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MULTIPLICITY AND CONCENTRATION OF NONTRIVIAL SOLUTIONS FOR GENERALIZED EXTENSIBLE BEAM EQUATIONS IN $\mathbb{R}^N$

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Abstract. In this article, we study a class of generalized extensible beam equations with a superlinear nonlinearity

$$\Delta^2 u - M\big(\|\nabla u\|_{L^2}^2\big)\Delta u + \lambda V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N, \quad u \in H^2(\mathbb{R}^N),$$

where $N \geq 3$, $M(t) = at^\delta + b$ with $a, \delta > 0$ and $b \in \mathbb{R}$, $\lambda > 0$ is a parameter, $V \in C(\mathbb{R}^N, \mathbb{R})$ and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$. Unlike most other papers on this problem, we allow the constant b to be non-positive, which has the physical significance. Under some suitable assumptions on $V(x)$ and $f(x,u)$, when a is small and λ is large enough, we prove the existence of two nontrivial solutions $u_{a,\lambda}^{(1)}$ and $u_{a,\lambda}^{(2)}$, one of which will blow up as the nonlocal term vanishes. Moreover, $u_{a,\lambda}^{(1)} \to u_\infty^{(1)}$ and $u_{a,\lambda}^{(2)} \to u_\infty^{(2)}$ strongly in $H^2(\mathbb{R}^N)$ as $\lambda \to \infty$, where $u_\infty^{(1)} \neq u_\infty^{(2)} \in H_0^2(\Omega)$ are two nontrivial solutions of Dirichlet BVPs on the bounded domain Ω. Also, the nonexistence of nontrivial solutions is also obtained for a large enough.

1. Introduction

We consider the nonlinear generalized extensible beam equations of the form

$$\begin{gathered} \Delta^2 u - M\big(\|\nabla u\|_{L^2}^2\big)\Delta u + \lambda V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{gathered} \tag{1.1}$$

where $N \geq 3, \Delta^2 u = \Delta(\Delta u), M(t) = at^\delta + b$ with $a, \delta > 0$ and $b \in \mathbb{R}$, $\lambda > 0$ is a parameter, and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$.

We assume that the potential $V(x)$ satisfies the following assumptions:

(A1) $V \in C(\mathbb{R}^N)$ and $V(x) \geq 0$ for all $x \in \mathbb{R}^N$;

(A2) there exists $c_0 > 0$ such that the set $\{V < c_0\} := \{x \in \mathbb{R}^N : V(x) < c_0\}$ has finite positive Lebesgue measure for $N \geq 4$ and

$$|\{V < c_0\}| < S_\infty^{-2}\big(1 + \frac{A_0^2}{2}\big)^{-1} \quad \text{for } N = 3,$$

where $|\cdot|$ is the Lebesgue measure, S_∞ is the best Sobolev constant for the imbedding of $H^2(\mathbb{R}^N)$ in $L^\infty(\mathbb{R}^N)$ for $N = 3$, and A_0 is defined in (1.7) below;

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(A3) $\Omega = \text{int}\{x \in \mathbb{R}^N : V(x) = 0\}$ is nonempty and has smooth boundary with $\overline{\Omega} = \{x \in \mathbb{R}^N : V(x) = 0\}$.

The above hypotheses, suggested by Bartsch et al. [3], imply that $\lambda V(x)$ represents a potential well whose depth is controlled by λ. If λ is sufficiently large, then $\lambda V(x)$ is known as the steep potential well. About its applications, we refer the reader to [15, 23, 24, 25, 26, 27, 35, 36] and references therein.

Equation (1.1) arises in an interesting physical context. In 1950, Woinowsky and Krieger [31] introduced the following extensible beam equation:

$$\rho u_{tt} + EIu_{xxxx} - \Big(\frac{Eh}{2I}\int_0^L |u_x|^2 dx + P_0\Big)u_{xx} = 0, \tag{1.2}$$

where L is the length of the beam in the rest position, E is the Young modulus of the material, I is the cross-sectional moment of inertia, ρ is the mass density, P_0 is the tension in the rest position and h is the cross-sectional area. This model is used to describe the transverse deflection $u(x,t)$ of an extensible beam of natural length L whose ends are held a fixed distance apart. Such problems are often referred to as being nonlocal because of the presence of the term $(\int_0^L |u_x|^2 dx)u_{xx}$, which indicates the change in the tension of the beam due to its extensibility. The qualitative and stable analysis of solutions for (1.2) can be traced back to the 1970s, for instance in the papers by Ball [2], Dickey [10] and Medeiros [21].

As a simplification of the von Karman plate equation, Berger [4] proposed the plate model describing large deflection of plate as follows

$$u_{tt} + \Delta^2 u - \Big(\int_\Omega |\nabla u|^2 dx + Q_0\Big)\Delta u = f(u, u_t, x), \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N (N = 1, 2)$ is a bounded domain with a sufficiently smooth boundary, the parameter Q_0 is in-plane forces applied to the plate ($Q_0 > 0$ represents outward pulling forces and $Q_0 < 0$ means inward extrusion forces) and the function f represents transverse loads which may depend on the displacement u and the velocity u_t. Apparently, when $N = 1$ and $f \equiv 0$ in (1.3), the corresponding equation becomes the extensible beam equation (1.2). Owing to its importance, the various properties of solutions for (1.3) have been treated by many researchers; see for example, [8, 9, 20, 22, 34]. More precisely, Patcheu [22] investigated the existence and decay property of global solutions to the Cauchy problem of (1.3) with $f(u, u_t, x) \equiv f(u_t)$ in the abstract form. Yang [34] studied the global existence, stability and the longtime dynamics of solutions to the initial boundary value problem (IBVP) of an extensible beam equation with nonlinear damping and source terms in any space dimensions, i.e., (1.3) with $f(u, u_t, x) = g(u_t) + h(u) + k(x)$.

In the previous two decades, the stationary form of (1.3), of the form similar to (1.1), has begun to attract attention, specially on the existence and multiplicity of nontrivial solutions, but the relevant results are rare. We refer the reader to [7, 12, 16, 19, 28, 29, 32, 33] and references therein. To be precise, Ma [19] studied the existence of nontrivial solutions for a class of extensible beam equations with nonlinear boundary conditions in dimension one. Wang et al. [28] concentrated on the Navier BVPs,

$$\begin{gathered}\Delta^2 u + \lambda(a\int_\Omega |\nabla u|^2 dx + b)\Delta u = f(x, u) \quad x \in \Omega,\\ u = \Delta u = 0 \quad x \in \partial\Omega,\end{gathered} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain and $\lambda, a, b > 0$. Applying mountain pass techniques and the truncation method, they obtained the existence of nontrivial solutions for (1.4) for λ small enough when $f(x,u)$ satisfies some superlinear assumptions. Cabada and Figueiredo [7] considered a class of generalized extensible beam equations with critical growth in $\mathbb{R}^N$,

$$\begin{gathered}\Delta^2 u - M(\|\nabla u\|_{L^2}^2)\Delta u + u = \lambda f(u) + |u|^{2^{**}-2}u \quad \text{in } \mathbb{R}^N,\\ u \in H^2(\mathbb{R}^N),\end{gathered}\tag{1.5}$$

where $M : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous increasing functions, $f \in C(\mathbb{R}, \mathbb{R})$, $2^{**} = \frac{2N}{N-4}$ with $N \geq 5$ and $\lambda > 0$ is a parameter. By using the minimax theorem and the truncation technique, the existence of nontrivial solutions of (1.5) is proved for λ sufficiently large. Later, Liang and Zhang [16] obtained the existence and multiplicity of nontrivial solutions for (1.5) via Lions' second concentration-compactness principle.

On the other hand, steep potential well has been applied to the study of the existence and multiplicity of nontrivial solutions for biharmonic equations without nonlocal term; see, for example, [14, 18, 26, 30, 35]. Specifically, Sun et al. [26] investigated the following biharmonic equations with p-Laplacian and steep potential well,

$$\begin{gathered}\Delta^2 u - \beta\Delta_p u + \lambda V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N,\\ u \in H^2(\mathbb{R}^N),\end{gathered}\tag{1.6}$$

where $N \geq 1$, $\beta \in \mathbb{R}$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $p \geq 2$ and $\lambda V(x)$ is a steep potential well. When f satisfies various superlinear or sublinear assumptions, they proved that (1.6) admits one or two nontrivial solutions, respectively.

Motivated by all results mentioned above, in the present paper we are concerned with a class of generalized extensible beam equations with steep potential well, i.e. (1.1). We focus our attention on the multiplicity and concentration of nontrivial solutions for (1.1). Distinguished from the existing literatures, (I) we allow the constant b to be non-positive, which has the physical significance; (II) we are interested in seeking two nontrivial solutions for (1.1) with a superlinear nonlinearity, one of which will blow up as the nonlocal term vanishes; (III) we would like to explore the phenomenon of concentrations of two different nontrivial solutions as $\lambda \to \infty$, which seems to be less involved in extensible beam equations.

It is noteworthy that in analysis, we have to face some challenges. First, since the constant $b \leq 0$ is allowed, how to construct an appropriate norm of the working space such that this norm is associated with the norm $\|\nabla u\|_{L^2} = (\int_{\mathbb{R}^N} |\nabla u|^2 dx)^{1/2}$ is crucial. Second, having considered the fact that the norms $\|\nabla u\|_{L^2}$ and $\|u\|_{H^2} = (\int_{\mathbb{R}^N}(|\Delta u|^2 + |\nabla u|^2 + u^2)dx)^{1/2}$ are not equivalent, how to verify that the energy functional of (1.1) is bounded below and coercive in $H^2(\mathbb{R}^N)$ is critical.

To overcome these difficulties, in this paper some new inequalities are established and new research techniques are introduced. By so doing, we obtain the existence of two nontrivial solutions for (1.1) by the minimax theory and the nonexistence of nontrivial solutions. Furthermore, we successfully figure out the concentrations of two different nontrivial solutions for (1.1) as $\lambda \to \infty$.

Before stating our results, we shall first introduce some notations. Denote the best Sobolev constant for the imbedding $H^2(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)(2 \leq r < +\infty)$ by S_r for $N = 4$. Let $A_0 > 0$ be a Gagliardo-Nirenberg constant satisfying the following

Gagliardo-Nirenberg inequality

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \le A_0^2 \Big(\int_{\mathbb{R}^N} |\Delta u|^2 dx \Big)^{1/2} \Big(\int_{\mathbb{R}^N} u^2 dx \Big)^{1/2}. \tag{1.7}$$

Set

$$\beta_N := \begin{cases} \big(1 + \frac{A_0^2}{2}\big)\big(1 + \overline{A}_N^{16/N} |\{a < c_0\}|^{4/N}\big) & \text{for } N = 3, 4, \\ \big(1 + \frac{A_0^2}{2}\big)\big(1 + \overline{B}_N^2 |\{V < c_0\}|^{4/N}\big) & \text{for } N \ge 4, \end{cases}$$

and

$$\Theta_{2,N} := \begin{cases} [\big(1 + \frac{A_0^2}{2}\big)^{-1} - S_\infty^2 |\{V < c_0\}|]^{-1} & \text{for } N = 3, \\ S_2^{-2}\big(1 + \frac{A_0^2}{2}\big) & \text{for } N = 4, \\ 1 + \frac{A_0^2}{2} & \text{for } N > 4. \end{cases}$$

Next we summarize our main results using the following assumptions:

- (A4) $f(x, s)$ is a continuous function on $\mathbb{R}^N \times \mathbb{R}$;
- (A5) there exists a constant $0 < d_0 < \alpha$ such that

  $$pF(x, s) - f(x, s)s \le d_0 s^2 \text{ for all } x \in \mathbb{R}^N \text{ and } s \in \mathbb{R},$$

  where

  $$\alpha = \begin{cases} \frac{1}{2}\delta\Theta_{2,N}^{-2}(2 + bA_0^2\beta_N) & \text{if } -2A_0^{-2}\beta_N^{-1} < b < 0, \\ \delta\Theta_{2,N}^{-2} & \text{if } b \ge 0, \end{cases}$$

  and $F(x, s) = \int_0^s f(x, t)dt$;
- (A6) for each $\epsilon \in (0, \frac{1}{2}(2 + bA_0^2\beta_N)\Theta_{2,N}^{-2})$, there exist constants $2 < p < \frac{2N}{N-2}$ and $C_{1,\epsilon}, C_{2,\epsilon} > 0$ satisfying $C_{1,\epsilon} > \frac{2\delta+2-p}{\delta p} C_{2,\epsilon}$ such that for all $x \in \mathbb{R}^N$ and $s \in \mathbb{R}$,

  $$C_{2,\epsilon} s^{p-1} - \gamma s \le f(x, s) \le \epsilon s + C_{1,\epsilon} s^{p-1}$$

  for some constant γ independent on ϵ.

