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BIFURCATION FROM THE FIRST EIGENVALUE OF THE p-LAPLACIAN WITH NONLINEAR BOUNDARY CONDITION

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Abstract. We consider the problem

$$\begin{gathered}\Delta_p u = |u|^{p-2}u \quad \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u + g(\lambda, x, u) \quad \text{on } \partial\Omega,\end{gathered}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary, $N \geq 2$, and Δ_p denotes the p-Laplacian operator. We give sufficient conditions for the existence of continua of solutions bifurcating from both zero and infinity at the principal eigenvalue of p-Laplacian with nonlinear boundary conditions. We also prove that those continua split on two, one containing strictly positive and the other containing strictly negative solutions. As an application we deduce results on anti-maximum and maximum principles for the p-Laplacian operator with nonlinear boundary conditions.

1. Introduction

We consider the following nonlinear boundary value problem for a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with smooth boundary

$$\begin{gathered}\Delta_p u = |u|^{p-2}u \quad \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u + g(\lambda, x, u) \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where $\Delta_p u := div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $1 < p < +\infty$, $\frac{\partial u}{\partial \nu}$ represents the exterior normal derivative of u and g is a given function satisfying some conditions to be specified. We are mainly concerned with the bifurcation from the first eigenvalue λ_1 of the eigenvalue problem associated with (1.1) given by

$$\begin{gathered}\Delta_p u = |u|^{p-2}u \quad \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u \quad \text{on } \partial\Omega,\end{gathered} \tag{1.2}$$

and its application to maximum and anti-maximum principles for

$$\begin{gathered}\Delta_p u = |u|^{p-2}u \quad \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u + h(x) \quad \text{on } \partial\Omega,\end{gathered} \tag{1.3}$$

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It is well known that the properties of eigenvalues play an important role in the study of the bifurcation problem, the maximum principle and the antimaximum principle for quasilinear equations. The eigenvalue problem (1.2) has been studied, for instance, in Fernandez Bonder and Rossi [12, 11] and Martinez and Rossi [16]. In [12] the authors proved that if $N \geq 2$ then (1.2) admits an infinite sequence of positive eigenvalues λ_n such that $\lambda_n \to +\infty$ as $n \to +\infty$ although in [16] it was proved that the first eigenvalue is simple, isolated and principal i.e. every eigenfunction associated with λ_1 has a constant sign. Results related to those in [12] and [16] were proved in [11] where the authors have considered a generalization of (1.2) with an indefinite weight function given on $\partial\Omega$.

Bifurcation problems for the p-Laplacian operator with Dirichlet boundary conditions received extensive attention in the 1990's. The reader is referred to [8, 13, 18, 19] for problems on bounded domains and to [10] for problems in unbounded domains. In 2001, D. Arcoya and J. Gámez [3] gave sufficient conditions for subcritical and supercritical bifurcations and gave a new approach to prove old and new results on anti-maximum and local maximum principles for the p-Laplacian when λ is close to λ_1. The case $p = 2$, (1.1) was studied by Arrieta, Pardo and Rodríguez-Bernal in [4] and by Pardo [17]. In [4], the authors proved that every eigenvalue of odd multiplicity is a bifurcation point of solutions from infinity and also presented some results on the maximum and anti-maximum principles.

The main purpose of this work is to extend the results proved in [4] for the semilinear case, i.e. (1.1) with $p = 2$, or those of [8, 13] for the p-Laplacian operator with Dirichlet boundary conditions.

In Section 2 we introduce basic notations and recall properties of the generalized topological degree of Browder-Petryshyn for non linear mappings. In Sections 3 and 4, we prove that, under suitable assumptions on the function g, the first eigenvalue λ_1 of $(\mathcal{P}_\lambda)$ is a bifurcation point respectively from zero and from infinity of the solutions of (1.1) using basic tools of topological degree. In Section 5 we prove the existence of unbounded positive and negative continua of solutions of (1.1) splitting from λ_1. We give in Section 6 conditions to have sub- and super-critical bifurcations and also give as application a non-variational proof of the anti-maximum and local maximum principles for problem (1.3). We show for instance a maximum principle for $\lambda \in (\lambda_1 - \delta, \lambda_1)$ and h satisfying $\int_{\partial\Omega} h\varphi_1 > 0$ instead of $h \geq 0, h \not\equiv 0$. Finally, in Section 7, we focus our attention on the constants appearing on the regularity results of weak solutions for the general problem given by

$$\begin{gathered} \Delta_p u = |u|^{p-2}u + f(\lambda, x, u) \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = h(\lambda, x, u) \quad \text{on } \partial\Omega, \end{gathered} \tag{1.4}$$

with f and h functions having subcritical growth.

2. Notation and preliminaries

We study the bifurcation of solutions for the quasilinear elliptic problem

$$\begin{gathered} \Delta_p u = |u|^{p-2}u \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u + g(\lambda, x, u) \quad \text{on } \partial\Omega. \end{gathered} \tag{2.1}$$

Throughout this article, Ω will be a smooth bounded domain in $\mathbb{R}^N$, $N > 1$, with a boundary $\partial\Omega$ of class $C^{2,\beta}$, $0 < \beta < 1$; ν its outer normal vector defined everywhere and $1 < p < +\infty$. We will denote by $d\sigma$ the surface measure (which is a $(N-1)$-dimensional Hausdorff measure). The set $W^{1,p}(\Omega)$ denotes the Sobolev space with its usual norm given by

$$\|u\|_{W^{1,p}(\Omega)} = \Big(\int_\Omega |\nabla u|^p + |u|^p\Big)^{1/p}$$

and $(W^{1,p}(\Omega))^*$ its topological dual. The critical Sobolev's exponent for the trace inclusion $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ will be denoted by p_*

$$p_* := \begin{cases} \frac{(N-1)p}{N-p} & \text{if } 1 < p < N, \\ +\infty & \text{if } p \geq N. \end{cases}$$

The weak convergence will be denoted by $\rightharpoonup$ and the strong one by $\to$. In the following, $W^{1,p}(\Omega)$ will be denoted by X and $(W^{1,p}(\Omega))^*$ by X^*.

Throughout this work the function g will satisfy the following condition:

(A1) $g : \mathbb{R} \times \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, that is, g is measurable in $x \in \partial\Omega$ for all $(\lambda, s) \in \mathbb{R} \times \mathbb{R}$ and continuous in $(\lambda, s) \in \mathbb{R} \times \mathbb{R}$ a.e. $x \in \partial\Omega$. Moreover, there exists $a \in [1, p_*)$ ($a \geq 1$ if $p \geq N$) such that for any bounded set $\mathcal{B} \subset \mathbb{R}$ there exist $C, D \in L^\infty(\partial\Omega)$ such that

$$|g(\lambda, x, s)| \leq C + D|s|^{a-1} \quad \text{a.e. } x \in \partial\Omega,\ \forall\lambda \in \mathcal{B},\ \forall s \in \mathbb{R}. \tag{2.2}$$

Solutions to (2.1) are understood in the weak sense, i.e., a function $u \in X$ is a weak solution of (2.1) if and only if

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v dx + \int_\Omega |u|^{p-2}uv dx = \lambda \int_{\partial\Omega} |u|^{p-2}uv\, d\sigma + \int_{\partial\Omega} g(\lambda, x, u)v\, d\sigma \tag{2.3}$$

for all $v \in X$. Notice that (A1) assures the integrability of the integrands in (2.3). Let us define the functionals

$$\begin{gathered} \langle J(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\, dx + \int_\Omega |u|^{p-2}uv\, dx, \\ \langle F(u), v\rangle = \int_{\partial\Omega} |u|^{p-2}uv\, d\sigma, \\ \langle G_\lambda(u), v\rangle = \int_{\partial\Omega} g(\lambda, x, u)v\, d\sigma, \end{gathered}$$

where $\langle\cdot,\cdot\rangle$ is the usual duality map defined on $X^* \times X$. Thus (2.3) is equivalent to

$$A_\lambda(u) := J(u) - \lambda F(u) - G_\lambda(u) = 0. \tag{2.4}$$

It is well known that J is continuous, $(p-1)$-homogeneous, odd, coercive, strictly monotonous and continuously invertible. The function F is continuous, odd, $(p-1)$-homogeneous and compact and for any fixed λ, G_λ is continuous and compact. The compactness of those maps is a consequence of the compact embedding of the trace map

$$X \hookrightarrow L^q(\partial\Omega).$$

Let us briefly recall some properties of the spectrum of Δ_p with nonlinear boundary conditions for the problem

$$\begin{gathered} \Delta_p u = |u|^{p-2}u \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial\nu} = \lambda|u|^{p-2}u \quad \text{on } \partial\Omega, \end{gathered} \tag{2.5}$$

see [22, 12, 16]. A real number λ is said to be an *eigenvalue* of (2.5) if and only if there exists $u \in X \setminus \{0\}$ such that

$$\int_\Omega (|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv)dx = \lambda \int_{\partial\Omega} |u|^{p-2}uv\, d\sigma$$

holds for all $v \in X$. The function u is called *eigenfunction associated* with the eigenvalue λ. It is well known that (2.5) admits an infinite sequence of positive eigenvalues λ_n such that $\lambda_n \to +\infty$ as $n \to +\infty$. Its first eigenvalue $\lambda_1 > 0$ is characterized by

$$\lambda_1 := \inf\Big\{ \frac{\int_\Omega |\nabla u|^p + \int_\Omega |u|^p}{\int_{\partial\Omega} |u|^p} : u \in X, \int_{\partial\Omega} |u|^p \neq 0 \Big\}. \tag{2.6}$$

It is also well know that λ_1 is simple and admits a normalized positive eigenfunction $\varphi_1 > 0$ in $\overline{\Omega}$. Furthermore any eigenfunction associated with eigenvalue different from λ_1 changes sign. We will denote by σ_p the set of all eigenvalues of (2.5) and we call it *spectrum of p-Laplacian with nonlinear boundary condition.* It is also known that λ_1 is isolated in the spectrum, which allows to define the second positive eigenvalue λ_2 of (2.5) as

$$\lambda_2 := \min\{\lambda \in \mathbb{R} : \lambda \text{ eigenvalue and } \lambda > \lambda_1\}.$$

In the next two sections, we will prove some bifurcation results at the first eigenvalue λ_1 from both zero and infinity. The main tool to prove our bifurcation results is the generalized topological degree of Browder-Petryshyn for non linear mappings of [5] (see also [21]) that we will apply for the operator A_λ defined in (2.4). Let us recall here some properties of this degree. Let V be a real separable reflexive Banach space, V^* its topological dual and $A : V \to V^*$ be a demi-continuous operator, that is, A satisfies that whenever $u_n \in V$ converges to some $u \in V$ then $Au_n \rightharpoonup Au$. We also assume that A satisfies the condition $\alpha(V)$, that is, for any sequence $u_n \in V$ satisfying $u_n \rightharpoonup u_0$ in V and

$$\limsup_{n\to+\infty} \langle A(u_n), u_n - u_0\rangle \leq 0,$$

then $u_n \to u_0 \in V$. Trivially every continuous map $A : V \to V^*$ is also demi-continuous. Note also that if A satisfies the condition $\alpha(V)$ then $A + K$ satisfies the condition $\alpha(V)$ for any compact operator $K : V \to V^*$.

Let $\{w_i\}_{i=1}^{+\infty}$ be an arbitrary complete subset of the space V and let us assume that for every n the elements $w_1 \dots w_n$ are linearly independent. Denote W_n the linear hull of the elements $w_1 \dots w_n$. We set

$$A_n(u) := \sum_{i=1}^{n} \langle A(u), w_i\rangle w_i.$$

For any arbitrary bounded open set $D \subset V$ such that $A(u) \neq 0$ for any $u \in \partial D$, the *degree of the mapping A at 0 with respect to $D \subset V$* is defined as follows:

$$\deg(A, D, 0) := \lim_{n\to+\infty} \deg_B(A_n, D \cap W_n, 0), \tag{2.7}$$

where $\deg_B$ denotes here the Browder degree. It is shown in [21][Chapter 2] that $\deg_B(A_n, D \cap W_n, 0)$ is constant for $n \geq n_0$ for some $n_0 \in \mathbb{N}$ and that the value in the limit (2.7) is independent of the choice of the system of functions $\{w_i\}$. One can also define the *index* of an isolated solution u_0 of the equation $A(u) = 0$ as

$$\operatorname{Ind}(A, u_0) := \lim_{\varepsilon\to 0} \deg(A, B_\varepsilon(u_0), 0).$$

Finally the usual properties of a degree as the additivity, excision and invariance under homotopy, hold for this generalisation of the degree of Browder. The following properties will be used.

Lemma 2.1. *Let $A : V \to V^*$ be a demi-continuous operator satisfying the $\alpha(V)$ condition.*

(i) [21, Chap. 2, Theorem 4.4] *Assume that there exists $r > 0$ such that $\langle A(u), u\rangle > 0$ for all $u \in V$, $\|u\|_V = r$. Then*
$$\deg(A, B_r(0), 0) = 1.$$

(ii) [9, Lemma 14.7] *Assume that A is a potential operator, i.e., there exists a continuous differentiable functional $B : V \to \mathbb{R}$ such that $B' = A$. Let u_0 be a local minimum of B and an isolated solution of $A(u) = 0$. Then*
$$\operatorname{Ind}(A, u_0) = 1.$$

Now let us now finally take $V = W^{1,p}(\Omega)$, denoted by X, the operator A_λ defined in (2.4) and check that A_λ satisfies the $\alpha(X)$ condition.

