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HÖLDER CONTINUITY OF BOUNDED GENERALIZED SOLUTIONS FOR NONLINEAR FOURTH-ORDER ELLIPTIC EQUATIONS WITH STRENGTHENED COERCIVITY AND NATURAL GROWTH TERMS

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Abstract. In this article we extend the author's previous results on the existence of bounded generalized solutions of a Dirichlet problem for nonlinear elliptic fourth-order equations with the principal part satisfying a strengthened coercivity condition, and a lower-order term having a "natural" growth with respect to the derivatives of the unknown function. Namely, we prove the Hölder continuity of bounded generalized solutions of such equations.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geq 3$. Then we consider the nonlinear fourth-order elliptic equations of divergent form

$$\sum_{|\alpha|=1,2} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, \nabla_2 u) + B(x, u, \nabla_2 u) = 0 \quad \text{in } \Omega, \tag{1.1}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n-dimensional multiindex with nonnegative integer components α_i, $i = 1, \dots, n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$ and $\nabla_2 u = \{D^\alpha u : |\alpha| = 1, 2\}$.

The main structural requirements for the coefficients A_α and B are the following strengthened coercivity condition: for a. e. $x \in \Omega$ and for every $s \in \mathbb{R}$ and $\xi = \{\xi_\alpha \in \mathbb{R} : |\alpha| = 1, 2\}$,

$$\sum_{|\alpha|=1,2} A_\alpha(x, s, \xi)\xi_\alpha \geq C\Big\{ \sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p \Big\} - f_1(x), \tag{1.2}$$

and the natural growth condition: for a. e. $x \in \Omega$ and for every $s \in \mathbb{R}$ and $\xi = \{\xi_\alpha \in \mathbb{R} : |\alpha| = 1, 2\}$

$$|B(x, s, \xi)| \leq b(|s|)\Big\{ \sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p \Big\} + f_2(x), \tag{1.3}$$

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where $C > 0$, $p \in (1, n/2)$, $q \in (2p, n)$, $b : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nondecreasing function, $f_{1,2} \geq 0$ and $f_{1,2} \in L^\tau(\Omega)$, $\tau > n/q$.

We recall that the strengthened coercivity condition goes back to Skrypnik [25], used because of the regularity problem of generalized solutions from the class $W^{m,p}(\Omega)$ for nonlinear elliptic equations of the divergent form

$$\sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha \mathcal{A}_\alpha(x, u, \dots, D^m u) = 0 \quad \text{in } \Omega. \tag{1.4}$$

In this problem the relation $n = mp$ is important. If $n < mp$, the Hölder continuity of solutions is a simple consequence of the embedding theorem (see, e.g., [12, Section 7.7]). For $n \geq mp$, the embedding theorems do not ensure the boundedness of an arbitrary solution $u \in W^{m,p}(\Omega)$.

In the case $m = 1$ and $n \geq p$, the properties of the boundedness and the continuity of generalized solutions are well known (see, e.g., [1, 2, 10, 13, 14], [19, Chapter IV, §7, and Chapter IX, §2]).

For $m \geq 2$ and $n = mp$, the boundedness of solutions is established in [11, 32, 33], and the continuity is proved in [24, Chapter II, § 3], [28, 34].

Eventually, in the case where $m \geq 2$ and $n > mp$, there are examples of equations of the form (1.4) with the smooth coefficients $\mathcal{A}_\alpha$ and unbounded solutions (see, e.g., [8, 21]). In this case, Skrypnik [25] separated a subclass of equations of the form (1.4) whose all generalized solutions are bounded and Hölder continuous. The separated subclass of equations is characterized by the strengthened coercivity condition under which the natural energy space is the space $W^{m,p}(\Omega) \cap W^{1,q}(\Omega)$ with $q > mp$. In particular, for $m = 2$, the structure of this class of equations is determined by the inequality of the form (1.2).

Article [25] initiated a research on local properties of solutions for nonlinear high-order elliptic equations with strengthened coercivity and sufficiently regular data. For example, there were established sufficient conditions for regularity of a boundary point [26] and for the removability of isolated singularities of solutions [7], obtained pointwise estimates for solutions of some model problems [5, 27], proved the Harnack inequality [6, 23].

At the same time, in the cited papers on the equations with the strengthened coercivity it was assumed that the coefficients $\mathcal{A}_\alpha$ satisfy the standard growth conditions for the space $W^{1,q}_{m,p}(\Omega) = W^{m,p}(\Omega) \cap W^{1,q}(\Omega)$. In this situation, the solvability of equation (1.4) is equivalent to the solvability of the operator equation $\mathcal{A}u = 0$ where the nonlinear operator $\mathcal{A} : W^{1,q}_{m,p}(\Omega) \to [W^{1,q}_{m,p}(\Omega)]^*$ is defined by the equality

$$\langle \mathcal{A}u, v\rangle = \sum_{|\alpha| \leq m} \int_\Omega \mathcal{A}_\alpha(x, u, ..., D^m u) D^\alpha v dx.$$

So, the standard theory of equations with pseudomonotone operators (see, e.g., [20]) is applicable. The case where the lower-order term has the natural (q, p)-growth like (1.3) is beyond the scope of this theory and requires separate consideration. In this regard, we refer to [29], [30], [31] where the existence and L^∞-estimates of solutions of the Dirichlet problem for nonlinear high-order elliptic equations with the strengthened coercivity and natural growth terms were established.

Existence and L^∞-estimates of bounded solutions of nonlinear second-order elliptic equations with natural growth lower-order terms were established for instance in [3, 9], and the Hölder continuity of the solutions was proved in [19, Chapter IX, §2]).

In the present article, we strengthen and supplement our previous results in [29], [30]. Namely, we prove the Hölder continuity in Ω of every generalized solution $u \in W^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ of equation (1.1) under the conditions (1.2), (1.3). For the proof, we use an analogue of Moser's method (see [19, Chapter IX], [22]) proposed in [25]. This method is based on obtaining of uniform L^k-estimates (at $k \to +\infty$) for an auxiliary function $\phi(u)$ such that its boundedness implies the Hölder continuity of the solution u. The new point in the proof (compared to [25]) is an additional requirement on the power exponent k in the test function in [25]. Namely, the use of the condition $k \geq k_0$ with a suitable $k_0 = k_0(\mathtt{data}) > 0$ leads to the absorption of the lower-order term of natural growth by the coercive principal part of the equation. In this regard, to complete the proof we need some further integral estimates of the solution associated with the use of the Lemma of John-Nirenberg [15].

We remark that a theory of existence and properties of solutions of nonlinear fourth-order elliptic equations with coefficients satisfying the strengthened coercivity condition and L^1-right-hand sides was developed in [16, 17], [18, Part I, Section 2] (see also [4] for equations with natural growth terms).

This article is organized as follows. In Section 2, we give the statement of the problem and present the main result (Theorem 2.3). In the same section, we provide examples of equations that satisfy all the hypotheses. In Section 3, we present some auxiliary results needed to the proof of Theorem 2.3 which is set out in Section 4.

2. Preliminaries and the statement of the main result

Let $n \in \mathbb{N}$, $n > 2$, and let Ω be a bounded domain of $\mathbb{R}^n$.

We shall use the following notation: $\mathbb{R}_+ = [0, +\infty)$; ∂S is the boundary of the set $S \subset \mathbb{R}^n$, $\overline{S} = S \cup \partial S$ is the closure of S; Λ is the set of all n-dimensional multi-indices α such that $|\alpha| = 1$ or $|\alpha| = 2$; $\mathbb{R}^{n,2}$ is the space of all mappings $\xi : \Lambda \to \mathbb{R}$; if $u \in W^{2,1}(\Omega)$, then $\nabla_2 u : \Omega \to \mathbb{R}^{n,2}$, and for every $x \in \Omega$ and for every $\alpha \in \Lambda$, $(\nabla_2 u(x))_\alpha = D^\alpha u(x)$. If $\tau \in [1, +\infty]$, then $\|\cdot\|_\tau$ is the norm in $L^\tau(\Omega)$. For every measurable set $E \subset \mathbb{R}^n$ we denote by $|E|$ n-dimensional Lebesgue measure of the set E.