Theorem 1.1. *Suppose that $N \ge 3, \delta \ge \frac{2}{N-2}, b > -2A_0^{-2}\beta_N^{-1}$ and assumptions (A1)–(A6) hold. Then there exists constants $\Lambda_1, a_* > 0$ such that for every $\lambda \ge \Lambda_1$ and $0 < a < a_*$, (1.1) admits at least two nontrivial solutions $u_{a,\lambda}^{(1)}$ and $u_{a,\lambda}^{(2)}$ satisfying $J_{a,\lambda}(u_{a,\lambda}^{(2)}) < 0 < J_{a,\lambda}(u_{a,\lambda}^{(1)})$. In particular, $u_{a,\lambda}^{(2)}$ is a ground state solution of (1.1). Furthermore, when $\delta > \frac{2}{N-2}$, for every $\lambda \ge \Lambda_1$,*

$$J_{a,\lambda}(u_{a,\lambda}^{(2)}) \to -\infty \quad \text{and} \quad \|u_{a,\lambda}^{(2)}\|_\lambda \to \infty \quad \text{as } a \to 0,$$

where $J_{a,\lambda}$ is the energy functional of (1.1) and $\|\cdot\|_\lambda$ is defined as (2.1).

Theorem 1.2. *Suppose that $N \ge 3, \delta \ge \frac{2}{N-2}, b > -2A_0^{-2}\beta_N^{-1}$ and conditions (A1), (A2) hold. In addition, we assume that the function f is a continuous function on $\mathbb{R}^N \times \mathbb{R}$ and satisfies*

- (A6') *for each $\epsilon \in (0, b\overline{S}^2 |\{V < c_0\}|^{-2/N})$, there exists constants $2 < p < \frac{2N}{N-2}$ and $C_{1,\epsilon} > 0$ such that for all $x \in \mathbb{R}^N$ $s \in \mathbb{R}$,*

  $$f(x, s) \le \epsilon s + C_{1,\epsilon} s^{p-1}.$$

Then there exists $a^* > 0$ such that for every $a > a^*$, $(K_{a,\lambda})$ does not admit any nontrivial solution for all $\lambda > bc_0^{-1}\overline{S}^2 |\{V < c_0\}|^{-2/N}$.

Theorem 1.3. *Assume that* $N \geq 5$. *Let* $u^{(1)}_{a,\lambda}$ *and* $u^{(2)}_{a,\lambda}$ *be the solutions obtained by Theorem 1.1. Then* $u^{(1)}_{a,\lambda} \to u^{(1)}_{\infty}$ *and* $u^{(2)}_{a,\lambda} \to u^{(2)}_{\infty}$ *in* $H^2(\mathbb{R}^N)$ *as* $\lambda \to \infty$, *where* $u^{(1)}_{\infty} \neq u^{(2)}_{\infty} \in H^2_0(\Omega)$ *are two nontrivial solutions of the Dirichlet BVP*

$$\begin{gathered} \Delta^2 u - M(\int_\Omega |\nabla u|^2 dx)\Delta u = f(x,u) \quad \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{1.8}$$

The remainder of this paper is organized as follows. After presenting some preliminary results in section 2, we prove Theorem 1.1 in section 3, and demonstrate proof of Theorem 1.2 in Sections 4. Sections 5 is dedicated to the proof of Theorem 1.3.

2. Preliminaries

Let

$$X = \{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\Delta u|^2 + V(x)u^2)dx < \infty\}$$

be equipped with the inner product and norm

$$\langle u, v\rangle = \int_{\mathbb{R}^N} (\Delta u \Delta v + V(x)uv)dx, \quad \|u\| = \langle u, u\rangle^{1/2}.$$

For $\lambda > 0$, we also need the following inner product and norm

$$\langle u, v\rangle_\lambda = \int_{\mathbb{R}^N} (\Delta u \Delta v + \lambda V(x)uv)dx, \quad \|u\|_\lambda = \langle u, u\rangle_\lambda^{1/2}. \tag{2.1}$$

It is clear that $\|u\| \leq \|u\|_\lambda$ for $\lambda \geq 1$. Now we set $X_\lambda = (X, \|u\|_\lambda)$.

By the Young and Gagliardo-Nirenberg inequalities, there exists a constant $A_0 > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \leq \frac{A_0^2}{2} \int_{\mathbb{R}^N} \big(|\Delta u|^2 + u^2\big)dx. \tag{2.2}$$

This shows that

$$\int_{\mathbb{R}^N} (|\Delta u|^2 + u^2)dx \leq \|u\|^2_{H^2} \leq \big(1 + \frac{A_0^2}{2}\big) \int_{\mathbb{R}^N} \big(|\Delta u|^2 + u^2\big)dx. \tag{2.3}$$

For $N = 3, 4$, applying condition (A1) and the Hölder, Young and Gagliardo-Nirenberg inequalities, there exists a sharp constant $\overline{A}_N > 0$ such that

$$\begin{aligned} &\int_{\mathbb{R}^N} u^2 dx \\ &\leq \frac{1}{c_0} \int_{\{V \geq c_0\}} V(x)u^2 dx + \Big(|\{V < c_0\}| \int_{\mathbb{R}^N} |u|^4 dx\Big)^{1/2} \\ &\leq \frac{1}{c_0} \int_{\mathbb{R}^N} V(x)u^2 dx + \frac{N\overline{A}_N^{\frac{16}{N}} |\{V < c_0\}|^{\frac{4}{N}}}{8} \int_{\mathbb{R}^N} |\Delta u|^2 dx + \frac{8-N}{8} \int_{\mathbb{R}^N} u^2 dx, \end{aligned}$$

which shows that

$$\int_{\mathbb{R}^N} u^2 dx \leq \frac{8}{Nc_0} \int_{\mathbb{R}^N} V(x)u^2 dx + \overline{A}_N^{16/N} |\{a < c_0\}|^{4/N} \int_{\mathbb{R}^N} |\Delta u|^2 dx. \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$\|u\|_{H^2}^2 \le \big(1+\frac{A_0^2}{2}\big)\max\big\{1+\overline{A}_N^{16/N}|\{a<c_0\}|^{4/N},\frac{8}{Nc_0}\big\}\|u\|^2. \tag{2.5}$$

Similarly, we obtain

$$\|u\|_{H^2}^2 \le \big(1+\frac{A_0^2}{2}\big)\big(1+\overline{A}_N^{16/N}|\{a<c_0\}|^{4/N}\big)\|u\|_\lambda^2 \tag{2.6}$$

for $\lambda \ge 8N^{-1}c_0^{-1}(1+\overline{A}_N^{16/N}|\{a<c_0\}|^{4/N})$. For $N>4$, by conditions (A1), (A2), and Hölder and Gagliardo-Nirenberg inequalities, there exists a constant $\overline{B}_N>0$ such that

$$\begin{aligned}\int_{\mathbb{R}^N}u^2dx &= \int_{\{V\ge c_0\}}u^2dx+\int_{\{V<c_0\}}u^2dx\\ &\le \frac{1}{c_0}\int_{\mathbb{R}^N}V(x)u^2dx+\overline{B}_N^2|\{V<c_0\}|^{4/N}\int_{\mathbb{R}^N}|\Delta u|^2dx.\end{aligned}$$

Combining the above inequality with (2.3) yields

$$\|u\|_{H^2}^2 \le \big(1+\frac{A_0^2}{2}\big)\max\big\{1+\overline{B}_N^2|\{V<c_0\}|^{4/N},\frac{1}{c_0}\big\}\|u\|^2. \tag{2.7}$$

Similarly, we have

$$\|u\|_{H^2}^2 \le \big(1+\frac{A_0^2}{2}\big)\big(1+\overline{B}_N^2|\{V<c_0\}|^{4/N}\big)\|u\|_\lambda^2 \tag{2.8}$$

for $\lambda \ge c_0^{-1}(1+\overline{B}_N^2|\{V<c_0\}|^{4/N})$. Set

$$\alpha_N=\begin{cases}\big(1+\frac{A_0^2}{2}\big)\max\{1+\overline{A}_N^{16/N}|\{a<c_0\}|^{4/N},\frac{8}{Nc_0}\} & \text{for } N=3,4,\\ \big(1+\frac{A_0^2}{2}\big)\max\{1+\overline{B}_N^2|\{V<c_0\}|^{4/N},\frac{1}{c_0}\} & \text{for } N\ge 5.\end{cases}$$

Thus, it follows from (2.5) and (2.7) that

$$\|u\|_{H^2}^2 \le \alpha_N\|u\|^2, \tag{2.9}$$

which implies that the imbedding $X\hookrightarrow H^2(\mathbb{R}^N)$ is continuous. If we set

$$\Lambda_N:=\begin{cases}8N^{-1}c_0^{-1}\big(1+\overline{A}_N^{16/N}|\{a<c_0\}|^{4/N}\big) & \text{for } N=3,4,\\ c_0^{-1}(1+\overline{B}_N^2|\{V<c_0\}|^{4/N}) & \text{for } N\ge 5,\end{cases}$$

then we have

$$\|u\|_{H^2}^2 \le \beta_N\|u\|_\lambda^2 \quad \text{for } \lambda\ge\Lambda_N, \tag{2.10}$$

where β_N is defined as (1.7). Furthermore, by (2.2), (2.3) and (2.10) one has

$$\int_{\mathbb{R}^N}|\nabla u|^2dx \le \frac{1}{2}A_0^2\beta_N\|u\|_\lambda^2 \quad \text{for } \lambda\ge\Lambda_N. \tag{2.11}$$

Since the imbedding $H^2(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$ is continuous, by (2.6), for any $r\in[2,+\infty)$ we have

$$\begin{aligned}\int_{\mathbb{R}^3}|u|^rdx &\le \|u\|_{L^\infty}^{r-2}\int_{\mathbb{R}^3}u^2dx\\ &\le S_\infty^{-(r-2)}\big(1+\frac{A_0^2}{2}\big)^{r/2}(1+\overline{A}_3^{16/3}|\{a<c_0\}|^{4/3})^{r/2}\|u\|_\lambda^r\end{aligned} \tag{2.12}$$

for $\lambda \geq \frac{8}{3c_0}(1+\overline{A}_3^{16/3}|\{a<c_0\}|^{4/3})$. Moreover, using the fact that the imbedding $H^2(\mathbb{R}^4) \hookrightarrow L^r(\mathbb{R}^4)$ $(2 \leq r < +\infty)$ is continuous and (2.6), for any $r \in [2,+\infty)$ one has

$$\int_{\mathbb{R}^4} |u|^r dx \leq S_r^{-r}\big(1+\frac{A_0^2}{2}\big)^{r/2}\big(1+\overline{A}_4^4|\{a<c_0\}|\big)^{r/2}\|u\|_\lambda^r \tag{2.13}$$

for $\lambda \geq 2c_0^{-1}(1+\overline{A}_4^4|\{a<c_0\}|)$, where S_r is the best Sobolev constant for the imbedding of $H^2(\mathbb{R}^4)$ in $L^r(\mathbb{R}^4)$ $(2 \leq r < +\infty)$. Finally, for $N>4$, from conditions (A1), (A2), (2.8) and Hölder and Gagliardo-Nirenberg inequalities again, it follows that for any $r \in [2, \frac{2N}{N-4})$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} |u|^r dx \\
&\leq C_0^{N(r-2)/4}\Big(\int_{\mathbb{R}^N} |u|^2 dx\Big)^{[2N-r(N-4)]/8}\Big(\int_{\mathbb{R}^N} |\Delta u|^2 dx\Big)^{N(r-2)/8} \\
&\leq C_0^{N(r-2)/4}\big(1+\frac{A_0^2}{2}\big)^{r/2}\|u\|_\lambda^r \quad \text{for } \lambda \geq \frac{1+C_0^2|\{V<c_0\}|^{4/N}}{c_0}.
\end{aligned} \tag{2.14}$$