Lemma 2.2. *Operators J and A_λ satisfy the condition $\alpha(X)$ for any $\lambda \in \mathbb{R}$.*

Proof. Since F and G_λ are compact maps, it is sufficient to check that J satisfies the $\alpha(X)$ condition. Assume that $u_n \rightharpoonup u_0$ in X and $\limsup_{n\to+\infty}\langle J(u_n), u_n - u_0\rangle \le 0$. Then u_n converges strongly to u_0 in $L^p(\Omega)$ and we have

$$\begin{aligned}
0 &\ge \limsup_{n\to+\infty}\langle J(u_n) - J(u_0), u_n - u_0\rangle \\
&= \limsup_{n\to+\infty} \int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_0|^{p-2}\nabla u_0\right)(\nabla u_n - \nabla u_0) \\
&\quad + \int_\Omega \left(|u_n|^{p-2}u_n - |u_0|^{p-2}u_0)(u_n - u_0\right).
\end{aligned}$$

For $\nabla u, \nabla v \in L^p(\Omega)^N$, we observe that

$$\begin{aligned}
&\int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\nabla(u - v) \\
&= \int_\Omega \left(|\nabla u|^p + |\nabla v|^p - |\nabla u|^{p-2}\nabla u\nabla v - |\nabla v|^{p-2}\nabla v\nabla u\right) \\
&\ge \int_\Omega (|\nabla u|^p + |\nabla v|^p) - \Big(\int_\Omega |\nabla u|^p\Big)^{1/p'}\Big(\int_\Omega |\nabla v|^p\Big)^{1/p} \\
&\quad - \Big(\int_\Omega |\nabla u|^p\Big)^{1/p}\Big(\int_\Omega |\nabla v|^p\Big)^{1/p'} \\
&= \Big[\Big(\int_\Omega |\nabla u|^p\Big)^{\frac{p-1}{p}} - \Big(\int_\Omega |\nabla v|^p\Big)^{\frac{p-1}{p}}\Big]\Big[\Big(\int_\Omega |\nabla u|^p\Big)^{1/p} - \Big(\int_\Omega |\nabla v|^p\Big)^{1/p}\Big] \ge 0.
\end{aligned}$$

From this two previous inequalities we deduce that $\int_\Omega |\nabla u_n|^p \to \int_\Omega |\nabla u_0|^p$ and consequently $u_n \to u_0$ in X. $\square$

3. Bifurcation from zero at the first eigenvalue

In this section we prove that there exists a bifurcation of solutions of problem (2.1) at $\lambda = \lambda_1$ using the previous generalization of the topological degree. Let us recall the standard definition of a bifurcation point for a generic family of nonlinear continuous maps $A_\lambda : V \to V^*$:

Definition 3.1. Let $E = \mathbb{R} \times V$ be equipped with the norm

$$\|(\lambda, u)\| := \big(|\lambda|^2 + \|u\|_V^2\big)^{1/2}. \tag{3.1}$$

A value $\bar\lambda \in \mathbb{R}$ is said to be a bifurcation point from zero of solutions of the problem $A_\lambda(u) = 0$ if there exists a sequence of solutions (λ_n, u_n) of $A_\lambda(u) = 0$ such that $(\lambda_n, u_n) \to (\bar\lambda, 0)$ in E.

Lemma 3.2. *Assume that*

$$g(\lambda, x, s) = o(|s|^{p-1}) \tag{3.2}$$

holds for s near $s = 0$ uniformly a.e. with respect to $x \in \partial\Omega$ and uniformly with respect to λ in any bounded subset of $\mathbb{R}$. Then

$$\lim_{\|u\|_X \to 0} \frac{\|G_\lambda(u)\|_{X^*}}{\|u\|_X^{p-1}} = 0$$

uniformly for λ in a bounded subset of $\mathbb{R}$.

Proof. Condition (3.2) implies that for any λ in a bounded set and for any $\varepsilon > 0$ there is a real number $\delta = \delta(\varepsilon) > 0$ such that, for a.e. $x \in \partial\Omega$, we have

$$|g(\lambda, x, s)| \le \varepsilon |s|^{p-1} \quad \text{for } |s| \le \delta.$$

Consider the subset of $\partial\Omega$ given by

$$\partial\Omega_\delta = \{x \in \partial\Omega : |u(x)| \le \delta\}.$$

By definition we have

$$\begin{aligned} \frac{\|G_\lambda(u)\|_{X^*}}{\|u\|_X^{p-1}} &= \sup_{\|w\|_X \le 1} \frac{1}{\|u\|_X^{p-1}} \Big| \int_{\partial\Omega} g(\lambda, x, u) w \Big| \\ &\le \sup_{\|w\|_X \le 1} \int_{\partial\Omega_\delta} \Big| \frac{g(\lambda, x, u)}{\|u\|_X^{p-1}} w \Big| + \sup_{\|w\|_X \le 1} \int_{\partial\Omega_\delta^c} \Big| \frac{g(\lambda, x, u)}{\|u\|_X^{p-1}} w \Big|. \end{aligned}$$

Set $v := u/\|u\|_X$. Then we have

$$\begin{aligned} \int_{\partial\Omega_\delta} \Big| \frac{g(\lambda, x, u)}{\|u\|_X^{p-1}} w \Big| &\le \varepsilon \int_{\partial\Omega_\delta} |w||v|^{(p-1)} \\ &\le \varepsilon \int_{\partial\Omega} |w||v|^{(p-1)} \\ &\le \varepsilon \Big(\int_{\partial\Omega} |w|^p \Big)^{1/p} \Big(\int_{\partial\Omega} |v|^p \Big)^{1/p'} \\ &\le c_p^p \varepsilon \|v\|_X^{p-1} \|w\|_X \le c_p^p \varepsilon \end{aligned} \tag{3.3}$$

where we have denoted, for any $1 \le q \le p_*$, $c_q = c(q, N, \Omega)$ the best constant of the trace embedding of X into $L^q(\partial\Omega)$. In the following, d will always denote a positive constant independent of ϵ, δ, and the functions $u, v, w, \dots$ appearing on the proof. We have

$$\begin{aligned}\int_{\partial\Omega_\delta^c} \Big|\frac{g(\lambda,x,u)}{\|u\|_X^{p-1}} w\Big| &\le \int_{\partial\Omega_\delta^c} \frac{(C+D|u|^a)|w|}{\|u\|_X^{p-1}} \\ &= \frac{1}{\|u\|_X^{p-1}} \int_{\partial\Omega_\delta^c} C|w| + \int_{\partial\Omega_\delta^c} \frac{D|u|^{a-1}|w|}{\|u\|_X^{p-1}} = I + II\,.\end{aligned}$$

We estimate I and II as follows:

$$\begin{aligned} I &\le \frac{\|C\|_{\infty,\partial\Omega}}{\|u\|_X^{p-1}} \operatorname{meas}(\partial\Omega_\delta^c)^{1/p_*'} \Big(\int_{\partial\Omega_\delta^c} |w|^{p_*}\Big)^{1/p_*} \\ &\le \frac{\|C\|_{\infty,\partial\Omega}}{\|u\|_X^{p-1}} \operatorname{meas}(\partial\Omega_\delta^c)^{1/p_*'} \Big(\int_{\partial\Omega} |w|^{p_*}\Big)^{1/p_*} \\ &\le c_{p_*} \frac{\|C\|_{\infty,\partial\Omega}}{\|u\|_X^{p-1}} \operatorname{meas}(\partial\Omega_\delta^c)^{1/p_*'}\,. \end{aligned}$$

Using Tchevichev's inequality we have , for any $1 < q \le p_*$,

$$\delta^q \operatorname{meas}(\partial\Omega_\delta^c) \le \int_{\partial\Omega_\delta^c} |u|^q,$$

and therefore, choosing $q = p_*$, it follows that

$$\begin{aligned} I &\le c_{p_*} \frac{\|C\|_{\infty,\partial\Omega}}{\|u\|_X^{p-1}} \frac{1}{\delta^{p_*/p_*'}} \Big(\int_{\partial\Omega_\delta^c} |u|^{p_*}\Big)^{1/p_*'} \\ &\le c_{p_*} \frac{\|C\|_{\infty,\partial\Omega}}{\|u\|_X^{p-1}} \frac{1}{\delta^{p_*/p_*'}} \Big(\int_{\partial\Omega} |u|^{p_*}\Big)^{1/p_*'} \\ &\le c_{p_*}^{p_*} \frac{\|C\|_{\infty,\partial\Omega}}{\|u\|_X^{p-1}} \frac{1}{\delta^{p_*/p_*'}} \|u\|_X^{p_*/p_*'} = d\delta^{-p_*/p_*'} \|u\|_X^{p_*-p}. \end{aligned} \tag{3.4}$$

On the other hand,

$$\begin{aligned} II &\le \|D\|_{\infty,\partial\Omega} \int_{\partial\Omega_\delta^c} |u|^{a-p} |v|^{p-1} |w| \\ &\le \|D\|_{\infty,\partial\Omega} \delta^{a-p_*} \int_{\partial\Omega_\delta^c} |u|^{p_*-p} |v|^{p-1} |w| \\ &\le \|D\|_{\infty,\partial\Omega} \delta^{a-p_*} \Big(\int_{\partial\Omega} |v|^{p^*}\Big)^{\frac{p-1}{p^*}} \Big(\int_{\partial\Omega} |w|^{p_*}\Big)^{1/p_*} \Big(\int_{\partial\Omega} |u|^{p_*}\Big)^{\frac{p_*-p}{p_*}} \\ &\le c_{p_*}^{p_*} \|D\|_{\infty,\partial\Omega} \delta^{a-p_*} \|v\|_X^{p-1} \|w\|_X \|u\|_X^{p_*-p} \le d\delta^{a-p_*} \|u\|_X^{p_*-p}. \end{aligned} \tag{3.5}$$

Thus, if $\|u\|_X \le \eta$ with $\eta = \big(\frac{\varepsilon}{\delta^{-p_*/p_*'} + \delta^{a-p_*}}\big)^{\frac{1}{p_*-p}}$, by adding (3.3), (3.4) and (3.5), we obtain

$$\frac{\|G_\lambda(u)\|_{X^*}}{\|u\|_X^{p-1}} \le d\varepsilon.$$

□

The following proposition is standard but we prove it here by the sake of completeness.

Proposition 3.3. *Assume* (3.2). *If* $(\bar\lambda, 0) \in E$ *is a bifurcation point of solutions of problem* (2.1) *then* $\bar\lambda$ *is an eigenvalue of* (2.5).

Proof. Since $(\bar{\lambda}, 0)$ is a bifurcation point from zero of solutions of (2.1) there is a sequence (λ_n, u_n) of nontrivial solutions of (2.1) such that $\lambda_n \to \bar{\lambda}$ in $\mathbb{R}$ and $\|u_n\|_X \to 0$ in X as $n \to +\infty$. Let $\tilde{u}_n := u_n/\|u_n\|_X$. Since the sequence $(\tilde{u}_n)$ is bounded in X, there exists a function $\tilde{u}_0 \in X$ and a subsequence, still denoted by $(\tilde{u}_n)$, such that $\tilde{u}_n \rightharpoonup \tilde{u}_0$, strongly in $L^p(\Omega)$, in $L^p(\partial\Omega)$ and such that $G_\lambda(\tilde{u_n}) \to G_\lambda(\tilde{u}_0)$ in X^*. Hence, using Lemma 3.2 we obtain

$$\limsup_{n\to+\infty} \langle J(\tilde{u}_n), v\rangle - \lambda_n \langle F(\tilde{u}_n), v\rangle = \limsup_{n\to+\infty} \frac{1}{\|u_n\|_X^{p-1}} \langle G_{\lambda_n}(u_n), v\rangle = 0 \tag{3.6}$$

for all $v \in X$. By taking $v = \tilde{u}_n - \tilde{u}_0$ in (3.6) it follows that

$$\limsup_{n\to+\infty} \langle J(\tilde{u}_n), \tilde{u}_n - \tilde{u}_0\rangle = 0$$

and, using that J satisfies the condition $\alpha(X)$, we conclude that $\tilde{u}_n \to \tilde{u}_0$ strongly in X. In particular $\|\tilde{u}_0\|_X = 1$ and $\tilde{u}_0 \neq 0$. Passing to the limit in (3.6) it comes

$$\langle J(\tilde{u}_0), v\rangle = \bar{\lambda}\langle F(\tilde{u}_0), v\rangle,$$

for all $v \in X$. Thus $\bar{\lambda}$ is an eigenvalue of (2.5) and $\tilde{u}_0$ an eigenfunction associated with $\bar{\lambda}$. □

The following theorem is the main result of this section. We prove that the first eigenvalue is a bifurcation point from zero of nontrivial solutions of (2.1).

Theorem 3.4. *Assume* (3.2). *Then there exists a maximal connected set $\mathcal{C}$ of nontrivial weak solutions of problem* (2.1) *which*

(i) *contains the point $(\lambda_1, 0)$ in its closure,*
(ii) *either $\mathcal{C}$ is unbounded in E or it contains in its closure a point $(\tilde{\lambda}, 0)$, where $\tilde{\lambda}$ is an eigenvalue of problem* (2.5) *different from λ_1.*

Proof. The proof consists in three steps:

Step 1. We claim that

$$\deg(A_\lambda, B_r(0), 0) = 1 \quad \forall \lambda \in (0, \lambda_1),\ \forall r > 0 \text{ sufficiently small.} \tag{3.7}$$

The proof of this claim is the following. First consider the operator $\hat{A}_\lambda(u) = J(u) - \lambda F(u)$. It follows from the variational characterization (2.6) of λ_1 that for all $\lambda \in (0, \lambda_1)$ and for all $u \in X$ with $\|u\|_X \neq 0$ we have $\langle \hat{A}_\lambda(u), u\rangle > 0$. Then by Lemma 2.1(i),

$$\deg(\hat{A}_\lambda, B_r(0), 0) = \operatorname{Ind}(\hat{A}_\lambda, 0) = 1, \quad \forall \lambda \in (0, \lambda_1),\ \forall r > 0. \tag{3.8}$$

For any fixed $\lambda \in (0, \lambda_1)$ we claim that, if r is sufficiently small, the equation

$$J(u) - \lambda F(u) - sG_\lambda(u) = 0$$

has no solution u with $\|u\|_X = r$ for all $s \in [0, 1]$. Assume by contradiction that for all $n > 0$ there exists u_n of norm $\frac{1}{n}$ and there exists $s_n \in [0, 1]$ such that

$$\langle J(u_n), v\rangle - \lambda\langle F(u_n), v\rangle - s_n\langle G_\lambda(u_n), v\rangle = 0. \tag{3.9}$$

Taking $w_n := \frac{u_n}{\|u_n\|_X}$ we infer that there exists w_0, s_0 such that, for a subsequence, $w_n \rightharpoonup w_0$, strongly in $L^p(\Omega)$, in $L^p(\partial\Omega)$, such that $G_\lambda(w_n) \to G_\lambda(w_0)$ in X^* and $s_n \to s_0$. Using (3.9) we deduce that

$$\langle J(w_0), v\rangle - \lambda\langle F(w_0), v\rangle \le \lim_n \frac{s_0}{\|u_n\|_X^{p-1}} \langle G_\lambda(u_n), v\rangle \tag{3.10}$$

for all $v \in X$. If $w_0 = 0$ then we will deduce, by taking $v = w_n$ in (3.9) and going to $+\infty$, that $J(w_n) \to 0$. Thus $w_n \to w_0 = 0$, a contradiction with $\|w_n\|_X = 1$. In particular for $v = w_0 \neq 0$ in (3.10) we have, using that $\lambda < \lambda_1$ on the left hand side, that

$$0 < \langle J(w_0), w_0\rangle - \lambda\langle F(w_0), w_0\rangle \leq \frac{s_0}{\|u_n\|_X^{p-1}}\langle G_\lambda(u_n), w_0\rangle \leq \frac{1}{\|u_n\|_X^{p-1}}\langle G_\lambda(u_n), w_0\rangle.$$

Since $\|w_0\|_X \leq 1$ and $\|u_n\|_X \to 0$, we have from Lemma 3.2 that

$$0 = \lim_n \frac{\|G_\lambda(u_n)\|_{X^*}}{\|u_n\|_X^{p-1}} \geq \langle J(w_0), w_0\rangle - \lambda\langle F(w_0), w_0\rangle > 0$$

which is a contradiction. Finally the claim (3.7) follows from (3.8) and the homotopy invariance of the degree.

