., 119 (2015), 37–45.
[13] Y. Chen, H. Gao; *Existence of positive solutions for nonlocal and nonvariational elliptic system*, Bull. Austral. Math. Soc., 72 (2005), no. 2, 271–281.
[14] Y. Chen, S. Levine, M. Rao; *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math. 66 (2006), no. 4, 1383–1406.
[15] M. Chipot, N. H. Chang; *On some model diffusion problems with a nonlocal lower order term,* Chin. Ann. Math. Ser. B, 24 (2003), no. 2, 147–166.
[16] M. Chipot, F. J. S. A. Corrêa; *Boundary layer solutions to functional*, Bull. Braz. Math. Soc., New Series 40 (2009), 381–393.
[17] M. Chipot, B. Lovat; *Some remarks on nonlocal elliptic and parabolic problems,* Nonlinear Anal., 30 (1997), no. 7, 4619–4627.
[18] M. Chipot, P. Roy; *Existence results for some functional elliptic equations*, Differential and Integral Equations, Vol. 27, n.3/4 (2014), 289–300.
[19] F. J. S. A. Corrêa, G. M. Figueiredo, F. P. M. Lopes; *On the Existence of Positive Solutions for a Nonlocal Elliptic Problem Involving the p-Laplacian and the Generalized Lebesgue Space* $L^{p(x)}(\Omega)$, Differential and Integral Equations, 21 (2008), no. 3-4, 305–324.
[20] F. J. S. A. Corrêa, F. P. M. Lopes; *Positive solutions for a class of nonlocal elliptic systems*, Comm. Appl. Nonlinear Anal., 14 (2007), no. 2, 67–77.
[21] M. Cuesta, L. Leadi; *Weighted eigenvalue problems for quasilinear elliptic operators with mixed Robin-Dirichlet boundary conditions*, J. Math. Anal. Appl., 422 (1) (2015) 1–26. doi: 10.1016/j.jmaa.2014.08.015.
[22] M. Cuesta, L. A. Leadi, P. Nshimirimana; *Bifurcation from the first eigenvalue of the p-Laplacian with nonlinear boundary condition*, Electron. J. Differential Equations, 2019 (2019) No. 32, 1–29.
[23] W. Deng, Y. Lie, C. Xie; *Blow-up and global existence for a nonlocal degenerate parabolic system*, J. Math. Anal. Appl., 277 (2003), no. 1, 199–217.
[24] P. Drábek; *Solvability and bifurcations of nonlinear equations*, Vol. 264 of Pitman Research Notes in Mathematics Series, Logman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1992.
[25] X. L. Fan; *Global* $C^{1,\alpha}$ *regularity for variable exponent elliptic equations in divergence form,* J. Differential Equations, 235 (2007), no. 2, 397–417.
[26] X. L. Fan; *On the sub-super solution method for* $p(x)$*-Laplacian equations*, J. Math. Anal. Appl., 330 (2007), no. 1, 665–682.
[27] X. L. Fan, Q. H. Zhang; *Existence of solution for p(x)-Laplacian Dirichlet problem*, Nonlinear Anal., 52 (2003), no. 8, 1843–1852.
[28] X. L. Fan, D. Zhao; *A class of De Giorgi type and Holder continuity*, Nonlinear Anal., 36 (1999), no. 3, Ser. A: Theory Methods, 295–318.
[29] X. L. Fan, Y. Z. Zhao, Q. H. Zhang; *A strong maximum principle for* $p(x)$*-Laplace equations*, Chinese J. Contemp. Math., 24 (2003), no. 3, 277–282.
[30] J. Fernández Bonder, J. D. Rossi; *A nonlinear eigenvalue problem with indefinite weights related to the Sobolev trace embedding*, Publ. Mat., 46 (1) (2002), 221–235.
[31] T. Godoy, J. P. Gossez, S. Paczka; *On the antimaximum principle for the p-Laplacian with indefinite weight*, Nonlinear Anal., 51 (3) (2002), 449–467. doi: 10.1016/s0362-546X(01)00839-2.
[32] T. Godoy, J. P. Gossez, S. Paczka; *Antimaximum principle for elliptic problems with weight*, Electron. J. Differential Equations, 1999 (1999) No. 22, 1–15.
[33] L. Leadi, A. Marcos; *A weighted eigencurve for Steklov problems with a potential*, NoDEA Nonlinear Differential Equations Appl. 20 (3) (2013) 687–713. doi: 10.1007/s00030-012-0175-0.
[34] J. Liu, Q. Zhang, C. Zhao; *Existence of positive solutions for* $p(x)$*-Laplacian equations with a singular nonlinear term*, Electron. J. Differ. Equ. 2014 (2014), no. 155, 21 pp.
[35] M. Mihailescu, V. Radulescu, D. Repovs; *On a non-homogeneous eigenvalue problem involving a potential: an Orlicz-Sobolev space setting*, J. Math. Pures Appl., (2010) (9) 93, no. 2, 132–148.

[36] G. Molica Bisci, V. Radulescu, R. Servadei; *Variational methods for nonlocal fractional problems*, Encyclopedia of Mathematics and its Applications, 162. Cambridge University Press, Cambridge, 2016.
[37] M. A. del Pino, R. F. Manásevich, A. E. Murúa; *Existence and multiplicity of solutions with prescribed period for a second order quasilinear ODE*, Nonlinear Anal., 18 (1) (1992), 79–92. doi: 10.1016/0362-546X(92)90048-J.
[38] P. Pucci, M. Xiang, B. Zhang; *Existence and multiplicity of entire solutions for fractional p-Kirchhoff equations.* Adv. Nonlinear Anal., 5 (2016), no. 1, 27–55.
[39] P. Pucci, Q. Zhang; *Existence of entire solutions for a class of variable exponent elliptic equations,* J. Differential Equations, 257 (2014), no. 5, 1529–1566.
[40] P. H. Rabinowitz; *Some global results for nonlinear eigenvalue problems*, J. Funct. Anal. 7 (1971), 487–513.
[41] V. Radulescu, D. Repovs; *Partial differential equations with variable exponents. Variational methods and qualitative analysis*, Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2015.
[42] M. Růžička; *Electrorheological Fluids: Modelling and Mathematical Theory*, Springer-Verlag, Berlin, 2000.
[43] G. C. G. dos Santos, G. M. Figueiredo; *Positive Solutions for a class of nonlocal problems involving Lebesgue generalized spaces: Scalar and system cases*, J. Elliptic Parabol. Equ., 2 (2016), no. 1-2, 235–266
[44] G. C. G. dos Santos, G. M. Figueiredo, L. S. Tavares; *A sub-supersolution method for a class of nonlocal problems involving the $p(x)$-Laplacian operator and applications.* Acta Appl. Math. 153 (2018), 171–187.
[45] J. Serrin; *Local behavior of solutions of quasi-linear equations*, Acta Math., 111 (1964) 247-302. doi: 10.1007/BF02391014.
[46] Y.-Q. Song, Y.-M. Chu, B.-Y. Liu, M.-K. Wang; *A note on generalized trigonometric and hyperbolic functions*, J. Math. Inequal., 8 (3) (2014) 630–642. doi:10.7153/jmi-08-46.
[47] J. L. Vázquez; *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim., 12 (3) (1984) 191–202. doi: 10.1007/BF01449041.

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