Set

$$\Theta_{r,N} := \begin{cases} S_\infty^{-(r-2)}\big(1+\frac{A_0^2}{2}\big)^{r/2}(1+\overline{A}_3^{16/3}|\{a<c_0\}|^{4/3})^{r/2} & \text{if } N=3, \\ S_r^{-r}\big(1+\frac{A_0^2}{2}\big)^{r/2}(1+\overline{A}_4^4|\{a<c_0\}|)^{r/2} & \text{if } N=4, \\ C_0^{N(r-2)/4}\big(1+\frac{A_0^2}{2}\big)^{r/2}(1+\overline{B}_N^2|\{V<c_0\}|^{4/N})^{r/2} & \text{if } N>4. \end{cases} \tag{2.15}$$

Thus, (2.12)–(2.15) show that for any $r \in [2, 2_*)$ and $\lambda \geq \Lambda_N$, it holds

$$\int_{\mathbb{R}^N} |u|^r dx \leq \Theta_{r,N}\|u\|_\lambda^r. \tag{2.16}$$

It is easily seen that (1.1) is variational and its solutions are critical points of the functional defined in X_λ by

$$J_{a,\lambda}(u) = \frac{1}{2}\|u\|_\lambda^2 + \frac{a}{2(1+\delta)}\|\nabla u\|_{L^2}^{2(1+\delta)} + \frac{b}{2}\|\nabla u\|_{L^2}^2 - \int_{\mathbb{R}^N} F(x,u)dx. \tag{2.17}$$

It is not difficult to prove that the functional $J_{a,\lambda}$ is of class C^1 in X_λ, and that

$$\begin{aligned}
\langle J'_{a,\lambda}(u), v\rangle &= \int_{\mathbb{R}^N} [\Delta u \cdot \Delta v + \lambda V(x)uv]dx + a\|\nabla u\|_{L^2}^{2\delta}\int_{\mathbb{R}^N} \nabla u \cdot \nabla v dx \\
&\quad + b\int_{\mathbb{R}^N} \nabla u \cdot \nabla v dx - \int_{\mathbb{R}^N} f(x,u)v dx.
\end{aligned} \tag{2.18}$$

Furthermore, we have the following results.

Lemma 2.1. *Suppose that $N \geq 3$ and $\delta \geq 2/(N-2)$. In addition, we assume that conditions $(V1)-(V2), (F1)$ and $(F3)$ hold. Then the energy functional $J_{a,\lambda}$ is bounded below and coercive on X_λ for all $a>0$ and*

$$\lambda \geq \Lambda_0 := \begin{cases} \max\{\Lambda_N, \frac{2\epsilon}{c_0}\} & \text{if } \delta > \frac{2}{N-2}, \\ \max\{\Lambda_N, \frac{2\epsilon}{c_0} + \frac{4C_{1,\epsilon}}{c_0 p}\big(\frac{2C_{1,\epsilon}(1+\delta)}{ap\overline{S}^{2N/(N-2)}}\big)^{\frac{(p-2)(N-2)}{2N-p(N-2)}}\} & \text{if } \delta = \frac{2}{N-2}. \end{cases}$$

Furthermore, for all $a>0$ and $\lambda \geq \Lambda_0$, there exists a constant $R_a > 0$ such that

$$J_{a,\lambda}(u) \geq 0 \quad \text{for all } u \in X_\lambda \text{ with } \|u\|_\lambda \geq R_a$$

Proof. Let $u \in X_\lambda$. Note that for any $2 \leq r \leq 2^* := \frac{2N}{N-2}$, it holds

$$\begin{aligned} &\int_{\mathbb{R}^N} |u|^r dx \\ &\leq \Big(\int_{\mathbb{R}^N} |u|^2 dx\Big)^{\frac{2^*-r}{2^*-2}} \Big(\int_{\mathbb{R}^N} |u|^{2^*} dx\Big)^{\frac{r-2}{2^*-2}} \\ &\leq \Big(\frac{1}{\lambda c_0}\int_{\mathbb{R}^N} \lambda V(x) u^2 dx + \overline{S}^{-2} |\{V < c_0\}|^{\frac{2}{N}} \|\nabla u\|_{L^2}^2\Big)^{\frac{2^*-r}{2^*-2}} \Big(\overline{S}^{-1}\|\nabla u\|_{L^2}\Big)^{\frac{N(r-2)}{2}}, \end{aligned} \tag{2.19}$$

where we have used the Hölder and Sobolev inequalities and $\overline{S}$ is the best Sobolev constant for the imbedding of $D^{1,2}(\mathbb{R}^N)$ in $L^{2^*}(\mathbb{R}^N)$. We now divide the proof into two separate cases:

Case A:

$$\int_{\mathbb{R}^N} \lambda V(x) u^2 dx \geq \lambda c_0 \big(\frac{4C_{1,\epsilon}}{p(\lambda c_0 - 2\epsilon)}\big)^{\frac{4}{(p-2)(N-2)}} (\overline{S}^{-1}\|\nabla u\|_{L^2})^{2^*}.$$

It follows from condition (A6) and (2.19) that

$$\begin{aligned} &J_{a,\lambda}(u) \\ &\geq \frac{1}{2}\|u\|_\lambda^2 + \frac{a}{2(1+\delta)}\|\nabla u\|_{L^2}^{2(1+\delta)} + \frac{b}{2}\|\nabla u\|_{L^2}^2 - \frac{\epsilon}{2}\int_{\mathbb{R}^N} u^2 dx - \frac{C_{1,\epsilon}}{p}\int_{\mathbb{R}^N} |u|^p dx \\ &\geq \frac{1}{4}\|u\|_\lambda^2 + \frac{a}{2(1+\delta)}\|\nabla u\|_{L^2}^{2(1+\delta)} + \frac{1}{2}(b - \epsilon \overline{S}^{-2} |\{V < c_0\}|^{\frac{2}{N}})\|\nabla u\|_{L^2}^2 \\ &\quad - \frac{C_{1,\epsilon}}{p\overline{S}^p} |\{V < c_0\}|^{\frac{2N-p(N-2)}{2N}} \|\nabla u\|_{L^2}^p. \end{aligned}$$

Since $\delta \geq \frac{2}{N-2}$, we have $1 + \delta > \frac{p}{2} > 1$. Then there exists a constant D_a such that

$$\begin{aligned} D_a &= \min_{t \geq 0}\Big[\frac{at^{1+\delta}}{2(1+\delta)} + \frac{t}{2}\Big(b - \frac{\epsilon|\{V < c_0\}|^{\frac{2}{N}}}{\overline{S}^2}\Big) - \frac{C_{1,\epsilon} t^{p/2}}{p\overline{S}^p} |\{V < c_0\}|^{\frac{2N-p(N-2)}{2N}}\Big] \\ &< 0, \end{aligned}$$

and $D_a \to -\infty$ as $a \to 0$. This and the above inequality leads to

$$J_{a,\lambda}(u) \geq \frac{1}{4}\|u\|_\lambda^2 + D_a \geq D_a,$$

which implies that $J_{a,\lambda}(u)$ is bounded below and coercive on X_λ for all $a > 0$ and $\lambda > \max\{\Lambda_N, \frac{2\epsilon}{c_0}\}$.

Case B:

$$\int_{\mathbb{R}^N} \lambda V(x) u^2 dx < \lambda c_0 \big(\frac{4C_{1,\epsilon}}{p(\lambda c_0 - 2\epsilon)}\big)^{\frac{4}{(p-2)(N-2)}} (\overline{S}^{-1}\|\nabla u\|_{L^2})^{2^*}.$$

By (2.19) one has

$$\begin{aligned} &\int_{\mathbb{R}^N} |u|^p dx \\ &\leq \Big(\frac{1}{\lambda c_0}\int_{\mathbb{R}^N} \lambda V(x) u^2 dx + \frac{|\{V < c_0\}|^{\frac{2}{N}}}{\overline{S}^2}\|\nabla u\|_{L^2}^2\Big)^{\frac{2^*-p}{2^*-2}} \Big(\overline{S}^{-1}\|\nabla u\|_{L^2}\Big)^{\frac{N(p-2)}{2}} \\ &\leq \overline{S}^{-2^*}\Big(\frac{4C_{1,\epsilon}}{p(\lambda c_0 - 2\epsilon)}\Big)^{\frac{2N-p(N-2)}{(p-2)(N-2)}} \|\nabla u\|_{L^2}^{2^*} + \frac{|\{V < c_0\}^{\frac{2N-p(N-2)}{2N}}}{\overline{S}^p}\|\nabla u\|_{L^2}^p. \end{aligned}$$

This and condition $(F3)$, give

$$
\begin{aligned}
&J_{a,\lambda}(u)\\
&\geq \frac{1}{2}\|u\|_\lambda^2 + \frac{a}{2(1+\delta)}\|\nabla u\|_{L^2}^{2(1+\delta)} + \frac{b}{2}\|\nabla u\|_{L^2}^2\\
&\quad - \frac{\epsilon}{2}\int_{\mathbb{R}^N} u^2 dx - \frac{C_{1,\epsilon}}{p}\int_{\mathbb{R}^N} |u|^p dx\\
&\geq \frac{1}{4}\|u\|_\lambda^2 + \frac{a}{2(1+\delta)}\|\nabla u\|_{L^2}^{2(1+\delta)} + \frac{1}{2}\big(b - \epsilon\overline{S}^{-2}|\{V<c_0\}|^{\frac{2}{N}}\big)\|\nabla u\|_{L^2}^2\\
&\quad - \frac{C_{1,\epsilon}}{p\overline{S}^{2^*}}\Big(\frac{4C_{1,\epsilon}}{p(\lambda c_0 - 2\epsilon)}\Big)^{\frac{2N-p(N-2)}{(p-2)(N-2)}}\|\nabla u\|_{L^2}^{2^*} - \frac{C_{1,\epsilon}}{p\overline{S}^p}|\{V<c_0\}|^{\frac{2N-p(N-2)}{2N}}\|\nabla u\|_{L^2}^p.
\end{aligned}
$$

If $\delta = \frac{2}{N-2}$, then for

$$
\lambda > \frac{2\epsilon}{c_0} + \frac{4C_{1,\epsilon}}{c_0 p}\Big[\frac{2C_{1,\epsilon}(1+\delta)}{ap\overline{S}^{2^*}}\Big]^{\frac{(p-2)(N-2)}{2N-p(N-2)}},
$$

there exists a constant $\overline{D}_a < D_a < 0$ such that

$$
\begin{aligned}
J_{a,\lambda}(u) &\geq \frac{1}{4}\|u\|_\lambda^2 + \frac{1}{2}(b - \epsilon\overline{S}^{-2}|\{V<c_0\}|^{\frac{2}{N}})\|\nabla u\|_{L^2}^2\\
&\quad + \Big[\frac{a}{2(1+\delta)} - \frac{C_{1,\epsilon}}{p\overline{S}^{2^*}}(\frac{4C_{1,\epsilon}}{p(\lambda c_0 - 2\epsilon)})^{\frac{2N-p(N-2)}{(p-2)(N-2)}}\Big]\|\nabla u\|_{L^2}^{2(1+\delta)}\\
&\quad - \frac{C_{1,\epsilon}}{p\overline{S}^p}|\{V<c_0\}|^{\frac{2N-p(N-2)}{2N}}\|\nabla u\|_{L^2}^p\\
&\geq \frac{1}{4}\|u\|_\lambda^2 + \overline{D}_a \geq \overline{D}_a.
\end{aligned}
$$

If $\delta > \frac{2}{N-2}$, then for $\lambda > \frac{2\epsilon}{c_0}$, there exists a constant $\widetilde{D}_a < 0$ such that

$$
\begin{aligned}
&J_{a,\lambda}(u)\\
&\geq \frac{1}{4}\|u\|_\lambda^2 + \frac{a}{2(1+\delta)}\|\nabla u\|_{L^2}^{2(1+\delta)} + \frac{1}{2}(b - \epsilon\overline{S}^{-2}|\{V<c_0\}|^{\frac{2}{N}})\|\nabla u\|_{L^2}^2\\
&\quad - \frac{C_{1,\epsilon}}{p\overline{S}^{2^*}}(\frac{4C_{1,\epsilon}}{p(\lambda c_0 - 2\epsilon)})^{\frac{2N-p(N-2)}{(p-2)(N-2)}}\|\nabla u\|_{L^2}^{2^*} - \frac{C_{1,\epsilon}}{p\overline{S}^p}|\{V<c_0\}|^{\frac{2N-p(N-2)}{2N}}\|\nabla u\|_{L^2}^p\\
&\geq \widetilde{D}_a.
\end{aligned}
$$

This indicates that $J_{a,\lambda}$ is bounded below and coercive on X_λ for all $a > 0$ and $\lambda \geq \Lambda_0$. Furthermore, for all $a > 0$ and $\lambda \geq \Lambda_0$, it is clear that there exists a constant $R_a > 0$ such that

$$
J_{a,\lambda}(u) \geq 0 \quad \text{for all } u \in X_\lambda \text{ with } \|u\|_\lambda \geq R_a.
$$

The proof is complete. □

Next, we give a useful theorem, which is the variant version of the mountain pass theorem. It can help us to find a so-called Cerami type (PS) sequence.