Step 2. First, from the definition of λ_2 we have $(\lambda_1, \lambda_2) \cap \sigma_p = \emptyset$. We claim that

$$\deg(A_\lambda, B_r(0), 0) = -1 \quad \forall \lambda \in (\lambda_1, \lambda_2),\ \forall r > 0 \text{ small enough}. \tag{3.11}$$

We start by evaluating $\operatorname{Ind}(\hat{A}_\lambda, 0)$ for any $\lambda \in (\lambda_1, \lambda_2)$ by using the procedure of [10]. Let us denote $\delta := \lambda_2 - \lambda_1$ and define, for a fixed number $K > 0$, $\Phi : \mathbb{R} \to \mathbb{R}$, a twice continuously differentiable function Φ as follows

$$\Phi(t) = \begin{cases} 0 & \text{if } t \leq K, \\ \frac{2\delta}{\lambda_1}(t - 2K) & \text{if } t \geq 3K, \end{cases} \tag{3.12}$$

and $\Phi(t)$ is positive and strictly convex in $(K, 3K)$. Define the functional $\Psi_\lambda : X \to \mathbb{R}$ by

$$\Psi_\lambda(u) = \frac{1}{p}\langle J(u), u\rangle - \frac{\lambda}{p}\langle F(u), u\rangle + \Phi\big(\frac{1}{p}\langle J(u), u\rangle\big). \tag{3.13}$$

It follows from Lemma 3.5 below that $\pm\ell\varphi_1$ are the global minima of Ψ_λ, where $\ell > 0$ is the unique positive constant such that $\Phi'\big(\frac{1}{p}\langle J(\ell\varphi_1), \ell\varphi_1\rangle\big) = \frac{\lambda-\lambda_1}{\lambda_1}$ and φ_1 is the (unique) positive eigenfunction associated with λ_1 satisfying $\|\varphi_1\|_X = 1$. Hence by Lemma 2.1(ii),

$$\operatorname{Ind}(\Psi'_\lambda, -\ell\varphi_1) = \operatorname{Ind}(\Psi'_\lambda, \ell\varphi_1) = 1. \tag{3.14}$$

On the other hand, let us show that for any $u \in X$ satisfying $\|u\|_X = \sigma$ with $\sigma > (3Kp)^{1/p}$, one has $\langle \Psi'_\lambda(u), u\rangle > 0$. Indeed

$$\begin{aligned}\langle \Psi'_\lambda(u), u\rangle &= \langle J(u), u\rangle - \lambda\langle F(u), u\rangle + \langle J(u), u\rangle\Phi'\Big(\frac{1}{p}\langle J(u), u\rangle\Big) \\ &\geq (1 + \frac{2\delta}{\lambda_1})\langle J(u), u\rangle - \lambda\langle F(u), u\rangle \\ &\geq (\lambda_1 - \lambda + 2\delta)\langle F(u), u\rangle \\ &\geq \delta\langle F(u), u\rangle > 0.\end{aligned}$$

Hence, by Lemma 2.1(i), we have

$$\deg(\Psi'_\lambda, B_\sigma(0), 0) = 1. \tag{3.15}$$

Thus, if we choose σ large enough in order to have $\pm\ell\varphi_1 \in B_\sigma(0)$, by the additivity property of the degree and the results (3.14) and (3.15) we deduce that

$$\operatorname{Ind}(\Psi'_\lambda, 0) = -1.$$

Furthermore since $\Phi = 0$ near 0 we have

$$\mathrm{Ind}(\hat{A}_\lambda, 0) = -1, \quad \forall \lambda \in (\lambda_1, \lambda_2).$$

Next we claim that for any r sufficiently small the equation

$$\langle J(u), v\rangle - \lambda \langle F(u), v\rangle - s\langle G_\lambda(u), v\rangle = 0 \quad \forall v \in X$$

has no solution u with $\|u\|_X = r$ for all $s \in [0,1]$. As in step 1, assume by contradiction that for all $n > 0$ there exists a solution u_n of norm $1/n$ and there exists $s_n \in [0,1]$ such that

$$J(u_n) - \lambda F(u_n) - s_n G_\lambda(u_n) = 0. \tag{3.16}$$

Taking $w_n := \frac{u_n}{\|u_n\|_X}$ there exists w_0, s_0 such that, up to a subsequence, $w_n \rightharpoonup w_0$, strongly in $L^p(\Omega)$, in $L^p(\partial\Omega)$, $G_\lambda(w_n) \to G_\lambda(w_0)$ in X^* and $s_n \to s_0$. If $w_0 = 0$ then we will deduce, by taking $v = \frac{w_n}{\|u_n\|_X^{p-1}}$ as test function in (3.16) and going to $+\infty$, that $J(w_n) \to 0$. That is, $w_n \to w_0 = 0$ which is a contradiction with $\|w_n\|_X = 1$. Hence $w_0 \neq 0$ and using again (3.16) for the test function $w_n - w_0$, we obtain, after dividing by $\|u_n\|_X^{p-1}$ and going to $+\infty$, that

$$\langle J(w_n), w_n - w_0\rangle \to 0.$$

We have just proved that $w_n \to w_0$ strongly in X. On the other hand we will have, passing to the limit in (3.16) after normalization, that

$$\langle J(w_0), v\rangle - \lambda\langle F(w_0), v\rangle = \lim_n \frac{s_0}{\|u_n\|_X^{p-1}} \langle G_\lambda(u_n), v\rangle = 0$$

for all $v \in X$. We have used here Lemma 3.2. Notice that is a contradiction since λ is not an eigenvalue. We have just prove the claim. Finally (3.11) follows from the homotopy invariance of the degree for $r > 0$ sufficiently small.

Step 3. Having proved (3.7) and (3.11) we can proceed step by step as in the original proof of Rabinowitz [19, Theorem 1.3, pp. 490-491], cf. also Drábek [9, Theorem 14.9, pp. 178-183] to get the desired result. □

Lemma 3.5. *Let Φ be defined in* (3.12) *and the functional Ψ_λ be defined in* (3.13) *for $\lambda \in (\lambda_1, \lambda_2)$. Then Ψ_λ is lower semicontinuous and coercive. The critical points of Ψ_λ are $0, \pm\ell\varphi_1$, where $\ell > 0$ is such that $\Phi'\big(\frac{1}{p}\langle J(\ell\varphi_1), \ell\varphi_1\rangle\big) = \frac{\lambda-\lambda_1}{\lambda_1}$ and φ_1 the positive eigenfunction associated with λ_1 with $\|\varphi_1\|_X = 1$. Moreover*

$$\min_X \Psi_\lambda = \Psi_\lambda(\pm\ell\varphi_1) < 0.$$

Proof. Clearly Ψ_λ is weakly lower semicontinuous. Indeed, assume $u_n \rightharpoonup u_0 \in X$. Then by the compact embedding of X in $L^p(\partial\Omega)$, we have

$$\langle F(u_n), u_n\rangle \to \langle F(u_0), u_0\rangle$$

and from the weak lower semicontinuity of the norm and the monotony of Φ, we have

$$\liminf_{n\to+\infty} \Big\{\frac{1}{p}\langle J(u_n), u_n\rangle + \Phi\Big(\frac{1}{p}\langle J(u_n), u_n\rangle\Big)\Big\} \geq \frac{1}{p}\langle J(u_0), u_0\rangle + \Phi\Big(\frac{1}{p}\langle J(u_0), u_0\rangle\Big).$$

and the result follows. Let us denote as before $\delta := \lambda_2 - \lambda_1$ and let us now show that Ψ_λ is coercive. Indeed, if $\|u\|_X \to +\infty$, two cases can occur. First if $\langle F(u), u\rangle$

is bounded, it follows immediately that $\Psi_\lambda(u) \to +\infty$ as $\|u\|_X \to +\infty$. If not, that is, if $\langle F(u), u\rangle \to +\infty$ as $\|u\|_X \to +\infty$, we obtain

$$\begin{aligned}\Psi_\lambda(u) &= \frac{1}{p}\langle J(u), u\rangle - \frac{\lambda}{p}\langle F(u), u\rangle + \Phi\Big(\frac{1}{p}\langle J(u), u\rangle\Big)\\ &\geq \frac{\lambda_1-\lambda}{p\lambda_1}\langle J(u), u\rangle + \Phi\Big(\frac{1}{p}\langle J(u), u\rangle\Big)\\ &\geq \frac{-\delta}{p\lambda_1}\langle J(u), u\rangle + \frac{2\delta}{\lambda_1}\Big(\frac{1}{p}\langle J(u), u\rangle - 2K\Big)\\ &= \frac{\delta}{p\lambda_1}\langle J(u), u\rangle - \frac{4\delta K}{\lambda_1} \to +\infty\end{aligned}$$

as $\|u\|_X \to +\infty$. In both cases we conclude that $\lim_{\|u\|_X\to+\infty}\Psi_\lambda(u) = +\infty$. Hence Ψ_λ is bounded from below and the minimum of Ψ_λ is achieved. Now we claim that the minimum is achieved at some eigenfunction associated with λ_1. First notice that

$$\langle \Psi_\lambda'(u), v\rangle = \langle J(u), v\rangle - \lambda\langle F(u), v\rangle + \langle J(u), v\rangle\Phi'\Big(\frac{1}{p}\langle J(u), u\rangle\Big)$$

for all $v \in X$. A critical point $u_0 \in X$ of Ψ_λ satisfies

$$\langle \Psi_\lambda'(u_0), v\rangle = \langle J(u_0), v\rangle - \lambda\langle F(u_0), v\rangle + \langle J(u_0), v\rangle\Phi'\Big(\frac{1}{p}\langle J(u_0), u_0\rangle\Big) = 0$$

for all $v \in X$. This implies

$$\langle J(u_0), v\rangle - \mu\langle F(u_0), v\rangle = 0 \quad \forall v \in X,$$

with

$$\mu = \frac{\lambda}{1 + \Phi'\big(\frac{1}{p}\langle J(u_0), u_0\rangle\big)}.$$

Since $\lambda \in (\lambda_1, \lambda_2)$ and $\Phi'(t) \geq 0$ for all $t \in \mathbb{R}$ then $\mu \leq \lambda \leq \lambda_2$. As μ is an eigenvalue then it must be $\mu = \lambda_1$ which implies

$$0 \neq \Phi'\Big(\frac{1}{p}\langle J(u_0), u_0\rangle\Big) = \frac{\lambda}{\lambda_1} - 1 = \frac{\lambda-\lambda_1}{\lambda_1} \neq \frac{2\delta}{\lambda_1}.$$

Consequently $\frac{1}{p}\langle J(u_0), u_0\rangle \in (K, 3K)$. Also, since $\mu = \lambda_1$, $u_0 = -\ell\varphi_1$ or $u_0 = \ell\varphi_1$. So, for $\lambda \in (\lambda_1, \lambda_2)$, Ψ_λ has precisely three isolated critical points $-\ell\varphi_1$, 0, $\ell\varphi_1$. In following, we show that the minimum of Ψ_λ is achieved in $-\ell\varphi_1$ and $\ell\varphi_1$. We claim that $\Psi_\lambda(\pm\ell\varphi_1) < 0$. Indeed one has

$$\Psi_\lambda(\ell\varphi_1) = \frac{\lambda_1-\lambda}{\lambda_1 p}\langle J(\ell\varphi_1), \ell\varphi_1\rangle + \Phi\Big(\frac{1}{p}\langle J(\ell\varphi_1), \ell\varphi_1\rangle\Big).$$

Since $\frac{1}{p}\langle J(\ell\varphi_1), \ell\varphi_1\rangle \in (K, 3K)$, the convexity of Φ implies that for all $t < 3K$,

$$\begin{aligned}&\Psi_\lambda(\ell\varphi_1)\\ &\leq \frac{\lambda_1-\lambda}{p\lambda_1}\langle J(\ell\varphi_1), \ell\varphi_1\rangle + \Phi(t) - \Phi'\Big(\frac{1}{p}\langle J(\ell\varphi_1), \ell\varphi_1\rangle\Big)\Big(t - \frac{1}{p}\langle J(\ell\varphi_1), \ell\varphi_1\rangle\Big)\\ &= \frac{\lambda_1-\lambda}{p\lambda_1}\langle J(\ell\varphi_1), \ell\varphi_1\rangle + \Phi(t) - \frac{\lambda-\lambda_1}{\lambda_1}\Big(t - \frac{1}{p}\langle J(\ell\varphi_1), \ell\varphi_1\rangle\Big)\\ &= \Phi(t) - \frac{\lambda-\lambda_1}{\lambda_1}t.\end{aligned}$$

In particular for $t = K$, one deduces that

$$\Psi_\lambda(\ell\varphi_1) \leq \Phi(K) - \frac{\lambda - \lambda_1}{\lambda_1} K = -\frac{\lambda - \lambda_1}{\lambda_1} K < 0.$$

Since Ψ_λ is even then we have $\Psi_\lambda(\pm\ell\varphi_1) < 0 = \Psi_\lambda(0)$ and the minima of Ψ_λ are then achieved in $-\ell\varphi_1$ and $\ell\varphi_1$. □

4. Bifurcation from infinity at the first eigenvalue

In this section we study the bifurcation from infinity. Let us first recall its definition.