Let $p \in (1, n/2)$ and $q \in (2p, n)$. We denote by $W^{1,q}_{2,p}(\Omega)$ the set of all functions in $W^{1,q}(\Omega)$ that have the second-order generalized derivatives in $L^p(\Omega)$. The set $W^{1,q}_{2,p}(\Omega)$ is a Banach space with the norm

$$\|u\| = \|u\|_{W^{1,q}(\Omega)} + \Big(\sum_{|\alpha|=2} \int_\Omega |D^\alpha u|^p dx\Big)^{1/p}.$$

We denote by $\mathring{W}^{1,q}_{2,p}(\Omega)$ the closure of the set $C_0^\infty(\Omega)$ in $W^{1,q}_{2,p}(\Omega)$.

We consider the equation

$$\sum_{\alpha \in \Lambda} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, \nabla_2 u) + B(x, u, \nabla_2 u) = 0 \quad \text{in } \Omega \tag{2.1}$$

under the following assumptions:

(A1) For every $\alpha \in \Lambda$, $A_\alpha : \Omega \times \mathbb{R} \times \mathbb{R}^{n,2} \to \mathbb{R}$ and $B : \Omega \times \mathbb{R} \times \mathbb{R}^{n,2} \to \mathbb{R}$ are Carathéodory functions, i.e. for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^{n,2}$, the functions

$A_\alpha(\cdot, s, \xi)$ and $B(\cdot, s, \xi)$ are measurable on Ω and, for almost every $x \in \Omega$, the functions $A_\alpha(x, \cdot, \cdot)$ and $B(x, \cdot, \cdot)$ are continuous in $\mathbb{R} \times \mathbb{R}^{n,2}$.

(A2) For almost every $x \in \Omega$ and for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^{n,2}$ the following inequalities hold:

$$\sum_{\alpha\in\Lambda} A_\alpha(x, s, \xi)\xi_\alpha \geq a(|s|)\Big\{ \sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p \Big\} - g_0(x), \tag{2.2}$$

$$\sum_{|\alpha|=1} |A_\alpha(x, s, \xi)|^{q/(q-1)} \leq a_1(|s|)\Big\{ \sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p \Big\} + g_1(x), \tag{2.3}$$

$$\sum_{|\alpha|=2} |A_\alpha(x, s, \xi)|^{p/(p-1)} \leq a_2(|s|)\Big\{ \sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p \Big\} + g_2(x), \tag{2.4}$$

$$|B(x, s, \xi)| \leq b(|s|)\Big\{ \sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p \Big\} + g_3(x), \tag{2.5}$$

where $a : \mathbb{R}_+ \to (0, +\infty)$ is a continuous nonincreasing function, $a_1, a_2, b : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous nondecreasing functions, g_0, g_1, g_2, g_3 are nonnegative summable functions on Ω.

Assumptions (A1) and (A2) provide the correct setting for the following definition.

Definition 2.1. A generalized solution of (2.1) is a function $u \in W^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ such that for every function $v \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega \Big\{ \sum_{\alpha\in\Lambda} A_\alpha(x, u, \nabla_2 u)D^\alpha v + B(x, u, \nabla_2 u)v \Big\} dx = 0. \tag{2.6}$$

Remark 2.2. Note that if in addition to assumptions (A1) and (A2) the functions a, a_1, a_2 are positive constants, the inequalities

$$\begin{gathered} \sum_{\alpha\in\Lambda} [A_\alpha(x, s, \xi) - A_\alpha(x, s, \xi')](\xi_\alpha - \xi'_\alpha) > 0, \\ B(x, s, \xi)s \geq -g_4(x), \quad g_4(x) \geq 0 \end{gathered} \tag{2.7}$$

hold for almost every $x \in \Omega$ and any $s \in \mathbb{R}$ and ξ, $\xi' \in \mathbb{R}^{n,2}$, $\xi \neq \xi'$, and the functions g_0, g_1, g_2, g_3, g_4 belong to $L^\tau(\Omega)$ with $\tau > n/q$, then there exists a generalized solution $u \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ of equation (2.1). This follows from [29, Theorem 2.1]. The result remains true if instead of (2.7), we assume that the function b in (2.5) is bounded and the left-hand side of equation (2.1) has an absorption term like $c|u|^{q-2}u$, $c > 0$ (see [30, Theorem 2.2]).

The main result of the present article is a theorem on the local Hölder continuity of any generalized solution $u \in W^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ of equation (2.1) under assumptions (A1) and (A2). Following [12, Chapter 4], we recall that a function $f : \mathcal{D} \to \mathbb{R}$, $\mathcal{D} \subset \mathbb{R}^n$ is *uniformly Hölder continuous with exponent* $\epsilon \in (0, 1)$ *in* $\mathcal{D}$ if the quantity

$$[f]_{\epsilon;\mathcal{D}} = \sup_{x,y\in\mathcal{D}, x\neq y} \frac{|f(x) - f(y)|}{|x - y|^\epsilon}$$

is finite; and *locally Hölder continuous with exponent* $\epsilon \in (0, 1)$ *in the domain* $\mathcal{D}$ if f is uniformly Hölder continuous with exponent $\epsilon \in (0, 1)$ on compact subsets of $\mathcal{D}$. We denote by $C^{0,\epsilon}(\mathcal{D})$ the set of all functions that are locally Hölder continuous with exponent $\epsilon \in (0, 1)$ in $\mathcal{D}$.

Now let us state the main result of this paper.

Theorem 2.3. *Assume that conditions* (A1) *and* (A2) *are satisfied with the functions g_0, g_1, g_2, g_3 belonging to $L^\tau(\Omega)$, $\tau > n/q$. Let $u \in W^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ be a generalized solution of equation* (2.1) *and* $\mathrm{M} = \|u\|_\infty$. *Then $u \in C^{0,\epsilon}(\Omega)$ and for any domain Ω' such that $\overline{\Omega'} \subset \Omega$, we have*

$$[u]_{\epsilon,\overline{\Omega'}} \le C$$

where $\epsilon = \epsilon(\mathtt{data})$ *and* $C = C(d, \mathtt{data})$ *are positive constants,* $d = \operatorname{dist}(\Omega', \partial\Omega)$ *and* $\mathtt{data} \equiv \big(n, p, q, \tau, |\Omega|, \mathrm{M}, a(\mathrm{M}), a_1(\mathrm{M}), a_2(\mathrm{M}), b(\mathrm{M}), \max_{0\le i\le 3} \|g_i\|_\tau\big)$.

Before proving Theorem 2.3, we give several auxiliary results and quote some examples where all the hypotheses are verified.