Lemma 2.2 ([11], Mountain Pass Theorem)**.** *Let E be a real Banach space with its dual space E^*, and suppose that $I \in C^1(E, R)$ satisfies*

$$
\max\{I(0), I(e)\} \leq \mu < \eta \leq \inf_{\|u\|=\rho} I(u),
$$

for some $\mu < \eta, \rho > 0$ and $e \in E$ with $\|e\| > \rho$. Let $c \geq \eta$ be characterized by

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq \tau \leq 1} I(\gamma(\tau)),$$

where $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}$ is the set of continuous paths joining 0 and e, then there exists a sequence $\{u_n\} \subset E$ such that

$$I(u_n) \to c \geq \eta \quad \text{and} \quad (1 + \|u_n\|)\|I'(u_n)\|_{E^*} \to 0 \quad \text{as } n \to \infty.$$

In what follows, we give two lemmas which ensure that the functional $J_{a,\lambda}$ has the mountain pass geometry.

Lemma 2.3. *Suppose that $b > -2A_0^{-2}\beta_N^{-1}$. In addition, assume that conditions $(V1)-(V2), (F1)$ and (A6) hold. Then there exists $\rho > 0$ such that for every $a > 0$ and $\lambda > \Lambda_N$,*

$$\inf\{J_{a,\lambda}(u) : u \in X_\lambda \text{ with } \|u\|_\lambda = \rho\} > \eta$$

for some $\eta > 0$.

Proof. By (2.11) and condition (A6), for all $u \in X_\lambda$ one has

$$\begin{aligned} &J_{a,\lambda}(u) \\ &\geq \frac{1}{2}\|u\|_\lambda^2 + \frac{a}{2(1+\delta)}\|\nabla u\|_{L^2}^{2(1+\delta)} + \frac{b}{2}\|\nabla u\|_{L^2}^2 - \frac{\epsilon}{2}\int_{\mathbb{R}^N} u^2 dx - \frac{C_{1,\epsilon}}{p}\int_{\mathbb{R}^N} |u|^p dx \\ &\geq \begin{cases} \frac{1}{2}(1 - \epsilon\Theta_{2,N}^2)\|u\|_\lambda^2 - \frac{C_{1,\epsilon}}{p}\Theta_{p,N}^p\|u\|_\lambda^p & \text{if } b \geq 0, \\ \frac{1}{2}(1 + \frac{bA_0^2}{2}\beta_N - \epsilon\Theta_{2,N}^2)\|u\|_\lambda^2 - \frac{C_{1,\epsilon}}{p}\Theta_{p,N}^p\|u\|_\lambda^p & \text{if } -2A_0^{-2}\beta_N^{-1} < b < 0. \end{cases} \end{aligned}$$

Let

$$g(t) = \frac{1}{2}(1 - \epsilon\Theta_{2,N}^2)t^2 - \frac{C_{1,\epsilon}\Theta_{p,N}^p}{p}t^p \quad \text{for } t \geq 0.$$

A direct calculation shows that

$$\max_{t \geq 0} g(t) = g(\bar{t}) = \frac{(p-2)}{2p}(1 - \epsilon\Theta_{2,N}^2)^{p/(p-2)}(C_{1,\epsilon}\Theta_{p,N}^p)^{-2/(p-2)},$$

where

$$\bar{t} = \Big[\frac{(1 - \epsilon\Theta_{2,N}^2)}{C_{1,\epsilon}\Theta_{p,N}^p}\Big]^{1/(p-2)}.$$

This shows that when $b \geq 0$, for every $u \in X_\lambda$ with $\|u\|_\lambda = \bar{t}$ we have $J_{a,\lambda}(u) \geq g(\bar{t}) > 0$. Choosing $\rho = \bar{t}$ and

$$\eta = \frac{(p-2)}{2p}(1 - \epsilon\Theta_{2,N}^2)^{p/(p-2)}(C_{1,\epsilon}\Theta_{p,N}^p)^{-2/(p-2)} > 0,$$

it is easy to see that the result holds. Similarly, when $-2A_0^{-2}\beta_N^{-1} < b < 0$, for every $u \in X_\lambda$ with

$$\|u\|_\lambda = \tilde{t} = \Big[\frac{(1 + \frac{bA_0^2}{2}\beta_N - \epsilon\Theta_{2,N}^2)}{C_{1,\epsilon}\Theta_{p,N}^p}\Big]^{1/(p-2)},$$

we can take $\rho = \tilde{t}$ and

$$\eta = \frac{(p-2)}{2p}\Big(1 + \frac{bA_0^2}{2}\beta_N - \epsilon\Theta_{2,N}^2\Big)^{p/(p-2)}(C_{1,\epsilon}\Theta_{p,N}^p)^{-2/(p-2)}$$

such that the result holds. This completes the proof. $\square$

Define

$$\Pi_\lambda = \sup_{u\in X_\lambda\setminus\{0\}} \frac{(\int_{\mathbb{R}^N} |u|^p dx)^{1/p}}{\|u\|_\lambda}. \tag{2.20}$$

It follows from (2.16) that

$$\Pi_\lambda \le \Theta_{p,N} \quad \text{for } \lambda \ge \Lambda_N. \tag{2.21}$$

Furthermore, by Appendix A there exist $\Lambda_1 \ge \Lambda_N$ and $\phi_\lambda \in X_\lambda\setminus\{0\}$ such that

$$\Pi_\lambda = \frac{(\int_{\mathbb{R}^N} |\phi_\lambda|^p dx)^{1/p}}{\|\phi_\lambda\|_\lambda} > 0 \quad \text{for every } \lambda \ge \Lambda_1, \tag{2.22}$$

and there exists a constant $\Pi_\infty > 0$ independent on λ such that

$$\Pi_\lambda \searrow \Pi_\infty \quad \text{as } \lambda \nearrow \infty. \tag{2.23}$$

Setting

$$a_* := \frac{2^{2+\delta} C_{2,\epsilon}\Pi_\infty^p (1+\delta)(p-2)}{\delta p A_0^{2(1+\delta)}\beta_N^{1+\delta}} \Big[\frac{C_{2,\epsilon}\Pi_\infty^p(2\delta+2-p)}{\delta p(1+\frac{bA_0^2}{2}\beta_N+\gamma\Theta_{2,N}^2)}\Big]^{\frac{2\delta+2-p}{p-2}}.$$

Lemma 2.4. *Assume that $b \in R$, and assumptions* (A1)–(A4), (A6) *hold. Let $\rho > 0$ be as in Lemma 2.3. Then for every $\lambda \ge \Lambda_1$ and $0 < a < a_*$, there exists $e \in X_\lambda$ satisfying*

$$\|e\|_\lambda > \rho \quad \text{and} \quad \|e\|_\lambda \to \infty \quad \text{as } a \to 0$$

such that

$$J_{a,\lambda}(e) < 0 \quad \text{and} \quad J_{a,\lambda}(e) \to -\infty \quad \text{as } a \to 0.$$

Proof. Let $\phi_\lambda \in X_\lambda\setminus\{0\}$ be as in (2.22) and let

$$\begin{aligned} I(t) &= I_{a,\lambda}(t\phi_\lambda) \\ &= \frac{t^2}{2}\|\phi_\lambda\|_\lambda^2 + \frac{at^{2(1+\delta)}}{2(1+\delta)}\|\nabla\phi_\lambda\|_{L^2}^{2(1+\delta)} + \frac{bt^2}{2}\|\nabla\phi_\lambda\|_{L^2}^2 \\ &\quad + \frac{\gamma t^2}{2}\int_{\mathbb{R}^N}\phi_\lambda^2 dx - \frac{C_{2,\epsilon}t^p}{p}\int_{\mathbb{R}^N}|\phi_\lambda|^p dx \text{ for } t>0. \end{aligned}$$

Then it follows from (2.11) and (2.16) that

$$\begin{aligned} I(t) \le \frac{A_0^{2(1+\delta)}\beta_N^{1+\delta}\|\phi_\lambda\|_\lambda^{2(1+\delta)}t^2}{2^{3+\delta}(1+\delta)} \Big[&at^{2\delta} \\ &+ \frac{2^{2+\delta}(1+\delta)(1+\frac{bA_0^2}{2}\beta_N+\gamma\Theta_{2,N}^2)}{A_0^{2(1+\delta)}\beta_N^{1+\delta}\|\phi_\lambda\|_\lambda^{2\delta}} - \frac{2^{3+\delta}(1+\delta)C_{2,\epsilon}\Pi_\infty^p}{pA_0^{2(1+\delta)}\beta_N^{1+\delta}\|\phi_\lambda\|_\lambda^{2(1+\delta)-p}}t^{p-2}\Big]. \end{aligned}$$

A direct calculation shows that there exists

$$t_{a,\lambda} := \Big(\frac{2^{2+\delta}C_{2,\epsilon}\Pi_\infty^p(1+\delta)(p-2)}{a\delta pA_0^{2(1+\delta)}\beta_N^{1+\delta}}\Big)^{1/(2\delta+2-p)}\|\phi_\lambda\|_\lambda^{-1} > 0$$

such that for every $0 < a < a_*$,

$$\begin{aligned} &at_{a,\lambda}^{2\delta} + \frac{2(1+\delta)(b+\gamma\Theta_{2,N}^2)}{A_0^2\beta_N\|\phi_\lambda\|_\lambda^{2\delta}} - \frac{4(1+\delta)C_{2,\epsilon}\Pi_\infty^p}{pA_0^2\beta_N\|\phi_\lambda\|_\lambda^{2(1+\delta)-p}}t_{a,\lambda}^{p-2} \\ &= \frac{2^{2+\delta}(1+\delta)\|\phi_\lambda\|_\lambda^{-2\delta}}{A_0^{2(1+\delta)}\beta_N^{1+\delta}}\Big[\Big(1+\frac{bA_0^2}{2}\beta_N+\gamma\Theta_{2,N}^2\Big) \end{aligned}$$

$$- \frac{C_{2,\epsilon}\Pi_\infty^p(2\delta+2-p)}{\delta p}\Big(\frac{2^{2+\delta}C_{2,\epsilon}\Pi_\infty^p(1+\delta)(p-2)}{a\delta p A_0^{2(1+\delta)}\beta_N^{1+\delta}}\Big)^{\frac{p-2}{2\delta+2-p}}\Big] < 0\,.$$