Definition 4.1. We say that $(\bar\lambda, +\infty)$ is a bifurcation point of solutions of problem (2.1) from infinity if every neighbourhood of $(\bar\lambda, +\infty)$ contains a solution of problem (2.1), i.e. there exists a sequence (λ_n, u_n) of solutions of problem (2.1) such that $\lambda_n \to \bar\lambda$ and $\|u_n\|_X \to +\infty$.

Lemma 4.2. *Assume that*

$$g(\lambda, x, s) = o(|s|^{p-1}) \tag{4.1}$$

holds for s large, uniformly a.e. with respect to $x \in \partial\Omega$ and uniformly for λ in any bounded set. Then

$$\lim_{\|u\|_X \to \infty} \frac{\|G_\lambda(u)\|_{X^*}}{\|u\|_X^{p-1}} = 0$$

uniformly for λ in a bounded subset of $\mathbb{R}$.

Proof. By condition (4.1) we mean that for any λ in a bounded set and for any $\varepsilon > 0$, there is a real $R = R(\varepsilon) > 0$ with R large enough such that for a.e. $x \in \partial\Omega$, we have

$$|g(\lambda, x, s)| \leq \varepsilon |s|^{p-1} \quad \text{for } |s| \geq R.$$

Consider now the subset of $\partial\Omega$ given by $\partial\Omega_R := \{x \in \partial\Omega : |u(x)| \geq R\}$ and let us compute separately

$$\sup_{\|w\|_X \leq 1} \int_{\partial\Omega_R} \Big| \frac{g(\lambda, x, u)}{\|u\|_X^{p-1}} w \Big| \quad \text{and} \quad \sup_{\|w\|_X \leq 1} \int_{\partial\Omega_R^c} \Big| \frac{g(\lambda, x, u)}{\|u\|_X^{p-1}} w \Big|.$$

By setting $v := u/\|u\|_X$, we have for any $\|w\|_X \leq 1$,

$$\int_{\partial\Omega_R} \Big| \frac{g(\lambda, x, u)}{\|u\|_X^{p-1}} w \Big| = \int_{\partial\Omega_R} \Big| \frac{g(\lambda, x, u)}{|u|^{p-1}} |v|^{p-1} w \Big| \leq c\varepsilon \|v\|_X^{p-1} \|w\|_X \leq c\varepsilon;$$

for some constant $c > 0$ independent of u, v, w, or λ. On the other hand, using (2.2),

$$\int_{\partial\Omega_R^c} |g(\lambda, x, u) w| \leq \int_{\partial\Omega_R^c} (C + D|u|^{a-1})|w| \leq c(1 + R^{a-1}) \int_{\partial\Omega_R^c} |w| \leq d(1 + R^{a-1}).$$

Combining this two inequalities we obtain that for all $\varepsilon > 0$ if $\|u\|_X \geq B_0 := \big(\frac{1}{\varepsilon} d(1 + R^{a-1})\big)^{\frac{1}{p-1}}$ then we have $\frac{\|G_\lambda(u)\|_{X^*}}{\|u\|_X^{p-1}} \leq \varepsilon$. The proof is complete. □

Proposition 4.3. *Assume* (4.1) *and the condition*

$$g(\lambda, \cdot, 0) \not\equiv 0 \quad \forall \lambda \in \mathbb{R}. \tag{4.2}$$

If $(\bar\lambda, \infty)$ is a bifurcation point of solutions of problem (2.1) *then $\bar\lambda$ is an eigenvalue of* (2.5).

Proof. Consider the following transformation: for $u \neq 0$ set $v := u.\|u\|_X^{-\frac{p}{p-1}}$, and then $u = v.\|v\|_X^{-p}$. If $u \in X$ is a solution of the problem (2.1) with $u \neq 0$ then we have

$$J(v) - \lambda F(v) - \tilde{G}_\lambda(v) = 0, \tag{4.3}$$

where

$$\tilde{G}_\lambda(v) := \begin{cases} \|v\|_X^{p(p-1)} G_\lambda(v.\|v\|_X^{-p}) & \text{if } v \neq 0, \\ 0 & \text{if } v = 0. \end{cases} \tag{4.4}$$

It is clear that $\tilde{G}_\lambda$ is continuous with respect to v. From assumptions (4.1) and Lemma 4.2 we infer that

$$\frac{\|\tilde{G}_\lambda(v)\|_{X^*}}{\|v\|_X^{p-1}} = \frac{\|G_\lambda(u)\|_{X^*}}{\|u\|_X^{p-1}} \to 0$$

as $\|v\|_X \to 0$. It follows immediately from this transformation that the pair $(\bar{\lambda}, +\infty)$ is bifurcation point of solutions of (2.1) if and only $(\bar{\lambda}, 0)$ is a bifurcation point of solutions for problem (4.3). Notice that the assumption (g_0) implies that (2.1) can not have a trivial solution $(\lambda, u) = (\lambda, 0)$ in E. Finally, the result comes from Proposition 3.3. □

The main result of this section reads as follows.

Theorem 4.4. *Assume that the function g satisfies* (4.1) *and* (4.2)*. Then there exists a maximal connected set $\tilde{\mathcal{C}}$ of nontrivial weak solutions of problem* (2.1) *which contains the point $(\lambda_1, +\infty)$ in its closure and it is either unbounded in E or it contains in its closure a point $(\tilde{\lambda}, +\infty)$ with $\tilde{\lambda} > \lambda_1$ an eigenvalue of problem* (2.5)*.*

Proof. Let $\mathcal{C} \subset E$ be maximal connected set of solutions of (4.3) given by Theorem 3.4 and define the set $\tilde{\mathcal{C}}$ to be the set of all pairs $(\lambda, u) \in E$ such that $u \neq 0$ and $\big(\lambda, \frac{u}{\|u\|_X^{\frac{p}{p-1}}}\big) \in \mathcal{C}$. □

5. Existence of continua of positive and negative solutions

In this section we prove the existence of a continua of solutions of (2.1) that bifurcate from $(\lambda_1, 0)$ in the positive and negative directions φ_1 and $-\varphi_1$, respectively. By this we mean that in a sufficiently small neighbourhood of $(\lambda_1, 0)$ these continua contain only solutions (λ, u) of problem (2.1) satisfying $u = t\varphi_1 + v$ with $\langle v, \varphi_1 \rangle_{L^2(\Omega)} = 0$, and

$$\|\frac{u - t\varphi_1}{t}\|_{C^1(\overline{\Omega})} \to 0$$

as $t \to 0$. Hence $u > 0$ in $\overline{\Omega}$ ($u < 0$ in $\overline{\Omega}$ respectively) if and only if $t > 0\,(t < 0)$, provided $|t| > 0$ is small enough.

Following the work by Dancer [7, Theorem 2], for the linear case with Dirichlet boundary conditions, we will prove similar results about the bifurcation branches obtained in Theorem 3.4 and Theorem 4.4. Similar results for the p-laplacian with Dirichlet boundary conditions can be found in [13, Theorem 3.7], [9] and [10] among others.

The following notation will be used.

$$\mathcal{S} := \overline{\{(\lambda, u) \in E : A_\lambda(u) = 0, u \neq 0\}}^E.$$

An easy consequence of this definition is the following result.

Lemma 5.1. *Let us assume* (3.2). *For all* $(\lambda, u) \in \mathcal{S}$ *with* $\lambda \in (-\infty, \lambda_2)$ *and* $\lambda \neq \lambda_1$ *we have* $u \not\equiv 0$ *on* $\partial\Omega$.

Proof. First we claim that if $u \equiv 0$ on $\partial\Omega$, then $u \equiv 0$ in Ω. Indeed, assume that there exists a sequence (λ_n, u_n) of solutions for (2.1) such that $u_n \to u$ in X and $\lambda_n \to \lambda$. First we have

$$\begin{gathered} -\Delta_p u_n + |u_n|^{p-2}u_n = 0 \quad \text{in } \Omega, \\ |\nabla u_n|^{p-2}\frac{\partial u_n}{\partial \nu} = \lambda_n |u_n|^{p-2}u_n + g(\lambda, x, u_n) \quad \text{on } \partial\Omega, \end{gathered} \tag{5.1}$$

and, since up to a subsequence, $u_n(x) \to u(x) = 0$ a.e. in $\partial\Omega$, we deduce that

$$\lim_{n\to+\infty} g(\lambda_n, x, u_n) = g(\lambda, x, 0) = 0.$$

Passing to the limit (in the weak sense) in (5.1) we have

$$\begin{gathered} -\Delta_p u + |u|^{p-2}u = 0 \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lim_{n\to+\infty} g(\lambda_n, x, u_n) = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

Hence $u \equiv 0$ in Ω. We now set $v_n := \frac{u_n}{\|u_n\|_X}$. Since $\|v_n\|_X = 1$ there exists a function v in X such that $v_n \rightharpoonup v$ in X and strongly in $L^p(\Omega)$ and in $L^p(\partial\Omega)$. Dividing (5.1) by $\|u_n\|_X^{p-1}$ and testing against v_n and using Lemma 3.2 we have

$$1 = \int_\Omega |\nabla v_n|^p + \int_\Omega |v_n|^p = \lambda_n \int_{\partial\Omega} |v_n|^p + \frac{1}{\|u_n\|_X^{p-1}}\langle G_\lambda(u_n), v_n\rangle \to \lambda \int_{\partial\Omega} |v|^p.$$

In particular $v \not\equiv 0$. Dividing (5.1) by $\|u_n\|_X^{p-1}$ and using Lemma 3.2 it follows that

$$\begin{gathered} -\Delta_p v + |v|^{p-2}v = 0 \quad \text{in } \Omega, \\ |\nabla v|^{p-2}\frac{\partial v}{\partial \nu} = \lambda |v|^{p-2}v \quad \text{on } \partial\Omega. \end{gathered}$$

Since $v \not\equiv 0$ then λ is an eigenvalue, which gives a contradiction. □

Given a real number $s > 0$, let us denote an open neighbourhood of $(\lambda_1, 0)$ in E by

$$B_s^E := \{(\lambda, u) \in E : \|u\|_X + |\lambda - \lambda_1| < s\}.$$

Our aim is to prove the following result.

Theorem 5.2. *Let* $g(\lambda, \cdot, \cdot) \in C^\gamma$ *for some* $\gamma > 0$*, uniformly for* λ *in a bounded set, satisfy hypothesis* (3.2). *Assume that there exists* $\delta > 0$ *such that*

$$\forall u \in X,\ 0 < \|u\|_X < \delta \Longrightarrow A_{\lambda_1}(u) \neq 0.$$

Then there are two maximal connected subsets $\mathcal{C}^+$ *and* $\mathcal{C}^-$ *of* $\mathcal{C}$ *(with* $\mathcal{C}$ *provided by Theorem 3.4) containing* $(\lambda_1, 0)$ *in their closures satisfying* $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-$ *and*

- (i) *there exists* $s > 0$ *small enough such that, if* $(\lambda, u) \in \mathcal{C}^\pm \cap B_s^E$*, we can write* $u = \pm t\varphi_1 + v$*, with* $v \in C^{1,\alpha}(\overline{\Omega})$ *satisfying* $\langle v, \varphi_1\rangle_{L^2(\partial\Omega)} = 0$ *and* $t > 0$ *such that*
$$|\lambda - \lambda_1| \to 0 \quad \textit{and} \quad \|v/t\|_{C^{1,\alpha}(\overline{\Omega})} \to 0 \quad \textit{as } t \to 0;$$
- (ii) $\mathcal{C}^\pm$ *are both unbounded. Moreover every solution* $u \in \mathcal{C}^+$ *(resp.* $\mathcal{C}^-$*) is strictly positive (resp. strictly negative) in* $\overline{\Omega}$.

As in the classical semilinear case, the proof of the existence of $\mathcal{C}^{\pm}$ and part (i) of this theorem is based on the following 3 lemmas 5.3, 5.4, 5.5. First, let us fix $\psi \in X^* = (W^{1,p}(\Omega))^*$ such that $\psi(\varphi_1) = 1$. We can take, as in [13], for example

$$\psi(\phi) = \|\varphi_1\|^{-2}_{L^2(\partial\Omega)} \int_{\partial\Omega} \phi\varphi_1 \, d\sigma \quad \forall \phi \in X.$$

Finally for any $\tau > 0$, we define

$$\mathcal{K}^{\pm}_{\tau} := \{(\lambda, u) \in E : \pm\psi(u) > \tau\|u\|_X\}.$$

In particular $\mathcal{K}^{\nu}_{\tau}$, $\nu = \pm$, are open convex cones, $\mathcal{K}^{-}_{\tau} = -\mathcal{K}^{+}_{\tau}$ and $\nu t\varphi_1 \in \mathcal{K}^{\nu}_{\tau}$ for any number $t > 0$. Finally we set $\mathcal{K}_{\tau} := \mathcal{K}^{+}_{\tau} \cup \mathcal{K}^{-}_{\tau}$.