Example 2.4. Let b, λ_1, $\lambda_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous nondecreasing functions and let $\mu : \mathbb{R}_+ \to (0, +\infty)$ be a continuous nonincreasing function. For every n-dimensional multiindex α, $|\alpha| = 1, 2$, we define the functions $A_\alpha : \Omega \times \mathbb{R} \times \mathbb{R}^{n,2} \to \mathbb{R}$ and $B : \Omega \times \mathbb{R} \times \mathbb{R}^{n,2} \to \mathbb{R}$ as follows:

$$A_\alpha(x,s,\xi) = \begin{cases} \big(\mu(|s|) + \lambda_1(|s|)\big)\Big(\sum_{|\beta|=1} \xi_\beta^2\Big)^{(q-2)/2} \xi_\alpha, \\ \quad \text{if } (x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^{n,2},\ |\alpha| = 1, \\ \big(\mu(|s|) + \lambda_2(|s|)\big)\Big(\sum_{|\beta|=2} \xi_\beta^2\Big)^{(p-2)/2} \xi_\alpha, \\ \quad \text{if } (x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^{n,2},\ |\alpha| = 2, \end{cases}$$

$$B(x,s,\xi) = b(|s|)\Big\{\sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p\Big\}, \quad (x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^{n,2}.$$

These functions satisfy assumptions (A1) and (A2). Suppose that there exists a generalized solution $u_0 \in W^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ of the equation

$$\begin{aligned} &\sum_{|\alpha|=2} D^\alpha\Big[\big(\mu(|u|) + \lambda_2(|u|)\big)\Big(\sum_{|\beta|=2} |D^\beta u|^2\Big)^{(p-2)/2} D^\alpha u\Big] \\ &- \sum_{|\alpha|=1} D^\alpha\Big[\big(\mu(|u|) + \lambda_1(|u|)\big)\Big(\sum_{|\beta|=1} |D^\beta u|^2\Big)^{(q-2)/2} D^\alpha u\Big] \\ &+ b(|u|)\Big(\sum_{|\alpha|=2} |D^\alpha u|^p + \sum_{|\alpha|=1} |D^\alpha u|^q\Big) = 0 \quad \text{in } \Omega. \end{aligned}$$

Then applying Theorem 2.3 to the solution u_0, we obtain that $u_0 \in C^{0,\epsilon}(\Omega)$ with some ϵ depending only on n, p, q, $|\Omega|$, $\mathrm{M} = \|u_0\|_\infty$, $\mu(\mathrm{M})$, $\lambda_1(\mathrm{M})$, $\lambda_2(\mathrm{M})$ and $b(\mathrm{M})$.

Example 2.5. We consider the equation

$$\begin{aligned} &\sum_{|\alpha|=2} D^\alpha\Big[\Big(\sum_{|\beta|=2} |D^\beta u|^2\Big)^{(p-2)/2} D^\alpha u\Big] - \sum_{|\alpha|=1} D^\alpha\Big[\Big(\sum_{|\beta|=1} |D^\beta u|^2\Big)^{(q-2)/2} D^\alpha u\Big] \\ &+ u b_1(|u|)\Big(\sum_{|\alpha|=2} |D^\alpha u|^p + \sum_{|\alpha|=1} |D^\alpha u|^q\Big) = f(x) \quad \text{in } \Omega \end{aligned} \tag{2.8}$$

where $f \in L^\tau(\Omega)$, $\tau > n/q$ and $b_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nondecreasing function, for example, $b_1(s) = s^\kappa$, $\kappa > 0$, or $b_1(s) = e^s$.

The coefficients and the right-hand side of this equation satisfy the conditions (A1), (A2) and all the assumptions in [29, Theorem 2.1] (see Remark 2.2). Therefore, there exists a generalized solution $u_1 \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ of equation (2.8) such that $\|u_1\|_\infty \le C$, where C is a positive constant depending only on n, p, q, τ, $\|f\|_\tau$ and $|\Omega|$. By Theorem 2.3, we have that $u_1 \in C^{0,\epsilon}(\Omega)$ with some ϵ depending only on n, p, q, τ, $\|f\|_\tau$ and $|\Omega|$.

Example 2.6. We consider the equation

$$\begin{aligned}&\sum_{|\alpha|=2} D^\alpha\Big[\Big(\sum_{|\beta|=2}|D^\beta u|^2\Big)^{(p-2)/2} D^\alpha u\Big] - \sum_{|\alpha|=1} D^\alpha\Big[\Big(\sum_{|\beta|=1}|D^\beta u|^2\Big)^{(q-2)/2} D^\alpha u\Big]\\ &+ c|u|^{q-2}u + b_2(x)\Big(\sum_{|\alpha|=2}|D^\alpha u|^p + \sum_{|\alpha|=1}|D^\alpha u|^q\Big) = f(x) \quad \text{in } \Omega\end{aligned}$$

where $c > 0$, $b_2 \in L^\infty(\Omega)$, $f \in L^\tau(\Omega)$ with $\tau > n/q$.

By [30, Theorem 2.2] (see also Remark 2.2) there exists a generalized solution $u_2 \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ of this equation such that $\|u_2\|_\infty \le C$, where C is a positive constant depending only on n, p, q, c, τ, $\|f\|_\tau$, $\|b_2\|_\infty$ and $|\Omega|$. By Theorem 2.3, we have that $u_2 \in C^{0,\epsilon}(\Omega)$ with some ϵ depending only on n, p, q, c, τ, $\|f\|_\tau$, $\|b_2\|_\infty$ and $|\Omega|$.

3. Auxiliary results

The following is the well-known Sobolev inequality for functions in $\mathring{W}^{1,q}(\Omega)$; see for example [12, Theorem 7.10].

Lemma 3.1. *Set $q^* = nq/(n-q)$. Then $\mathring{W}^{1,q}(\Omega) \subset L^{q*}(\Omega)$. Furthermore, there exists a positive constant $c_{n,q}$ depending only on n and q such that, for every function $u \in \mathring{W}^{1,q}(\Omega)$,*

$$\Big(\int_\Omega |u|^{q^*}dx\Big)^{1/q^*} \le c_{n,q}\Big(\sum_{|\alpha|=1}\int_\Omega |D^\alpha u|^q dx\Big)^{1/q}. \tag{3.1}$$

We denote by $B_\rho(y) := \{x \in \mathbb{R}^n : |x-y| < \rho\}$ the open ball with center y and radius $\rho > 0$; when not important, or clear from the context, we shall omit denoting the center as follows: $B_\rho \equiv B_\rho(y)$.

Lemma 3.2. *Let $f \in W^{1,q}(B_\rho)$. Suppose there exists a measurable subset $G \subset B_\rho$ and positive constants C' and C'' such that*

$$|G| \ge C'\rho^n, \quad \max_G |f| \le C''.$$

Then

$$\int_{B_\rho} |f|^q dx \le C\rho^q\Big(\sum_{|\alpha|=1}\int_{B_\rho}|D^\alpha f|^q dx + \rho^{n-q}\Big)$$

where C is a positive constant depending only on n, q, C', C''.

The proof of the above lemma is given in [24, Chapter 1, §2, Lemma 4].

The following lemma is due to John and Nirenberg [15] (see also [12, Theorem 7.21]).

Lemma 3.3. *Let $f \in W^{1,1}(\mathcal{O})$ where $\mathcal{O}$ is a convex domain in $\mathbb{R}^n$. Suppose there exists a positive constant K such that*

$$\sum_{|\alpha|=1} \int_{\mathcal{O}\cap B_\rho} |D^\alpha f| dx \le K\rho^{n-1} \quad \text{for all balls } B_\rho.$$

Then there exist positive constants σ_0 and C depending only on n such that

$$\int_{\mathcal{O}} \exp\Big(\frac{\sigma}{K}|f-(f)_{\mathcal{O}}|\Big) dx \le C(\operatorname{diam}\mathcal{O})^n$$

where $\sigma = \sigma_0|\mathcal{O}|(\operatorname{diam}\mathcal{O})^{-n}$, $(f)_{\mathcal{O}} = \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} f dx$.

The following result is discussed in [12, Lemma 8.23].

Lemma 3.4. *Let ω be a non-decreasing function on an interval $(0, R_0]$ satisfying, for all $R \le R_0$, the inequality*

$$\omega(\vartheta R) \le \theta\omega(R) + \varphi(R)$$

where φ is also non-decreasing function and $0 < \vartheta, \theta < 1$. Then, for any $\delta \in (0, 1)$ and $R \le R_0$, we have

$$\omega(R) \le C\Big(\Big(\frac{R}{R_0}\Big)^\epsilon \omega(R_0) + \varphi(R^\delta R_0^{1-\delta})\Big)$$

where $C = C(\vartheta, \theta)$ and $\epsilon = \epsilon(\vartheta, \theta, \delta)$ are positive constants.