This implies $I(t_{a,\lambda}) = I_{a,\lambda}(t_{a,\lambda}\phi_\lambda) < 0$ for $0 < a < a_*$ and

$$I_{a,\lambda}(t_{a,\lambda}\phi_\lambda) \to -\infty \quad \text{as } a \to 0.$$

Choosing $e = t_{a,\lambda}|\phi_\lambda|$. Clearly,

$$\|e\|_\lambda = \|t_{a,\lambda}\phi_\lambda\|_\lambda = \Big[\frac{2^{2+\delta}C_{2,\epsilon}\Pi_\infty^p(1+\delta)(p-2)}{a\delta p A_0^{2(1+\delta)}\beta_N^{1+\delta}}\Big]^{1/(2\delta+2-p)} \to \infty \quad \text{as } a \to 0.$$

Note that for $0 < a < a_*$, by (2.21) and (2.23),

$$\Big[\frac{2^{2+\delta}C_{2,\epsilon}\Pi_\infty^p(1+\delta)(p-2)}{a\delta p A_0^{2(1+\delta)}\beta_N^{1+\delta}}\Big]^{\frac{1}{2\delta+2-p}} > \Big[\frac{\delta p(1+\frac{bA_0^2}{2}\beta_N+\gamma\Theta_{2,N}^2)}{C_{2,\epsilon}\Theta_{p,N}^p(2\delta+2-p)}\Big]^{\frac{1}{p-2}},$$

by using (2.21). This and (A6) lead to

$$\|e\|_\lambda > \rho := \begin{cases} \big(\frac{1-\epsilon\Theta_{2,N}^2}{C_{1,\epsilon}\Theta_{p,N}^p}\big)^{1/(p-2)} & \text{if } b \ge 0,\\ \big(\frac{1+\frac{bA_0^2}{2}\beta_N-\epsilon\Theta_{2,N}^2}{C_{1,\epsilon}\Theta_{p,N}^p}\big)^{\frac{1}{p-2}} & \text{if } -2A_0^{-2}\beta_N^{-1} < b < 0, \end{cases}$$

where $\rho > 0$ is as in Lemma 2.3. Moreover, by condition (A6), it holds $J_{a,\lambda}(e) \le I_{a,\lambda}(e) < 0$ for $0 < a < a_*$. The proof is complete. $\square$

We define

$$c_\lambda = \inf_{\gamma\in\Gamma_\lambda}\max_{0\le t\le 1} J_{a,\lambda}(\gamma(t)),$$
$$c_0(\Omega) = \inf_{\gamma\in\overline{\Gamma}_\lambda(\Omega)}\max_{0\le t\le 1} J_{a,\lambda}|_{H_0^2(\Omega)}(\gamma(t)),$$

where $J_{a,\lambda}|_{H_0^2(\Omega)}$ is a restriction of $J_{a,\lambda}$ on $H_0^2(\Omega)$,

$$\Gamma_\lambda = \{\gamma \in C([0,1],X_\lambda) : \gamma(0)=0, \gamma(1)=e\},$$
$$\overline{\Gamma}_\lambda(\Omega) = \{\gamma \in C([0,1],H_0^2(\Omega)) : \gamma(0)=0, \gamma(1)=e\}.$$

Note that for $u \in H_0^2(\Omega)$,

$$\begin{aligned} J_{a,\lambda}|_{H_0^2(\Omega)}(u) &= \frac12\int_\Omega |\Delta u|^2dx + \frac{a}{2(1+\delta)}(\int_\Omega |\nabla u|^2dx)^{2(1+\delta)} \\ &\quad + \frac{b}{2}\int_\Omega |\nabla u|^2dx - \int_\Omega F(x,u)dx \end{aligned}$$

and $c_0(\Omega)$ independent of λ. Moreover, if conditions (A4) and (A6) hold, then by the proofs of Lemmas 2.3 and 2.4, we can conclude that $J_{a,\lambda}|_{H_0^2(\Omega)}$ satisfies the mountain pass hypothesis as in Theorem 2.2.

Since $H_0^2(\Omega) \subset X_\lambda$ for all $\lambda > 0$, one can see that $0 < \eta \le c_\lambda \le c_0(\Omega)$ for all $\lambda \ge \Lambda_N$. Take $D_0 > c_0(\Omega)$. Then

$$0 < \eta \le c_\lambda \le c_0(\Omega) < D_0 \quad \text{for all } \lambda \ge \Lambda_N.$$

By Lemmas 2.3, 2.4 and Theorem 2.2, we obtain that for each $\lambda \ge \Lambda_N$, there exists a sequence $\{u_n\} \subset X_\lambda$ such that

$$J_{a,\lambda}(u_n) \to c_\lambda > 0 \quad \text{and} \quad (1+\|u_n\|_\lambda)\|J'_{a,\lambda}(u_n)\|_{X_\lambda^{-1}} \to 0 \quad \text{as } n \to \infty.$$

3. Proof of Theorem 1.1

Recall that a C^1-functional $J_{a,\lambda}$ satisfies Cerami condition at level c ($(C)_c$-condition for short) if every sequence $\{u_n\} \subset X_\lambda$ satisfying

$$J_{a,\lambda}(u_n) \to c \text{ and } (1 + \|u_n\|_\lambda)\|J'_{a,\lambda}(u_n)\|_{X_\lambda^{-1}} \to 0,$$

has a convergent subsequence, and such sequence is called a $(C)_c$-sequence.

Lemma 3.1. *Assume that $N \geq 1$, $\delta > 0$ and $b > -2A_0^{-2}\beta_N^{-1}$. In addition, assume that conditions* (A1)–(A4), (A6) *hold. Then $\{u_n\}$ is bounded in X_λ for each $\lambda \geq \Lambda_0$, where $\{u_n\}$ is a $(C)_c$-sequence.*

Proof. Following the argument in Lemma 2.1, we can conclude that the $(C)_c$-sequence $\{u_n\}$ is bounded in X_λ for each $\lambda \geq \Lambda_0$. □

Proposition 3.2. *Assume that $b > -2A_0^{-2}\beta_N^{-1}$, and that conditions* (A1)–(A6) *hold. Then for each $D > 0$, there exists $\Lambda_1 := \Lambda_1(D) \geq \Lambda_0 > \Lambda_N$ such that $J_{a,\lambda}$ satisfies the $(C)_c$-condition in X_λ for all $c < D$ and $\lambda > \Lambda_1$.*

Proof. Let $\{u_n\}$ be a $(C)_c$-sequence with $c < D$. By Lemma 3.1, $\{u_n\}$ is bounded in X_λ and there exists $D_0 > 0$ such that $\|u_n\|_\lambda \leq D_0$. Then there exist a subsequence $\{u_n\}$ and u_0 in X_λ such that

$$\begin{gathered} u_n \rightharpoonup u_0 \quad \text{weakly in } X_\lambda, \\ u_n \to u_0 \quad \text{strongly in } L^r_{\rm loc}(\mathbb{R}^N), \text{ for } 2 \leq r < 2_*, \\ u_n \to u_0 \quad \text{a.e. in } \mathbb{R}^N. \end{gathered}$$

Moreover, (2.11) and (2.16) imply that the imbedding $X_\lambda \hookrightarrow W^{1,2}(\mathbb{R}^N)$ is continuous, which shows that

$$u_n \rightharpoonup u_0 \quad \text{weakly in } W^{1,2}(\mathbb{R}^N).$$

Similar to the proof of [13, Lemma 4.4], one can easily obtain that

$$\nabla u_n(x) \to \nabla u_0(x) \quad \text{a.e. in } \mathbb{R}^N.$$

Thus, it follows from Brezis-Lieb lemma [6] that

$$\int_{\mathbb{R}^N} |\nabla(u_n - u_0)|^2 dx = \int_{\mathbb{R}^N} |\nabla u_n|^2 dx - \int_{\mathbb{R}^N} |\nabla u_0|^2 dx + o(1). \tag{3.1}$$

Now we prove that $u_n \to u_0$ strongly in X_λ. Let $v_n = u_n - u_0$. Then $v_n \rightharpoonup 0$ in X_λ. By condition (A2), we have

$$\int_{\mathbb{R}^N} v_n^2 dx = \int_{\{V \geq c_0\}} v_n^2 dx + \int_{\{V < c_0\}} v_n^2 dx \leq \frac{1}{\lambda c_0}\|v_n\|_\lambda^2 + o(1). \tag{3.2}$$

Using (3.2) and the Hölder and Sobolev inequalities, for any $\lambda > \Lambda_N$, we have the following estimates:

For $N = 3$,

$$\begin{aligned} \int_{\mathbb{R}^N} |v_n|^r dx &\leq \|v_n\|_{L^\infty}^{r-2} \int_{\mathbb{R}^N} v_n^2 dx \leq \frac{S_\infty^{r-2}}{\lambda c_0}\|v_n\|_{H^2}^{r-2}\|v_n\|_\lambda^2 + o(1) \\ &\leq \frac{S_\infty^{r-2}}{\lambda c_0}\Big[\big(1 + \frac{A_0^2}{2}\big)^{-1} - S_\infty^2|\{V < c_0\}|\Big]^{-(r-2)/2}\|v_n\|_\lambda^r + o(1). \end{aligned} \tag{3.3}$$

For $N=4$,

$$\begin{aligned}
\int_{\mathbb{R}^N}|v_n|^r dx &\le \Big(\int_{\mathbb{R}^N} v_n^2 dx\Big)^{1/2}\Big(\int_{\mathbb{R}^N} v_n^{2(r-1)}dx\Big)^{1/2} \\
&\le \Big(\frac{1}{\lambda c_0}\|v_n\|_\lambda^2+o(1)\Big)^{1/2} S_{2(r-1)}^{-(r-1)}\big(1+\frac{A_0^2}{2}\big)^{(r-1)/2}\|v_n\|_\lambda^{r-1} \\
&= \frac{S_{2(r-1)}^{-(r-1)}}{\sqrt{\lambda c_0}}\big(1+\frac{A_0^2}{2}\big)^{(r-1)/2}\|v_n\|_\lambda^r+o(1).
\end{aligned} \tag{3.4}$$

For $N>4$,

$$\begin{aligned}
\int_{\mathbb{R}^N}|v_n|^r dx &\le \Big(\int_{\mathbb{R}^N}|v_n|^2 dx\Big)^{\frac{2_*-r}{2_*-2}}\Big(\int_{\mathbb{R}^N}|v_n|^{2_*}dx\Big)^{\frac{r-2}{2_*-2}} \\
&\le C_0^{\frac{2_*(r-2)}{2_*-2}}\big(\frac{1}{\lambda c_0}\big)^{\frac{2_*-r}{2_*-2}}\|v_n\|_\lambda^r+o(1).
\end{aligned} \tag{3.5}$$

Set

$$\Psi_r := \begin{cases} \frac{S_\infty^{r-2}}{\lambda c_0}[(1+\frac{A_0^2}{2})^{-1}-S_\infty^2|\{V<c_0\}|]^{-(r-1)/2} & \text{if } N=3,\\ \frac{S_{2(r-1)}^{-(r-1)}}{\sqrt{\lambda c_0}}\big(1+\frac{A_0^2}{2}\big)^{(r-1)/2} & \text{if } N=4,\\ C_0^{\frac{2_*(r-2)}{2_*-2}}\big(\frac{1}{\lambda c_0}\big)^{(2_*-r)/(2_*-2)} & \text{if } N>4. \end{cases}$$

Clearly, $\Psi_r\to 0$ as $\lambda\to\infty$. Inequalities (3.3)–(3.5) indicate that

$$\int_{\mathbb{R}^N}|v_n|^r dx \le \Psi_r\|v_n\|_\lambda^r+o(1). \tag{3.6}$$

Following the argument in [23], it is easy to verify that

$$\int_{\mathbb{R}^N}F(x,v_n)dx=\int_{\mathbb{R}^N}F(x,u_n)dx-\int_{\mathbb{R}^N}F(x,u_0)dx+o(1) \tag{3.7}$$

and

$$\sup_{\|h\|_\lambda=1}\int_{\mathbb{R}^N}[f(x,v_n)-f(x,u_n)+f(x,u_0)]h(x)dx=o(1).$$