Lemma 5.3. *For every $0 < \tau < 1$ there exists a number $0 < s_0 = s_0(\tau)$ such that*

$$\mathcal{S} \setminus \{(\lambda_1, 0)\} \cap \overline{B}^E_{s_0} \subset \mathcal{K}_{\tau}.$$

Moreover, if $(\lambda, u) \in \mathcal{S} \setminus \{(\lambda_1, 0)\} \cap \overline{B}^E_{s_0}$ and we write $u = t\varphi_1 + v$, with $v \in C^{1,\alpha}(\overline{\Omega})$ satisfying $\psi(v) = 0$ and $|t| > \tau\|u\|_X$ then

$$|\lambda - \lambda_1| \to 0 \text{ and } \|v/t\|_{C^{1,\alpha}(\overline{\Omega})} \to 0 \quad \text{as } t \to 0.$$

Lemma 5.4. *Let τ be sufficiently small. Suppose $\delta_1, \delta_2 > 0$ are such that $0 < \delta_1 + \delta_2 < s_0$ and $A_\lambda(u) \neq 0$ if $\|u\|_X = \delta_1$ and $|\lambda - \lambda_1| \leq \delta_2$. We have*

1. *if $0 < \sigma < \delta_2$ and $\beta = \beta(\sigma) > 0$ is sufficiently small, then $0 < \|u\|_X < \beta$ imply $A_{\lambda_1 \pm \sigma}(u) \neq 0$;*
2. *$\deg(A_{\lambda_1+\sigma}, W^{\mu}, 0) - \deg(A_{\lambda_1-\sigma}, W^{\mu}, 0) = 1$, where*

$$W^{\mu} := \{u \in X : \exists \lambda \in \mathbb{R} : (\lambda, u) \in K^{\mu}_{\tau} \text{ and } \beta < \|u\|_X < \delta_1\}.$$

Lemma 5.5. *Let τ be sufficiently small. For any $0 < \epsilon < s_0$ we define T^{-}_{ϵ} to be the component of $\overline{\mathcal{C}} \setminus (B^E_{\epsilon} \cap \mathcal{K}^{+}_{\tau})$ containing $(\lambda_1, 0)$. If T^{-}_{ϵ} is bounded in E then*

$$\partial B^E_{\epsilon} \cap \mathcal{K}^{+}_{\tau} \cap T^{-}_{\epsilon} \neq \emptyset.$$

Proof of Lemma 5.3. Suppose that for some $\tau > 0$ such a number s_0 does not exist. Then we can find a decreasing sequence $0 < s_n \leq 1$ with $s_n \to 0$ and another sequence $(\lambda_n, u_n) \in \mathcal{S} \setminus (\lambda_1, 0) \cap \overline{B}^E_{s_n}$ such that $|\psi(u_n)| \leq \tau\|u_n\|_X$ for all n. Notice that we must have $u_n \not\equiv 0$ in $\partial\Omega$ for all n large enough because of Lemma 5.1. Since $s_n \to 0$ then $\|u_n\|_X \to 0$ and $\lambda_n \to \lambda_1$. Set $w_n := \frac{u_n}{\|u_n\|_X}$. We have that $w_n \rightharpoonup \pm\varphi_1$ because $w_n \rightharpoonup w$ for some $w \in X$ that will be a solution of

$$-\Delta_p w + |w|^{p-2}w = 0 \quad \text{in } \Omega,$$
$$|\nabla w|^{p-2}\frac{\partial w}{\partial \nu} = \lambda|w|^{p-2}w \quad \text{on } \partial\Omega.$$

Furthermore $w \not\equiv 0$ because

$$\begin{aligned} 1 &= \int_{\Omega} |\nabla w_n|^p + \int_{\Omega} |w_n|^p \\ &= \lambda_n \int_{\partial\Omega} |w_n|^p + \frac{1}{\|u_n\|^{p-1}_X} \int_{\partial\Omega} g(\lambda, x, u_n)w_n \to \lambda_1 \int_{\partial\Omega} |w|^p. \end{aligned}$$

Consequently, $\psi(w_n) \to \pm 1$ and therefore $1 \leq |\psi(w_n)| \leq \tau$. We have just proved that the first statement of the lemma is true for all $0 < \tau < 1$.

To prove the second statement let (λ_n, u_n) be a sequence such that $(\lambda_n, u_n) \in \mathcal{S} \cap \overline{B}^E_{s_0} \subset K_\tau$ and write $u_n = t_n\varphi_1 + v_n$. Here t_n is defined by $t_n := \frac{\int_{\partial\Omega} u_n\varphi_1}{\int_{\partial\Omega}\varphi_1^2}$ so we have that $|t_n| > \tau\|u_n\|_X$. Then, if $t_n \to 0$, it follows that $\|u_n\|_X \to 0$ and we can prove as previously that $\frac{u_n}{\|u_n\|_X} \to \pm\varphi_1$ in X. Consequently,

$$\frac{u_n}{\|u_n\|_X} = \frac{t_n\varphi_1}{\|u_n\|_X} + \frac{v_n}{\|u_n\|_X}$$

which tends to $\pm\varphi_1$. In particular, we have

$$\int_{\partial\Omega} \frac{u_n}{\|u_n\|_X}\varphi_1 + \int_{\partial\Omega} \frac{v_n}{\|u_n\|_X}\varphi_1 \to \pm \int_{\partial\Omega} \varphi_1^2.$$

On another hand, we have $\int_{\partial\Omega} \frac{u_n}{\|u_n\|_X}\varphi_1 = \frac{t_n}{\|u_n\|_X}$ and therefore $\frac{t_n}{\|u_n\|_X} \to \pm 1$. Hence $\frac{v_n}{\|u_n\|_X} \to 0$ in X and consequently

$$\frac{u_n}{t_n} \to \varphi_1, \quad \frac{v_n}{t_n} = \frac{v_n}{\|u_n\|_X}\frac{\|u_n\|_X}{t_n} \to 0 \text{ strongly in } X.$$

Using Lemma 7.3, and more precisely, using (7.8), we have $\|\frac{u_n}{\|u_n\|_X}\|_{C^{1,\alpha}(\overline{\Omega})} \le C$. Thus

$$\|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \le C\|u_n\|_X.$$

Hence for some $\alpha' < \alpha$, that we still denote α, we have $u_n/t_n \to \varphi_1$ in $C^{1,\alpha}(\overline{\Omega})$ and also $\|\frac{v_n}{t_n}\|_{C^{1,\alpha}(\overline{\Omega})} \to 0$. □

Proof of Lemma 5.4. (1) is trivially true. The proof of (2) is the same as the one of [7], we outlined it for completeness. We define

$$G^*_\lambda(u) := \begin{cases} G_\lambda(u) & \text{if } \psi(u) < -\eta\|u\|\|_X, \\ \frac{\psi(u)}{-\eta\|u\|_X} G_\lambda(u) & \text{if } 0 > \psi(u) > -\eta\|u\|\|_X, \\ -G_\lambda(u) & \text{if } \psi(u) > 0, \end{cases}$$
$$A^*_\lambda := J - \lambda F - G^*_\lambda.$$

It is clear that A^*_λ satisfies the $\alpha(X)$ condition. By our hypothesis, the equation $A_{\lambda_1+\delta}(u) = 0$ has no solution on $\partial B_{\delta_1}, \partial B_\beta$, or in $B_{\delta_1} \setminus W^+ \cup W^- \cup B_\beta$ by lemma 5.3. Hence

$$\deg(A^*_{\lambda_1+\sigma}, B_{\delta_1}, 0) = \deg(A^*_{\lambda_1+\sigma}; B_\beta, 0) + \deg(A^*_{\lambda_1+\sigma}, W^-, 0) + \deg(A^*_{\lambda_1+\sigma}, W^+, 0).$$

Since A^*_λ is odd we also have $\deg(A^*_{\lambda_1+\sigma}, W^-, 0) = \deg(A^*_{\lambda_1+\sigma}, W^+, 0)$ so

$$2\deg(A^*_{\lambda_1+\sigma}, W^-, 0) = \deg(A^*_{\lambda_1+\sigma}, B_{\delta_1}, 0) - \deg(A^*_{\lambda_1+\sigma}, B_\beta, 0).$$

Similarly,

$$2\deg(A^*_{\lambda_1-\sigma}, W^-, 0) = \deg(A^*_{\lambda_1-\sigma}, B_{\delta_1}, 0) - \deg(A^*_{\lambda_1-\sigma}, B_\beta, 0).$$

On the one hand one can prove as in step 2 of the proof of Theorem 3.4 that

$$\deg(A^*_{\lambda_1-\sigma}, B_\beta, 0) = -\deg(A^*_{\lambda_1+\sigma}, B_\beta, 0) = 1$$

On the other hand, for $\|u\|_X = \delta_1$ and $|\lambda - \lambda_1| < \delta_2$ we have $A_\lambda(u) \ne 0$ by our assumptions, so the homotopy $\sigma \in (-\delta_1, \delta_1) \to A_{\lambda_1+\sigma}$ is admissible on B_{δ_1} and whence

$$\deg(A_{\lambda_1-\sigma}, B_{\delta_1}, 0) = \deg(A_\lambda, B_{\delta_1}, 0) = \deg(A_{\lambda_1+\sigma}, B_{\delta_1}, 0)$$

holds for all $\lambda \in (\lambda_1 - \delta_1, \lambda_1 + \delta_1)$. Then we conclude, using that $A^*_\lambda = A_\lambda$ along W^-,

$$\deg(A_{\lambda_1+\sigma}, W^-, 0) - \deg(A_{\lambda_1-\sigma}, W^-, 0) = 1.$$

□

Proof of Lemma 5.5. Since we are assuming that zero is an isolated solution of $A_{\lambda_1}(u) = 0$ then one can use Lemma 2 of [7] without no changes. □

Proof Theorem 5.2. One follows step by step the proof of Theorem 2 of [7] [13, Theorem 3.7]. First one defines the sets

$$D^\nu_\epsilon := \text{ component of } (\lambda_1, 0) \cup (\mathcal{S} \cap \overline{B}^E_\epsilon \cap K^\nu_\tau) \text{ containing } (\lambda_1, 0),$$
$$C^\nu_\epsilon \text{ component of } \overline{\mathcal{C} \setminus D^{-\nu}_\epsilon}, \quad \mathcal{C}^\nu = \cup_{0<\epsilon<s_0} C^\nu_\epsilon$$

It is proved in [19][Lemma 6.1] that the definition of $\mathcal{C}^\nu$ is independent of τ, that those sets are connected and that $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-$.

(i) We will only prove the result for $\nu = +$. First, let us prove that $C^+_\varepsilon \subset K^+_\tau$ if $0 < \tau < 1$. Assume by contradiction that there exists a sequence of solutions (λ_n, u_n) with $(\lambda_n, u_n) \in C^+_\varepsilon$, $\lambda_n \to \lambda_1$, $\|u_n\|_X \to 0$ and such that $(\lambda_n, u_n) \notin K^+_\tau$. Since $(\lambda_n, u_n) \in C^+_\varepsilon$ it follows that $(\lambda_n, u_n) \notin D^-_\epsilon$ and we have $\psi(u_n) \geq -\tau\|u_n\|_X$ and since $(\lambda_n, u_n) \notin K^+_\tau$ we will get

$$-\tau\|u_n\|_X \leq \psi(u_n) \leq \tau\|u_n\|_X \tag{5.2}$$

Set $\tilde{u}_n := \frac{u_n}{\|u_n\|_X}$, then $\|\tilde{u}_n\|_X = 1$ and we can prove that $\tilde{u}_n \to \pm\varphi_1$ in $C^{1,\alpha}(\overline{\Omega})$ as in the proof of Lemma 5.3. From the definition of ψ, we have

$$\psi(\tilde{u}_n) \to \psi(\pm\varphi_1) = \pm 1. \tag{5.3}$$

Consequently we obtain from (5.2) and (5.3) that $\tau \geq 1$ in contradiction with the hypothesis $\tau \in (0, 1)$. We have proved that $C^+_\varepsilon \subset K^+_\tau$ if $0 < \tau < 1$. To complete the proof of (i) let $(\lambda, u) \in C^+_\varepsilon$ for $0 < \epsilon < s_0$ and write $u = t\varphi_1 + v$ with $v \in C^{1,\alpha}(\overline{\Omega})$ and $\langle v, \varphi_1\rangle_{L^2(\partial\Omega)} = 0$. Thus $\psi(u) = \psi(t\varphi_1 + v) = t$ and, since $C^+_\varepsilon \subset K^+_\tau$, we have that $\psi(u) > \tau\|u\|_X$ and in particular $t > 0$. The asymptotic behaviour as $t \to 0$ has been already proved in Lemma 5.3.

(ii) Let us first show that solutions $\mathcal{C}$ are either positive or negative. Indeed it follows from (i) and the fact that $\varphi_1 > 0$ on $\overline{\Omega}$ that $u > 0$ in $\overline{\Omega}$ if $t = \psi(u)$ is small. We claim now that $u > 0$ in $\overline{\Omega}$ for all $u \in \mathcal{C}^+$. Indeed, if not, there would exist $(\hat{\lambda}, \hat{u}) \in \mathcal{C}^+$ such that $\hat{u}(x) \leq 0$ at some point $x \in \overline{\Omega}$. Since $\mathcal{C}^+$ is connected and the solutions are positive if the norm is small, we conclude the existence of some $(\tilde{\lambda}, \tilde{u})$ in C such that $\tilde{u} \not\equiv 0$, $\tilde{u} \geq 0$ and $\tilde{u}(x_0) = 0$ for some $x_0 \in \overline{\Omega}$. If $x_0 \in \Omega$ then we have a contradiction with the Harnack's inequality, see [20]; if $x_0 \in \partial\Omega$ then it follows from the boundary condition of (2.1) and $g(\lambda, x, 0) = 0$ that $\frac{\partial u}{\partial \nu}(x_0) = 0$, which contradicts the maximum principle of Vazquez, see [23]. As the consequence of the definitive sign of the solutions the case $\overline{\mathcal{C}^+} \cap \overline{\mathcal{C}^-} \neq \{(\lambda_1, 0)\}$ can not occur. Hence it follows from Theorem 2 of Dancer in [7] that $\overline{\mathcal{C}^+}$ and $\overline{\mathcal{C}^-}$ must be unbounded. □

Corollary 5.6. *Let $g(\lambda, \cdot, \cdot) \in C^\gamma$ for some $\gamma > 0$, uniformly for λ in a bounded set, satisfy the hypothesis* (3.2)*. Moreover assume that the function g satisfies the hypothesis: There exists $\delta > 0$ such that for all $s \in]-\delta, \delta[$, $s \neq 0$ we have*

$$g(\lambda_1, x, s)s < 0 \quad \text{a.e. } x \in \partial\Omega. \tag{5.4}$$

Then the conclusions of Theorem 5.2 hold.

Proof. If $u \not\equiv 0$ in Ω, it follows from (2.1) that $u \not\equiv 0$ on $\partial\Omega$. Let us take $u \in X$, $u \not\equiv 0$, a solution of problem (2.1). We take $v = u$ in its weak form to get

$$\int_\Omega |\nabla u|^p + \int_\Omega |u|^p - \lambda_1 \int_{\partial\Omega} |u|^p = \int_{\partial\Omega} g(\lambda_1, x, u)u. \tag{5.5}$$

Then it follows from variational characterization of the first eigenvalue λ_1 of (2.5) that the term of the left hand of (5.5) is positive. On another hand, from Lemma 7.3, there exists $\delta' > 0$ such that if $\|u\|_X < \delta'$ implies $\|u\|_{\infty,\partial\Omega} < \delta$. Then we have $\int_{\partial\Omega} g(\lambda, x, u)u < 0$, a contradiction with (5.5). □

Remark 5.7. The results on existence of positive and negative continua of solutions are proved by Girg-Takáč in [13] for the Dirichlet problem under more restrictive conditions on g, including the restriction that g does not depend of λ.