4. Proof of Theorem 2.3

Suppose that conditions (2.2)–(2.5) hold with the functions

$$g_0, g_1, g_2, g_3 \in L^\tau(\Omega), \quad \tau > n/q.$$

Let $u \in W^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ be a generalized solution of equation (2.1). We set $\mathrm{M} = \|u\|_\infty$, thus

$$|u| \le \mathrm{M} < +\infty \quad \text{on } \Omega. \tag{4.1}$$

By c_i, $i = 0, 1, \dots$, we shall denote positive constants depending only on

$$\texttt{data} \equiv \big(n,\ p,\ q,\ \tau,\ |\Omega|,\ \mathrm{M},\ a(\mathrm{M}), a_1(\mathrm{M}), a_2(\mathrm{M}), b(\mathrm{M}), \max_{0\le i\le 3} \|g_i\|_\tau\big).$$

Furthermore, let Ω' be an arbitrary open subset of Ω such that $\overline{\Omega'} \subset \Omega$ and $d = \operatorname{dist}(\Omega', \partial\Omega)$. We fix $x_0 \in \Omega'$. For every $R \in \big(0, \min\{1, d/4\}\big)$, we set

$$\mu(R) = \min_{B_R(x_0)} u, \quad M(R) = \max_{B_R(x_0)} u, \quad \omega(R) = M(R) - \mu(R).$$

Here the symbols min and max of course stands for *essential infimum and supremum.*

We fix a positive number r such that

$$r < \min\Big\{1 - \frac{n}{q\tau}, \frac{q-2p}{q-p}\Big\}. \tag{4.2}$$

For every $R \in \big(0, \min\{1, d/4\}\big)$, we shall establish the inequality

$$\omega(R) \le \theta\omega(2R) + R^r \tag{4.3}$$

with a constant $\theta \in (0, 1)$ depending only on `data`. This inequality and Lemma 3.4 imply the validity of Theorem 2.3.

To prove (4.3), we fix R such that $0 < R < \min\{1, d/4\}$ and set

$$G_1(R) = \Big\{x \in B_{3R/2}(x_0) : u(x) \le \frac{\mu(2R) + M(2R)}{2}\Big\},$$
$$G_2(R) = B_{3R/2}(x_0) \setminus G_1(R),$$

and define a function $v_0 : B_{2R}(x_0) \to \mathbb{R}$ as follows:

$$v_0(x) = \begin{cases} 1 + \ln \frac{2\omega(2R)}{M(2R) - u(x) + R^r} & \text{if } |G_1(R)| \ge \frac{|B_{3R/2}(x_0)|}{2}, \\ 1 + \ln \frac{2\omega(2R)}{u(x) - \mu(2R) + R^r} & \text{if } |G_2(R)| \ge \frac{|B_{3R/2}(x_0)|}{2}. \end{cases} \tag{4.4}$$

It is easy to see that (4.3), and hence Theorem 2.3, follows from the estimate

$$\|v_0\|_{L^\infty(B_R(x_0))} \le c_0, \tag{4.5}$$

For definiteness we assume that the function v_0 is defined by the first line in (4.4). We can also assume that

$$\omega(2R) \ge \frac{eR^r}{2}, \tag{4.6}$$

and therefore, $v_0 \ge 1$ a.e. in $B_{2R}(x_0)$, otherwise inequality (4.3) holds.

To derive inequality (4.5), we need some integral estimates of the solution u. We set

$$\Phi = \sum_{|\alpha|=1} |D^\alpha u|^q + \sum_{|\alpha|=2} |D^\alpha u|^p.$$

Lemma 4.1. *Let $B_\rho \subset \Omega$ and let $\zeta \in C_0^\infty(\Omega)$ be a function such that*

$$\zeta = 0 \ \text{ in } \Omega \setminus B_\rho \quad \text{and} \quad 0 \le \zeta \le 1. \tag{4.7}$$

Then there exists a positive constant c_1 such that

$$\int_{B_\rho} \Phi\, \zeta^q dx \le c_1 \rho^n \Big(1 + \rho^{-q} + \max_{B_\rho} \Big\{ \sum_{|\alpha|=1} |D^\alpha \zeta|^q + \sum_{|\alpha|=2} |D^\alpha \zeta|^p \Big\}\Big). \tag{4.8}$$

Proof. For every $x \in \Omega$ we set $v_1(x) = e^{\lambda u(x)} \zeta^q(x)$ where

$$\lambda = 2b(\mathrm{M})/a(\mathrm{M}). \tag{4.9}$$

Simple calculations show that $v_1 \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ and the following assertions hold:

(a) for every n-dimensional multi-index α, $|\alpha| = 1$,

$$D^\alpha v_1 = \lambda e^{\lambda u} \zeta^q D^\alpha u + q e^{\lambda u} \zeta^{q-1} D^\alpha \zeta \quad \text{a. e. in } \Omega,$$

(b) for every n-dimensional multi-index α, $|\alpha| = 2$,

$$\begin{aligned} &|D^\alpha v_1 - \lambda e^{\lambda u} \zeta^q D^\alpha u| \\ &\le \lambda^2 e^{\lambda u} \zeta^q \sum_{|\beta|=1} |D^\beta u|^2 + 2q\lambda e^{\lambda u} \zeta^{q-1} \Big\{ \sum_{|\beta|=1} |D^\beta u| \Big\} \Big\{ \sum_{|\beta|=1} |D^\beta \zeta| \Big\} \\ &+ q(q-1) e^{\lambda u} \zeta^{q-2} \sum_{|\beta|=1} |D^\beta \zeta|^2 + q e^{\lambda u} \zeta^{q-1} |D^\alpha \zeta| \quad \text{a.e. in } \Omega. \end{aligned}$$

Since $v_1 \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$, by (2.6), we have

$$\int_\Omega \Big\{ \sum_{\alpha\in\Lambda} A_\alpha(x,u,\nabla_2 u) D^\alpha v_1 + B(x,u,\nabla_2 u) v_1 \Big\} dx = 0.$$

From this equality, using (2.2), (2.5), (4.1), (4.7), (4.9) and assertions (a) and (b), we deduce that

$$b(\mathrm{M}) \int_{B_\rho} \Phi e^{\lambda u} \zeta^q dx \le I_1 + I_2 + I_3 + I_4 + e^{\lambda \mathrm{M}} \int_{B_\rho} (g_3 + \lambda g_0) dx \tag{4.10}$$

where

$$I_1 = q \sum_{\alpha\in\Lambda} \int_{B_\rho} |A_\alpha(x,u,\nabla_2 u)|\, |D^\alpha \zeta|\, e^{\lambda u} \zeta^{q-1} dx,$$

$$I_2 = \lambda^2 \sum_{|\alpha|=2} \sum_{|\beta|=1} \int_{B_\rho} |A_\alpha(x,u,\nabla_2 u)|\, |D^\beta u|^2\, e^{\lambda u} \zeta^q dx,$$

$$I_3 = q^2 \sum_{|\alpha|=2} \sum_{|\beta|=1} \int_{B_\rho} |A_\alpha(x,u,\nabla_2 u)|\, |D^\beta \zeta|^2\, e^{\lambda u} \zeta^{q-2} dx,$$

$$I_4 = 2\lambda q \sum_{|\alpha|=2} \sum_{|\beta|=1} \sum_{|\gamma|=1} \int_{B_\rho} |A_\alpha(x,u,\nabla_2 u)|\, |D^\beta u|\, |D^\gamma \zeta|\, e^{\lambda u} \zeta^{q-1} dx.$$

Let us obtain suitable estimates for the addends in the right-hand side of (4.10).