Thus, using (3.1), (3.7) and Brezis-Lieb Lemma [6], we deduce that

$$\begin{aligned}
J_{a,\lambda}(u_n)-J_{a,\lambda}(u_0) &= \frac12\|v_n\|_\lambda^2+\frac{a}{2(1+\delta)}(\|\nabla u_n\|_{L^2}^{2(1+\delta)}-\|\nabla u_0\|_{L^2}^{2(1+\delta)}) \\
&\quad+\frac{b}{2}\|\nabla v_n\|_{L^2}^2-\int_{\mathbb{R}^N}F(x,v_n)dx+o(1).
\end{aligned} \tag{3.8}$$

Moreover, it follows from the boundedness of the sequence $\{u_n\}$ in X_λ and (2.11) that there exists a constant $A>0$ such that

$$\|\nabla u_n\|_{L^2}^2\to A \quad \text{as } n\to\infty.$$

This indicates that for any $\varphi\in C_0^\infty(\mathbb{R}^N)$,

$$\begin{aligned}
o(1) &= \langle J'_{a,\lambda}(u_n),\varphi\rangle \\
&= \int_{\mathbb{R}^N}\Delta u_n\Delta\varphi dx+\int_{\mathbb{R}^N}\lambda V(x)u_n\varphi dx+a\|\nabla u_n\|_{L^2}^{2\delta}\int_{\mathbb{R}^N}\nabla u_n\nabla\varphi dx \\
&\quad+b\int_{\mathbb{R}^N}\nabla u_n\nabla\varphi dx-\int_{\mathbb{R}^N}f(x,u_n)\varphi dx \\
&\to \int_{\mathbb{R}^N}\Delta u_0\Delta\varphi dx+\int_{\mathbb{R}^N}\lambda V(x)u_0\varphi dx+aA^\delta\int_{\mathbb{R}^N}\nabla u_0\nabla\varphi dx
\end{aligned}$$

$$+ b\int_{\mathbb{R}^N} \nabla u_0 \nabla\varphi dx - \int_{\mathbb{R}^N} f(x,u_0)\varphi dx \quad \text{as } n\to\infty,$$

which shows that

$$\|u_0\|_\lambda^2 + (aA^\delta + b)\int_{\mathbb{R}^N} |\nabla u_0|^2 dx - \int_{\mathbb{R}^N} f(x,u_0)u_0 dx = o(1).$$

Note that

$$\begin{aligned} o(1) &= \langle J'_{a,\lambda}(u_n), u_n\rangle \\ &= \|u_n\|_\lambda^2 + a\|\nabla u_n\|_{L^2}^{2(1+\delta)} + b\|\nabla u_n\|_{L^2}^2 - \int_{\mathbb{R}^N} f(x,u_n)u_n dx. \end{aligned}$$

Combining the above two equalities gives

$$\begin{aligned} o(1) &= \|u_n\|_\lambda^2 + a\|\nabla u_n\|_{L^2}^{2(1+\delta)} + b\|\nabla u_n\|_{L^2}^2 - \int_{\mathbb{R}^N} f(x,u_n)u_n dx \\ &\quad - \|u_0\|_\lambda^2 - (aA^\delta + b)\int_{\mathbb{R}^N} |\nabla u_0|^2 dx + \int_{\mathbb{R}^N} f(x,u_0)u_0 dx \\ &= \|v_n\|_\lambda^2 + a\|\nabla u_n\|_{L^2}^{2(1+\delta)} - a\|\nabla u_n\|_{L^2}^{2\delta}\|\nabla u_0\|_{L^2}^2 \\ &\quad + b\|\nabla v_n\|_{L^2}^2 - \int_{\mathbb{R}^N} f(x,v_n)v_n dx + o(1) \\ &= \|v_n\|_\lambda^2 + a\|\nabla u_n\|_{L^2}^{2\delta}\|\nabla v_n\|_{L^2}^2 + b\|\nabla v_n\|_{L^2}^2 - \int_{\mathbb{R}^N} f(x,v_n)v_n dx + o(1). \end{aligned} \tag{3.9}$$

By Lemma 2.1, there exists a constant $K<0$ such that

$$J_{a,\lambda}(u_0) \ge K. \tag{3.10}$$

Thus, by (A5) and (3.8)–(3.10), one has

$$\begin{aligned} D - K &\ge c - J_{a,\lambda}(u_0) \\ &\ge J_{a,\lambda}(u_n) - J_{a,\lambda}(u_0) + o(1) \\ &\ge \frac{1}{2}\|v_n\|_\lambda^2 + \frac{a}{2(1+\delta)}\big(\|\nabla u_n\|_{L^2}^{2(1+\delta)} - \|\nabla u_0\|_{L^2}^{2(1+\delta)}\big) \\ &\quad + \frac{b}{2}\|\nabla v_n\|_{L^2}^2 - \int_{\mathbb{R}^N} F(x,v_n)dx + o(1) \\ &\ge \frac{\delta}{2(1+\delta)}\|v_n\|_\lambda^2 + \frac{b\delta}{2(1+\delta)}\|\nabla v_n\|_{L^2}^2 - \frac{d_0}{2(1+\delta)}\int_{\mathbb{R}^N} v_n^2 dx \\ &\quad + \frac{a}{2(1+\delta)}(\|\nabla u_n\|_{L^2}^{2\delta} - \|\nabla u_0\|_{L^2}^{2\delta})\|\nabla u_0\|_{L^2}^2 + o(1) \\ &\ge \frac{\delta - d_0\Theta_{2,N}^2}{2(1+\delta)}\|v_n\|_\lambda^2 + \frac{b\delta}{2(1+\delta)}\|\nabla v_n\|_{L^2}^2 \\ &\quad + \frac{a}{2(1+\delta)}\big(\|\nabla u_n\|_{L^2}^{2\delta} - \|\nabla u_0\|_{L^2}^{2\delta}\big)\|\nabla u_0\|_{L^2}^2 + o(1), \end{aligned}$$

which implies that there exists a constant $\widehat{D} = \widehat{D}(a,D) > 0$ such that

$$\|v_n\|_\lambda^2 \le \widehat{D} + o(1) \quad \text{for every } \lambda > \Lambda_N. \tag{3.11}$$

It follows (A6), (3.6) and (3.11) that

$$o(1) = \|v_n\|_\lambda^2 + a\|\nabla u_n\|_{L^2}^{2\delta}\|\nabla v_n\|_{L^2}^2 + b\|\nabla v_n\|_{L^2}^2 - \int_{\mathbb{R}^N} f(x,v_n)v_n dx$$

$$
\begin{aligned}
&\geq \|v_n\|_\lambda^2 + b\|\nabla v_n\|_{L^2}^2 - \epsilon \int_{\mathbb{R}^N} v_n^2 dx - C_{1,\epsilon} \int_{\mathbb{R}^N} |v_n|^p dx \\
&\geq \begin{cases} \|v_n\|_\lambda^2 - \epsilon \Psi_r^2 \widehat{D} - C_{1,\epsilon} \Psi_r^p \widehat{D}^{p/2} & \text{if } b \geq 0, \\ \frac{1}{2}(2 + bA_0^2 \beta_N)\|v_n\|_\lambda^2 - \epsilon \Psi_r^2 \widehat{D} - C_{1,\epsilon} \Psi_r^p \widehat{D}^{p/2} & \text{if } -2A_0^{-2}\beta_N^{-1} < b < 0, \end{cases}
\end{aligned}
$$

which implies that there exists $\Lambda_1 := \Lambda_1(a, D) \geq \Lambda_0 > \Lambda_N$ such that for each $\lambda > \Lambda_1$,

$$v_n \to 0 \text{ strongly in } X_\lambda.$$

This completes the proof. □

Theorem 3.3. *Assume that $N \geq 3$, $\delta \geq \frac{2}{N-2}$ and $b > -2A_0^{-2}\beta_N^{-1}$. In addition, we assume that conditions* (A1)–(A6) *are satisfied. Then for each $0 < a < a_*$ and $\lambda > \Lambda_1$, $J_{a,\lambda}$ has a nonzero critical point $u_{a,\lambda}^{(1)} \in X_\lambda$ such that $J_{a,\lambda}(u_{a,\lambda}^{(1)}) = c_\lambda > 0$.*

Proof. By Theorem 2.2, Lemmas 2.3 and 2.4, for every $\lambda > \Lambda_1$ and $0 < a < a^*$, there exists a sequence $\{u_n\} \subset X_\lambda$ satisfying

$$J_{a,\lambda}(u_n) \to c_\lambda > 0 \quad \text{and} \quad (1 + \|u_n\|_\lambda)\|J'_{a,\lambda}(u_n)\|_{X_\lambda^{-1}} \to 0, \quad \text{as } n \to \infty.$$

By Lemma 3.1, one has $\{u_n\}$ is bounded in X_λ. Then it follows from Proposition 3.2 and the fact of $0 < \eta \leq c_\lambda \leq c_0(\Omega)$ that $J_{a,\lambda}$ satisfies the $(C)_\alpha$-condition in X_λ for all $c_\lambda < D$ and $\lambda > \Lambda_1$. This indicates that there exist a subsequence $\{u_n\}$ and $u_{a,\lambda}^{(1)} \in X_\lambda$ such that $u_n \to u_{a,\lambda}^{(1)}$ strongly in X_λ. The proof is complete. □

Lemma 3.4. *Suppose that $N \geq 3$, $\delta \geq \frac{2}{N-2}$ and $b > -2A_0^{-2}\beta_N^{-1}$. In addition, assume that conditions (A1)-(A4),(F1), (A6) hold. Then for every $0 < a < a_*$ and $\lambda > \Lambda_1$ one has*

$$-\infty < \theta_a =: \inf\{J_{a,\lambda}(u) : u \in X_\lambda \text{ with } \rho < \|u\|_\lambda < R_a\} < \frac{\kappa}{2} < 0. \tag{3.12}$$

The proof of the above lemma follows directly from Lemmas 2.1 and 2.4.

Theorem 3.5. *Suppose that $N \geq 3$, $\delta \geq 2/(N-2)$ and $b > -2A_0^{-2}\beta_N^{-1}$. In addition, assume that conditions* (A1)–(A6) *hold. Then for every $0 < a < a_*$ and $\lambda > \Lambda_1$, $J_{a,\lambda}$ has a nonzero critical point $u_{a,\lambda}^{(2)} \in X_\lambda$ such that*

$$J_{a,\lambda}(u_{a,\lambda}^{(2)}) = \theta_a < 0,$$

where $\widehat{\theta}_a$ is as in (3.12)*. Furthermore, when $\delta > \frac{2}{N-2}$, for every $\lambda > \Lambda_1$ it holds*

$$J_{a,\lambda}(u_{a,\lambda}^{(2)}) \to -\infty \quad \text{and} \quad \|u_{a,\lambda}^{(2)}\|_\lambda \to \infty \text{ as } a \to 0,$$

Proof. It follows from Lemmas 3.1, 3.4 and the Ekeland variational principle that there exists a minimizing bounded sequence $\{u_n\} \subset X_\lambda$ with $\rho < \|u_n\|_\lambda < R_a$ such that

$$J_{a,\lambda}(u_n) \to \theta_a \quad \text{and} \quad J'_{a,\lambda}(u_n) \to 0 \text{ as } n \to \infty.$$

Similar to the proof of Theorem 3.3, there exist a subsequence $\{u_n\}$ and $u_{a,\lambda}^{(2)} \in X_\lambda$ with $\rho < \|u_{a,\lambda}^{(2)}\|_\lambda < R_a$ such that $u_n \to u_{a,\lambda}^{(2)}$ strongly in X_λ, which implies that $J'_{a,\lambda}(u_{a,\lambda}^{(2)}) = 0$ and $J_{a,\lambda}(u_{a,\lambda}^{(2)}) = \theta_a < 0$. Furthermore, by Lemmas 2.1 and 2.4 we have

$$J_{a,\lambda}(u_{a,\lambda}^{(2)}) \leq J_{a,\lambda}(e) \to -\infty \quad \text{as } a \to 0.$$