To obtain similar results for the bifurcation from infinity, we consider the standard transformation $u \mapsto v := u.\|u\|^{-\frac{p}{p-1}}$ as in Theorem 4.4. Thus we obtain the following result.

Theorem 5.8. *Assume that the function g satisfies* (4.1) *and* (4.2) *for every $\lambda \in \mathbb{R}$. Assume that $g \in C^\gamma$ for some $\gamma > 0$, uniformly for λ in a bounded set. Furthermore, assume that there exists $\delta > 0$ such that*

$$J(v) = \lambda_1 F(v) + \tilde{G}_{\lambda_1}(v)$$

has no nontrivial solution $v \in X$, $0 < \|v\|_X < \delta$, where $\tilde{G}_\lambda$ is given by (4.4). *Then there are two maximal connected subsets such that $\tilde{\mathcal{C}} = \tilde{\mathcal{C}}^+ \cap \tilde{\mathcal{C}}^-$ of $\mathcal{S}$ (with $\tilde{\mathcal{C}}$ provided from Theorem 4.4) containing $(\lambda_1, +\infty)$ in their closure. Moreover*

(i) *there exists $s > 0$ such that, if $(\lambda, u) \in \tilde{\mathcal{C}}^\pm$ satisfying $|\lambda - \lambda_1| < s$, $\|u\|_X > s$ we can write $u = \pm t\varphi_1 + v$, with $v \in C^{1,\alpha}(\overline{\Omega})$ satisfying $\langle v, \varphi_1\rangle_{L^2(\partial\Omega)} = 0$ and $t > 0$ such that*

$$|\lambda - \lambda_1| \to 0 \text{ and } \|v/t\|_{C^{1,\alpha}(\overline{\Omega})} \to 0 \text{ as } t \to +\infty;$$

(ii) *$\tilde{\mathcal{C}}^\pm$ are both unbounded. Moreover every solution $u \in \tilde{\mathcal{C}}^+$ (resp. $\tilde{\mathcal{C}}^-$) is positive (resp. negative) in $\overline{\Omega}$.*

The proof of the above theorem is straight forward and we omit it.

Corollary 5.9. *Let $g \in C^\gamma$ for some $\gamma > 0$, uniformly for λ in a bounded set, satisfy the hypothesis* (4.1) *and* (4.2). *Assume further that*

$$g(\lambda_1, x, s)s < 0 \quad \text{a.e. } x \in \partial\Omega,\ \forall |s| > \delta \text{ for some } \delta > 0. \tag{5.6}$$

Then the conclusions of Theorem 5.8 hold.

6. Sub and super critical bifurcation

In this section we study whether the previous bifurcation are placed to the right or to the left of λ_1.

Definition 6.1. (1) If $\bar{\lambda}$ is a *bifurcation point from zero* of solutions $(\lambda, u) \in \mathbb{R} \times X$ of $A_\lambda(u) = 0$, we say that such bifurcation is *subcritical* (respectively *supercritical*) if there exists a neighbourhood $\mathbb{V}$ of $(\bar{\lambda}, 0)$ in $\mathbb{R} \times X$, such that every nontrivial solution $(\lambda, u) \in \mathbb{V}$ satisfies $\lambda < \bar{\lambda}$ (respectively $\lambda > \bar{\lambda}$).

(2) Similarly, we say that a *bifurcation point at $\lambda = \bar{\lambda}$ from infinity* of solutions $(\lambda, u) \in \mathbb{R} \times X$ of $A_\lambda(u) = 0$ is *subcritical* (respectively *supercritical*) if there exists

$\varepsilon, M > 0$, $\mathbb{W} = [\bar{\lambda} - \varepsilon, \bar{\lambda} + \varepsilon] \times (X \setminus B_X(0, M))$, such that every solution $(\lambda, u) \in \mathbb{W}$ satisfies $\lambda < \bar{\lambda}$ (respectively $\lambda > \bar{\lambda}$). Here $B_X(0, M)$ denotes the open ball of X of center 0 and radius M.

The following lemma will be used to get the conditions of subcritical or supercritical bifurcation.

Lemma 6.2. *Let $u \in X \cap C(\overline{\Omega})$ be a solution of (2.1) strictly positive in $\overline{\Omega}$. Then*

$$\frac{\int_{\partial\Omega} g(\lambda, x, u) \frac{\varphi_1^p}{u^{p-1}}}{\int_{\partial\Omega} \varphi_1^p} \leq \lambda_1 - \lambda \leq \frac{\int_{\partial\Omega} g(\lambda, x, u) u}{\int_{\partial\Omega} u^p}. \tag{6.1}$$

Proof. By taking $v = u$ in weak form of problem (2.1) we obtain

$$\int_{\partial\Omega} g(\lambda, x, u) u = \int_{\Omega} |\nabla u|^p + \int_{\Omega} u^p - \lambda \int_{\partial\Omega} u^p \geq (\lambda_1 - \lambda) \int_{\partial\Omega} u^p.$$

On the other hand if $v = \frac{\varphi_1^p}{u^{p-1}}$ we obtain

$$\begin{aligned}
&\int_{\partial\Omega} g(\lambda, x, u) \frac{\varphi_1^p}{u^{p-1}} - (\lambda_1 - \lambda) \int_{\partial\Omega} \varphi_1^p \\
&= \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \Big(\frac{\varphi_1^p}{u^{p-1}} \Big) + \int_{\Omega} u^{p-1} \Big(\frac{\varphi_1^p}{u^{p-1}} \Big) - \int_{\Omega} |\nabla \varphi_1|^p - \int_{\Omega} \varphi_1^p \\
&= \int_{\Omega} p |\nabla u|^{p-2} \frac{\varphi_1^{p-1}}{u^{p-1}} \nabla u \nabla \varphi_1 - \int_{\Omega} (p-1) \frac{\varphi_1^p}{u^p} |\nabla u|^p - \int_{\Omega} |\nabla \varphi_1|^p \\
&= - \int_{\Omega} L(\varphi_1, u) \leq 0,
\end{aligned}$$

where $L(\varphi_1, u)$ is the expression of Picone's identity (see [1]).

Let $w \geq 0, v > 0$ be two continuous functions in Ω differentiable a.e. Denote

$$\begin{gathered}
L(w, v) = |\nabla w|^p + (p-1) \frac{w^p}{v^p} |\nabla v|^p - p \frac{w^{p-1}}{v^{p-1}} |\nabla v|^{p-2} \nabla v \nabla w \,, \\
R(w, v) = |\nabla w|^p - |\nabla v|^{p-2} \nabla \big(\frac{w^p}{v^{p-1}} \big) \nabla v.
\end{gathered}$$

Then (i) $L(w, v) = R(w, v)$, (ii) $L(w, v) \geq 0$ a.e., and (iii) Assume that w/v belongs to $W^{1,1}_{\rm loc}(\Omega)$. Then $L(w, v) = 0$ a.e. in Ω if and only if $w = kv$ for some $k \in \mathbb{R}$.

Therefore we obtain

$$\frac{\int_{\partial\Omega} g(\lambda, x, u) \frac{\varphi_1^p}{u^{p-1}}}{\int_{\partial\Omega} \varphi_1^p} \leq (\lambda_1 - \lambda).$$

□

The following theorem give sufficient conditions to have subcritical and supercritical bifurcation from infinity of positive solutions of (2.1).

Theorem 6.3. *Assume (4.1), (4.2) and that $g \in C^\gamma$ for some $\gamma > 0$ uniformly for λ in a bounded set. Assume that there exist $s_0 > 0$, $\alpha \in \mathbb{R}$ and $\underline{B} \in L^1(\partial\Omega)$ (respectively $\overline{B} \in L^1(\partial\Omega)$) such that*

1. *$g(\lambda, x, s) s^\alpha \geq \underline{B}(x)$ (respectively $g(\lambda, x, s) s^\alpha \leq \overline{B}(x)$) for all $s > s_0$, and for λ in neighbourhood of λ_1, a.e. $x \in \partial\Omega$;*

(2) *for* $\underline{A}_\alpha(x) := \liminf_{(\lambda,s)\to(\lambda_1,+\infty)} g(\lambda,x,s)s^\alpha$*, one has* $\int_{\partial\Omega} \underline{A}_\alpha \varphi_1^{1-\alpha} > 0$ *(respectively* $\overline{A}_\alpha(x) := \limsup_{(\lambda,s)\to(\lambda_1,+\infty)} g(\lambda,x,s)s^\alpha$*, one has* $\int_{\partial\Omega} \overline{A}_\alpha \varphi_1^{1-\alpha} < 0$*).*

Then the bifurcation from infinity at $\lambda = \lambda_1$ *of positive solutions of* (2.1) *is subcritical (respectively supercritical).*

Proof. Let (λ_n, u_n) be a sequence of strictly positive solutions (2.1) converging to $(\lambda_1, +\infty)$ and assume by contradiction that $\lambda_n > \lambda_1$. Using the inequality (6.1), we have

$$\int_{\partial\Omega} g(\lambda_n, x, u_n) \frac{\varphi_1^p}{u_n^{p-1}} \le (\lambda_1 - \lambda_n) \int_{\partial\Omega} \varphi_1^p < 0.$$

and therefore

$$\int_{\partial\Omega} g(\lambda_n, x, u_n) u_n^\alpha \varphi_1^p \frac{\|u_n\|_X^{p-1+\alpha}}{u_n^{p-1+\alpha}} < 0. \tag{6.2}$$

We know from Theorem 5.2 that $v_n := \frac{u_n}{\|u_n\|_X} \to \varphi_1 > 0$ in $C^{1,\alpha}(\overline{\Omega})$. Hence $\varphi_1^p \frac{\|u_n\|_X^{p-1+\alpha}}{u_n^{p-1+\alpha}} \ge C > 0$ for n large and C independent of n and we can use hypothesis (1) to estimate

$$g(\lambda_n, x, u_n) u_n^\alpha \varphi_1^p \frac{\|u_n\|_X^{p-1+\alpha}}{u_n^{p-1+\alpha}} \ge C\underline{B} \in L^1(\partial\Omega).$$

By Fatou's Lemma and hypothesis (2) we obtain

$$\int_{\partial\Omega} \underline{A}_\alpha \varphi_1^{1-\alpha} \le \liminf \int_{\partial\Omega} g(\lambda_n, x, u_n) u_n^\alpha \varphi_1^p \frac{\|u_n\|_X^{p-1+\alpha}}{u_n^{p-1+\alpha}},$$

so

$$\liminf \int_{\partial\Omega} g(\lambda_n, x, u_n) u_n^\alpha \varphi_1^p \frac{\|u_n\|_X^{p-1+\alpha}}{u_n^{p-1+\alpha}} \ge \int_{\partial\Omega} \underline{A}_\alpha \varphi_1^{1-\alpha} > 0.$$

This inequality implies that for n large enough

$$\int_{\partial\Omega} g(\lambda_n, x, u_n) u_n^\alpha \varphi_1^p \frac{\|u_n\|_X^{p-1+\alpha}}{u_n^{p-1+\alpha}} > 0,$$

in contradiction with (6.2). □

We can prove similarly the following theorem about the bifurcation from zero.

Theorem 6.4. *Assume* (3.2) *and that* $g(\lambda,\cdot,\cdot) \in C^\gamma(\partial\Omega \times \mathbb{R})$ *for some* $\gamma > 0$ *and all* λ *in a bounded set. Assume that there exist* $s_0 > 0$, $\alpha \in \mathbb{R}$ *and* $\underline{B} \in L^1(\partial\Omega)$ *(respectively* $\overline{B} \in L^1(\partial\Omega)$*) such that*

(1) $g(\lambda,x,s)s^\alpha \ge \underline{B}(x)$ *(respectively* $g(\lambda,x,s)s^\alpha \le \overline{B}(x)$*) for all* $0 < s < s_0$*, and for* λ *in neighbourhood of* λ_1*, a.e.* $x \in \partial\Omega$*;*
(2) *for* $\underline{A}_\alpha(x) := \liminf_{(\lambda,s)\to(\lambda_1,0^+)} g(\lambda,x,s)s^\alpha$*, one has* $\int_{\partial\Omega} \underline{A}_\alpha \varphi_1^{1-\alpha} > 0$ *(respectively* $\overline{A}_\alpha(x) := \limsup_{(\lambda,s)\to(\lambda_1,0^+)} g(\lambda,x,s)s^\alpha$*, one has* $\int_{\partial\Omega} \overline{A}_\alpha \varphi_1^{1-\alpha} < 0$*).*

Then the bifurcation from zero at $\lambda = \lambda_1$ *of positive solutions of* (2.1) *is subcritical (respectively supercritical).*

6.1. **Application to the anti-maximum principle.** In this paragraph we will give a new proof of the well known anti-maximum principle of problem (6.3), we also obtain the local maximum principle of (6.3). These results were proved by [2] using variational method, and by [3] (for Dirichlet boundary conditions) and by [4] (linear problem with Steklov boundary condition). We have borrowed some ideas from those papers. We consider the simple problem

$$\begin{gathered}\Delta_p u = |u|^{p-2}u \quad \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u + h(x) \quad \text{on } \partial\Omega,\end{gathered} \tag{6.3}$$

Let us denote here by S the set of $h \in L^\infty(\partial\Omega)$ such that the problem (6.3) with $\lambda = \lambda_1$ has solution in X. Notice that $0 \in S$ and that $S \neq L^\infty(\partial\Omega)$ as a result of the following lemma.

Lemma 6.5. *If $h \geq 0$ or $h \leq 0$ and $h \neq 0$ then there exists no solution of* (6.3) *with $\lambda = \lambda_1$.*

Proof. The proof is standard: if for instance $h \geq 0$ and such a solution exists, take as test function $v = u^-$ to obtain

$$\int_\Omega (|\nabla u^-|^p + |u^-|^p) = \lambda_1 \int_{\partial\Omega} |u^-|^p - \int_{\partial\Omega} hu^- \leq \lambda_1 \int_{\partial\Omega} |u^-|^p$$

which implies $u^- = c\varphi_1 \Rightarrow u = -c\varphi_1 \Rightarrow h = 0$, a contradiction. We have used here that $\varphi_1 > 0$ in $\overline{\Omega}$. $\square$

In the next theorem we assume that $h \in C^{\gamma_0}(\partial\Omega)$ which seems a quite strong hypothesis. The reason is that we need some estimates of the C^α-norm of the solutions. Since, to our knowledge, there is not a L^p- regularity theory for the $p-$laplacian operator neither with Dirichlet or Steklov (or Newmann) boundary conditions that would ensure C^α regularity on $\overline{\Omega}$, we use instead the well established $C^{1,\alpha}$-regularity results of Lieberman [15].