Estimate for I_1. Using the Young's inequality with the exponents $q/(q-1)$ and q, (2.3), (4.1) and (4.7), we obtain

$$\begin{aligned}
&q \sum_{|\alpha|=1} \int_{B_\rho} |A_\alpha(x,u,\nabla_2 u)|\, |D^\alpha \zeta|\, e^{\lambda u} \zeta^{q-1} dx \\
&\le \frac{b(\mathrm{M})}{16} \int_{B_\rho} \Phi e^{\lambda u} \zeta^q dx + c_2 \int_{B_\rho} g_1 dx + c_2 \rho^n \max_{B_\rho} \sum_{|\alpha|=1} |D^\alpha \zeta|^q.
\end{aligned}$$

Using the Young's inequality with the exponents $p/(p-1)$ and p, (2.4), (4.1) and (4.7), we obtain

$$\begin{aligned}
&q \sum_{|\alpha|=2} \int_{B_\rho} |A_\alpha(x,u,\nabla_2 u)|\, |D^\alpha \zeta|\, e^{\lambda u} \zeta^{q-1} dx \\
&\le \frac{b(\mathrm{M})}{16} \int_{B_\rho} \Phi e^{\lambda u} \zeta^q dx + c_3 \int_{B_\rho} g_2 dx + c_3 \rho^n \max_{B_\rho} \sum_{|\alpha|=2} |D^\alpha \zeta|^p.
\end{aligned}$$

We set

$$\Phi_\zeta = \sum_{|\alpha|=1} |D^\alpha \zeta|^q + \sum_{|\alpha|=2} |D^\alpha \zeta|^p.$$

From the last two inequalities it follows that

$$I_1 \le \frac{b(\mathrm{M})}{8} \int_{B_\rho} \Phi e^{\lambda u} \zeta^q dx + c_4 \int_{B_\rho} (g_1 + g_2) dx + c_4 \rho^n \max_{B_\rho} \Phi_\zeta. \tag{4.11}$$

Estimates for I_2, I_3 and I_4. It is obvious that

$$\frac{p-1}{p}+\frac{2}{q}+\frac{q-2p}{qp}=1,\quad q-1=(p-1)\frac{q}{p}+(\frac{q}{p}-1). \tag{4.12}$$

Using this equalities, the Young's inequality, (2.4), (4.1) and (4.7), we obtain

$$I_2\le\frac{b(\mathrm{M})}{8}\int_{B_\rho}\Phi e^{\lambda u}\zeta^q dx+c_5\int_{B_\rho}g_2dx+c_5\rho^n, \tag{4.13}$$

$$I_3\le\frac{b(\mathrm{M})}{8}\int_{B_\rho}\Phi e^{\lambda u}\zeta^q dx+c_6\int_{B_\rho}g_2dx+c_6\rho^n\max_{B_\rho}\Phi_\zeta+c_6\rho^n, \tag{4.14}$$

$$I_4\le\frac{b(\mathrm{M})}{8}\int_{B_\rho}\Phi e^{\lambda u}\zeta^q dx+c_7\int_{B_\rho}g_2dx+c_7\rho^n\max_{B_\rho}\Phi_\zeta+c_7\rho^n, \tag{4.15}$$

From (4.10), (4.11), (4.13)–(4.15) it follows that

$$\frac{b(\mathrm{M})}{2}\int_{B_\rho}\Phi e^{\lambda u}\zeta^q dx\le c_8\Big(\int_{B_\rho}gdx+\rho^n\max_{B_\rho}\Phi_\zeta+\rho^n\Big)$$

where $g=g_0+g_1+g_2+g_3$. By Hölder's inequality and the inequality $\tau>n/q$ we have

$$\int_{B_\rho}gdx\le\|g\|_\tau|B_\rho|^{(\tau-1)/\tau}\le c_9\rho^{n-q}.$$

The last two inequalities and (4.1) imply inequality (4.8). The proof is complete. $\square$

Lemma 4.2. *Let $B_\rho\subset B_{2R}(x_0)$ and let $\zeta\in C_0^\infty(\Omega)$ be a function such that condition (4.7) be satisfied. Then there exists a positive constant c_{10} such that*

$$\begin{aligned}&\int_{B_\rho}\frac{\Phi\,\zeta^q dx}{(M(2R)-u+R^r)^q}\\&\le c_{10}\rho^n\Big(\rho^{-q}+\max_{B_\rho}\Big\{\sum_{|\alpha|=1}|D^\alpha\zeta|^q+\sum_{|\alpha|=2}|D^\alpha\zeta|^p\Big\}\\&\quad+\rho^{2p-q}\max_{B_\rho}\sum_{|\alpha|=2}|D^\alpha\zeta|^p+\rho^{-q(q-2p)/(q-p)}\max_{B_\rho}\sum_{|\alpha|=1}|D^\alpha\zeta|^{qp/(q-p)}\Big).\end{aligned} \tag{4.16}$$

Proof. For every $x\in B_{2R}(x_0)$, we set $U(x)=M(2R)-u(x)+R^r$,

$$v_2(x)=\begin{cases}\zeta^q(x)[U(x)]^{1-q} & \text{if } x\in B_{2R}(x_0),\\ 0 & \text{if } x\in\Omega\setminus B_{2R}(x_0).\end{cases}$$

Simple calculations show that

$$v_2\in\mathring{W}^{1,q}_{2,p}(\Omega)\cap L^\infty(\Omega)$$

and the following assertions hold:

(c) for every n-dimensional multi-index α, $|\alpha|=1$,

$$D^\alpha v_2=qU^{1-q}\zeta^{q-1}D^\alpha\zeta+(q-1)U^{-q}\zeta^qD^\alpha u\quad\text{a. e. in } B_{2R}(x_0),$$

(d) for every n-dimensional multi-index α, $|\alpha|=2$,

$$\begin{aligned}&\Big|D^\alpha v_2-(q-1)U^{-q}\zeta^qD^\alpha u\Big|\\&\le q\,U^{1-q}\zeta^{q-1}|D^\alpha\zeta|+q(q-1)\,U^{1-q}\zeta^{q-2}\sum_{|\beta|=1}|D^\beta\zeta|^2\end{aligned}$$

$$
\begin{aligned}
&+ 2q(q-1)\, U^{-q}\zeta^{q-1}\Big\{\sum_{|\beta|=1}|D^\beta u|\Big\}\Big\{\sum_{|\beta|=1}|D^\beta \zeta|\Big\}\\
&+ q(q-1)\, U^{-1-q}\zeta^{q}\sum_{|\beta|=1}|D^\beta u|^2 \quad \text{a.e. in } B_{2R}(x_0).
\end{aligned}
$$

Putting the function v_2 into (2.6) instead of v and using (2.2), (2.5), (4.1), (4.7) and assertions (c) and (d), we obtain

$$
\begin{aligned}
&a(\mathrm{M})\int_{B_\rho}\Phi\, U^{-q}\zeta^q dx\\
&\le I_1'+I_2'+I_3'+I_4'+I_5'+\int_{B_\rho}(g_0+(2\mathrm{M}+1)g_3)U^{-q}dx,
\end{aligned}
\tag{4.17}
$$

where

$$
\begin{gathered}
I_1' = q\sum_{\alpha\in\Lambda}\int_{B_\rho}|A_\alpha(x,u,\nabla_2 u)|\,|D^\alpha\zeta|\,U^{1-q}\zeta^{q-1}dx,\\
I_2' = q\sum_{|\alpha|=2}\sum_{|\beta|=1}\int_{B_\rho}|A_\alpha(x,u,\nabla_2 u)|\,|D^\beta u|^2 U^{-1-q}\zeta^{q}dx,\\
I_3' = q\sum_{|\alpha|=2}\sum_{|\beta|=1}\int_{B_\rho}|A_\alpha(x,u,\nabla_2 u)|\,|D^\beta \zeta|^2 U^{1-q}\zeta^{q-2}dx,\\
I_4' = 2q\sum_{|\alpha|=2}\sum_{|\beta|=1}\sum_{|\gamma|=1}\int_{B_\rho}|A_\alpha(x,u,\nabla_2 u)|\,|D^\beta u|\,|D^\gamma\zeta| U^{-q}\zeta^{q-1}dx,\\
I_5' = b(\mathrm{M})\int_{B_\rho}\Phi\, U^{1-q}\zeta^q dx.
\end{gathered}
$$

Next we obtain suitable estimates for I_1', I_2', I_3', I_4', I_5'.