This implies $\|u_{a,\lambda}^{(2)}\|_\lambda \to \infty$ as $a \to 0$. The proof is complete. □

We are now ready to prove Theorem 1.1. By Theorems 3.3 and 3.5, for every $0 < a < a_*$ and $\lambda > \Lambda_1$, there exist two nontrivial solutions $u^{(1)}_{a,\lambda}$ and $u^{(2)}_{a,\lambda}$ of $(K_{a,\lambda})$ such that

$$J_{a,\lambda}(u^{(2)}_{a,\lambda}) = \theta_a < \frac{\kappa}{2} < 0 < \eta < c_\lambda = J_{a,\lambda}(u^{(1)}_{a,\lambda}),$$

which implies that $u^{(1)}_{a,\lambda} \neq u^{(2)}_{a,\lambda}$. Furthermore, when $\delta > \frac{2}{N-2}$, for every $\lambda > \Lambda_1$ it holds

$$J_{a,\lambda}(u^{(2)}_{a,\lambda}) \to -\infty \quad \text{and} \quad \|u^{(2)}_{a,\lambda}\|_\lambda \to \infty \text{ as } a \to 0.$$

Since $J_{a,\lambda}(u) \geq 0$ on $\{u \in X_\lambda$ with $\|u\|_\lambda \leq \rho \cup \|u\|_\lambda \geq R_a\}$ by Lemmas 3.4 and 2.3, we conclude that $u^{(2)}_{a,\lambda}$ is a ground state solution of $(K_{a,\lambda})$. This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2. Let u_0 be a nontrivial solution of $(K_{a,\lambda})$. Then

$$\|u_0\|^2_\lambda + a\|\nabla u_0\|^{2(1+\delta)}_{L^2} + b\|\nabla u_0\|^2_{L^2} - \int_{\mathbb{R}^N} f(x,u_0)u_0 dx = 0.$$

We now divide the proof into two separate cases:

Case A:

$$\int_{\mathbb{R}^N} \lambda V(x)u_0^2 dx \geq \lambda c_0 (\frac{C_{1,\epsilon}}{\lambda c_0 - \epsilon})^{\frac{4}{(p-2)(N-2)}} (\overline{S}^{-1}\|\nabla u_0\|_{L^2})^{\frac{2N}{N-2}}.$$

It follows from the condition (A6) and (2.19) that

$$\begin{aligned}
0 &= \|u_0\|^2_\lambda + a\|\nabla u_0\|^{2(1+\delta)}_{L^2} + b\|\nabla u_0\|^2_{L^2} - \int_{\mathbb{R}^N} f(x,u_0)u_0 dx \\
&\geq \|u_0\|^2_\lambda + a\|\nabla u_0\|^{2(1+\delta)}_{L^2} + b\|\nabla u_0\|^2_{L^2} - \epsilon\Big(\frac{1}{\lambda c_0}\int_{\mathbb{R}^N} \lambda V(x)u_0^2 dx \\
&\quad + \overline{S}^{-2}|\{V<c_0\}|^{\frac{2}{N}}\|\nabla u_0\|^2_{L^2}\Big) - \frac{C_{1,\epsilon}}{\overline{S}^{\frac{N(p-2)}{2}}}\Big(\frac{1}{\lambda c_0}\int_{\mathbb{R}^N} \lambda V(x)u_0^2 dx \\
&\quad + \overline{S}^{-2}|\{V<c_0\}|^{\frac{2}{N}}\|\nabla u_0\|^2_{L^2}\Big)^{\frac{2N-p(N-2)}{4}} \|\nabla u_0\|^{\frac{N(p-2)}{2}}_{L^2} \\
&\geq a\|\nabla u_0\|^{2(1+\delta)}_{L^2} + \Big(b - \epsilon\overline{S}^{-2}|\{V<c_0\}|^{2/N}\Big)\|\nabla u_0\|^2_{L^2} \\
&\quad - \frac{C_{1,\epsilon}}{\overline{S}^p}|\{V<c_0\}|^{\frac{2N-p(N-2)}{2N}}\|\nabla u_0\|^p_{L^2} + \Big(\int_{\mathbb{R}^N}\lambda V(x)u_0^2 dx\Big)^{\frac{2N-p(N-2)}{4}} \\
&\quad \times \Big[\frac{\lambda c_0 - \epsilon}{\lambda c_0}\Big(\int_{\mathbb{R}^N}\lambda V(x)u^2 dx\Big)^{\frac{(p-2)(N-2)}{4}} \\
&\quad - \frac{C_{1,\epsilon}}{\overline{S}^{\frac{N(p-2)}{2}}}(\frac{1}{\lambda c_0})^{\frac{2N-p(N-2)}{4}}\|\nabla u_0\|^{\frac{N(p-2)}{2}}_{L^2}\Big] \\
&\geq a\|\nabla u_0\|^{2(1+\delta)}_{L^2} + \big(b - \epsilon\overline{S}^{-2}|\{V<c_0\}|^{2/N}\big)\|\nabla u_0\|^2_{L^2} \\
&\quad - \frac{C_{1,\epsilon}}{\overline{S}^p}|\{V<c_0\}|^{\frac{2N-p(N-2)}{2N}}\|\nabla u_0\|^p_{L^2} > 0\,,
\end{aligned}$$

provided that

$$a > \frac{p-2}{2\delta}\Big[\frac{2(\delta+1)-p}{2\delta(b-\epsilon\overline{S}^{-2}|\{V<c_0\}|^{2/N})}\Big]^{\frac{2(\delta+1)-p}{p-2}}\Big(\frac{C_{1,\epsilon}}{\overline{S}^p}|\{V<c_0\}|^{\frac{2N-p(N-2)}{2N}}\Big)^{\frac{2\delta}{p-2}}.$$

This is a contradiction.

Case B:

$$\int_{\mathbb{R}^N}\lambda V(x)u_0^2dx < \lambda c_0\Big[\frac{C_{1,\epsilon}}{(\lambda c_0-\epsilon)}\Big]^{\frac{4}{(p-2)(N-2)}}(\overline{S}^{-1}\|\nabla u_0\|_{L^2})^{\frac{2N}{N-2}}.$$

By (2.19) one has

$$\begin{aligned}
&\int_{\mathbb{R}^N}|u_0|^pdx\\
&\le \Big(\frac{1}{\lambda c_0}\int_{\mathbb{R}^N}\lambda V(x)u_0^2dx + \frac{|\{V<c_0\}|^{2/N}}{\overline{S}^2}\|\nabla u_0\|_{L^2}^2\Big)^{\frac{2^*-p}{2^*-2}}\Big(\overline{S}^{-1}\|\nabla u_0\|_{L^2}\Big)^{\frac{N(p-2)}{2}}\\
&< \overline{S}^{-2^*}\Big(\frac{C_{1,\epsilon}}{\lambda c_0-\epsilon}\Big)^{\frac{2N-p(N-2)}{(p-2)(N-2)}}\|\nabla u_0\|_{L^2}^{2^*} + \overline{S}^{-p}|\{V<c_0\}|^{\frac{2N-p(N-2)}{2N}}\|\nabla u_0\|_{L^2}^p.
\end{aligned}$$

Using this and (A6), we have

$$\begin{aligned}
0 &= \|u_0\|_\lambda^2 + a\|\nabla u_0\|_{L^2}^{2(1+\delta)} + b\|\nabla u_0\|_{L^2}^2 - \int_{\mathbb{R}^N} f(x,u_0)u_0dx\\
&> a\|\nabla u_0\|_{L^2}^{2(1+\delta)} + (b-\epsilon\overline{S}^{-2}|\{V<c_0\}|^{\frac{2}{N}})\|\nabla u_0\|_{L^2}^2\\
&\quad - \frac{C_{1,\epsilon}^{\frac{4}{(p-2)(N-2)}}}{\overline{S}^{2^*}}(\lambda c_0-\epsilon)^{-\frac{2N-p(N-2)}{(p-2)(N-2)}}\|\nabla u_0\|_{L^2}^{2^*}\\
&\quad - \frac{C_{1,\epsilon}}{\overline{S}^p}|\{V<c_0\}|^{\frac{2N-p(N-2)}{2N}}\|\nabla u_0\|_{L^2}^p.
\end{aligned}$$

If $\delta = 2/(N-2)$, then for

$$\begin{aligned}
a &> \frac{p-2}{2\delta}\Big[\frac{2(\delta+1)-p}{2\delta(b-\epsilon\overline{S}^{-2}|\{V<c_0\}|^{2/N})}\Big]^{\frac{2(\delta+1)-p}{p-2}}\Big(\frac{C_{1,\epsilon}}{\overline{S}^p}|\{V<c_0\}|^{\frac{2N-p(N-2)}{2N}}\Big)^{\frac{2\delta}{p-2}}\\
&\quad + \frac{C_{1,\epsilon}}{\overline{S}^{2^*}}\Big(\frac{C_{1,\epsilon}}{\lambda c_0-\epsilon}\Big)^{\frac{2N-p(N-2)}{(p-2)(N-2)}},
\end{aligned}$$

we have

$$\begin{aligned}
0 &> a\|\nabla u_0\|_{L^2}^{2(1+\delta)} + (b-\epsilon\overline{S}^{-2}|\{V<c_0\}|^{\frac{2}{N}})\|\nabla u_0\|_{L^2}^2\\
&\quad - \frac{C_{1,\epsilon}}{\overline{S}^{2^*}}\Big(\frac{C_{1,\epsilon}}{\lambda c_0-\epsilon}\Big)^{\frac{2N-p(N-2)}{(p-2)(N-2)}}\|\nabla u_0\|_{L^2}^{2^*}\\
&\quad - \frac{C_{1,\epsilon}}{\overline{S}^p}|\{V<c_0\}|^{\frac{2N-p(N-2)}{2N}}\|\nabla u_0\|_{L^2}^p > 0.
\end{aligned}$$

This is a contradiction. If $\delta > 2/(N-2)$, then we consider the following two cases:

(i)

$$\|\nabla u_0\|_{L^2}^2 \ge \overline{S}^2|\{V<c_0\}|^{\frac{N-2}{N}}\Big(\frac{\lambda c_0-\epsilon}{C_{1,\epsilon}}\Big)^{\frac{2}{p-2}}.$$

Then we have

$$0 > a\|\nabla u_0\|_{L^2}^{2(1+\delta)} + (b-\epsilon\overline{S}^{-2}|\{V<c_0\}|^{\frac{2}{N}})\|\nabla u_0\|_{L^2}^2$$

$$
\begin{aligned}
&- \frac{C_{1,\epsilon}}{\overline{S}^{2^*}}\Big(\frac{C_{1,\epsilon}}{(\lambda c_0-\epsilon)}\Big)^{\frac{2N-p(N-2)}{(p-2)(N-2)}}\|\nabla u_0\|_{L^2}^{2^*} - \frac{C_{1,\epsilon}}{\overline{S}^p}|\{V<c_0\}|^{\frac{2N-p(N-2)}{2N}}\|\nabla u_0\|_{L^2}^p \\
&\geq a\|\nabla u_0\|_{L^2}^{2(1+\delta)} + \big(b-\epsilon\overline{S}^{-2}|\{V<c_0\}|^{\frac{2}{N}}\big)\|\nabla u_0\|_{L^2}^2 \\
&- \frac{2C_{1,\epsilon}}{\overline{S}^{2^*}}(\frac{C_{1,\epsilon}}{(\lambda c_0-\epsilon)})^{\frac{2N-p(N-2)}{(p-2)(N-2)}}\|\nabla u_0\|_{L^2}^{2^*} > 0\,,
\end{aligned}
$$

provided that

$$
\begin{aligned}
a > \;& \frac{2^{\frac{2+\delta(N-2)}{2}} C_{1,\epsilon}^{\frac{2\delta}{p-2}}(\lambda c_0-\epsilon)^{-\frac{[2N-p(N-2)]\delta}{2(p-2)}}}{\delta(N-2)\overline{S}^{N\delta}} \\
&\times \Big[\frac{\delta(N-2)-2}{\delta(N-2)(b-\epsilon\overline{S}^{-2}|\{V<c_0\}|^{\frac{2}{N}})}\Big]^{\frac{\delta(N-2)-2}{2}}.
\end{aligned}
$$

This is a contradiction.