Theorem 6.6. *For every $h \in C^{\gamma_0}(\partial\Omega)\setminus S$ with $\int_{\partial\Omega} h\varphi_1 > 0$ there exists $\delta = \delta(h) > 0$ such that for any solution u of* (6.3) *we have*

- (i) *$\lambda \in (\lambda_1, \lambda_1 + \delta)$ implies $u < 0$ in $\overline{\Omega}$,*
- (ii) *$\lambda \in (\lambda_1 - \delta, \lambda_1)$ implies $u > 0$ in $\overline{\Omega}$.*

A similar result can be stated for functions $h \in C^{\gamma_0}(\partial\Omega) \setminus S$ with $\int_{\partial\Omega} h\varphi_1 < 0$ using that u is a solution of (6.3) if and only if $-u$ is a solution of (6.3) with $-h$ instead of h.

Proof. (i) Assume by contradiction that for some $h \in C^\gamma(\partial\Omega)\setminus S$ with $\int_{\partial\Omega} h\varphi_1 > 0$ there is a sequence (λ_n, u_n) of solutions of (6.3) with $\lambda = \lambda_n$, $\lambda_n > \lambda_1$, $\lambda_n \to \lambda_1$ and $u_n(x_n) \geq 0$ for some $x_n \in \overline{\Omega}$. First we claim that $\|u_n\|_X \to +\infty$. Indeed assume by contradiction that $\|u_n\|_X < C$ for some constant $C > 0$. Then there exists a function $u \in X$ and a subsequence (u_n) such that $u_n \rightharpoonup u$ in X, strongly in $L^p(\Omega), L^p(\partial\Omega)$ and a.e. Since (λ_n, u_n) solves (6.3) with $\lambda = \lambda_n$, we have

$$\langle J(u_n), w\rangle - \lambda_n \langle F(u_n), w\rangle - \langle H(u_n), w\rangle = 0 \tag{6.4}$$

for all $w \in X$, where $\langle H(u), v\rangle := \int_{\partial\Omega} hv$. Testing against $w = u_n - u$ and using that the operator $J - \lambda F - H$ satisfies the $\alpha(X)$-condition we conclude that $u_n \to u$ in X. Hence passing to the limit in (6.4) we obtain that u satisfies (6.3) with $\lambda = \lambda_1$, in contradiction with the fact that $h \in L^\infty(\partial\Omega) \setminus S$. Thus we have proved that $\|u_n\|_X \to +\infty$. Dividing now (6.4) by $\|u_n\|_X^{p-1}$ we obtain

$$\langle J(v_n), w\rangle - \lambda_n \langle F(v_n), w\rangle - \frac{1}{\|u_n\|_X^{p-1}} \langle H(u_n), w\rangle = 0$$

for all $w \in X$, where $v_n := \frac{u_n}{\|u_n\|_X}$. Passing again to the limit this last identity we will get that $v_n \to \varphi_1$ or $v_n \to -\varphi_1$ in X. Using that

$$\|\lambda_n |v_n|^{p-2} v_n + \frac{h}{\|u_n\|_X^{p-1}}\|_{p_*} \le C$$

we infer from Proposition 7.1 that $\|v_n\|_\infty + \|v_n\|_{\infty,\partial\Omega} \le C$, for some $C > 0$ independent of n. Then, noticing that $\tilde{h}(\lambda, x, s) := \lambda_n |s|^{p-2} s + \frac{h(x)}{\|u_n\|_X^{p-1}}$ is of class C^γ with $\gamma = \inf\{p-1, \gamma_0\}$ and that $\|\tilde{h}\|_{C^\gamma} \le \tilde{C}$ for some constant independent of n, if follows by Theorem 7.2 that $\|v_n\|_{C^{1,\alpha}(\overline{\Omega})} \le C$ for some $0 < \alpha < 1$ and $C > 0$ independent of n. Hence, up to a subsequence, the sequence v_n converges strongly to either φ_1 or $-\varphi_1$ in $C^{1,\alpha'}(\overline{\Omega})$ for any fixed $0 < \alpha' < \alpha < 1$. Since $u_n(x_n) > 0$ then it must be that $v_n \to \varphi_1$. Then, for n large enough, we have $u_n > 0$ in $\overline{\Omega}$ and it follows from Lemma 6.2 that

$$\int_{\partial\Omega} h \frac{\varphi_1^p}{u_n^{p-1}} \le (\lambda_1 - \lambda_n) \int_{\partial\Omega} \varphi_1^p. \tag{6.5}$$

Observe that

$$\operatorname{sgn}\Big(\int_{\partial\Omega} h \frac{\varphi_1^p}{u_n^{p-1}}\Big) = \operatorname{sgn}\Big(\int_{\partial\Omega} h\varphi_1^p \frac{\|u_n\|_X^{p-1}}{u_n^{p-1}}\Big).$$

Since $\int_{\partial\Omega} h\varphi_1^p \frac{\|u_n\|_X^{p-1}}{u_n^{p-1}} \to \int_{\partial\Omega} h\varphi_1$ and $\int_{\partial\Omega} h\varphi_1 > 0$ by hypothesis, then for n large enough $\int_{\partial\Omega} h \frac{\varphi_1^p}{u_n^{p-1}} > 0$ and we obtain from (6.5) that $\lambda_n < \lambda_1$ for n large, a contradiction.

(ii) Assume now by contradiction that for some $h \in C^\gamma(\partial\Omega) \setminus S$ with $\int_{\partial\Omega} h\varphi_1 > 0$, there is a sequence (λ_n, u_n) of solutions of (6.3) with $\lambda = \lambda_n$ and $\lambda_n < \lambda_1$, $\lambda_n \to \lambda_1$ and there exists some $x_n \in \overline{\Omega}$ with $u_n(x_n) \le 0$. Arguing as above, we prove that the sequence $v_n := \frac{u_n}{\|u_n\|_X} \to -\varphi_1$ in $C^{1,\alpha}(\overline{\Omega})$ so $u_n < 0$ for n large. Since from one hand we have from the equation (taking $w = u_n$) that

$$(\lambda_1 - \lambda_n) \int_{\partial\Omega} |u_n|^p \le \int_{\partial\Omega} h u_n \tag{6.6}$$

and, from the other hand $\int_{\partial\Omega} h \frac{u_n}{\|u_n\|_X} \to -\int_{\partial\Omega} h\varphi_1 < 0$ then $\int_{\partial\Omega} h u_n < 0$ for n large enough, and we obtain a contradiction with (6.6). □

7. Appendix: Regularity results on weak solution

In this part, we focus our attention on the regularity of weak solutions of the problem

$$\begin{gathered} \Delta_p u = |u|^{p-2} u + f(\lambda, x, u) \quad \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = h(\lambda, x, u) \quad \text{on } \partial\Omega, \end{gathered} \tag{7.1}$$

In [6] and [14] the authors studied the boundedness of weak solutions of a problem similar to (2.1). We want to extend here the results on boundedness use for a more general non linearity f and h. We will assume that

(A2) $f:\mathbb{R}\times\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function and for a given bounded set $\Lambda\subset\mathbb{R}$ there exists $1\le b\le p^*$ and $B_1\in L^{\bar s}(\Omega), B_2\in L^s(\Omega)$ with $\bar s\ge \frac{p^*}{p^*-1}$, $s\ge \frac{p^*}{p^*-b}$ such that
$$|f(\lambda,x,u)|\le B_1+B_2|u|^{b-1},\quad \forall\lambda\in\Lambda,\ \text{a.e. } x\in\Omega,\ \forall u\in\mathbb{R}.$$

(A3) $h:\mathbb{R}\times\partial\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function and for a given bounded set $\Lambda\subset\mathbb{R}$ there exists $1\le a\le p_*$, and $A_1\in L^{\bar r}(\partial\Omega), A_2\in L^r(\partial\Omega)$ with $\bar r\ge\frac{p_*}{p_*-1}$, $r\ge\frac{p_*}{p_*-a}$ such that
$$|h(\lambda,x,u)|\le A_1+A_2|u|^{a-1},\quad \forall\lambda\in\Lambda,\ \text{a.e. } x\in\partial\Omega,\ \forall u\in\mathbb{R}.$$

These estimates will allow us to get the $C^{1,\alpha}(\bar\Omega)$-regularity of the weak solutions in the case that $g(\lambda,.,.)\in C^\gamma$, for some $\gamma>0$ uniformly for λ in a bounded set (see Theorem 7.2 below).

Proposition 7.1. *Assume* (A2), (A3) *are satisfied. Let λ be in bounded set $\Lambda\subset\mathbb{R}$. Then every weak solution u of* (7.1) *lies in $L^\infty(\partial\Omega)\cap L^\infty(\Omega)$ and, if $\|u\|_{q_1}+\|u\|_{q_2,\partial\Omega}\le C_0$, there exists a constant $C>0$ that depends on $\sup|\Lambda|$, $\|B_1\|_{\bar s}$, $\|B_2\|_s$, $\|A_1\|_{\bar r,\partial\Omega}$, $\|A_2\|_{r,\partial\Omega}$, p, q_1, q_2, α, b, Ω, and C_0; where $q_1=p^*$ if $p<N$ and $q_1=2p$ if $p=N$; $q_2=p_*$ if $p<N$ and $q_2=2p$ if $p=N$; such that*
$$\|u\|_\infty+\|u\|_{\infty,\partial\Omega}\le C.$$

Proof. If $p>N$, the answer follows from the classical Sobolev embedding $X\hookrightarrow L^\infty(\Omega)$ and from the trace embedding $X\hookrightarrow L^\infty(\partial\Omega)$. Thus it suffices to consider the case $p\le N$.

Let us first assume that the solution u is non-negative. Define the function $v_M(x):=\min\{u(x),M\}$ for $M>0$. For $k>0$ define $\phi(x)=v_M^{kp+1}(x)$ then one has $\nabla\phi(x)=(kp+1)v_M^{kp}\nabla v_M(x)$. It is clear that $\phi\in X\cap L^\infty(\Omega)\cap L^\infty(\partial\Omega)$ so we can use ϕ as test function in (7.1) to get
$$\begin{aligned}&(kp+1)\int_\Omega|\nabla u|^{p-2}\nabla u\nabla v_M v_M^{kp}+\int_\Omega u^{p-1}v_M^{kp+1}\\&=\int_\Omega f(\lambda,x,u)v_M^{kp+1}+\int_{\partial\Omega}h(\lambda,x,u)v_M^{kp+1}\end{aligned}$$
which implies
$$\begin{aligned}&\frac{(kp+1)}{(k+1)^p}\int_\Omega|\nabla v_M^{k+1}|^p+\int_\Omega v_M^{p(k+1)}\\&\le\|B_1\|_{\bar s}\Big(\int_\Omega v_M^{\bar s'(kp+1)}dx\Big)^{1/\bar s'}+\|B_2\|_s\Big(\int_\Omega(|u|^{b-1}v_M^{kp+1})^{s'}dx\Big)^{1/s'}\\&\quad+\|A_1\|_{\bar r,\partial\Omega}\Big(\int_{\partial\Omega}v_M^{\bar r'(kp+1)}\,d\sigma\Big)^{1/\bar r'}+\|A_2\|_{r,\partial\Omega}\Big(\int_{\partial\Omega}\big(u^{a-1}v_M^{kp+1}\big)^{r'}\Big)^{1/r'}\end{aligned}\tag{7.2}$$
By the trace and Sobolev's embedding there exists $C_1>0$ such that
$$\|v_M^{k+1}\|_{q_2,\partial\Omega}\le C_1^p\|v_M^{k+1}\|_X,\quad \|v_M^{k+1}\|_{q_1}\le C_1^p\|v_M^{k+1}\|_X$$
Thus, letting $M\to+\infty$, using Fatou's Lemma and Holder's inequality we obtain
$$\begin{aligned}\|u\|_{(k+1)q_1}+\|u\|_{(k+1)q_2,\partial\Omega}&\le(d\ell_kc_k)^{\frac{1}{p(k+1)}}\Big(\|u\|_{(kp+a)r',\partial\Omega}^{\frac{kp+a}{p(k+1)}}+\|u\|_{(kp+b)s'}^{\frac{kp+b}{p(k+1)}}\\&\quad+\|u\|_{(kp+1)\bar r',\partial\Omega}^{\frac{kp+1}{p(k+1)}}+\|u\|_{(kp+1)\bar s',}^{\frac{kp+1}{p(k+1)}}+1\Big).\end{aligned}\tag{7.3}$$

where $c_k^{-1} := \frac{(kp+1)}{(k+1)^p} < 1$, $d > 0$ is some constant independent of k and $\ell_k > 0$is a constant(coming from Holder's inequality). Let us set

$$k_1 := \min\big\{\frac{q_1 - bs'}{ps'}, \frac{q_2 - ar'}{pr'}, \frac{q_1 - \bar{s}'}{p\bar{s}'}, \frac{q_2 - \bar{r}'}{p\bar{r}'}\big\},$$
$$\xi_0 := \|u\|_{q_2,\partial\Omega} + \|u\|_{q_1}, \quad \xi_1 := \|u\|_{q_2(k_1+1),\partial\Omega} + \|u\|_{q_1(k_1+1)}.$$

Using Holder's inequality and that

$$\max\{(k_1 p + b)s', (k_1 p + 1)\bar{s}'\} \le q_1,$$
$$\max\{(k_1 p + a)r', (k_1 p + 1)\bar{r}'\} \le q_2 \le q_1 .$$

It follows from (7.3) that

$$\xi_1 \le (d\ell_{k_1} c_{k_1})^{\frac{1}{p(k_1+1)}} \big(\xi_0 + 1\big)^{\frac{q_1}{p(k_1+1)}}$$