Estimate for I_1'. Using the Young's inequality with the exponents $q/(q-1)$ and q, (2.3), (4.1) and (4.7), we obtain

$$
\begin{aligned}
&q\sum_{|\alpha|=1}\int_{B_\rho}|A_\alpha(x,u,\nabla_2 u)|\,|D^\alpha\zeta|\,U^{1-q}\zeta^{q-1}dx\\
&\le \frac{a(\mathrm{M})}{20}\int_{B_\rho}\Phi\, U^{-q}\zeta^q dx + c_{11}\int_{B_\rho}g_1 U^{-q}dx + c_{11}\rho^n\max_{B_\rho}\sum_{|\alpha|=1}|D^\alpha\zeta|^q.
\end{aligned}
\tag{4.18}
$$

We use the Young's inequality with the exponents $p/(p-1)$ and p, (2.4), (4.1) and (4.7) to obtain

$$
\begin{aligned}
&q\sum_{|\alpha|=2}\int_{B_\rho}|A_\alpha(x,u,\nabla_2 u)|\,|D^\alpha\zeta|\,U^{1-q}\zeta^{q-1}dx\\
&\le \frac{a(\mathrm{M})}{20}\int_{B_\rho}\Phi\, U^{-q}\zeta^q dx + c_{12}\int_{B_\rho}g_2 U^{-q}dx + c_{12}\sum_{|\alpha|=2}\int_{B_\rho}|D^\alpha\zeta|^p U^{p-q}\zeta^{q-p}dx,
\end{aligned}
$$

whence, taking into account the inequalities $U \geq R^r$, $\rho/2 < R < 1$ and (4.2), we derive

$$\begin{aligned} &q \sum_{|\alpha|=2} \int_{B_\rho} |A_\alpha(x,u,\nabla_2 u)|\,|D^\alpha \zeta| U^{1-q}\zeta^{q-1} dx \\ &\leq \frac{a(\mathrm{M})}{20} \int_{B_\rho} \Phi\, U^{-q}\zeta^q dx + c_{12} \int_{B_\rho} g_2 U^{-q} dx + c_{13}\rho^{n-q+2p} \max_{B_\rho} \sum_{|\alpha|=2} |D^\alpha\zeta|^p. \end{aligned} \tag{4.19}$$

Summing inequalities (4.18) and (4.19), we obtain

$$\begin{aligned} I_1' \leq{}& \frac{a(\mathrm{M})}{10} \int_{B_\rho} \Phi U^{-q}\zeta^q dx + c_{14} \int_{B_\rho} (g_1+g_2) U^{-q} dx \\ &+ c_{14}\rho^n \max_{B_\rho} \sum_{|\alpha|=1} |D^\alpha\zeta|^q + c_{14}\rho^{n-q+2p} \max_{B_\rho} \sum_{|\alpha|=2} |D^\alpha\zeta|^p. \end{aligned} \tag{4.20}$$

Estimate for I_2'. We use (4.1), the first equality in (4.12), Young's inequality, (2.4) and (4.7) to obtain

$$I_2' \leq \frac{a(\mathrm{M})}{10} \int_{B_\rho} \Phi\, U^{-q}\zeta^q dx + c_{15} \int_{B_\rho} g_2 U^{-q} dx + c_{15} \int_{B_\rho} U^{-q(q-p)/(q-2p)}\zeta^q dx.$$

Estimating the last integral in this inequality by means of the inequalities $U \geq R^r$, $\rho/2 < R < 1$ and (4.2), we obtain

$$I_2' \leq \frac{a(\mathrm{M})}{10} \int_{B_\rho} \Phi\, U^{-q}\zeta^q dx + c_{15} \int_{B_\rho} g_2 U^{-q} dx + c_{16}\rho^{n-q}. \tag{4.21}$$

Estimates for I_3' and I_4'. Using the reasoning similar the proof of (4.21), we obtain

$$\begin{aligned} I_3' \leq{}& \frac{a(\mathrm{M})}{10} \int_{B_\rho} \Phi\, U^{-q}\zeta^q dx + c_{17} \int_{B_\rho} g_2 U^{-q} dx \\ &+ c_{17}\rho^n \max_{B_\rho} \sum_{|\alpha|=1} |D^\alpha\zeta|^q + c_{17}\rho^{n-q}, \end{aligned} \tag{4.22}$$

$$\begin{aligned} I_4' \leq{}& \frac{a(\mathrm{M})}{10} \int_{B_\rho} \Phi\, U^{-q}\zeta^q dx + c_{18} \int_{B_\rho} g_2 U^{-q} dx \\ &+ c_{18}\rho^{n-q(q-2p)/(q-p)} \max_{B_\rho} \sum_{|\alpha|=1} |D^\alpha\zeta|^{qp/(q-p)}. \end{aligned} \tag{4.23}$$

Estimate for I_5'. We use Young's inequality and Lemma 4.1, to obtain

$$I_5' \leq \frac{a(\mathrm{M})}{10} \int_{B_\rho} \Phi\, U^{-q}\zeta^q dx + c_{19}\rho^n \Big(\rho^{-q} + \max_{B_\rho} \Big\{ \sum_{|\alpha|=1} |D^\alpha\zeta|^q + \sum_{|\alpha|=2} |D^\alpha\zeta|^p \Big\}\Big).$$

Collecting (4.17), (4.20)–(4.23) and the above inequality, we obtain

$$\begin{aligned} &\frac{a(\mathrm{M})}{2} \int_{B_\rho} \Phi\, U^{-q}\zeta^q dx \\ &\leq c_{20}\rho^{n-q} + c_{20}\rho^n \max_{B_\rho} \Big\{ \sum_{|\alpha|=1} |D^\alpha\zeta|^q + \sum_{|\alpha|=2} |D^\alpha\zeta|^p \Big\} \end{aligned}$$

$$
\begin{aligned}
&+ c_{20}\rho^{n-q+2p} \max_{B_\rho} \sum_{|\alpha|=2} |D^\alpha \zeta|^p + c_{20}\rho^{n-q(q-2p)/(q-p)} \max_{B_\rho} \sum_{|\alpha|=1} |D^\alpha \zeta|^{qp/(q-p)} \\
&+ c_{20} \int_{B_\rho} gU^{-q} dx
\end{aligned}
$$

where $g = g_0 + g_1 + g_2 + g_3$. Finally, to obtain (4.16), we estimate the last integral in this inequality by means of Holder's inequality and the relations (4.2), $U \geq R^r$ and $\rho/2 < R < 1$. The proof is complete. □

Lemma 4.3. *For every $\kappa \geq 1$ there is a positive constant $c = c(\mathtt{data}, \kappa)$ such that $\lim_{k\to+\infty} c(\mathtt{data}, \kappa) = +\infty$ and*

$$
\int_{B_{3R/2}(x_0)} v_0^\kappa dx \leq cR^n. \tag{4.24}
$$

Proof. First, we estimate from above the average integral

$$
(v_0)_{B_{3R/2}(x_0)} = \frac{1}{|B_{3R/2}(x_0)|} \int_{B_{3R/2}(x_0)} v_0 dx
$$

by a constant depending only on $\mathtt{data}$. For this purpose we choose a function $\zeta_1 \in C_0^\infty(\Omega)$ such that

$$
\begin{gathered}
0 \leq \zeta_1 \leq 1 \text{ in } \Omega, \quad \zeta_1 = 1 \text{ in } B_{3R/2}(x_0), \quad \zeta_1 = 0 \text{ in } \Omega \setminus B_{2R}(x_0), \\
|D^\alpha \zeta_1| \leq K_1 R^{-|\alpha|} \quad \text{for } |\alpha| = 1, 2,
\end{gathered}
$$

where K_1 is an absolute constant, not depending on R. Taking into account the facts that $1 \leq v_0 \leq 1 + \ln 4$ on $G_1(R)$ and $|G_1(R)| \geq |B_{3R/2}(x_0)|/2$, and using Holder's inequality, Lemmas 3.2 and 4.2 and the properties of the function ζ_1, we obtain

$$
\begin{aligned}
(v_0)_{B_{3R/2}(x_0)} &\leq c_{21} R^{-n/q} \Big(\int_{B_{3R/2}(x_0)} v_0^q dx \Big)^{1/q} \\
&\leq c_{22} R^{1-n/q} \Big(\int_{B_{3R/2}(x_0)} \frac{\Phi \zeta_1^q dx}{(M(2R) - u + R^r)^q} + R^{n-q} \Big)^{1/q} \\
&\leq c_{23}.
\end{aligned} \tag{4.25}
$$