(ii)

$$
\|\nabla u_0\|_{L^2}^2 < \overline{S}^2|\{V<c_0\}|^{\frac{N-2}{N}}(\frac{\lambda c_0-\epsilon}{C_{1,\epsilon}})^{\frac{2}{p-2}}.
$$

Then

$$
\begin{aligned}
0 &> a\|\nabla u_0\|_{L^2}^{2(1+\delta)} + \big(b-\epsilon\overline{S}^{-2}|\{V<c_0\}|^{\frac{2}{N}}\big)\|\nabla u_0\|_{L^2}^2 \\
&- \frac{C_{1,\epsilon}}{\overline{S}^{2^*}}\Big(\frac{C_{1,\epsilon}}{(\lambda c_0-\epsilon)}\Big)^{\frac{2N-p(N-2)}{(p-2)(N-2)}}\|\nabla u_0\|_{L^2}^{2^*} - \frac{C_{1,\epsilon}}{\overline{S}^p}|\{V<c_0\}|^{\frac{2N-p(N-2)}{2N}}\|\nabla u_0\|_{L^2}^p \\
&\geq a\|\nabla u_0\|_{L^2}^{2(1+\delta)} + \big(b-\epsilon\overline{S}^{-2}|\{V<c_0\}|^{\frac{2}{N}}\big)\|\nabla u_0\|_{L^2}^2 \\
&- 2\frac{C_{1,\epsilon}}{\overline{S}^p}|\{V<c_0\}|^{\frac{2N-p(N-2)}{2N}}\|\nabla u_0\|_{L^2}^p > 0\,,
\end{aligned}
$$

provided that

$$
a > \frac{p-2}{2\delta}\Big[\frac{2(\delta+1)-p}{2\delta(b-\epsilon\overline{S}^{-2}|\{V<c_0\}|^{2/N})}\Big]^{\frac{2(\delta+1)-p}{p-2}}\Big(\frac{2C_{1,\epsilon}|\{V<c_0\}|^{\frac{2N-p(N-2)}{2N}}}{\overline{S}^p}\Big)^{\frac{2\delta}{p-2}}.
$$

We also get a contradiction. Therefore, there exists a constant $a^*>0$ such that for every $a>a^*$, $(K_{a,\lambda})$ does not admit any nontrivial solution for all $\lambda>\Lambda_N$. This completes the proof of Theorem 1.2.

5. Concentration of solutions

In this section, we investigate the concentration for solutions and give the proof of Theorem 1.3.

Proof of Theorem 1.3. Following the arguments in [3], we first choose a positive sequence $\{\lambda_n\}$ such that $\Lambda_1<\lambda_1\leq\lambda_2\leq...\leq\lambda_n\to\infty$ as $n\to\infty$. Let $u_n^{(i)}:=u_{a,\lambda_n}^{(i)}$ with $i=1,2$ be the critical points of J_{a,λ_n} obtained in Theorem 1.1. Since

$$
J_{a,\lambda_n}(u_n^{(2)}) < \frac{\kappa}{2} < 0 < \eta < c_{\lambda_n} = J_{a,\lambda_n}(u_n^{(1)}) < D, \tag{5.1}
$$

by Lemma 2.1 one has

$$
\|u_n^{(i)}\|_{\lambda_n} \leq C_0, \tag{5.2}
$$

where the constant $C_0 > 0$ is independent of λ_n. This implies that $\|u_n^{(i)}\|_{\lambda_1} \le C_0$. Thus, there exist $u_\infty^{(i)} \in X$ $(i = 1, 2)$ such that

$$\begin{gathered} u_n^{(i)} \rightharpoonup u_\infty^{(i)} \quad \text{weakly in } X_{\lambda_1}, \\ u_n^{(i)} \to u_\infty^{(i)} \quad \text{strongly in } L^r_{\rm loc}(\mathbb{R}^N) \text{ for } 2 \le r < 2_*, \\ u_n^{(i)} \to u_\infty^{(i)} \quad \text{a.e. in } \mathbb{R}^N. \end{gathered}$$

Following the proof of Proposition 3.2, we conclude that

$$u_n^{(i)} \to u_\infty^{(i)} \quad \text{strongly in } X_{\lambda_1}.$$

This shows that $u_n^{(i)} \to u_\infty^{(i)}$ strongly in $H^2(\mathbb{R}^N)$ by (2.10) and that

$$\|\nabla u_n^{(i)}\|_{L^2}^2 \to \|\nabla u_\infty^{(i)}\|_{L^2}^2 \quad \text{as } n \to \infty \tag{5.3}$$

by (2.11) and (3.1). Fatou's Lemma leads to

$$\int_{\mathbb{R}^N} V(x)(u_\infty^{(i)})^2 dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} V(x)\big(u_n^{(i)}\big)^2 dx \le \liminf_{n\to\infty} \frac{\|u_n^{(i)}\|_{\lambda_n}^2}{\lambda_n} = 0,$$

which implies that $u_\infty^{(i)}(x) = 0$ a.e. in $\mathbb{R}^N \backslash \overline{\Omega}$. Moreover, fixing $\phi \in C_0^\infty(\mathbb{R}^N \backslash \overline{\Omega})$, we have

$$\int_{\mathbb{R}^N\backslash\overline{\Omega}} \nabla u_\infty^{(i)}(x)\phi(x)dx = -\int_{\mathbb{R}^N\backslash\overline{\Omega}} u_\infty^{(i)}(x)\nabla\phi(x)dx = 0.$$

This indicates that

$$\nabla u_\infty^{(i)}(x) = 0 \quad \text{a.e. in } \mathbb{R}^N\backslash\overline{\Omega}.$$

Since $\partial\Omega$ is smooth, $u_\infty^{(i)} \in H^2(\mathbb{R}^N\backslash\overline{\Omega})$ and $\nabla u_\infty^{(i)} \in H^1(\mathbb{R}^N\backslash\overline{\Omega})$, it follows from Trace Theorem that there are constants $\overline{C}, \widetilde{C} > 0$ such that

$$\begin{gathered} \|u_\infty^{(i)}\|_{L^2(\partial\Omega)} \le \overline{C}\|u_\infty^{(i)}\|_{H^2(\mathbb{R}^N\backslash\overline{\Omega})} = 0, \\ \|\nabla u_\infty^{(i)}\|_{L^2(\partial\Omega)} \le \widetilde{C}\|\nabla u_\infty^{(i)}\|_{H^1(\mathbb{R}^N\backslash\overline{\Omega})} = 0. \end{gathered}$$

These inequalities show that $u_\infty^{(i)} \in H_0^2(\Omega)$.

Since $\langle J'_{a,\lambda_n}(u_n^{(i)}), \varphi\rangle = 0$ for any $\varphi \in C_0^\infty(\Omega)$, combining (5.3), it is not difficult to check that

$$\int_\Omega \Delta u_\infty^{(i)} \Delta\varphi dx + \Big[a\big(\int_\Omega |\nabla u_\infty^{(i)}|^2 dx\big)^\delta + b\Big] \int_\Omega \nabla u_\infty^{(i)} \cdot \nabla\varphi dx = \int_\Omega f(x, u_\infty^{(i)})\varphi dx;$$

that is, $u_\infty^{(i)}$ are the weak solutions of the equation

$$\Delta^2 u - M\big(\int_\Omega |\nabla u|^2 dx\big)\Delta u = f(x,u) \quad \text{in } \Omega,$$

where $M(t) = at^\delta + b$. Since $u_n^{(i)} \to u_\infty^{(i)}$ strongly in X, using (5.1) and the fact that η and κ are independent of λ_n gives

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\Delta u_\infty^{(1)}|^2 dx + \frac{a}{2(1+\delta)}\Big(\int_\Omega |\nabla u_\infty^{(1)}|^2 dx\Big)^{\delta+1} \\ &+ \frac{b}{2}\int_\Omega |\nabla u_\infty^{(1)}|^2 dx - \int_\Omega F(x, u_\infty^{(1)})dx \ge \eta > 0 \end{aligned}$$

and

$$\frac{1}{2}\int_\Omega |\Delta u_\infty^{(2)}|^2 dx + \frac{a}{2(1+\delta)}\big(\int_\Omega |\nabla u_\infty^{(2)}|^2 dx\big)^{\delta+1}$$

$$+\frac{b}{2}\int_\Omega |\nabla u_\infty^{(2)}|^2dx-\int_\Omega F(x,u_\infty^{(2)})dx\le\frac{\kappa}{2}<0.$$

These imply that $u_\infty^{(i)}\neq 0 (i=1,2)$ and $u_\infty^{(1)}\neq u_\infty^{(2)}$. The proof is complete. □

6. Appendix A

Consider the biharmonic equation

$$\begin{gathered}\Delta^2u+\lambda V(x)u=|u|^{p-2}u \quad \text{in } \mathbb{R}^N,\\ u\in H^2(\mathbb{R}^N),\end{gathered}\tag{6.1}$$

where $N\ge 3$, $2<p<\frac{2N}{N-2}$, $\lambda>0$ is a parameter and $V(x)$ satisfies conditions (A1)–(A3).

Equation (6.1) is variational and its solutions are critical points of the functional defined in X_λ by

$$\mathcal{J}_\lambda(u)=\frac{1}{2}\|u\|_\lambda^2-\frac{1}{p}\int_{\mathbb{R}^N}|u|^pdx,$$

where $\|u\|_\lambda$ is defined as (2.1). It is easily seen that the functional J_λ is of class C^1 in X_λ, and that

$$\langle\mathcal{J}_\lambda'(u),v\rangle=\int_{\mathbb{R}^N}[\Delta u\cdot\Delta v+\lambda V(x)uv]dx-\int_{\mathbb{R}^N}|u|^{p-2}uv\,dx.\tag{6.2}$$

We define the Nehari manifold by

$$\mathcal{N}=\{u\in X_\lambda\backslash\{0\}:\langle\mathcal{J}_\lambda'(u),u\rangle=0\}.$$

Clearly, $J_\lambda(u)$ is bounded below and coercive on N, since $p>2$. Following the standard argument, we conclude that there exists a positive constant $\Lambda_1\ge\Lambda_N$ such that (6.1) admits a positive ground state solution $\phi_\lambda\in H^2(\mathbb{R}^N)$ for every $\lambda\ge\Lambda_1$. Similar to the argument in [17, Theorem 22], we obtain that Π_λ defined as (2.20) is achieved and

$$\Pi_\lambda=\frac{(\int_{\mathbb{R}^N}|\phi_\lambda|^pdx)^{1/p}}{\|\phi_\lambda\|_\lambda}>0 \quad \text{for every } \lambda\ge\Lambda_1.$$

Furthermore, similar to the proof of Theorem 1.3, it follows that $\phi_\lambda\to\phi_\infty$ in $H^2(\mathbb{R}^N)$ and in $L^p(\mathbb{R}^N)$ as $\lambda\to\infty$, where $0\neq\phi_\infty\in H_0^2(\Omega)$ is the weak solution of biharmonic equations as follows

$$\Delta^2u=|u|^{p-2}u \quad \text{in } \Omega.$$

This implies that

$$\Pi_\lambda\to\Pi_\infty:=\frac{(\int_\Omega|\phi_\infty|^pdx)^{1/p}}{(\int_\Omega|\Delta\phi_\infty|^2dx)^{1/2}}>0 \quad \text{as } \lambda\nearrow\infty.$$

Note that

$$\Pi_\lambda=\sup_{u\in X_\lambda\backslash\{0\}}\frac{\big(\int_{\mathbb{R}^N}|u|^pdx\big)^{1/p}}{\|u\|_\lambda}$$

is decreasing on λ. Hence, we have

$$\Pi_\lambda\searrow\Pi_\infty \text{ as } \lambda\nearrow\infty.$$

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