Define $k_0 = 1$ and successively

$$k_n := \min\Big\{\frac{(k_{n-1}+1)q_1 - bs'}{ps'}, \frac{(k_{n-1}+1)q_1 - \bar{s}'}{p\bar{s}'}, \frac{(k_{n-1}+1)q_2 - ar'}{pr'}, \frac{(k_{n-1}+1)q_2 - \bar{r}'}{p\bar{r}'}\Big\},$$
$$c_n := c_{k_n}\ell_{k_n}, \quad \xi_n := \|u\|_{q_2(k_n+1),\partial\Omega} + \|u\|_{q_1(k_n+1)}.$$

By iteration we obtain

$$\xi_n \le (dc_n)^{\frac{1}{(k_n+1)p}} \, (\xi_{n-1} + 1)^{\frac{q_1}{(k_n+1)p}} .$$

Observe that $\lim_{n\to+\infty} k_n = +\infty$ and therefore $u \in L^n(\Omega) \cap L^n(\partial\Omega)$ for all $n > 1$. To obtain an uniform bound for u, we define a new sequence

$$q_{n+1} := q_1\Big(\frac{q_n}{tp} + \frac{1}{p'}\Big)$$

where t is any fixed number satisfying $1 < t < \frac{q_1}{p}$. Let us show the new estimate

$$\big(\|u\|_{q_{n+1}} + \|u\|_{q_{n+1},\partial\Omega}\big)^{q_{n+1}} \le \mathcal{C}\Big(\frac{q_{n+1}^p}{q_n}\Big)^{q_1/p}\big(\|u\|_{q_n} + \|u\|_{q_n,\partial\Omega}\big)^{q_n q_1/tp} \tag{7.4}$$

for some constant $\mathcal{C} > 0$. Indeed, since we know that $u \in L^i(\Omega) \cap L^i(\partial\Omega, \sigma)$ for all $i > 1$, we can use $u^{q_n/t}$ as a test function to get from the right hand side of the equation (7.1)

$$\begin{aligned} R &= \int_\Omega f(\lambda, x, u)u^{q_n/t} + \int_{\partial\Omega} g(\lambda, x, u)u^{q_n/t} \\ &\le \int_\Omega [B_1 + B_2 u^{b-1}]u^{q_n/t} + \int_{\partial\Omega} [A_1 + A_2 u^{a-1}]u^{q_n/t} \\ &\le D_1\big(\|u\|_{q_n} + \|u\|_{q_n,\partial\Omega}\big)^{q_n/t} \end{aligned}$$

with $D_1 = \|B_1 + B_2 u^{b-1}\|_{t'} + \|A_1 + A_2 u^{a-1}\|_{t',\partial\Omega}$. We stress here that the constant D_1 can be estimated from above by a constant depending on $\|B_1\|_{\bar{s}}$, $\|B_2\|_s$, $\|A_1\|_{\bar{r},\partial\Omega}$, $\|A_2\|_{r,\partial\Omega}$ and on C_0; thanks to (7.4). Consequently $\mathcal{C}$ depends on the aforementioned data. From the left hand side (the gradient term) of (7.1) we obtain, after using Sobolev's embedding,

$$C_1 \frac{q_n}{t}(\frac{q_n}{tp} + \frac{1}{p'})^{-p}\big(\|u\|_{q_{n+1}} + \|u\|_{q_{n+1},\partial\Omega}\big)^{q_{n+1}\frac{p}{q_1}} \le L,$$

where

$$L = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla(u^{q_n/t}) + \int_\Omega |u|^{p-2}u(u^{q_n/t})\,.$$

The equality $R = L$ gives the estimate (7.4). For simplicity we denote

$$\theta_n := q_n \ln(\|u\|_{q_n} + \|u\|_{q_n,\partial\Omega}), \quad B_n := \ln(\mathcal{C}\Big(\frac{q_{n+1}^p}{q_n}\Big)^{q_1}).$$

By induction we obtain from (7.4) that $\theta_{n+1} \le B_n + (q_1/tp)\theta_n$. Then

$$\theta_n \le (q_1/tp)^n\theta_0 + \sum_{i=1}^n (q_1/tp)^i B_{n-i}. \tag{7.5}$$

A simple estimate gives

$$d_0 \le \sum_{i=1}^n (q_1/tp)^i B_{n-i} \le d_1 (q_1/tp)^n$$

for some $d_0, d_1 > 0$, and also we have $q_n \ge d_2(q_1/tp)^n$ for some $d_2 > 0$. Then from (7.5)

$$\frac{\theta_n}{q_n} \le \frac{(\theta_0 + d_1)(q_1/tp)^n}{q_n} \le \frac{\theta_0 + d_1}{d_2}$$

and hence

$$\|u\|_{q_n} + \|u\|_{q_n,\partial\Omega} \le e^{\frac{\theta_0+d_1}{d_2}}\,.$$

We conclude by letting $n \to +\infty$. If u changes sign we can proceed in the same way as in the previous case to show that $u^\pm \in L^\infty(\Omega)$. □

With Proposition 7.1, we can now use the Lieberman's theorem on regularity in [15] to get the $C^{1,\alpha}-$regularity on $\overline{\Omega}$. We need here that both $f(\lambda,\cdot,\cdot)$ is bounded in Ω and $h(\lambda,\cdot,\cdot)$ is bounded and Lipschitz continuous.

Theorem 7.2. *Assume that f and h satisfy* (A2), (A3) *with $B_1, B_2 \in L^\infty(\Omega)$ and $A_1, A_2 \in L^\infty(\partial\Omega)$. Assume further that $h(\lambda,\cdot,\cdot) \in C^\gamma$ for some $\gamma > 0$ uniformly for λ in a bounded set. Then, if u is a solution of problem* (7.1) *with $\|u\|_{q_1} + \|u\|_{q_2,\partial\Omega} \le C_0$ then there exists a constant $\tilde{C} > 0$, depending on $\sup|\Lambda|$, $\|B_1\|_\infty$, $\|B_2\|_\infty$, $\|A_1\|_{\infty,\partial\Omega}$, $\|A_2\|_{\infty,\partial\Omega}$, p, q_1, q_2, a, b, Ω, and C_0; and there exists $0 < \alpha < 1$ depending on the previous data, such that*

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \le \tilde{C}.$$

7.1. **A priori estimates.** In the following, we will prove that each of these three norms, $\|u\|_{\infty,\partial\Omega}$, $\|u\|_X$ and $\|u\|_{C^{1,\alpha}(\overline{\Omega})}$, can be used to distinguish between solutions (λ_n, u_n) with u_n having arbitrary small or large norm and λ in bounded set of $\mathbb{R}$.

Lemma 7.3. *Assume* (3.2). *Let $\{(\lambda_n, u_n)\}_{n=1}^\infty$ be a sequence of solutions of* (2.1) *with λ_n in bounded subset of $\mathbb{R}$. Then the following three statements are equivalent, as $n \to +\infty$,*

(i) $\|u_n\|_{\infty,\partial\Omega} \to 0$;
(ii) $\|u_n\|_X \to 0$;
(iii) *if $g(\lambda,\cdot,\cdot) \in C^\gamma$, for some $\gamma > 0$ uniformly for λ in a bounded set, then* $\|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \to 0$.

Proof. Clearly, it follows from the definitions of the different norms that (iii) implies (i) and (ii).

We now prove that (i) implies (ii). In fact let (λ_n, u_n) be a sequence of solutions for (2.1) such that $\|u_n\|_{\infty,\partial\Omega} \to 0$. Then it follows from the weak form of (2.1) with $v = u_n$ that

$$\begin{aligned} \|u_n\|_X^p &= \int_\Omega |\nabla u_n|^p + \int_\Omega |u_n|^p \\ &= \lambda_n \int_{\partial\Omega} |u_n|^p + \int_{\partial\Omega} g(\lambda, x, u_n)u_n \\ &\le C\|u_n\|_{\infty,\partial\Omega}^p + \|u_n\|_{\infty,\partial\Omega} \int_{\partial\Omega} (A_1 + A_2|u_n|^{a-1}) \end{aligned} \tag{7.6}$$

which tends to zero. Hence the result follows. We notice that inequality (7.6) does not use condition (3.2).

We now prove that (ii) implies (i). In fact, let us $\tilde{u}_n := \frac{u_n}{\|u_n\|_X}$ and from (2.1) we obtain that

$$\begin{aligned} &\int_\Omega |\nabla \tilde{u}_n|^{p-2} \nabla \tilde{u}_n \nabla v + \int_\Omega |\tilde{u}_n|^{p-2} \tilde{u}_n v \\ &= \lambda \int_{\partial\Omega} |\tilde{u}_n|^{p-2} \tilde{u}_n v + \frac{1}{\|u_n\|_X^{p-1}} \int_{\partial\Omega} g(\lambda, x, \tilde{u}_n)v, \end{aligned}$$

for all $v \in X$. This implies

$$\int_\Omega |\nabla z|^{p-2} \nabla z \nabla v + \int_\Omega |z|^{p-2} z v = \lambda \int_{\partial\Omega} |z|^{p-2} z v + \frac{\int_{\partial\Omega} g(\lambda, x, z\|u_n\|_X)v}{\|u_n\|_X^{p-1}},$$

for all $v \in X$. It follows from conditions (A1) and (3.2) that

$$|g(\lambda, x, z)| \le a_1|z|^{p-1} + b_1|z|^{a-1} \quad \forall z \in \mathbb{R}$$

Indeed if $a \le p$, then

$$g(\lambda, x, z) \le C|z|^{p-1} \quad \forall z \in \mathbb{R}$$

if $a > p$, then

$$\begin{aligned} \frac{|g(\lambda, x, z\|u_n\|_X)|}{\|u_n\|_X^{p-1}} &\le a_1|z|^{p-1} + b_1|z|^{a-1}\|u_n\|_X^{a-p} \\ &\le a_1|z|^{p-1} + b_1|z|^{a-1} \le c_1 + d_1 b|z|^{a-1} \end{aligned}$$

for some constants c_1, d_1 independent of n.

It follows from Proposition 7.1 that $\tilde{u}_n \in L^\infty(\partial\Omega)$ and there exists a constant $\kappa > 0$, κ which does not depend on n, such that $\|\tilde{u}_n\|_{\infty,\partial\Omega} \le \kappa$. Then

$$\|u_n\|_{\infty,\partial\Omega} \le \kappa \|u_n\|_X, \tag{7.7}$$

and the result follows.

To show that (i) implies (iii), let us set $y_n := \frac{u_n}{\|u_n\|_{\infty,\partial\Omega}}$ which implies that $\|y_n\|_{\infty,\partial\Omega} = 1$. Since (λ_n, u_n) solves problem (2.1), then we have

$$\begin{gathered} -\Delta_p y_n + |y_n|^{p-2} y_n = 0 \quad \text{in } \Omega, \\ |\nabla y_n|^{p-2} \frac{\partial y_n}{\partial \nu} = \lambda_n |y_n|^{p-2} y_n + \frac{g(\lambda_n, x, u_n)}{\|u_n\|_{\infty,\partial\Omega}^{p-1}} \quad \text{on } \partial\Omega, \end{gathered}$$

Set

$$h_n := \lambda_n |y_n|^{p-2} y_n + \frac{g(\lambda_n, x, u_n)}{\|u_n\|_{\infty,\partial\Omega}^{p-1}} \quad \text{on } \partial\Omega$$

then we have

$$\|h_n\|_{\infty,\partial\Omega} \leq \sup |\lambda_n| |y_n|^{p-1} + \frac{|g(\lambda_n, x, u_n)|}{|u_n|^{p-1}} \frac{|u_n|^{p-1}}{\|u_n\|_{\infty,\partial\Omega}^{p-1}} \leq c_1 + d_1 \frac{g(\lambda_n, x, u_n)}{|u_n|^{p-1}}$$

for some constant $C_1, d_1 > 0$ independent of n. We deduce from (3.2) that $\|h_n\|_{\infty,\partial\Omega}$ is uniformly bounded and then from Theorem 7.2 we conclude that $y_n \in C^{1,\alpha}(\overline{\Omega})$ and there exists a constant $C > 0$ such that $\|y_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq C$ and the result follows.

The case (ii) implies (iii). Indeed, using (2) we have that $\tilde{u}_n := \frac{u_n}{\|u_n\|_X}$ is bounded in $L^\infty(\partial\Omega)$ by a constant independent of n. Therefore it follows from Theorem 7.2 that $\tilde{u}_n \in C^{1,\alpha}(\overline{\Omega})$ and there exists a constant C independent of n such that

$$\|\tilde{u}_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq C \tag{7.8}$$

and we obtain the desired result. □

As in the previous lemma, we prove, in the following, similar results when $\|u\|_X \to +\infty$.

Lemma 7.4. *Assume that g satisfies* (4.1). *Let $\{(\lambda_n, u_n)\}_{n=1}^\infty$ be a sequence of solutions of problem* (2.1) *such that λ_n is a bounded subset of $\mathbb{R}$. Then the following three statements are equivalent, as $n \to +\infty$*

(i) $\|u_n\|_{\infty,\partial\Omega} \to +\infty$;
(ii) $\|u_n\|_X \to +\infty$;
(iii) *if $g(\lambda, \cdot, \cdot) \in C^\gamma$ for some $\gamma > 0$ uniformly for λ in a bounded set then* $\|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \to +\infty$.

Proof. That (i) and (ii) imply (iii) follows directly from the definition of these norms.

To prove that (i) implies (ii), assume, by contradiction, that the sequence $\{u_n\}_{n=1}^\infty$ contains a bounded subsequence, still denoted in the same way, which is bounded in X. Then it follows from Sobolev's embedding that $\|u_n\|_{L^{q_1}(\Omega)}$ and $\|u_n\|_{L^{q_2}(\partial\Omega)}$ are bounded by some constant independent on n. From results on regularity (see Proposition 7.1) we have that $u_n \in L^\infty(\partial\Omega)$ and there exists a constant $C > 0$ independent of n such that $\|u_n\|_{\infty,\partial\Omega} \leq C$ in contradiction with (i).

Now we prove that (iii) implies (i). As in previous case, assume, by contradiction, that the sequence $\{u_n\}_{n=1}^\infty$ contains a bounded subsequence in $L^\infty(\partial\Omega)$, still denoted in the same way. Then it follows from (7.6) that u_n is bounded in X. From Theorem 7.2 there exists a constant C such that $\|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq C$ from where we obtain a contradiction. □

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