Next, let $B_{2\rho} \subset B_{2R}(x_0)$, and let $\zeta_2 \in C_0^\infty(\Omega)$ be a function such that

$$
\begin{gathered}
0 \leq \zeta_2 \leq 1 \text{ in } \Omega, \quad \zeta_2 = 1 \text{ in } B_\rho, \quad \zeta_2 = 0 \text{ in } \Omega \setminus B_{2\rho}, \\
|D^\alpha \zeta_2| \leq K_2 \rho^{-|\alpha|} \quad \text{for } |\alpha| = 1, 2,
\end{gathered}
$$

where K_2 is an absolute constant, not depending on ρ. Using Holder's inequality, Lemma 4.2 and the properties of the function ζ_2, we derive that

$$
\begin{aligned}
\sum_{|\alpha|=1} \int_{B_\rho} |D^\alpha v_0| dx &\leq c_{24} \rho^{n-n/q} \Big(\sum_{|\alpha|=1} \int_{B_\rho} |D^\alpha v_0|^q dx \Big)^{1/q} \\
&\leq c_{24} \rho^{n-n/q} \Big(\int_{B_{2\rho}} \frac{\Phi\, \zeta_2^q dx}{(M(2R) - u + R^r)^q} \Big)^{1/q} \\
&\leq c_{25} \rho^{n-1}.
\end{aligned}
$$

Hence, by Lemma 3.3, we have

$$
\int_{B_{3R/2}(x_0)} \exp\Big(c_{26} |\, v_0 - (v_0)_{B_{3R/2}(x_0)} | \Big) dx \leq c_{27} R^n. \tag{4.26}
$$

Now let $\kappa \geq 1$. Then inequalities (4.25) and (4.26) imply (4.24). The proof is complete. $\square$

Now we are ready to prove inequality (4.5).

Lemma 4.4. *There is a positive constant c_0 such that inequality (4.5) holds.*

Proof. We proceed the proof in four steps.

Step 1. We fix a function $\psi_0 \in C_0^\infty(\mathbb{R})$ such that

$$0 \leq \psi_0 \leq 1 \text{ on } \mathbb{R}, \quad \psi_0 = 1 \text{ in } [-1,1], \quad \psi_0 = 0 \text{ in } \mathbb{R} \setminus (-3/2, 3/2).$$

For any $x \in \Omega$ we set $\psi(x) = \psi_0\big(\frac{|x-x_0|}{R}\big)$,

$$\tilde{v}(x) = \begin{cases} [v_0(x)]^k \psi^t(x) [U(x)]^{1-q} & \text{if } x \in B_{2R}(x_0), \\ 0 & \text{if } x \in \Omega \setminus B_{2R}(x_0), \end{cases}$$

where $U = M(2R) - u + R^r$,

$$k \geq \overline{k} := \max\{q, (4\mathrm{M}+1) b(\mathrm{M})/a(\mathrm{M})\}, \tag{4.27}$$

$$\nu := \max\{q, 2qp/(q-2p)\} < t \leq C_0 k, \tag{4.28}$$

and $C_0 = C_0(n,p,q,\tau) > 1$ is a constant that will be specified below. Simple calculations show that

$$\tilde{v} \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$$

and the following assertions hold:

(a1) for every $\alpha \in \Lambda$, $|\alpha| = 1$,

$$|D^\alpha \tilde{v} - (q-1) v_0^k \psi^t U^{-q} D^\alpha u - k v_0^{k-1} \psi^t U^{-q} D^\alpha u| \leq \frac{c_{28} k v_0^k \psi^{t-1}}{R\, U^{q-1}} \quad \text{a.e. in } \Omega,$$

(a2) for every $\alpha \in \Lambda$, $|\alpha| = 2$,

$$\begin{aligned} &|D^\alpha \tilde{v} - (q-1) v_0^k \psi^t U^{-q} D^\alpha u - k v_0^{k-1} \psi^t U^{-q} D^\alpha u| \\ &\leq \frac{c_{28} k^2 v_0^k \psi^{t-2}}{U^{q-1}} \Big\{ \frac{1}{R^2} + \sum_{|\beta|=1} \frac{|D^\beta u|^2}{U^2} \Big\} \quad \text{a.e. in } \Omega, \end{aligned}$$

where $c_{28} > 0$ depends only on C_0, $\max_{\mathbb{R}} |\psi_0'|$ and $\max_{\mathbb{R}} |\psi_0''|$.

Putting the function $\tilde{v}$ in (2.6) instead of v and using (2.2), (2.5), and assertions (a1) and (a2), we obtain

$$\begin{aligned} &(q-1) a(\mathrm{M}) \int_{B_{2R}(x_0)} \Phi\, U^{-q} v_0^k \psi^t dx + k a(\mathrm{M}) \int_{B_{2R}(x_0)} \Phi\, U^{-q} v_0^{k-1} \psi^t dx \\ &\leq b(\mathrm{M}) \int_{B_{2R}(x_0)} \Phi\, U^{1-q} v_0^k \psi^t dx + \int_{B_{2R}(x_0)} k g_4 v_0^k \psi^t U^{-q} dx + \mathcal{I}_1 + \mathcal{I}_2, \end{aligned} \tag{4.29}$$

where $g_4 = 2g_0 + (2M+1) g_3$,

$$\mathcal{I}_1 = \frac{c_{28} k}{R} \sum_{|\alpha|=1} \int_{B_{2R}(x_0)} |A_\alpha(x,u,\nabla_2 u)| U^{1-q} v_0^k \psi^{t-1} dx, \tag{4.30}$$

$$\mathcal{I}_2 = c_{28} k^2 \sum_{|\alpha|=2} \int_{B_{2R}(x_0)} |A_\alpha(x,u,\nabla_2 u)| U^{1-q} v_0^k \psi^{t-2} \Big\{ \frac{1}{R^2} + \sum_{|\beta|=1} \frac{|D^\beta u|^2}{U^2} \Big\} dx. \tag{4.31}$$

Step 2. We show that the first term in the right-hand side of inequality (4.29) is absorbed by the second term in its left-hand side. For this we need the inequality

$$Uv_0 \le 4\mathrm{M}+1 \quad \text{a.e. in } B_{2R}(x_0). \tag{4.32}$$

To prove it, we consider the function

$$\chi(s) = (s+R^r)\ln\frac{2\omega(2R)}{s+R^r}, \quad s\in[0,\omega(2R)].$$

By (4.6), we have that $\chi \ge 0$ in $[0,\omega(2R)]$ and $\hat{s} := 2e^{-1}\omega(2R) - R^r \in [0,\omega(2R)]$. By standard techniques of differential calculus we obtain

$$\max_{s\in[0,\,\omega(2R)]}\chi(s) = \chi(\hat{s}) = 2e^{-1}\omega(2R) \le 2\mathrm{M}.$$

Now inequality (4.32) follows from the relations $R\le 1$ and

$$Uv_0 = M(2R) - u + R^r + \chi(M(2R)-u) \le 4\mathrm{M}+1 \quad \text{a.e. in } B_{2R}(x_0).$$

Using (4.32), the first term on the right-hand side of inequality (4.29) is estimated in the following way

$$b(\mathrm{M})\int_{B_{2R}(x_0)}\Phi\, U^{1-q}v_0^k\psi^t dx \le (4\mathrm{M}+1)b(\mathrm{M})\int_{B_{2R}(x_0)}\Phi U^{-q}v_0^{k-1}\psi^t dx. \tag{4.33}$$

Now (4.27), (4.29) and (4.33) imply the inequality

$$(q-1)a(\mathrm{M})\int_{B_{2R}(x_0)}\Phi\, U^{-q}v_0^k\psi^t dx \le k\int_{B_{2R}(x_0)} g_4 v_0^k\psi^t U^{-q}dx + \mathcal{I}_1 + \mathcal{I}_2\,. \tag{4.34}$$

Step 3. Let us estimate from above the quantities $\mathcal{I}_1$ and $\mathcal{I}_2$, which are defined by (4.30) and (4.31) respectively. We use (2.3) and the Young's inequality

$$|yz| \le \varepsilon|y|^{q/(q-1)} + \varepsilon^{1-q}|z|^q,$$

where

$$y = |A_\alpha(x,u,\nabla_2 u)|U^{1-q}\psi^{(q-1)t/q}, \quad |\alpha|=1, \quad z = k\,\psi^{(t-q)/q}/R,$$

and ε is an appropriate positive number, to obtain

$$\begin{aligned}\mathcal{I}_1 \le{}& \frac{(q-1)a(\mathrm{M})}{4}\int_{B_{2R}(x_0)}\Phi U^{-q}v_0^k\psi^t dx \\ &+ c_{29}\int_{B_{2R}(x_0)} g_1 v_0^k\psi^t U^{-q}dx + \frac{c_{29}k^q}{R^q}\int_{B_{2R}(x_0)} v_0^k\psi^{t-q}dx,\end{aligned} \tag{4.35}$$

where $c_{29}>0$ depends only on c_{28}, q, $a(\mathrm{M})$ and $a_1(\mathrm{M})$.

Using the first equality in (4.12), Young's inequality and (4.2), we establish that if $\varepsilon>0$, α, $\beta\in\Lambda$, $|\alpha|=2$ and $|\beta|=1$, then

$$\begin{aligned}&|A_\alpha(x,u,\nabla_2 u)|U^{1-q}\psi^{t-2}\cdot k^2R^{-2} \\ &\le \varepsilon|A_\alpha(x,u,\nabla_2 u)|^{p/(p-1)}U^{-q}\psi^t + \varepsilon(1+\varepsilon^{-qp/(q-2p)})k^qR^{-q}\psi^{t-2qp/(q-2p)} \quad \text{on } \Omega,\end{aligned} \tag{4.36}$$

$$\begin{aligned}&|A_\alpha(x,u,\nabla_2 u)|U^{1-q}\psi^{t-2}\cdot k^2|D^\beta u|^2U^{-2} \\ &\le \varepsilon|A_\alpha(x,u,\nabla_2 u)|^{p/(p-1)}U^{-q}\psi^t + \varepsilon|D^\beta u|^qU^{-q}\psi^t \\ &\quad + \varepsilon^{1-qp/(q-2p)}k^{2qp/(q-2p)}R^{-q}\psi^{t-2qp/(q-2p)} \quad \text{on } \Omega.\end{aligned} \tag{4.37}$$

From (2.4), (4.31), (4.36), (4.37), taking into account (4.28) and the suitable choice of ε, we deduce the estimate

$$\begin{aligned}\mathcal{I}_2 \le \frac{(q-1)a(\mathrm{M})}{4}\int_{B_{2R}(x_0)} \Phi U^{-q} v_0^k \psi^t dx \\ + c_{30}\int_{B_{2R}(x_0)} g_2 v_0^k \psi^t U^{-q} dx + \frac{c_{30}k^\nu}{R^q}\int_{B_{2R}(x_0)} v_0^k \psi^{t-\nu} dx,\end{aligned} \tag{4.38}$$

where $c_{30} > 0$ depends only on c_{28}, q, p, $a(\mathrm{M})$ and $a_2(\mathrm{M})$.

From (4.34), (4.35), (4.38), (4.28) and (4.2) it follows that

$$\begin{aligned}&\int_{B_{2R}(x_0)} \Phi\, U^{-q} v_0^k \psi^t dx \\ &\le \frac{c_{31}k^\nu}{R^q}\int_{B_{2R}(x_0)} v_0^k \psi^{t-\nu} dx + \frac{c_{31}k^\nu}{R^{q-n/\tau}}\int_{B_{2R}(x_0)} (g_1+g_2+g_4) v_0^k \psi^{t-\nu} dx.\end{aligned}$$

Estimating the last two integrals by Hölder's inequality with the exponents τ and $\tau/(\tau-1)$ and taking into account (4.27) and (4.28), we obtain that for every $k \ge \overline{k}$ and $t \in (\nu, C_0 k]$ the following inequality holds (see also [25, inequality (39)]):

$$\int_{B_{2R}(x_0)} \Phi\, U^{-q} v_0^k \psi^t dx \le \frac{c_{32}k^\nu}{R^{q-n/\tau}}\Big(\int_{B_{2R}(x_0)} (v_0^k \psi^{t-\nu})^{\tau/(\tau-1)} dx\Big)^{(\tau-1)/\tau}. \tag{4.39}$$

Step 4. We set

$$J(k,t) = \frac{1}{R^n}\int_{B_{2R}(x_0)} v_0^k \psi^t dx, \quad k \in \mathbb{R},\ t > 0,$$
$$\theta = \frac{\tau}{\tau-1}\cdot\frac{q}{q^*}, \quad \tilde{\nu} = \frac{(q+\nu)q^*}{q}.$$

The following assertion holds: if $k \ge \overline{k}q^*/q$ and $\tilde{\nu} < t \le C_0 k$, then

$$J(k,t) \le c_{33}k^{\tilde{\nu}}[J(k\theta, t\theta - \tilde{\nu})]^{1/\theta}. \tag{4.40}$$

Let $k \ge \overline{k}q^*/q$ and $\tilde{\nu} < t \le C_0 k$. Then, applying inequality (3.1) to the function $v_0^{k/q^*}\psi^{t/q^*}$, we obtain

$$\begin{aligned}J(k,t) \le \frac{c_{34}k^{q^*}}{R^n}\Big(\int_{B_{2R}(x_0)} \Phi\, U^{-q} v_0^{kq/q^*}\psi^{tq/q^*-q} dx \\ + \frac{1}{R^q}\int_{B_{2R}(x_0)} v_0^{kq/q^*}\psi^{tq/q^*-q} dx\Big)^{q^*/q}.\end{aligned}$$

From this inequality, estimating the first addend in the brackets by means of (4.39) and the second addend by means of Hölder's inequality, we deduce (4.40).

We choose a number $i_0 \in \mathbb{N}$ such that $\theta^{-i_0} > \overline{k}q^*/q$ and set $C_0 = \tilde{\nu}/(1-\theta)$,

$$k_i = \theta^{-i_0-i}, \quad t_i = \frac{\tilde{\nu}(\theta^{-i_0-i}-1)}{1-\theta}, \quad J_i = J(k_i, t_i), \quad i = 0,1,2,\dots.$$

Then (4.40) and the inequality $\theta < 1$ imply that for every $i = 0,1,2,\dots$,

$$J_i^{1/k_i} \le c_{35} J_0^{\theta^{i_0}}.$$

By Lemma 4.3 we have $J_0^{\theta^{i_0}} \leq c_{36}$. From the last two inequalities it follows that

$$\|v_0\|_{L^\infty(B_R(x_0))} = \lim_{i\to\infty} \Big(\frac{1}{R^n}\int_{B_R(x_0)} v_0^{k_i}\,dx\Big)^{1/k_i} \leq \limsup_{i\to\infty} J_i^{1/k_i} \leq c_0.$$

The proof is complete. □

Inequality (4.5) implies (4.3), and hence according to Lemma 3.4, the assertions of Theorem 2.3 hold.

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